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Elastic modelling of surface waves in single and multicomponent systems – Lecture notes

CISM Course: Surface waves in geomechanics, Udine, September 2004

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Abstract

The main aim of this article is to present a review of most important acoustic surface waves which are described by linear one- and two-component models. It has been written for the CISM-course: "*Surface waves in Geomechanics*" (Udine, September 6-10, 2004). Among the waves in one-component linear elastic media we present the classical Rayleigh waves on a plane boundary, Rayleigh waves on a cylindrical surface, Love waves, Stoneley waves (solid/solid and fluid/solid interface). In the second part of the article we discuss two two-component models of porous materials (Biot's model and a simple mixture model). We indicate basic differences of the models and demonstrate qualitative similarities. We introduce as well some fundamental notions yielding the description of surface waves in two-component systems (saturated porous materials) and review certain (porous materials with impermeable boundaries) asymptotic results for such waves. However, the full discussion of this subject including numerous results of computer calculations can be found in the article of B. Albers [3] also included in this volume.

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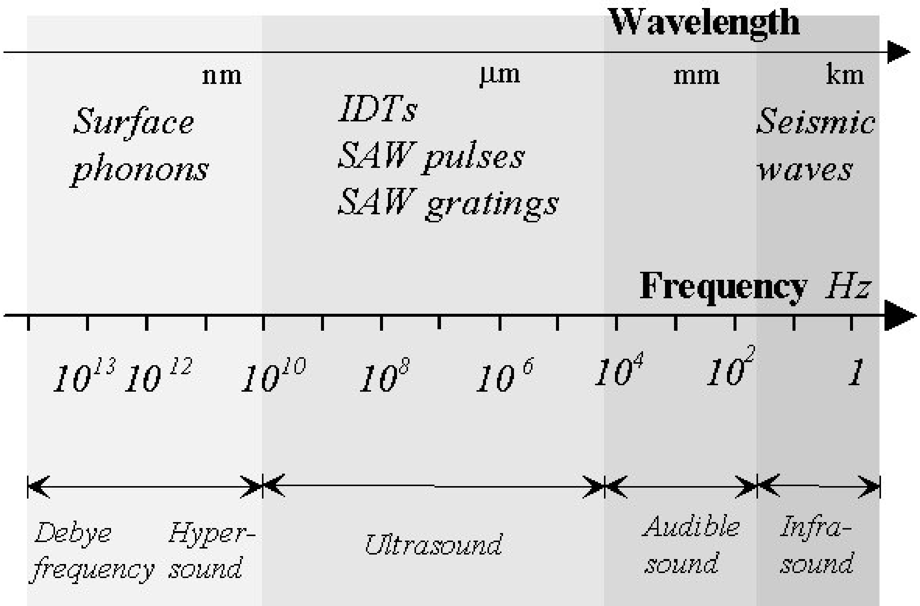
1 Introduction

Surface waves exist in an extremely wide range of frequencies over some 10 orders of magnitude. This is schematically shown in the diagram below [32].



Acoustic surface waves (SAWs) were discovered in 1885 by Rayleigh [53] and they were the main subject of studies for some decades due to their appearance in the form of seismic waves. The development of interdigital transducers (IDTs) and focused short laser pulses around 1960 extended their study and applications from the low frequency range to the ultrasound. Recent research work goes beyond this frequency range even to the hypersound region corresponding to quantized lattice vibrations (surface phonons).

Box 1: Rayleigh, *John William Strutt, 3rd Baron*

Born: Terling, Essex, November 12, 1842,
Died: Witham, Essex, June 30, 1919

"Strutt at the age of thirty-one inherited his father's title, so that he is almost invariably referred to as Lord Rayleigh" (I. Asimov; *Asimov's Biographical Encyclopedia of Science and Technology*, Doubleday & Co., Inc., Garden City, N.Y., 1964). In 1873 he was elected to the Royal Society, in 1879 suceeded Maxwell as director of the Cavendish Laboratory at Cambridge. All his life he was interested in wave motion of all varieties: electromagnetic waves, black-body radiation, sound waves, water waves. Important discoveries in chemistry but the Nobel Prize in the year 1904 in physics. Since 1905 he was the president of the Royal Society, and in 1908 he became chancellor of Cambridge University.

This article is devoted to the presentation of models of basic types of surface waves in homogeneous elastic materials and in two-component poroelastic materials. As the main field of application of these models is assumed to be geomechanics [39], we concentrate on models appropriate for the frequency regime of ultra-, audible- and infrasound. The reader interested in applications of surface waves to testing electronic materials should consult the review article of R. M. White [74]. Modelling, numerical evaluation and experimental verification of surface waves in heterogeneous materials, both elastic and viscoelastic, do not enter at all these notes in spite of the fact that they are particularly important in nondestructive testing of soils (e.g. [37], [55]). However, they are the subject of detailed presentations in other articles of this volume ([27], [34], [38], [54]). Theoretical and numerical details concerning the propagation of surface waves in two-component porous materials are also presented separately [3]. For this reason, we concentrate here solely on the explanation of some guidelines for these subjects without going much into details.

We leave out entirely the problem of nonlinear surface waves. This subject develops in the recent years very vehemently (e.g. compare [50]) but it requires different theoretical methods than these applied for linear waves. However, some hints on this subject can be found in one of the articles of this volume [43].

In spite of the vast literature on surface waves, there exists no comprehensive presentation of this subject for scientists working on engineering and materials sciences and applying the technique of wave propagation in, for example, nondestructive testing. This article aims at the theoretical description of the most popular surface waves, in particular Rayleigh, Love, Stoneley, in one-component elastic media as well as the appropriate extensions to the saturated porous materials. We introduce as well some terminology used in this theory which is peculiar for seismology, geotechnics, and nondestructive testing in materials science. Due to the lack of space, we leave out the important problem of sources of waves. However, an extensive presentation of this subject can be found elsewhere ([2], [8], [56]).

Each surface wave is presented in a self-contained manner and the corresponding Section can be read independently of other Sections of the article. It means that the derivation of the fundamental equation – dispersion relation, follows from the field equations and boundary conditions for each case anew. In the literature, this is sometimes unified by the ray method and the so-called transition and reflection conditions, following from boundary conditions, which altogether yield the method of the **constructive interference** (e.g. [2], [13], [67]). We do not apply this method in this article in order to have an opportunity for the discussion of the physical inside of both field equations and of boundary conditions for such different systems as the contact of a linear elastic solid with an ideal fluid vs., say, the contact of porous materials with a permeable boundary with an ideal fluid.

2 Water waves and classical Rayleigh waves

2.1 Water waves in an ideal incompressible fluid model

There exists a vast literature on this subject (e.g. [8], [25], [16], [75]). In contrast to all other waves considered in this article which are produced by an unspecified source far away from the space-time point $(\mathbf{x}, \mathbf{t})$ of analysis, water waves result from the action of gravity. We present them here because they are well known to all who observed the motion of water on the beach [66], and, simultaneously, they possess all properties characteristic for surface waves.

We consider an ideal (inviscid) incompressible fluid which means that its mass density ρ is constant. We assume that the motion is irrotational, i.e. the velocity $\mathbf{v}(\mathbf{x}, t)$ possesses a potential ϕ. Then the mass conservation yields

$$\nabla^2 \phi = 0, \quad \mathbf{v} = \operatorname{grad} \phi, \quad \rho = const. \tag{1}$$

Simultaneously, the momentum balance equation has the form

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \operatorname{grad} \mathbf{v} \right) = - \operatorname{grad} p + \rho \mathbf{b}, \tag{2}$$

where p – pressure, $\mathbf{b}$ – mass force. Integration of this equation with respect to spacial variable $\mathbf{x}$ yields the Bernoulli equation

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} v^2 + \chi + \frac{p}{\rho} = C(t), \quad \mathbf{b} = - \operatorname{grad} \chi, \tag{3}$$

where $C(t)$ is an arbitrary function of time and we have used the identity $\mathbf{v} \cdot \operatorname{grad} \mathbf{v} = \frac{1}{2} \operatorname{grad} v^2$, $v^2 \equiv \mathbf{v} \cdot \mathbf{v}$, which holds for irrotational velocity fields.

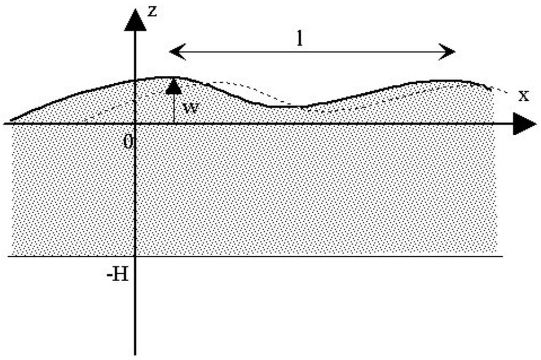


Fig.1: *Water profile at the instant of time t*

We investigate the motion of the fluid shown schematically in Fig. 1. For the two-dimensional motion assumed in this problem the upper boundary surface is described by the relations

$$f_s(\mathbf{x},t) = z - w(x,t) = 0 \quad \Rightarrow \tag{4}$$
$$\Rightarrow \qquad \mathbf{n} = \frac{\operatorname{grad} f_s}{|\operatorname{grad} f_s|} \approx \mathbf{e}_z, \quad \mathbf{n}\cdot\mathbf{v}_s = -\frac{\frac{\partial f_s}{\partial t}}{|\operatorname{grad} f_s|} \approx \frac{\partial w}{\partial t}\mathbf{e}_z,$$

where $\mathbf{n}$ is the unit outward normal vector of the surface, $\mathbf{v}_s$ is the velocity of this surface, and $\mathbf{e}_z$ the unit vector in the direction of z-axis. The approximation in the above relations means that we linearize the problem with respect to the elevation function w and its derivatives.

Let p_a denote the atmospheric pressure acting on the fluid. Then, under the assumption that we can neglect the surface tension, the continuity of the mass flux on the surface (the surface is **material** with respect to the fluid) and the Bernoulli equation (3) yield

$$(\mathbf{v}-\mathbf{v}_s)\cdot\mathbf{n} = 0, \tag{5}$$
$$p_a = -\rho g w - \rho\left(\frac{\partial w}{\partial t}\right)_{z=w} + \rho C(t),$$

where the nonlinear contribution $\frac{1}{2}v^2$ has been neglected[1].

Instead of the potential ϕ, we use the potential $\phi' = \phi + \frac{p_a}{\rho}t - \int C(t)\,dt$ because $\mathbf{v} = \operatorname{grad}\phi \equiv \operatorname{grad}\phi'$. Then, omitting the prime,

$$w = -\frac{1}{g}\left.\frac{\partial\phi}{\partial t}\right|_{z=0}, \tag{6}$$

where we place the boundary at $z=0$ instead of $z=w$ which yields the error of the same order of magnitude as in other approximations.

[1] Approximations which we make in this derivation are based on the comparison with the wave-length l ($=\frac{2\pi}{k}$, k – wave number), wave period T ($=\frac{2\pi}{\omega}$, ω – frequency), and the amplitude a. Namely

$$|\mathbf{v}| \sim \frac{a}{T}, \quad \left|\frac{\partial\mathbf{v}}{\partial t}\right| \sim \frac{a}{T^2}, \quad |\mathbf{v}\cdot\operatorname{grad}\mathbf{v}| \sim \frac{a^2}{T^2 l} \quad \Rightarrow$$

v^2 can be neglected if $a \ll l$. Simultaneously

$$|w| \sim a, \quad \left|\frac{\partial w}{\partial x}\right| \sim \frac{a}{l} \ll 1.$$

Simultaneously, the first condition (5) and the relation $(4)_3$ yield

$$\frac{\partial w}{\partial t} = \left.\frac{\partial \phi}{\partial z}\right|_{z=0}. \tag{7}$$

Combination of (6) and (7) yields the following kinematic boundary condition for ϕ

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = 0. \tag{8}$$

The second boundary condition which must be fulfilled by solutions of equation (1) is formulated at the bottom

$$\mathbf{v} \cdot \mathbf{e}_z\big|_{z=-H} = \left.\frac{\partial \phi}{\partial z}\right|_{z=-H} = 0. \tag{9}$$

Box 2: *Water waves*

Conditions satisfied by the velocity potential ϕ

$$\begin{aligned} \nabla^2 \phi &= 0 \quad \text{in the fluid,} \\ \frac{\partial \phi}{\partial z} &= 0 \quad \text{on the bottom } z = -H, \\ \frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} &= 0 \quad \text{on the free surface } z = 0, \end{aligned}$$

$$\begin{aligned} w &= -\frac{1}{g}\frac{\partial \phi}{\partial t} \quad \text{on the free surface } z = 0, \\ \frac{\partial w}{\partial t} &= \frac{\partial \phi}{\partial z} \quad \text{on the free surface } z = 0. \end{aligned}$$

Among the last three conditions two are independent.

Dispersion relation

$$\omega^2 = gk \operatorname{th} kH.$$

Now we are in the position to make an ansatz for solutions of the problem. We seek it in the form of the wave progressive in x-direction

$$\phi(x, z, t) = \left(Ae^{kz} + Be^{-kz}\right)\cos(kx - \omega t). \tag{10}$$

Then, according to (6), the elevation w satisfies the one-dimensional wave equation.

Boundary conditions (8) and (9) yield

$$\begin{aligned} -\omega^2 (A + B) + gk (A - B) &= 0, \\ Ae^{-kH} - Be^{kH} &= 0. \end{aligned} \tag{11}$$

Consequently, from the determinant of this homogeneous set we obtain the following **dispersion relation**

$$\omega^2 = gk \,\mathrm{th}\,(kH)\,. \tag{12}$$

Simultaneously, the potential can be written in the form

$$\phi = \phi_{\max} \frac{\mathrm{ch}\, k\,(H+z)}{\mathrm{ch}\, kH} \cos\left(kx - \omega t\right), \tag{13}$$

where $\phi_{\max}$ is a constant of integration.

According to (6) we obtain for the elevation

$$w = -w_{\max} \sin\left[k\left(x - c_{ph}t\right)\right], \quad c_{ph} := \frac{\omega}{k}, \quad w_{\max} := \frac{\omega}{g}\phi_{\max}. \tag{14}$$

Hence the elevation changes in the x-direction as it were the wave moving with the **phase velocity**

$$c_{ph} = \frac{\omega}{k} = \sqrt{\frac{gl}{2\pi}\,\mathrm{th}\left(2\pi\frac{H}{l}\right)}, \quad l := \frac{2\pi}{k}, \tag{15}$$

where l is the wavelength. The phase velocity of this wave depends on the frequency ω (or on the wave number k) and, therefore, the wave is called **dispersive**. This property is characteristic for all surface waves which propagate in systems with a characteristic **length scale** (e.g. the depth of the layer, the characteristic length of heterogeneous materials whose properties depend on the location in space, etc.).

In order to find orbits of material points we use the following relation for displacements

$$\begin{aligned}
\frac{\partial u_x}{\partial t} &= v_x = \frac{\partial \phi}{\partial x} \quad \Rightarrow \quad u_x = -\frac{k}{\omega}\phi_{\max}\frac{\mathrm{ch}\, k\,(H+z)}{\mathrm{ch}\, kH}\cos\left(kx - \omega t\right), \\
\frac{\partial u_z}{\partial t} &= v_z = \frac{\partial \phi}{\partial z} \quad \Rightarrow \quad u_z = -\frac{k}{\omega}\phi_{\max}\frac{\mathrm{sh}\, k\,(H+z)}{\mathrm{ch}\, kH}\sin\left(kx - \omega t\right).
\end{aligned} \tag{16}$$

Elimination of time yields

$$\frac{u_x^2}{\alpha_x^2} + \frac{u_z^2}{\alpha_z^2} = 1, \tag{17}$$

where

$$\alpha_x := \frac{k}{\omega}\phi_{\max}\frac{\mathrm{ch}\, k\,(H+z)}{\mathrm{ch}\, kH}, \quad \alpha_z := \frac{k}{\omega}\phi_{\max}\frac{\mathrm{sh}\, k\,(H+z)}{\mathrm{ch}\, kH}. \tag{18}$$

Consequently, the orbit of each particle is an ellipse with semiaxes α_x, α_z. The largest ellipse appears at $z = 0$, and at the bottom $z = -H$, it degenerates into a straight line. The orbits are schematically shown in Fig.2.

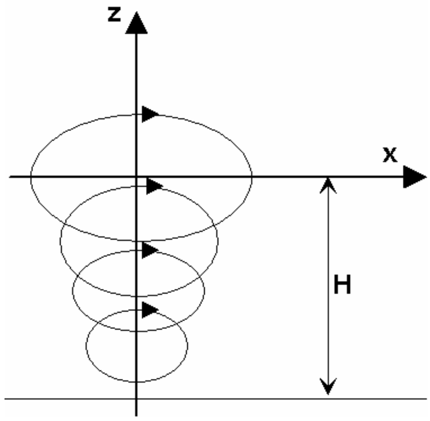


Fig.2: *Orbits of particles given by (17). Particles travel in the clockwise (prograde) direction.*

In the short-wave limit (the deep water!): $kH \to \infty$, we have $\operatorname{th} kH \approx 1$, i.e. $\omega^2 \approx gk$. Consequently, the phase velocity is given by the relation

$$c_{ph} = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \frac{g}{\omega}. \tag{19}$$

Simultaneously, the velocity potential becomes

$$\phi \approx \phi_{\max} e^{kz} \cos(kx - \omega t). \tag{20}$$

Hence the motion of the fluid is negligible at the depth of about a wavelength $l = \frac{2\pi}{k}$. For this reason, these waves are called **surface waves**.

The above dispersive wave gives rise to a structure of propagation which has a very important bearing. The arrival of such waves to receivers is observed in form of wave packages rather than in the form of single monochromatic waves or wave fronts. In order to illustrate this property on our simple example of deep water waves we consider the wave consisting of a narrow band of frequencies near the middle frequency ω_0 rather than a single frequency considered above. The solution (20) must be now replaced by the Fourier integral which accounts for all frequencies entering the band

$$\begin{aligned} \phi(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{\max}(\omega) e^{kz} \cos(kx - \omega t) \, d\omega \approx \\ &\approx \frac{1}{2\pi} \phi_{\max}(\omega_0) e^{k_0 z} \int_{-\infty}^{\infty} \cos(kx - \omega t) \, d\omega, \end{aligned} \tag{21}$$

where

$$k_0 = \frac{\omega_0^2}{g}. \tag{22}$$

Integration yields for small $\left| \frac{\Delta\omega}{\omega_0} \right|$

$$\phi = \frac{\Delta\omega}{2\pi} \phi_{\max}(\omega_0) e^{k_0 z} M(\omega_0, \Delta\omega) \cos[k_0(x - c_{ph} t)], \tag{23}$$

where

$$M(\omega_0, \Delta\omega) := \frac{\sin\left[\frac{\Delta\omega}{c_g}(x - c_g t)\right]}{\frac{\Delta\omega}{c_g}(x - c_g t)}, \tag{24}$$

and we have the relations

$$\begin{aligned} k - k_0 &= \frac{1}{g}\left(\omega^2 - \omega_0^2\right) \approx \frac{1}{c_g}(\omega - \omega_0), \\ c_{ph} &= \frac{g}{\omega_0}, \quad c_g := \left.\frac{d\omega}{dk}\right|_{\omega=\omega_0} = \frac{1}{2} c_{ph} = \frac{g}{2\omega_0}. \end{aligned} \tag{25}$$

The quantity M is called the **modulator**, and as shown on an example in Fig. 3, it has an extremum at $x - c_g t = 0$. The modulator is an envelope of the band of waves and propagates with the **group velocity** $c_g = \frac{d\omega}{dk}$. The **carrier** which in our example is described by the cosine function in (23) describes the motion at the frequency ω_0 with the phase velocity c_{ph}. In Figure 3 we show the wave in two instances of time. Clearly, due to different velocities, the shape of the full wave plotted as the solid line moved differently from the envelope of the modulator indicated by dotted curves.

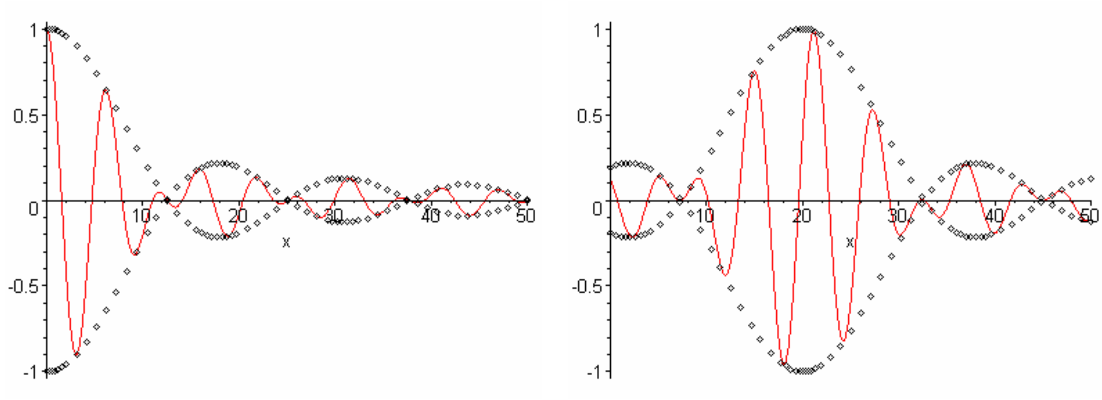


Fig. 3: *Narrow band wave: configurations at two different instances of time (arbitrary units)*

Due to its practical importance, this form of propagation shall be discussed further in this article in some details.

2.2 Remark on the choice of the independent variable in dispersion relations

It is easy to observe in the above example that in a general dispersion relation $f(\omega, k) = 0$ we may choose the frequency ω as an independent variable and calculate the wave number k as a function of ω, or, *vice versa*, we can choose the wave number k (or, equivalently, the wave length $l = \frac{2\pi}{k}$) as the independent variable and calculate the frequency ω as a function of k. A transformation from one choice to the other yields the same results provided both ω and k are real and the relation between these variables is monotonous. In the seismological literature these properties are taken for granted and, on the same page, one can find sometimes ω and sometimes k chosen as independent. Obviously, in the above presented example this is legitimate because the dispersion relation (12) is invertible.

However, problems with complex solutions of the dispersion relation cannot be treated so carelessly. Such solutions appear, for instance, in problems with damping. We present in this remark two very simple examples in order to illustrate the problem. In practical applications to surface waves this problem appears, for instance, in heterogeneous systems and in two-component systems with a relative motion of components (diffusion).

Let us first consider the following one-dimensional partial differential equation for the unknown function u

$$\frac{\partial^2 u}{\partial t^2} + \eta \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{26}$$

where η is a positive constant (viscosity) and c is a real constant. Obviously, in the particular case $\eta = 0$ this equation becomes the classical linear wave equation.

We seek the solution of the above equation in the form of a monochromatic wave

$$u = U e^{i(kx - \omega t)}, \tag{27}$$

where U, k, ω are constant. Substitution of (27) in (26) yields the dispersion relation

$$\omega^2 + i\eta\omega + c^2 k^2 = 0. \tag{28}$$

We consider two cases: the given real frequency ω, and the given real wave number k.

1) The given real frequency ω. Then the solution of the dispersion relation has the form

$$\begin{aligned} \operatorname{Re} k &= \frac{\omega}{c}\frac{1}{\sqrt{2}}\sqrt{1+\sqrt{1+\left(\frac{\eta}{\omega}\right)^2}}, \\ \operatorname{Im} k &= \frac{\frac{\eta}{c}}{\sqrt{2}\sqrt{1+\sqrt{1+\left(\frac{\eta}{\omega}\right)^2}}}. \end{aligned} \tag{29}$$

It is clear that $\operatorname{Re} k$ is always different from zero and inspection of (27) shows that $\frac{\omega}{\operatorname{Re} k}$ defines the phase velocity while $\operatorname{Im} k$ is the **attenuation** of the wave, i.e. the rate of decay of the amplitude in space due to the damping η.

2) The given real wave number k. Then the solution of the dispersion relation is as follows

$$\begin{aligned} \operatorname{Re}\omega &= \begin{cases} ck & \text{for} \quad k > \frac{\eta}{2c}, \\ 0 & \text{for} \quad k \le \frac{\eta}{2c}, \end{cases}, \\ \operatorname{Im}\omega &= \begin{cases} -\frac{\eta}{2} \quad \text{for} \quad k > \frac{\eta}{2c}, \\ -\frac{\eta}{2} \mp ck\sqrt{\frac{\eta^2}{4c^2k^2}-1} \quad \text{for} \quad k \le \frac{\eta}{2c}. \end{cases} \end{aligned} \tag{30}$$

Hence, the wave propagates with the phase velocity $\frac{\operatorname{Re}\omega}{k}$ solely for sufficiently large wave numbers (i.e. for sufficiently short waves $l < \frac{\pi c}{\eta}$). The wave number $k_c = \frac{\eta}{2c}$ is **critical**. Below this number the equation (26) describes a pure damping and waves cannot propagate at all.

Secondly let us consider the following one-dimensional partial differential equation

$$\frac{\partial^2 u}{\partial t^2} + \gamma\frac{\partial u}{\partial x} - c^2\frac{\partial^2 u}{\partial x^2} = 0, \tag{31}$$

where γ is a positive constant (spacial damping) and c is a real constant.

Again we consider a solution in the form (27). We obtain immediately the following dispersion relation

$$\omega^2 - i\gamma k - c^2k^2 = 0. \tag{32}$$

As before we consider two cases: the given real frequency ω, and the given real wave number k.

1) The given real frequency ω. We obtain the following relations

$$\begin{aligned} \operatorname{Re} k &= \begin{cases} \pm\frac{\omega}{c}\sqrt{1-\frac{\gamma^2}{4c^2\omega^2}} \quad \text{for} \quad \omega > \frac{\gamma}{2c}, \\ 0 \quad \text{for} \quad \omega \le \frac{\gamma}{2c}, \end{cases}, \\ \operatorname{Im} k &= \begin{cases} -\frac{\gamma}{2c^2} \quad \text{for} \quad \omega > \frac{\gamma}{2c}, \\ -\frac{\gamma}{2c^2} \pm \frac{\omega}{c}\sqrt{\frac{\gamma^2}{4c^2\omega^2}-1} \quad \text{for} \quad \omega > \frac{\gamma}{2c}. \end{cases} \end{aligned} \tag{33}$$

Hence, in contrast to the previous case, the given real frequency yields a critical damping $\omega_c = \frac{\gamma}{2c}$. For higher frequencies, we have the propagation of a wave with the phase velocity $\frac{\omega}{\operatorname{Re} k}$ and the decay of the amplitude in space while in the range of lower frequencies the wave cannot propagate and we have a pure damping.

2) The given real wave number k. Then it follows

$$
\begin{aligned}
\operatorname{Re}\omega &= \frac{ck}{\sqrt{2}}\sqrt{1+\sqrt{1+\left(\frac{\gamma}{c^2k}\right)^2}}, \\
\operatorname{Im}\omega &= -\frac{1}{\sqrt{2}}\frac{\frac{\gamma}{c}}{\sqrt{1+\sqrt{1+\left(\frac{\gamma}{c^2k}\right)^2}}}.
\end{aligned} \tag{34}
$$

Here the wave propagates for any choice of k with the phase velocity $\frac{\operatorname{Re}\omega}{k}$.

The above simple examples show that these problems in which we have a kind of a viscosity contribution corresponding to a lower time derivative in the governing equations the choice of the wave number k as the independent variable in the dispersion relation leads to critical phenomena specified by the critical wave number k_c. In their range the relation $k \to \omega$ is not invertible. This is not the case when we choose the frequency ω as the independent variable.

On the other hand, problems in which we deal with spacial heterogeneities yielding the presence of lower spacial derivatives in the governing equations lead to critical phenomena specified by the critical frequency ω_c. In their range the relation $\omega \to k$ is not invertible. It is not the case when we choose the wave number k as the independent variable.

In this article we deal primarily with homogeneous problems in which we may have the damping caused by diffusion (viscosity). Consequently, it is natural to choose the frequency ω as the independent variable. We do not pay attention to this choice solely in cases when the relation between ω and k is unconditionally invertible.

Certainly, equations which describe simultaneously viscosity and heterogeneity possess critical damping related to the frequency as well as to the wave number.

2.3 Rayleigh waves on plane boundaries of linear elastic homogeneous materials

The surface wave described in Section 2.1 is not typical for solids. As we see in the rest of this article, models of surface waves appear primarily as a combination of bulk waves which, in turn, follow as solutions of **hyperbolic** field equations.

We illustrate this statement by the classical example of the Rayleigh wave [53] (for example, see as well: [71], [1]).

Let us consider the linear elastic material described by the following equations for the unknown fields of velocity $\mathbf{v}(\mathbf{x}, t)$ and deformation $\mathbf{e}(\mathbf{x}, t)$

◊ momentum balance

$$
\rho \frac{\partial \mathbf{v}}{\partial t} = \operatorname{div} \mathbf{T}, \tag{35}
$$

◊ Hooke's law (constitutive relation for the Cauchy stress tensor $\mathbf{T}$)

$$
\mathbf{T} = \lambda \operatorname{tr} \mathbf{e} \mathbf{1} + 2\mu \mathbf{e}, \tag{36}
$$

where $\mathbf{e}$ is the Almansi-Hamel tensor of small deformations (i.e. $\|\mathbf{e}\| \ll 1^2$), λ, μ are the Lamé's moduli,

$\Diamond$ kinematic compatibility condition

$$\frac{\partial \mathbf{e}}{\partial t} = \operatorname{sym}\operatorname{grad}\mathbf{v}. \tag{37}$$

Let us first seek bulk waves, described by the above equations in the infinite medium. It is sufficient to exploit the case of the monochromatic wave

$$\begin{aligned} \mathbf{v} &= \mathbf{V}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \equiv \mathbf{V}\exp\left[ik\left(\frac{1}{k}\mathbf{k}\cdot\mathbf{x} - c_{ph}t\right)\right], \quad k := \sqrt{\mathbf{k}\cdot\mathbf{k}}, \\ \mathbf{e} &= \mathbf{E}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \equiv \mathbf{E}\exp\left[ik\left(\frac{1}{k}\mathbf{k}\cdot\mathbf{x} - c_{ph}t\right)\right], \quad c_{ph} := \frac{\omega}{k}, \end{aligned} \tag{38}$$

where ω is the given frequency, $\mathbf{k}$ the **wave vector**, $k = \sqrt{\mathbf{k}\cdot\mathbf{k}}$ is the **wave number**, and $\mathbf{V}, \mathbf{E}$ are constant **amplitudes**.

Substitution in (37) yields the following relation between the amplitudes

$$-\omega\mathbf{E} = \frac{1}{2}\left(\mathbf{k}\otimes\mathbf{V} + \mathbf{V}\otimes\mathbf{k}\right). \tag{39}$$

Hence the momentum balance (35) leads to the following equation

$$-\omega\rho\mathbf{V} = \lambda\operatorname{tr}\mathbf{E}\mathbf{k} + 2\mu\mathbf{E}\mathbf{k} = -\lambda\frac{1}{\omega}\mathbf{V}\cdot\mathbf{k}\mathbf{k} - \mu\frac{1}{\omega}\left(\mathbf{V}\cdot\mathbf{k}\mathbf{k} + k^2\mathbf{V}\right). \tag{40}$$

We split this equation into the component parallel to $\mathbf{k}$ and perpendicular to $\mathbf{k}$

$$\left(\rho\omega^2 - (\lambda + 2\mu)\,k^2\right)\mathbf{V}\cdot\mathbf{k} = 0, \tag{41}$$

$$\left(\rho\omega^2 - \mu k^2\right)\left(\mathbf{V} - \frac{1}{k^2}\mathbf{V}\cdot\mathbf{k}\mathbf{k}\right) = 0. \tag{42}$$

Obviously, we obtain two solutions

$$1)\ \mathbf{V} = \frac{1}{k^2}\mathbf{V}\cdot\mathbf{k}\mathbf{k} \quad \Rightarrow \quad \left(\frac{\omega}{k}\right)^2 = \frac{\lambda + 2\mu}{\rho}, \tag{43}$$

$$2)\ \mathbf{V}\cdot\mathbf{k} = 0 \quad \Rightarrow \quad \left(\frac{\omega}{k}\right)^2 = \frac{\mu}{\rho}. \tag{44}$$

The first solution describes **longitudinal waves** or P **waves** (P for primary; the amplitude is parallel to the direction of propagation: $\mathbf{V}\|\mathbf{k}$) whose phase velocity is

$$c_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \tag{45}$$

[2] The norm of the tensor $\mathbf{e}$ is defined by its eigenvalues which are identical with the principal stretches $\lambda_{(i)}$

$$\det\left(\mathbf{e} - \lambda_{(i)}\mathbf{1}\right) = 0, \quad i = 1, 2, 3,$$

$$\|\mathbf{e}\| := \max\left\{\left|\lambda_{(1)}\right|, \left|\lambda_{(2)}\right|, \left|\lambda_{(3)}\right|\right\}.$$

and this is not dependent on the frequency. Consequently, the wave is **non-dispersive** – each monochromatic wave propagates with the same velocity.

The second solution describes **transversal waves** or S **waves** (S for secondary or shear; the amplitude is perpendicular to the direction of propagation: $\mathbf{V}\cdot\mathbf{k}=\mathbf{0}$) whose phase velocity is

$$c_T=\sqrt{\frac{\mu}{\rho}}. \tag{46}$$

Consequently, this wave is also **non-dispersive.**

The system of equations of linear elasticity is **hyperbolic** provided $\mu>0, \lambda+2\mu>0$. Then both velocities of the above described bulk waves are **real**.

Let us mention in passing that the separation of the eigenvalue problem presented above can be done in many other ways. Depending on a particular problem, they appear in the literature on the classical elasticity as well as in other problems of mechanics. Let us mention two of them.

If we introduce the field of **displacement** $\mathbf{u}(\mathbf{x},t)$ the problem is described by equations (35), (36) due to the following relations

$$\mathbf{v}=\frac{\partial\mathbf{u}}{\partial t},\quad \mathbf{e}=\frac{1}{2}\left(\operatorname{grad}\mathbf{u}+\left(\operatorname{grad}\mathbf{u}\right)^T\right), \tag{47}$$

and the relation (37) is identically satisfied. Simultaneously, the vector $\mathbf{u}$ can be represented as a sum of a **potential** part $\mathbf{u}_L$ and a **solenoidal** part $\mathbf{u}_T$ which, as can be shown by a straightforward calculation, satisfy the following relations

$$\begin{aligned}\mathbf{u} &= \mathbf{u}_L+\mathbf{u}_T,\quad \operatorname{curl}\mathbf{u}_L=0,\quad \operatorname{div}\mathbf{u}_T=0,\\ \frac{\partial^2\mathbf{u}_L}{\partial t^2} &= c_L^2\nabla^2\mathbf{u}_L,\quad \frac{\partial^2\mathbf{u}_T}{\partial t^2}=c_T^2\nabla^2\mathbf{u}_T.\end{aligned} \tag{48}$$

Hence we obtain two wave equations whose solutions have the form of two waves discussed before.

One can use as well the following identity satisfied by any differentiable vector field (the so-called Helmholtz decomposition theorem)

$$\mathbf{u}=\operatorname{grad}\varphi+\operatorname{curl}\boldsymbol{\psi}, \tag{49}$$

where $\varphi,\boldsymbol{\psi}$ are the so-called **scalar and vector potentials**. Again one can easily show that they satisfy the following equations

$$\frac{\partial^2\varphi}{\partial t^2}=c_L^2\nabla^2\varphi,\quad \frac{\partial^2\boldsymbol{\psi}}{\partial t^2}=c_T^2\nabla^2\boldsymbol{\psi}. \tag{50}$$

We obtain again the same result.

We proceed to construct a solution for a semiinfinite linear elastic medium, i.e. a medium with a **boundary**. The presence of a boundary leads to important wave effects. First of all, there are bulk wave reflection and transmission phenomena. We shall not discuss them in any details in this article. An interested reader should consult, for instance, [2], [67].

In order to analyze the problem we need boundary conditions on the plane boundary. We choose the Cartesian coordinates with the z-axis perpendicular to the boundary. The boundary is defined by $z = 0$. We consider the case of the boundary free of loading. Hence

$$\mathbf{Tn}\big|_{z=0} = 0, \quad \mathbf{n} = -\mathbf{e}_z, \quad \mathbf{u}\big|_{z\to\infty} = 0, \tag{51}$$

where $\mathbf{e}_z$ is the unit basis vector of the z-axis. It means that the z-axis is oriented into the medium.

We seek the solution by splitting the displacement $\mathbf{u}$ into the potential and solenoidal parts, $\mathbf{u}_L, \mathbf{u}_T$. Then we make the following ansatz

$$\begin{aligned} \mathbf{u}_L &= A_L e^{-\gamma z} e^{i(kx-\omega t)} \mathbf{e}_x + B_L e^{-\gamma z} e^{i(kx-\omega t)} \mathbf{e}_z, \\ \mathbf{u}_T &= A_T e^{-\beta z} e^{i(kx-\omega t)} \mathbf{e}_x + B_T e^{-\beta z} e^{i(kx-\omega t)} \mathbf{e}_z, \end{aligned} \tag{52}$$

where A_L, A_T, B_L, B_T are constant amplitudes. It means that we anticipate a progressive wave solution in the x-direction and the decay of the solution in the z-direction provided both γ, β are positive. If this should not be the case a solution in the form of the surface wave would not exist. Obviously, the form of the solution (52) indicates that particles move in the xz-plane. This is the characteristic feature of Rayleigh waves.

Substitution of the above solution in conditions $(48)_{2,3}$ characterizing the potential and solenoidal part of displacement yields the compatibility conditions

$$\frac{\gamma^2}{k^2} = 1 - \frac{c_R^2}{c_L^2}, \quad \frac{\beta^2}{k^2} = 1 - \frac{c_R^2}{c_T^2}, \quad c_R := \frac{\omega}{k}. \tag{53}$$

Now the wave equations $(48)_{4,5}$ lead to the following form of the solution

$$\begin{aligned} B_L &= i\frac{\gamma}{k} A_L \quad \Rightarrow \quad \mathbf{u}_L = \left(\mathbf{e}_x + i\frac{\gamma}{k}\mathbf{e}_z\right) A_L e^{-\gamma z} e^{i(kx-\omega t)}, \\ B_T &= i\frac{k}{\beta} A_T \quad \Rightarrow \quad \mathbf{u}_T = \left(\mathbf{e}_x + i\frac{k}{\beta}\mathbf{e}_z\right) A_T e^{-\beta z} e^{i(kx-\omega t)}. \end{aligned} \tag{54}$$

In order to find the amplitudes we apply the boundary conditions (51). Obviously, the last condition, the so-called **Sommerfeld condition**, is satisfied identically provided both γ and β are positive. The remaining relations yield

$$\begin{aligned} \left(c_L^2 - 2c_T^2\right) \frac{\partial \mathbf{u}\cdot\mathbf{e}_x}{\partial x} + c_L^2 \frac{\partial \mathbf{u}\cdot\mathbf{e}_z}{\partial z}\bigg|_{z=0} &= 0, \\ \frac{\partial \mathbf{u}\cdot\mathbf{e}_x}{\partial z} + \frac{\partial \mathbf{u}\cdot\mathbf{e}_z}{\partial x}\bigg|_{z=0} &= 0. \end{aligned} \tag{55}$$

Substitution of (54) gives rise to the following homogeneous set for the unknown constants

$$\begin{aligned} \left(2 - \frac{c_R^2}{c_T^2}\right) A_L + 2A_T &= 0, \\ 2\frac{\gamma\beta}{k^2} A_L + \left(2 - \frac{c_R^2}{c_T^2}\right) A_T &= 0. \end{aligned} \tag{56}$$

Hence, the determinant leads to the following **Rayleigh dispersion relation**

$$\mathcal{P}_R := \left(2 - \frac{c_R^2}{c_T^2}\right)^2 - 4\sqrt{1 - \frac{c_R^2}{c_T^2}}\sqrt{1 - \frac{c_R^2}{c_L^2}} = 0. \tag{57}$$

Clearly, solutions c_R of this equation are independent of the frequency ω. In other words, Rayleigh waves in a semiinfinite medium are **nondispersive**.

We show (e.g. see [8]) that this equation possesses one real solution $c_R < c_T$. Namely, it can be easily written in the form

$$f(y) := y^3 - 8y^2 + 8(3 - 2n)y - 16(1 - n) = 0, \quad y := \frac{c_R^2}{c_T^2}, \quad n := \frac{c_T^2}{c_L^2} \tag{58}$$

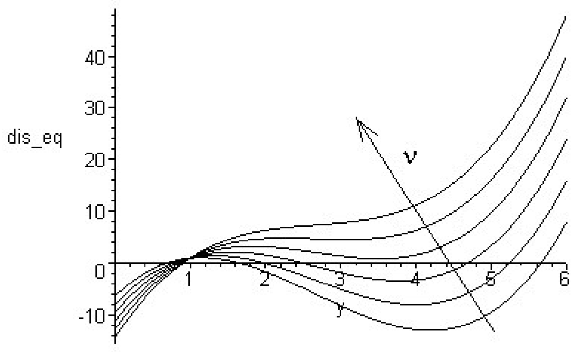
This function is concave in the interval $(0,1)$ and $f(0) = -16(1-n) < 0$, $f(1) = 1$. Consequently, there exists one root in the interval determined by the condition $c_R < c_T$. However, it may possess two other real roots bigger than 1 for $c_T/c_L \equiv \sqrt{\mu/(\lambda + 2\mu)}$ bigger than app. 0.57. It is instructive to calculate this coefficient in terms of the Poisson's ratio ν

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \equiv \frac{1}{2}\frac{1 - 2\frac{\mu}{\lambda + 2\mu}}{1 - \frac{\mu}{\lambda + 2\mu}} \quad \Rightarrow \quad n \equiv \frac{\mu}{\lambda + 2\mu} = \frac{1}{2}\frac{1 - 2\nu}{1 - \nu}, \tag{59}$$

i.e.

$$0 < \nu \leq \frac{1}{2} \quad \Rightarrow \quad \frac{1}{2} > \frac{\mu}{\lambda + 2\mu} \geq 0. \tag{60}$$

Consequently, the value $\sqrt{\mu/(\lambda + 2\mu)} = \sqrt{0.5} \approx 0.707$ corresponds to the minimum value of ν equal to zero. For growing ν the fraction $\mu/(\lambda + 2\mu)$ decays. Hence the range $0.570 < \sqrt{\mu/(\lambda + 2\mu)} < 0.707$ is physically not empty. The function $f(y)$ is plotted in Fig. 4 for Poisson's number $\nu:\ -0.25, 0, 0.1667, 0.2875, 0.375, 0.4444$. These correspond to the fraction $\mu/(\lambda + 2\mu)$ decaying from 0.6 to 0.1 by 0.1.



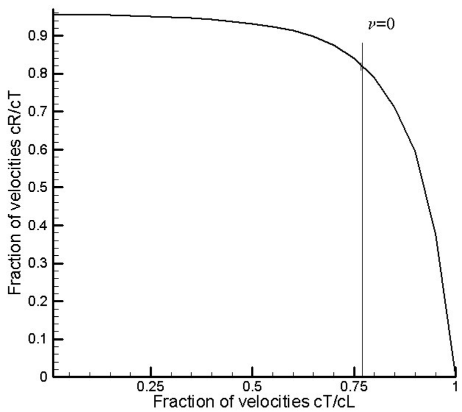


Fig. 4 (left): *Plots of the function $f(y)$ for values of the fraction $\left(\frac{c_T}{c_L}\right)^2$ from* 0.1 *(upper curve) to* 0.6 by 0.1 *(i.e. for the Poisson's number ν between* 0.4444 *and* -0.25).

Fig: 5 (right): *Dimensionless velocity of Rayleigh waves $\frac{c_R}{c_T}$ as a function of the fraction $\frac{c_T}{c_L}$.*

There exist some hints (e.g. [46]) that both the complex roots appearing for $\nu \lessapprox 0.3$ as well as the roots bigger than one may possess a physical bearing and describe **leaky** waves intermediate between surface waves and bulk waves which loose its energy to both Rayleigh wave and two usual body waves. However, the velocity of such waves should rather not exceed the biggest velocity of the bulk waves c_L and this is satisfied if the lower root, say y_0, satisfies the condition $1 < y_0 < \frac{1}{n} \equiv \frac{\lambda+2\mu}{\mu}$. Even this possibility is not excluded entirely anymore and there are some claims that supersonic Rayleigh waves may indeed appear. We discuss some aspects of this issue in Sec. 4.4. Some details can be also found in the books of Brekhovskikh and Godin [12], Viktorov [72], and possible practical applications in nondestructive testing in [7].

The solution slower than the shear velocity is shown in Fig. 5, where the fraction c_R/c_T is plotted as a function of the fraction of bulk velocities c_T/c_L. Clearly, the value of the Rayleigh velocity lies very near the velocity of shear waves for most values of the material parameters.

Let us consider the orbits of particles of Rayleigh waves. According to $(48)_1$, and (54) the displacement is given by the relations

$$\begin{aligned} u_x &= \left(A_L e^{-\gamma z} + A_T e^{-\beta z}\right) e^{i(kx-\omega t)}, \\ u_z &= i\left(\frac{\gamma}{k} A_L e^{-\gamma z} + \frac{k}{\beta} A_T e^{-\beta z}\right) e^{i(kx-\omega t)}. \end{aligned} \tag{61}$$

We choose the motion in which the x-component propagates as $\cos(kx - \omega t)$. Then taking real parts of the above relations and eliminating time we obtain

$$\frac{(\operatorname{Re} u_x)^2}{\alpha_x^2} + \frac{(\operatorname{Re} u_z)^2}{\alpha_z^2} = 1, \tag{62}$$

where

$$\begin{aligned} \alpha_x &:= A_L e^{-\gamma z} + A_T e^{-\beta z}, \\ \alpha_z &:= \frac{\gamma}{k} A_L e^{-\gamma z} + \frac{k}{\beta} A_T e^{-\beta z}, \end{aligned} \tag{63}$$

and

$$\operatorname{Re} u_x = \alpha_x \cos(kx - \omega t), \quad \operatorname{Re} u_z = -\alpha_z \sin(kx - \omega t). \tag{64}$$

Hence, as in the case of water waves (comp. (16)), the orbit of each particle is an ellipse with semiaxes $|\alpha_x|, |\alpha_z|$. However, the motion is anticlockwise (retrograde). The semiaxes decay exponentially with the depth (see: Fig. 6).

Obviously, the vector of the amplitude of the Rayleigh wave lies on the xz-plane. This plane is defined by the direction of propagation x (called in seismology the P direction, for primary), and the direction of decay z (called in seismology SV-direction: for shear vertical). It is called in seismology the **plane of incidence** or the **sagittal plane** in materials sciences. In the general case, the progressive wave solution in the x-direction may also possess an amplitude in the direction perpendicular to the plane of incidence – the so-called SH-direction (for shear horizontal). This part of the solution would have the form

$$u_y(x, z, t) = C e^{-\delta z} e^{i(kx-\omega t)}. \tag{65}$$

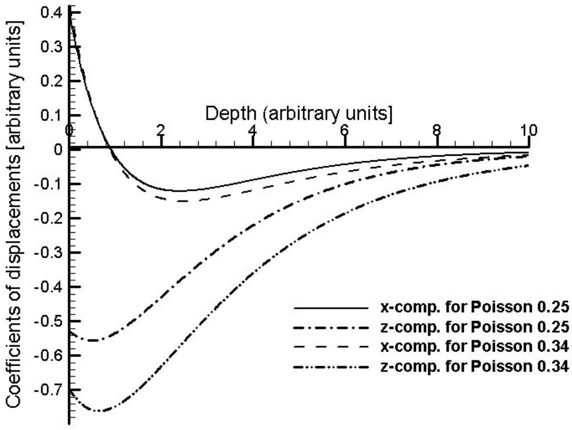


Fig. 6: *Coefficients of components of the displacement* α_x, α_z *as functions of the depth* z *for two values of the Poisson's ratio:* $\nu = 0.25, 0.34$ (i.e. $c_T/c_L = 0.5773, 0.4924$).

Such waves are called Love waves. It is easy to check that the boundary condition (51) yields immediately $C = 0$. Consequently, Love waves do not exist in homogeneous semiinfinite media. As we see in the next Section they do exist in layers on semiinfinite media.

2.4 Rayleigh waves on cylindrical boundaries

The problem of surface waves on cylindrical surfaces is quite common in geotechnics and appears, for instance, in the analysis of waves in boreholes. We present a simple example of a solid cylinder in order to see the influence of curvature on propagation of surface waves.

Let us investigate the problem of propagation of surface waves in an infinite cylinder of the radius R, i.e. waves which satisfy the following conditions:

1) the surface of the cylinder is free of stresses,

2) waves propagate in the circumferential direction and decay in the radial direction.

This problem has been solved in 1958 by Viktorov [69], [70] and it demonstrates the influence of the curvature of the surface on the propagation of surface waves.

It is convenient to use the potentials φ and $\boldsymbol{\psi}$ in this case. As the problem is two-dimensional the vector potential has solely one component ψ_z in the direction of the axis of cylinder. These potentials satisfy the equations (50) which have the following form in **polar coordinates**

$$\begin{aligned} \frac{\partial^2 \varphi}{\partial t^2} &= c_L^2 \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \varphi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} \right\}, \\ \frac{\partial^2 \psi_z}{\partial t^2} &= c_T^2 \left\{ \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi_z}{\partial \theta^2} \right\}, \end{aligned} \tag{66}$$

where r, θ are the radius and the angle coordinate, respectively.

It is convenient to consider the extension of the problem to the infinite interval for the angle θ: $-\infty < \theta < \infty$. Then the axis $r = 0$ is the branch cut of the infinite order. This extension allows to consider the propagation of surface waves on an infinite plane of repetitions of the circumferential surface of the cylinder. In practical applications it may correspond, for instance, to spiral surface waves whose amplitude is weakly dependent on the z-variable.

In polar coordinates the physical radial and circumferential components of displacement are given by relations

$$u_r = \frac{\partial \varphi}{\partial r} + \frac{1}{r}\frac{\partial \psi_z}{\partial \theta}, \quad u_\theta = \frac{1}{r}\frac{\partial \varphi}{\partial \theta} - \frac{\partial \psi_z}{\partial r}. \tag{67}$$

We seek solutions in the form

$$\varphi = \Phi(r) e^{i(p\theta - \omega t)}, \quad \psi_z = \Psi(r) e^{i(p\theta - \omega t)}, \tag{68}$$

where p plays the role of the angular wave number. It is related to the wave length l by the relation $p = 2\pi R/l$. The usual wave number is then given by the relation $k = 2\pi/l \equiv p/R$. Due to the extension of domain for angle θ, the variable p can be considered as continuous. Certainly, the limit $p \to 0$ corresponds to very long waves ($l \gg R$) and the limit $p \to \infty$ is the limit of short waves.

Substitution of the above ansatz in the equations (66) yields the following Bessel equations

$$\begin{aligned} \frac{d^2\Phi}{d\xi_L^2} + \frac{1}{\xi_L}\frac{d\Phi}{d\xi_L} + \left(1 - \frac{p^2}{\xi_L^2}\right)\Phi &= 0, \quad \xi_L := r\frac{\omega}{c_L} \equiv \frac{r}{R}p\frac{c}{c_L}, \\ \frac{d^2\Psi}{d\xi_T^2} + \frac{1}{\xi_T}\frac{d\Psi}{d\xi_T} + \left(1 - \frac{p^2}{\xi_T^2}\right)\Psi &= 0, \quad \xi_T := r\frac{\omega}{c_T} \equiv \frac{r}{R}p\frac{c}{c_T}, \quad c := \frac{\omega R}{p}. \end{aligned} \tag{69}$$

The solution of this set which remains finite in the middle points of the cylinder has the following form

$$\Phi = AJ_p(\xi_L), \quad \Psi = BJ_p(\xi_T), \tag{70}$$

where A, B are constants and J_p is the Bessel function of order p.

The constants appearing in the solution should be determined from the boundary condition on the free surface $r = R$. These boundary conditions – the boundary is free of stresses – have the following form in the cylindrical coordinates

$$\begin{aligned} &\left(c_L^2 - 2c_T^2\right)\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}\right) + \\ &+2c_T^2\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial^2 \psi_z}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \psi_z}{\partial \theta}\right) = 0, \quad \text{for} \quad r = R, \end{aligned} \tag{71}$$

$$\frac{2}{r}\frac{\partial^2\varphi}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \varphi}{\partial \theta} + \frac{1}{r^2}\frac{\partial^2 \psi_z}{\partial \theta^2} - \frac{\partial^2 \psi_z}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_z}{\partial r} = 0,$$

i.e. the radial stress and the shear stress are zero at the circumferential surface $r = R$.

Substitution of (70) in (68) and, subsequently, in (71) yields a homogeneous set of equations for the constants A, B

$$\begin{aligned} b_{11}A + b_{12}iB &= 0, \\ b_{21}A + b_{22}iB &= 0, \end{aligned} \tag{72}$$

where[3]

$$b_{11} := -2pJ_p^L - \frac{c}{c_T}\frac{c_L}{c_T}p\left(2p+1-2\frac{c_T^2}{c_L^2}\right)J_{p+1}^L + \frac{c^2}{c_T^2}p^2 J_{p+2}^L,$$

$$b_{12} := 2\left[\left(p^2-1\right)J_p^T - \frac{c}{c_T}p^2 J_{p+1}^T\right], \quad b_{21} := p\left(2p-1\right)J_p^L - 2\frac{c}{c_T}\frac{c_T}{c_L}p^2 J_{p+1}^L,$$

$$b_{22} = 2p\left(p-1\right)J_p^T - \frac{c}{c_T}p\left(2p-1\right)J_{p+1}^T + \frac{c^2}{c_T^2}J_{p+2}^T, \tag{73}$$

$$J_p^L := J_p\left(p\frac{c}{c_L}\right), \quad J_p^T := J_p\left(p\frac{c}{c_T}\right),$$

and similarly for the other Bessel functions.

Consequently, for the existence of nontrivial solutions the determinant of the set (72) must be zero and we obtain the following dispersion relation for this problem

$$\mathcal{P}_{RC}(c,p) := b_{11}b_{22} - b_{12}b_{21} = 0. \tag{74}$$

This relation is plotted in Fig. 7 for the data $c_T/c_L = 0.4924$ which corresponds to Poisson's ratio $\nu = 0.34$. This value has been chosen to coincide with the value chosen by Viktorov [72] in his numerical example. Not visible in the Figure 7 are infinitely many zero points for relative velocities bigger than $c/c_T > c_L/c_T = 2.0310$.

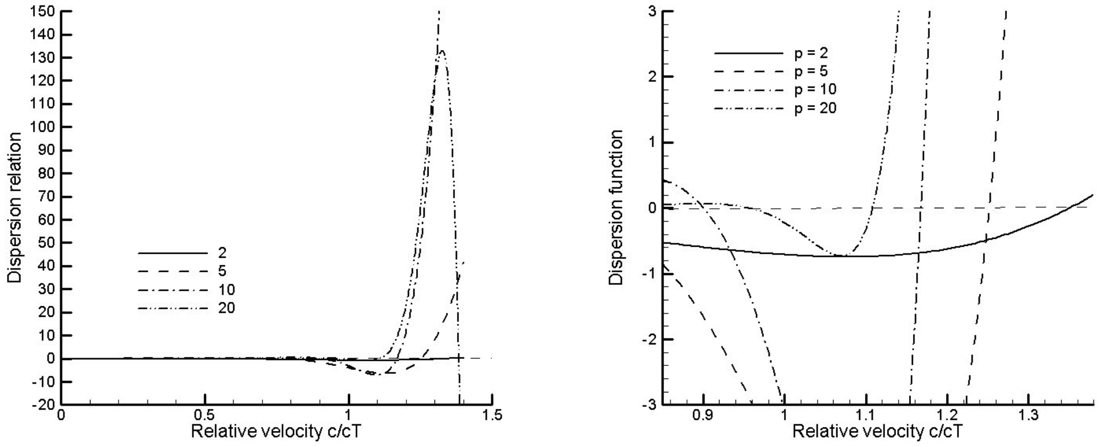


Fig. 7: *Zeros of the dispersion relation (74) for four different values of the wave number p :* $2, 5, 10, 20$. *The drawing on the right is the magnification of the part of the picture on the left.*

Rayleigh waves on the cylinder are the waves whose velocity c is bigger than c_T and correspond to the first zero points above the point $\frac{c}{c_T} = 1$. This is demonstrated on the

[3] We have used here the following identity for Bessel functions

$$\frac{dJ_p(x)}{dx} = \frac{p}{x}J_p(x) - J_{p+1}(x).$$

right hand side of Fig. 7 which is a magnification of the diagram on the left hand side. These solutions of the dispersion relation exist for $p > 1$ (i.e. $l < 2\pi R$) and decay to the Rayleigh velocity on the plane boundary for $p \to \infty$. This is a purely geometrical effect. The velocity of surface waves can be bigger than the velocity of shear waves because the path of the surface wave is longer than the path of the shear wave in the case of curved boundaries. For instance, for points of the boundary lying on the same diameter, the shear wave covers the distance $2R$ and the surface wave the distance πR. Consequently, the arrival time of the shear wave is shorter than the arrival time of the surface wave and, in this way, we fulfil the condition for constructive interference.

This effect was demonstrated by Viktorov who claims as well that an influence of the Poisson's ratio (i.e. relative bulk velocity $\frac{c_T}{c_L}$) is rather small.

Numerous zero points in this problem indicate that, in addition to Rayleigh waves, some additional waves may exist due to interactions of bulk waves with the surface. We shall discuss some aspects of this property of dispersion relations in Sec. 4.4.

The problem of propagation of surface waves on cylindrical surfaces has an important bearing in nondestructive testing of shafts. These circumferential waves belong then to the class of **guided** waves (e.g. [1], [41]). The dispersion relation for surface waves in a hollow cylinder with an inner shaft was obtained numerically by Valle, Qu and Jacobs [68]. They compare the first five modes of propagation of these waves with the corresponding modes for the solid and hollow cylinders and with the Rayleigh wave on the plane boundary. It comes out that the behavior of waves in the layered cylinder coincides with this of the solid cylinder in the low frequency regime and it differs substantially in the range of high frequencies. In this range, the first mode tends asymptotically to the Rayleigh mode on the plane surface, while the second mode approaches in this limit a Rayleigh mode appearing on sliding interfaces. Further references can be found in the original work [68].

3 Waves in a layer of an ideal fluid and Love waves on plane boundaries

3.1 Layer of an ideal compressible fluid on a semiinfinite rigid body

In order to appreciate the influence of heterogeneities on the propagation of surface waves, we investigate first a simple example of a layer of an **ideal fluid** $-\infty < x < \infty, 0 \leq z \leq H$. The upper surface $z = H$ (z-axis is oriented upward in this case) is free of loading and the lower surface $z = 0$ is in contact with a **rigid body.** The problem is described by the equations of mass and momentum conservation

$$\frac{\partial \rho}{\partial t} + \rho_0 \operatorname{div} \mathbf{v} = 0, \quad \rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\operatorname{grad} p, \quad p = p_0 + \kappa (\rho - \rho_0), \tag{75}$$

where ρ_0, p_0 are reference constant values of the mass density and pressure, respectively, and κ denotes a constant compressibility coefficient of the fluid.

Simple manipulations lead to the following wave equation for the pressure p

$$\frac{\partial^2 p}{\partial t^2} = \kappa \nabla^2 p, \quad (x, z) \in (-\infty, \infty) \times (0, H). \tag{76}$$

The solution of this equation must satisfy the following boundary conditions

$$p\left(x, z=H, t\right)=0, \quad v_z\left(x, z=0, t\right)=0. \tag{77}$$

As before, we seek the solution in the form of a monochromatic wave of the frequency ω

$$p=\left(Ae^{irkz}+Be^{-irkz}\right)e^{i(kx-\omega t)}. \tag{78}$$

Then, due to the momentum balance, the second boundary condition can be replaced by the following one

$$\frac{\partial p}{\partial z}\left(x, z=0, t\right)=0. \tag{79}$$

Substitution of (78) in the equation (76) yields the compatibility relation

$$r^2=\frac{c_{ph}^2}{c^2}-1, \quad c_{ph}:=\frac{\omega}{k}, \quad c:=\sqrt{\kappa}. \tag{80}$$

Simultaneously the evaluation of boundary conditions with the ansatz (78) yields the set of homogeneous algebraic relations for the constants A and B

$$\begin{aligned} Ae^{irkH}+Be^{-irkH} &= 0, \\ A-B &= 0. \end{aligned} \tag{81}$$

Consequently, the determinant of this set must be equal to zero and we obtain the **dispersion relation**

$$\cos\left(rkH\right)=0. \tag{82}$$

In order to obtain nontrivial solutions we have to require that r is real. This means, however, that the dispersion relation given in term of the periodic function yields infinitely many solutions. Each solution is called a **mode of propagation**. This is the characteristic feature of **heterogeneous** systems.

Simultaneously, it follows from (82) that the phase velocities c_{ph} are bigger than the velocity of propagation c appearing in the wave equation for the pressure (76)($\kappa\equiv c^2$), and that they go to infinity as the frequency approaches certain critical values. If we require that waves of the form (78) do exist then this seems to violate the basic property of the hyperbolic problem. This result follows from the assumption that the foundation of the fluid is a **rigid body** in which all disturbances propagate with an **infinite velocity**. As we see further a modification of the boundary condition $(77)_2$ for the case of contact with an elastic body which we make for the so-called Love waves eliminates this paradox.

In details, solution of the equation (82) yields immediately the following relation between the phase velocity and the frequency

$$c_{ph}=\frac{c}{\sqrt{1-\frac{\omega_{cr}^2}{\omega^2}}}, \quad \omega_{cr}:=\left(n+\frac{1}{2}\right)\pi\frac{c}{H}, \quad n=1,2,\ldots \tag{83}$$

This relation is illustrated in Figure 8.

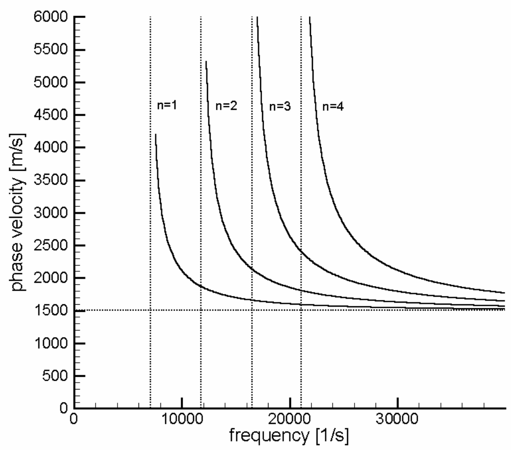


Figure 8: *Phase velocity for a layer of an ideal fluid. Numerical data:* $c = 1500\frac{m}{s}, H = 1m$. *Modes:* $n = 1, 2, 3, 4$ *are shown in the Figure.*

3.2 Love waves on plane boundaries

The paradox of infinite phase velocities does not appear anymore in the case of surface waves which propagate in an elastic layer over an elastic half-space. Transversal waves in such a system have been described in 1911 by Love [42]. We proceed to present briefly these results. They form the simplest illustration of the problem of surface waves in heterogeneous solids.

We consider the propagation of a wave whose amplitude has solely an $\mathbf{e}_y$-component $u_y \equiv \mathbf{u} \cdot \mathbf{e}_y$ (perpendicular to the (x, z)-plane; hence it corresponds to SH amplitude for waves with P-SV incident plane in seismological terminology). The body consists of a layer of thickness H in the z-direction in which the mass density is ρ' and the velocity of shear waves is c'_T. This layer is connected to the elastic half-space $z \leq 0$ whose mass density is ρ and the velocity of shear waves c_T. We seek the solution of wave equations

$$\begin{aligned} \frac{\partial^2 u'_y}{\partial t^2} &= c'^2_T \nabla^2 u'_y, \quad 0 < z < H, \\ \frac{\partial^2 u_y}{\partial t^2} &= c^2_T \nabla^2 u_y, \quad z < 0, \end{aligned} \tag{84}$$

in the form

$$\begin{aligned} u'_y &= \left(A' e^{iks'z} + B' e^{-iks'z} \right) e^{i(kx-\omega t)} \equiv \\ &\equiv 2 \left(\operatorname{Re} A' \cos ks'z - \operatorname{Im} A' \sin ks'z \right) e^{i(kx-\omega t)}, \\ u_y &= B e^{ksz} e^{i(kx-\omega t)}, \end{aligned} \tag{85}$$

i.e. in the form of a monochromatic wave which propagates in the direction of the x-axis with the frequency ω, wave number k in this direction, and with the phase velocity $c := \frac{\omega}{k}$. The wave should decay in the z-direction, i.e. s must be positive. We check now if the ansatz (85) can fulfil equations (84), and the following boundary conditions

1) shear stress on the plane $z = H$ is equal to zero, i.e.

$$\frac{\partial u'_y}{\partial z} (x, z = H, t) = 0, \tag{86}$$

2) shear stress and the displacement must be continuous on the interface $z = 0$

$$\begin{aligned} \rho' c_T'^2 \frac{\partial u_y'}{\partial z} (x, z = 0, t) &= \rho c_T^2 \frac{\partial u_y}{\partial z} (x, z = 0, t), \\ u_y' (x, z = 0, t) &= u_y (x, z = 0, t). \end{aligned} \tag{87}$$

Substitution of the ansatz (85) in equations (84) yields

$$s'^2 = \frac{c^2}{c_T'^2} - 1, \quad s^2 = 1 - \frac{c^2}{c_T^2}, \quad c \equiv \frac{\omega}{k}. \tag{88}$$

Boundary condition (86) leads immediately to the following relation for the displacement in the layer

$$u_y' = 2 \operatorname{Re} A' \frac{\cos (ks' (H - z))}{\cos (ks' H)} e^{i(kx - \omega t)}. \tag{89}$$

Then the boundary conditions (87) yield a homogeneous set of two algebraic relations for the constants $\operatorname{Re} A', B$. Consequently, its determinant must be zero and this condition yields the **Love dispersion relation**

$$\omega = \frac{c}{Hs'} \left[\arctan \left(\frac{\rho c_T^2 s}{\rho' c_T'^2 s'} \right) + n\pi \right], \quad n = 1, 2, 3, \ldots, \tag{90}$$

where both s, and s' must be real, i.e.

$$c_T' \leq c \leq c_T. \tag{91}$$

This is the condition for the *existence* of Love waves. Hence the Love waves can propagate solely in layers which are **softer than the foundation**. In addition there exist infinitely many modes of propagation whose existence is limited from below by corresponding **critical frequencies**. All these modes are dispersive because the phase velocities depend on the frequency given by the inverse relation to (90).

BOX 4: *Love waves*

Dispersion relation

$$\omega = \frac{c}{Hs'} \left[\arctan \left(\frac{\rho c_T^2 s}{\rho' c_T'^2 s'} \right) + n\pi \right], \quad n = 1, 2, 3, \ldots,$$

where

$$s^2 = 1 - \frac{c^2}{c_T^2}, \quad s'^2 = \frac{c^2}{c_T'^2} - 1, \quad c_T' \leq c \leq c_T.$$

Displacement

$$\begin{aligned} u_y' &= u_{y0}' \frac{\cos (ks' (H - z))}{\cos (ks' H)} e^{i(kx - \omega t)}, \\ u_{y0}' &= 2 \operatorname{Re} A'. \end{aligned}$$

In Fig. 9 we show an example of the solution of relation (90) for the following data:

$$c_T = 5\frac{km}{s}, \quad c_T' = 3\frac{km}{s}, \quad \frac{\rho'}{\rho} = 0.875, \quad H = 10km. \tag{92}$$

We plot the relative velocities c_{ph}/c_T as functions of the frequency ω for four values of n.

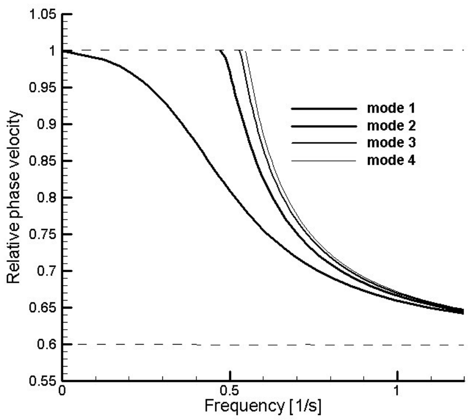


Fig. 9: *Relative phase velocities for four modes ($n = 0, 1, 2, 3$) of the Love wave given by relation* (90)

The data were chosen to be identical with the data used in the Box 7.2. by Aki and Richards [2]. For these data the phase and group velocities of the first mode are shown in Fig. 10.

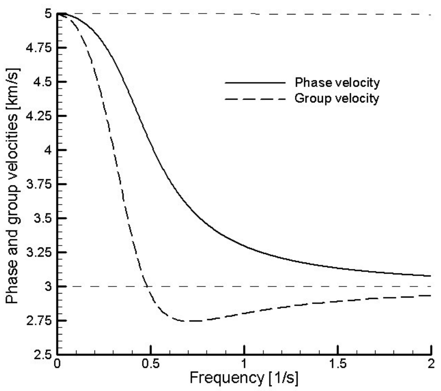


Fig. 10: *The phase and group velocities of the first mode of Love wave*

The group velocity as a function of the phase velocity was calculated by means of the following relation

$$c_g = \frac{\frac{d\omega}{dc_{ph}} c_{ph}^2}{\frac{d\omega}{dc_{ph}} c_{ph} - \omega}, \tag{93}$$

which follows immediately from the definitions. Then we can directly use the relation (90) and invert variables $c_{ph} \to \omega$ in the graphical program. In this way, we avoid a numerical differentiation.

For completeness, we present in Fig. 11 the wave number k as a function of frequency for the first mode of the Love wave. Clearly, the relation is monotonous and, according to our earlier remarks, we may choose ω as well as k as independent.

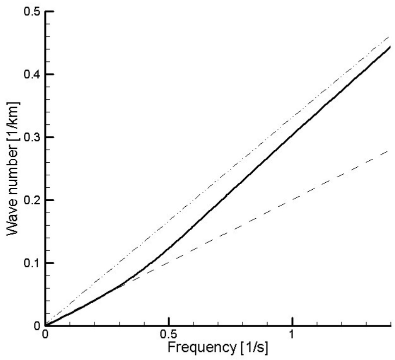


Fig. 11: *The dispersion relation $k = k(\omega)$ for the first mode of the Love wave. The dashed line corresponds to the initial velocity, i.e. $k_0 = 5\omega$, and dashdotted line to the limit velocity in infinity $k_\infty = 3\omega$.*

3.3 Arrival times for packages of waves – example

Bearing above results in mind, we investigate an example for the procedure of calculating the arrival times of a package of dispersive waves. This problem of propagation of packages of dispersive waves was already indicated in Sec. 2.1. Instead of assuming a narrow band structure we consider a problem in which an arbitrary displacement $u(x, z, t)$ has the following Fourier representation

$$u(x, z, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A(\omega; z) e^{i(kx-\omega t)} d\omega, \ k = k(\omega), \tag{94}$$

where a dependence on the variable z is parametric. In order to evaluate this integral, we make the assumption that the amplitude $A(\omega; z)$ is a slowly varying function of ω in comparison with variations caused by the phase Φ. This assumption is schematically illustrated in Figure 12.

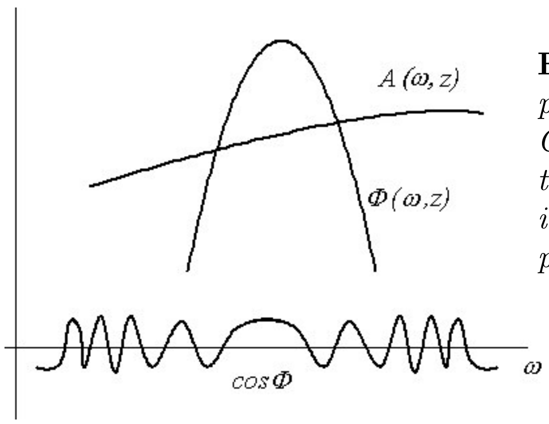


Fig. 12: *Illustration of the principle of stationary phase. Changes of the amplitude A, the trigonometric contribution to the integral $\cos\Phi$ (or $\sin\Phi$), and the phase Φ*

$$\Phi := kx - \omega t.$$

In such a case, similarly to the method of saddle point of integration, we can assume that the essential contribution to the integral comes from the vicinity of **stationary points** of the phase Φ. For a chosen point x and a chosen instant of time t these points, say ω_s, are defined by the relation

$$\frac{d\Phi}{d\omega}\left(x,t,\omega_s\right)=0 \quad \Rightarrow \quad \frac{dk}{d\omega}\left(\omega_s\right)x-t=0 \quad \Rightarrow \quad c_g\left(\omega_s\right)=\frac{x}{t}. \tag{95}$$

If there is a single such point then we can expand the phase

$$\Phi \approx k_s x-\omega_s t+\frac{1}{2}\left.\frac{d^2k}{d\omega^2}\right|_{\omega=\omega_s}\left(\omega-\omega_s\right)^2, \tag{96}$$

and the integral (94) becomes

$$u\left(x,z,t\right)\approx\frac{1}{2\pi}A\left(\omega_s;z\right)e^{i\left(k_s x-\omega_s t\right)}\int_{-\infty}^{\infty}\exp\left[-i\frac{1}{2c_g^2}\frac{dc_g}{d\omega}x\left(\omega-\omega_s\right)^2\right]d\omega, \tag{97}$$

where $k_s=k\left(\omega_s\right)$ and c_g as well as $\frac{dc_g}{d\omega}$ are evaluated at ω_s. Easy integration[4] yields

$$u\left(x,z,t\right)=\frac{A\left(\omega_s;z\right)c_g}{\sqrt{2\pi\left|\frac{dc_g}{d\omega}\right|x}}e^{i\left(k_s x-\omega_s t-\varsigma\frac{\pi}{4}\right)}, \quad \varsigma:=\operatorname{sign}\frac{dc_g}{d\omega}. \tag{98}$$

In the case of many stationary points the above formula transforms into the sum of contributions for each point.

If the derivative of the group velocity at the stationary point vanishes we have to make an expansion to the third order and it follows

$$u\left(x,z,t\right)=\frac{A\left(\omega_s;z\right)}{\left(\frac{9}{2}\frac{1}{c_g^2}\frac{d^2c_g}{d\omega^2}x\right)^{\frac{1}{3}}\Gamma\left(\frac{2}{3}\right)}e^{i\left(k_s x-\omega_s t\right)}. \tag{99}$$

The phase determined in the above described manner is called the **Airy phase.** We proceed to present a numerical example of its evaluation from a given dispersion relation. We do so for the above presented Love surface waves. We choose the data (92) and the distance from the source $x=500$ km. These are the same data as those chosen by Aki and Richards [2] in their presentation (Box 7.2).

[4]The integral can be written as follows

$$\int_{-\infty}^{\infty}\exp\left[-i\frac{1}{2c_g^2}\frac{dc_g}{d\omega}x\left(\omega-\omega_s\right)^2\right]d\omega=2c_g\sqrt{2\left(\left|\frac{dc_g}{d\omega}\right|x\right)^{-1}}\int_0^{\infty}\exp\left(-i\varsigma\sigma^2\right)d\sigma=$$

$$=2c_g\sqrt{2\left(\left|\frac{dc_g}{d\omega}\right|x\right)^{-1}}\left(1-\varsigma i\right)\int_0^{\infty}\cos\sigma^2 d\sigma=c_g\sqrt{2\pi\left(\left|\frac{dc_g}{d\omega}\right|x\right)^{-1}}e^{-\varsigma i\frac{\pi}{4}},$$

where we have used the formula for the Fresnel integral

$$\int_0^{\infty}\cos\sigma^2 d\sigma=\frac{\sqrt{2\pi}}{4}.$$

Box 5: *Arrival times of wave packages*

Procedure for calculation of arrivals:

1) determination of the dispersion relation $k = k(\omega)$, phase velocity $c_{ph} = \frac{\omega}{k}$, and the group velocity $c_g = \left(\frac{dk}{d\omega}\right)^{-1}$,

2) for a chosen point x, determination of the phase $\Phi = k(\omega)x - \omega t$ for different instances of time t,

3) determination of stationary points ω_s :
$\frac{d\Phi}{d\omega}(\omega_s) = 0 \quad \Rightarrow \quad \omega_s = \omega_s(t; x)$,

4) calculation of the derivative $\frac{dc_g}{d\omega}(\omega = \omega_s)$,

5) calculation of the amplitude $u(t; x)$ as a function of time.

Let us begin from the plots of the phase Φ for different values of time (Fig.13). The wave number k and the frequency ω are related by the implicit dispersion relation (90).

Obviously, the first signal arrives at $x = 500$ km after $t = 100$ seconds because the fastest disturbance corresponds to the monochromatic wave traveling with the velocity 5 km/s (see: Fig. 10). The last arrival at the instant approximately $t = 181$ seconds corresponds to the minimum group velocity $\min c_g \approx 2.74$ km/s.

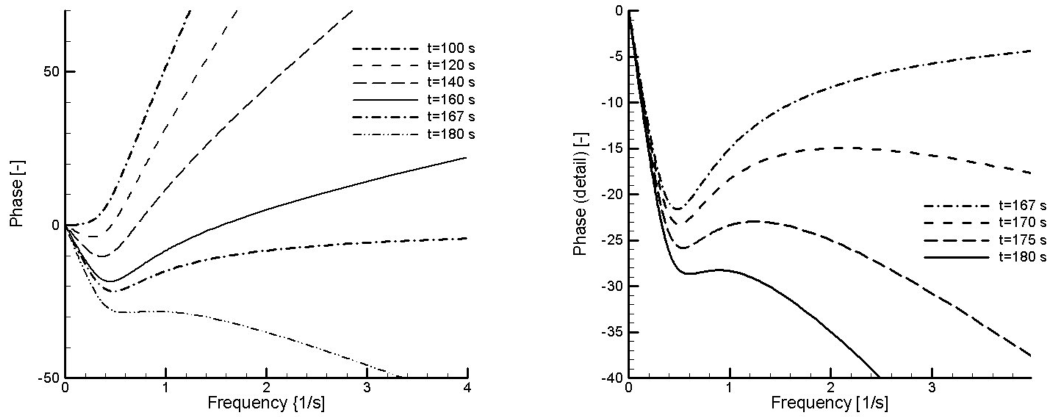


Fig. 13: *Plots of the phase $\Phi := kx - \omega t$ as a function of frequency ω for the data (92),* $x = 500$ km, *and different instances of time t. On the right hand side, we present some details in the important region of time (see: text).*

By means of the above curves for the phase we can find the location of extrema of phases for different times. These are shown in Fig. 14. There is no minimum before the first arrival at $t = 100$ sec. Then the value of frequency ω_s at which the phase possesses a minimum grows to the limit value at the arrival time with the minimum group velocity

$t = 181$ sec. In the time range between the arrival time with the limit phase velocity for $\omega \to \infty$ (i.e. $500/3 = 167$ sec.) and the largest arrival time $t = 181$ sec the phase possesses also a maximum. Corresponding frequencies are also plotted in Fig. 14.

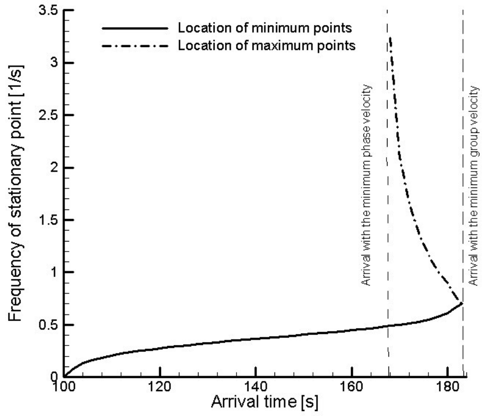


Fig. 14: *Frequencies ω_s of stationary points of the phase Φ as a function of arrival times*

The monotonous relation between the time t and the frequency ω_s of the phase minimum is sometimes used by seismologists to replace on seismograms the time scale by a scale based on the frequency.

The rest of steps required in the procedure presented in Box 5 we perform numerically. We choose the time step to be 2 seconds and calculate the group velocity and its derivative at frequencies ω_s for different instances of time, and subsequently the amplitude. The results are presented in Fig.15.

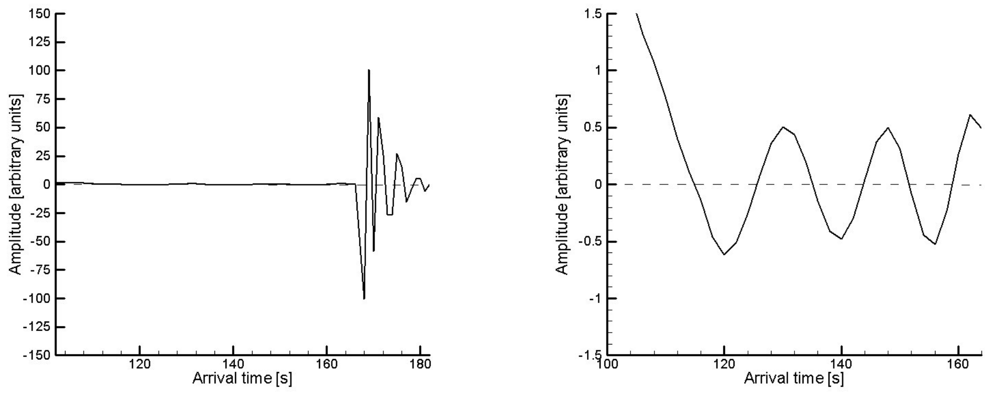


Fig. 15: *Amplitude as a function of arrival time. The left figure presents the amplitude in the whole range of existence of a nonzero amplitude while the right figure is a magnification of the amplitude in the range before the arrival with the limit group velocity* $c_g = 3$ km/s ($t \approx 167$ sec).

The quality of this calculation is rather poor and we can easily spot reasons for numerical problems. First of all, in order to evaluate the integral (94) we have used solely the approximation given by the formula (98). This is, of course, very bad in the vicinity of points $t = 100$ sec and $t = 167$ sec, where $\frac{dc_g}{d\omega}$ is equal to zero. It is visible in particular at the first instant of arrival time $t = 100$ sec where the amplitude should be zero (Fig. 13 right). The procedure of calculation should be changed in such regions to this based on the formula (99) or to even higher order approximations. Simultaneously, in the range of the Airy phase where the variations of the amplitude are very strong the time step should be made much smaller to get a sufficient accuracy.

However, in spite of their bad quality the above results reflect the most important features of the arrival of a package of dispersive waves. An example of a real seismogram from the earthquake Oklahoma is shown in Fig. 16.

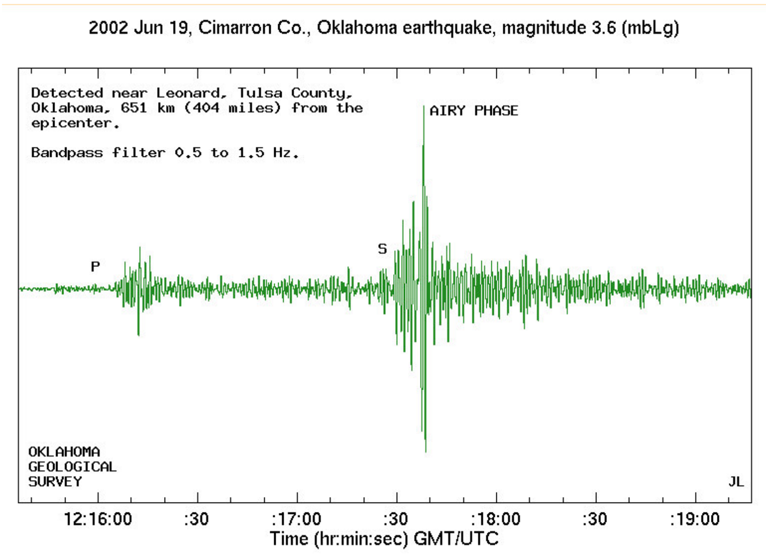


Fig. 16: *Arrivals recorded by Oklahoma (Tulsa County) earthquake in 2002 illustrating the notion of the Airy phase* (Oklahoma Geological Survey, A State Agency for Research and Public Services).

3.4 Rayleigh waves in a layer of elastic material

Now we investigate a problem similar to the propagation of Love waves, i.e. the propagation in a semiinfinite elastic body with a layer of thickness H and different material properties but we assume the amplitude of the wave to lie in P-SV-plane rather than in the SH-direction. This is the same assumption as in the case of Rayleigh waves on the plane boundary of the semiinfinite medium.

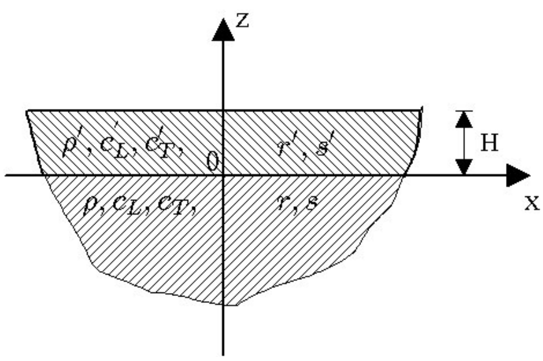


Fig. 17: *The geometry considered in the Rayleigh wave problem in the medium with layer*

The material properties are as follows. The mass density of the semiinfinite medium is ρ, the velocities of the longitudinal and transversal waves c_L, c_T, and in the layer they are ρ', c'_L, c'_T. We use the potentials for the description of displacements, i.e.

$$u_x = \frac{\partial \varphi}{\partial x} + \frac{\partial \psi_y}{\partial z}, \quad u_z = \frac{\partial \varphi}{\partial z} - \frac{\partial \psi_y}{\partial x} \quad \text{for} \quad z \leq 0,$$

$$u'_x = \frac{\partial \varphi'}{\partial x} + \frac{\partial \psi'_y}{\partial z}, \quad u'_z = \frac{\partial \varphi'}{\partial z} - \frac{\partial \psi'_y}{\partial x} \quad \text{for} \quad 0 \leq z \leq H. \tag{100}$$

Anticipating the existence of a surface wave in the semiinfinite medium, we make the following ansatz for solution

$$\varphi = A e^{krz} e^{i(kx-\omega t)}, \quad \psi_y = B e^{ksz} e^{i(kx-\omega t)},$$

$$\begin{aligned} \varphi' &= \left(A'_1 \sin\left(kr'z\right) + A'_2 \cos\left(kr'z\right)\right) e^{i(kx-\omega t)}, \\ \psi'_y &= \left(B'_1 \sin\left(ks'z\right) + B'_2 \cos\left(ks'z\right)\right) e^{i(kx-\omega t)}. \end{aligned} \tag{101}$$

These potentials fulfil wave equations of the form (50) with the appropriate material parameters. This requirement yields

$$r^2 = 1 - \frac{c^2}{c_L^2}, \quad s^2 = 1 - \frac{c^2}{c_T^2}, \quad c := \frac{\omega}{k},$$

$$r'^2 = \frac{c^2}{c_L'^2} - 1, \quad s'^2 = \frac{c^2}{c_T'^2} - 1. \tag{102}$$

Solutions of the problem should fulfil the following boundary conditions

1) The stress vector on the free surface $z = H$ must be zero

$$\begin{aligned} \rho' c_L'^2 \left(\frac{\partial^2 \varphi'}{\partial x^2} + \frac{\partial^2 \varphi'}{\partial z^2}\right) - \rho' c_T'^2 \left(\frac{\partial^2 \varphi'}{\partial x^2} + \frac{\partial^2 \psi'_y}{\partial x \partial z}\right) &= 0, \\ \rho' c_T'^2 \left(2\frac{\partial^2 \varphi'}{\partial x \partial z} + \frac{\partial^2 \psi'_y}{\partial z^2} - \frac{\partial^2 \psi'_y}{\partial x^2}\right) &= 0, \end{aligned} \tag{103}$$

2) The displacement and the stress vector must be continuous on the interface $z = 0$

$$\frac{\partial \varphi}{\partial x} + \frac{\partial \psi_y}{\partial z} = \frac{\partial \varphi'}{\partial x} + \frac{\partial \psi'_y}{\partial z}, \quad \frac{\partial \varphi}{\partial z} - \frac{\partial \psi_y}{\partial x} = \frac{\partial \varphi'}{\partial z} - \frac{\partial \psi'_y}{\partial x}, \tag{104}$$

$$\begin{aligned} &\rho c_L^2 \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2}\right) - \rho c_T^2 \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \psi_y}{\partial x \partial z}\right) = \\ &= \rho' c_L'^2 \left(\frac{\partial^2 \varphi'}{\partial x^2} + \frac{\partial^2 \varphi'}{\partial z^2}\right) - \rho' c_T'^2 \left(\frac{\partial^2 \varphi'}{\partial x^2} + \frac{\partial^2 \psi'_y}{\partial x \partial z}\right), \end{aligned} \tag{105}$$

$$\rho c_T^2 \left(2\frac{\partial^2 \varphi}{\partial x \partial z} + \frac{\partial^2 \psi_y}{\partial z^2} - \frac{\partial^2 \psi_y}{\partial x^2}\right) = \rho' c_T'^2 \left(2\frac{\partial^2 \varphi'}{\partial x \partial z} + \frac{\partial^2 \psi'_y}{\partial z^2} - \frac{\partial^2 \psi'_y}{\partial x^2}\right), \tag{106}$$

3) The solution vanishes for $z \to -\infty$.

These conditions form a homogeneous set of algebraic equations for constants $iA, B, iA'_1, B'_1, iA'_2, B'_2$. The matrix of coefficients has the following form

$$\mathbf{D}_L := \tag{107}$$

$$= \begin{bmatrix} 0 & 1 & s' & 0 & -1 & -s \\ r' & 0 & 0 & -1 & -r & 1 \\ 0 & \frac{\rho'}{\rho}\left(\frac{c^2}{c_T^2} - 2\frac{c_T'^2}{c_T^2}\right) & -2\frac{\rho'}{\rho}\frac{c_T'^2}{c_T^2}s' & 0 & 2-\frac{c^2}{c_T^2} & 2s \\ 2\frac{\rho'}{\rho}\frac{c_T'^2}{c_T^2}r' & 0 & 0 & 2-\frac{c^2}{c_T'^2} & -2r & -\left(2-\frac{c^2}{c_T^2}\right) \\ \left(2-\frac{c^2}{c_T'^2}\right)S_{r'} & \left(2-\frac{c^2}{c_T'^2}\right)C_{r'} & s'C_{s'} & -s'S_{s'} & 0 & 0 \\ 2r'C_{r'} & -2r'S_{r'} & \left(2-\frac{c^2}{c_T'^2}\right)S_{s'} & \left(2-\frac{c^2}{c_T'^2}\right)C_{s'} & 0 & 0 \end{bmatrix},$$

where for typographical reasons we have introduced the notation

$$S_{r'} = \sin kr'H, \quad C_{r'} = \cos kr'H, \quad S_{s'} = \sin ks'H, \quad C_{s'} = \cos ks'H. \tag{108}$$

Consequently, the determinant of this matrix yields the dispersion relation

$$\mathcal{P}_{RL} := \det \mathbf{D}_L = 0. \tag{109}$$

Solutions of this kind of equations are discussed in other contributions to this volume and, for this reason, we skip here a complicated analysis. However, two properties of this problem are immediately visible in the above relations. First of all, if there exists a solution of the above dispersion relation then, similarly to Love waves, it is **dispersive** and there exist infinitely many modes due to the contributions of trigonometric functions to the equation (109). Secondly, the relations (102) and the boundary condition in infinity yield the following necessary condition for the existence of solutions in the form of surface waves

$$c_L \geq c_T \geq c \geq c_L' \geq c_T'. \tag{110}$$

Further we discuss in some details the problem of existence of solutions solely for the much simpler problem of Stoneley waves.

4 Stoneley waves

4.1 Interface of two semiinfinite elastic solids

Properties of waves on an interface were considered for the first time by Stoneley in 1924 [62]. The interface investigated in this work is a classical ideal plane boundary between two linearly elastic isotropic solids and boundary conditions follow from the so-called jump conditions on the ideal singular surface. This means that all surface effects like surface tension, boundary layers, intermediate layers of a different material, etc. are neglected. In contrast to Love waves, amplitudes of the wave should decay with the distance from the interface in both directions: for the coordinate z going to $+\infty$ and to $-\infty$ (see: Fig. 18). Formally, one can construct the solution of the problem by taking the infinite limit of depth H of the layer in the solution for Rayleigh waves in a semiinfinite medium with the layer. However, following the work of Stoneley, we formulate the problem anew.

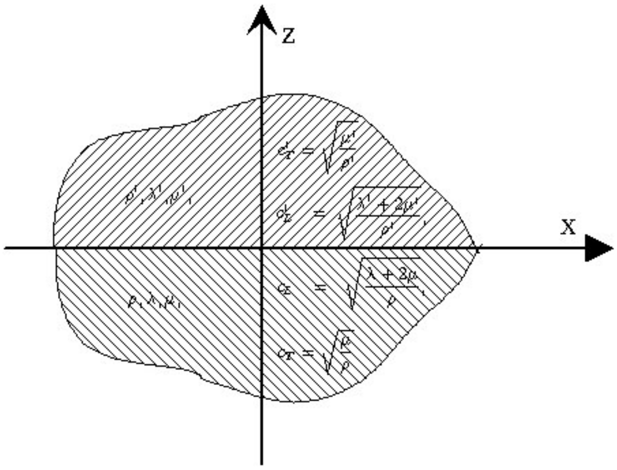


Fig: 18: *Interface of two elastic solids – Stoneley wave*

We consider two semiinfinite elastic media whose properties are described, as before, by the mass density ρ', the velocities of propagation of bulk waves: c'_L, c'_T in the upper part $(z > 0)$, and, respectively, ρ, c_L, c_T in the lower part $(z < 0)$. The bulk waves satisfy the equations (50) for potentials with appropriate constants in each part of the system. For technical reasons, it is convenient to choose the potentials $\varphi, \psi \equiv \psi_y, \varphi', \psi' \equiv \psi'_y$ for the components u_x, u_z, u'_x, u'_z of the displacement and the components of displacement u_y, u'_y as unknown. For the two-dimensional problem, the latter would be given solely by the difference of derivatives of the two components of the vector potentials: $\psi_x(x, z), \psi_z(x, z)$ and, correspondingly, by $\psi'_x(x, z), \psi'_z(x, z)$. Certainly, this can be replaced by combined functions u_y, u'_y.

Consequently, we make the following ansatz for the solution

$$\begin{aligned} \varphi &= Ae^{krz}e^{ik(x-ct)} \quad \text{for} \quad z<0, \quad \varphi' = A'e^{-kr'z}e^{ik(x-ct)} \quad \text{for} \quad z>0, \\ \psi &= Be^{ksz}e^{ik(x-ct)} \quad \text{for} \quad z<0, \quad \psi' = B'e^{-ks'z}e^{ik(x-ct)} \quad \text{for} \quad z>0, \qquad (111) \\ u_y &= Ce^{kqz}e^{ik(x-ct)} \quad \text{for} \quad z<0, \quad u'_y = C'e^{-kqs'z}e^{ik(x-ct)} \quad \text{for} \quad z>0. \end{aligned}$$

Hence, the wave is supposed to propagate in the x-direction, and the problem is assumed to be two-dimensional.

Substitution in equations (50) yields the following compatibility conditions for the exponents of potentials

$$\begin{aligned} r^2 &= 1 - \frac{c^2}{c_L^2}, \quad s^2 = 1 - \frac{c^2}{c_T^2}, \qquad (112) \\ r'^2 &= 1 - \frac{c^2}{c_L'^2}, \quad s'^2 = 1 - \frac{c^2}{c_T'^2}, \end{aligned}$$

As we see in a moment, the relations for q and q' are immaterial.

Boundary conditions follow from the continuity of displacements on the interface

$$[[\mathbf{u}]] \equiv \mathbf{u}' - \mathbf{u} = 0 \quad \text{for} \quad z = 0, \qquad (113)$$

and from the jump condition for the stress vector on the plane boundary $z = 0$ as well as from the Sommerfeld conditions for $|z| \to \infty$

$$\begin{aligned} [[\mathbf{T}]]\,\mathbf{n} &\equiv (\mathbf{T}' - \mathbf{T})\,\mathbf{n} = 0, \quad \text{for} \quad z = 0, \qquad (114) \\ \lim_{z\to\infty} \mathbf{u}' &= 0, \quad \lim_{z\to-\infty} \mathbf{u} = 0, \end{aligned}$$

where $\mathbf{n} = \mathbf{e}_z$ is the unit vector normal to the interface.

Hence, for real k, Stoneley waves may exist solely if r, s, q, r', s', q' are nonnegative.

The boundary condition for shear stresses in y-direction yields immediately

$$C = C' = 0, \tag{115}$$

i.e. a Stoneley wave with the amplitude in SH-direction (horizontal polarization) does not exist. The amplitude must lie in the vertical P-SV incident plane.

Bearing the relations (49) in mind, we obtain for $z = 0$

$$\begin{aligned} (iA' - s'B') - (iA + sB) &= 0, \\ (-r'iA' + B') - (riA + B) &= 0, \end{aligned} \tag{116}$$

$$\begin{gathered} \left[-\rho' c_L'^2 \left(1 - r'^2\right) + 2\rho' c_T'^2\right](iA') - 2\rho' c_T'^2 s' B' = \\ = \left[-\rho c_L^2 \left(1 - r^2\right) + 2\rho c_T^2\right](iA') + 2\rho c_T^2 sB, \\ \rho' c_T'^2 \left[-2iA'r' + B'\left(1 + s'^2\right)\right] = \rho c_T^2 \left[2iAr + B\left(1 + s^2\right)\right]. \end{gathered} \tag{117}$$

As usual, this set of homogeneous algebraic equations yields the dispersion relation.

It can be written in the following compact form

$$\mathbf{D}_{St}\mathbf{X} = 0, \quad \mathbf{X} := \left[iA', B', iA, B\right]^T, \tag{118}$$

where the matrix $\mathbf{D}_{St}$ is defined as follows

$$\mathbf{D}_{St} := \tag{119}$$

$$= \begin{bmatrix} -1 & \sqrt{1 - \dfrac{c^2}{c_T'^2}} & 1 & \sqrt{1 - \frac{c^2}{c_T^2}} \\ \sqrt{1 - \dfrac{c^2}{c_L'^2}} & -1 & \sqrt{1 - \dfrac{c^2}{c_L^2}} & 1 \\ -\frac{\rho'}{\rho}\frac{c_T'^2}{c_T^2}\left(2 - \dfrac{c^2}{c_T^2}\right) & 2\frac{\rho'}{\rho}\frac{c_T'^2}{c_T^2}\sqrt{1 - \dfrac{c^2}{c_T'^2}} & 2 - \dfrac{c^2}{c_T^2} & 2\sqrt{1 - \dfrac{c^2}{c_T^2}} \\ 2\frac{\rho'}{\rho}\frac{c_T'^2}{c_T^2}\sqrt{1 - \dfrac{c^2}{c_L'^2}} & -\frac{\rho'}{\rho}\frac{c_T'^2}{c_T^2}\left[2 - \dfrac{c^2}{c_T'^2}\right] & 2\sqrt{1 - \dfrac{c^2}{c_L^2}} & 2 - \dfrac{c^2}{c_L^2} \end{bmatrix}.$$

The determinant of this matrix gives rise to the dispersion relation for Stoneley waves

$$\mathcal{P}_{St} := \det \mathbf{D}_{St} = 0. \tag{120}$$

One important property of Stoneley waves follows immediately from the inspection of matrix $\mathbf{D}_{St}$. Namely, if there exists a real solution $\frac{c}{c_T}$ of the equation (120) then it is independent of the frequency. Hence, like Rayleigh waves, Stoneley waves are **nondispersive.**

The question of existence of Stoneley waves is far from being trivial. Already in the original work [62] Stoneley has shown that for certain combinations of material parameters these waves may not exist (see also: [1], [15]).

In Figure 19, we quote some existence results of J. G. Scholte [57] for two limit cases of material parameters. Solutions for the Stoneley wave exist solely in the range between the two curves A and B.

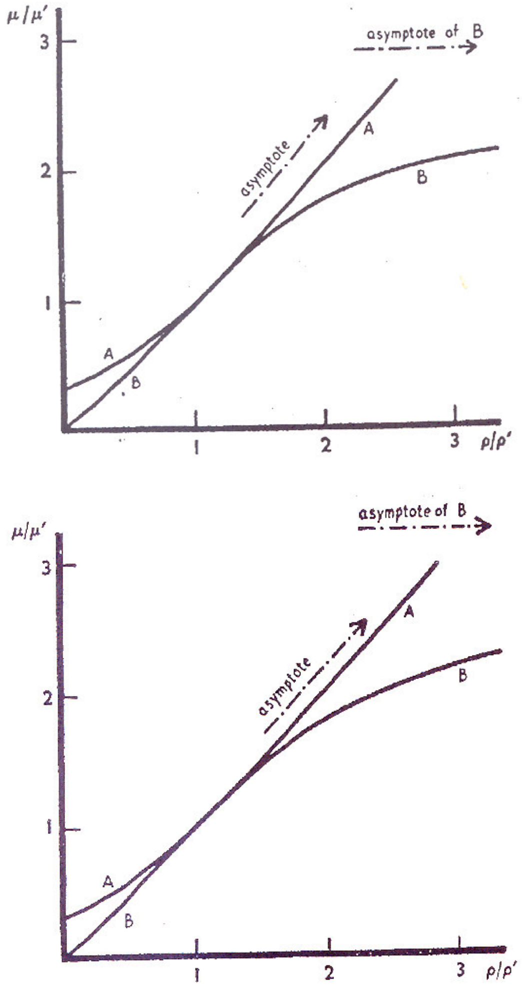


Fig. 19: *Existence results of Scholte [57] for two limit values of material parameters:*

the upper Figure corresponds to the case $\frac{\lambda}{\mu} = \frac{\lambda'}{\mu'} = 1$, *i.e.* $\mathfrak{m} = \frac{1}{\sqrt{3}}, \mathfrak{c}_L = \mathfrak{c}_T$;

the lower Figure corresponds to the case $\lambda = \lambda' \rightarrow \infty$, *i.e.* $\mathfrak{m} = 0, \mathfrak{c}_L = 1$.

An extensive study of existence of this wave was also carried through by Ginzbarg and Strick whose paper [30] contains many graphs of ranges of existence.

4.2 Interface of a semiinfinite elastic solid and a semiinfinite ideal fluid

The interface between the linear elastic solid and the ideal fluid yields the problem of surface waves which can be obtained from the above problem of Stoneley waves as a particular case. Equations for this problem were formulated by Scholte in 1947 [58] and, for this reason, these waves are sometimes called Scholte waves.

We derive the dispersion relation again from the governing equations indicating some points which are common with modelling of surface waves for porous materials with permeable boundaries.

We choose the same coordinates as before and assume that the fluid in the range $z > 0$ is ideal, i.e. we have the following linear equations for fields $\rho', \mathbf{v}'$ in this region

$$\frac{\partial \rho'}{\partial t} + \rho'_0 \operatorname{div} \mathbf{v}' = 0, \quad \rho'_0 \frac{\partial \mathbf{v}'}{\partial t} = -\operatorname{grad} p', \quad p' = p'_0 + \kappa' \left(\rho' - \rho'_0\right), \tag{121}$$

where κ' is the compressibility coefficient. Consequently

$$\frac{\partial^2 \mathbf{v}'}{\partial t^2} = c_L'^2 \,\mathrm{grad}\,\mathrm{div}\,\mathbf{v}', \quad c_L'^2 = \kappa'. \tag{122}$$

It is customary to introduce for technical reasons the displacement vector $\mathbf{u}'$ also for the fluid component

$$\mathbf{v}' = \frac{\partial \mathbf{u}'}{\partial t} \quad \Rightarrow \quad \frac{\partial^2 \mathbf{u}'}{\partial t^2} = c_L'^2 \,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}' + \mathbf{C}(\mathbf{x}), \tag{123}$$

where $\mathbf{C}(\mathbf{x})$ is an arbitrary function of the spacial variable. We can redefine the displacement in the following way

$$\mathbf{u}' \to \mathbf{u}' + \mathbf{U}(\mathbf{x}), \quad \mathrm{grad}\,\mathrm{div}\,\mathbf{U} = -\mathbf{C}, \tag{124}$$

and eliminate this arbitrary function entirely because we are interested solely in the time derivative of $\mathbf{u}'$.

It is easy to see that only the potential part of $\mathbf{u}'$ is of interest. Hence

$$\mathbf{u}' = \mathrm{grad}\,\varphi' \quad \Rightarrow \quad \frac{\partial^2 \varphi'}{\partial t^2} = c_L'^2 \nabla^2 \varphi', \tag{125}$$

where an arbitrary function of time was incorporated in the potential.

Hence, the solution for monochromatic waves in this case can be written in the form (111), (112) in which $B' = 0$.

We have to modify also the boundary conditions. Instead of the continuity of motion on the interface between two solids we have now solely the condition of continuity of the normal velocity

$$\mathbf{v}' \cdot \mathbf{n} = \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{n} \quad \text{for} \quad z = 0. \tag{126}$$

For the ideal fluid the tangential component of the velocity can be arbitrary and, consequently, we have a scalar condition instead of the vector condition (113).

Condition for stresses must be modified as well because now we have the shear stress in the fluid identically zero and the normal component of stresses is reduced to the pressure. The latter has to be written in terms of the displacement potential φ'. Namely, it follows by the integration of the mass balance $(121)_1$

$$\rho' = \rho_0' (1 - \mathrm{div}\,\mathbf{u}') = \rho_0' \left(1 - \nabla^2 \varphi'\right) \quad \Rightarrow \quad p' = p_0' - \rho_0' c_L'^2 \nabla^2 \varphi'. \tag{127}$$

Boundary conditions for stresses at the boundary $z = 0$ have now the form

$$\rho_0' c_L'^2 \nabla^2 \varphi' = \rho c_L^2 \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} \right) - 2\rho c_T^2 \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial z} \right), \tag{128}$$

$$2\frac{\partial^2 \varphi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial^2 \psi}{\partial x^2} = 0.$$

Substitution of relations (111) yields again the homogeneous set of equations

$$\mathbf{D}_{Sch} \mathbf{X} = 0, \quad \mathbf{X} := \left[iA', iA, B\right]^T, \tag{129}$$

where

$$\mathbf{D}_{Sch} := \begin{bmatrix} \sqrt{1-\frac{c^2}{c_L'^2}} & \sqrt{1-\frac{c^2}{c_L^2}} & 1 \\ \frac{\rho_0'}{\rho}\frac{c^2}{c_T^2} & 2-\frac{c^2}{c_T^2} & 2\sqrt{1-\frac{c^2}{c_T^2}} \\ 0 & 2\sqrt{1-\frac{c^2}{c_L^2}} & 2-\frac{c^2}{c_T^2} \end{bmatrix}. \tag{130}$$

The determinant of this matrix yields immediately the following

Box 6: *Stoneley (Scholte) dispersion relation*

$$\sqrt{1-\frac{c^2}{c_L'^2}}\mathcal{P}_R + \frac{\rho_0'}{\rho}\frac{c^4}{c_T^4}\sqrt{1-\frac{c^2}{c_L^2}} = 0, \tag{131}$$

where

$$\mathcal{P}_R := \left(2-\frac{c^2}{c_L^2}\right)^2 - 4\sqrt{1-\frac{c^2}{c_L^2}}\sqrt{1-\frac{c^2}{c_T^2}}. \tag{132}$$

The last relation is identical with the dispersion function (57) defining Rayleigh waves.

It is obvious that velocity c is independent of the frequency ω and, consequently, also this surface wave is **nondispersive**.

The problem of existence of Stoneley-Scholte waves has been rather intensively investigated (e.g. [63], [64]). It can be shown that a single real solution exists if $c_L' < c_R$, where c_R is Rayleigh velocity and it is smaller than c_L', i.e. it is the smallest velocity of propagation appearing in the system.

One of the important properties of the Stoneley-Scholte wave is that the energy is carried by this wave primarily in the fluid. This can be proven by the analysis of amplitudes which we do not present here.

Apart from this real velocity, there exist complex solutions of the dispersion relation of Box 6. These were discussed in details by Ansell [6]. Some properties of complex roots yielding leaky waves [51] shall be briefly presented Subsection 4.4.

Propagation of surface waves in the system described above but either with a weak anisotropy of the solid or the fluid under a hydrostatic pressure has been investigated by Norris and Sinha [49]. They have used a perturbation method and found explicit corrections to the velocity of Scholte waves.

4.3 Interface of a semiinfinite elastic solid and a layer of an ideal fluid

In this case, one can consider waves intermediate between Stoneley and Rayleigh varying the height H of the layer. We quote here solely a finite result which can be derived in the same way as the above presented examples of surface waves. We obtain [72]

$$\sqrt{\frac{c^2}{c_L'^2}-1}\mathcal{P}_R+\frac{\rho'}{\rho}\frac{c^4}{c_T^4}\sqrt{1-\frac{c^2}{c_L^2}}\tan\left(\frac{\omega H}{c}\sqrt{\frac{c^2}{c_L'^2}-1}\right)=0. \tag{133}$$

where $\mathcal{P}_R$ is again the Rayleigh dispersion function.

Clearly these waves are dispersive, there exist infinitely many surface modes whose velocity is bigger than the velocity in the fluid. The above dispersion relation possesses also complex solutions which we discuss in the next Subsection.

4.4 Few remarks on leaky waves

It has been seen already in the Rayleigh dispersion relation that for some values of material parameters some solutions $k(\omega)$ may be complex. This means that the factor $e^{i(kx-\omega t)}=e^{-\operatorname{Im}kx}e^{i(\operatorname{Re}kx-\omega t)}$ of the ansatz for the solution yields a decay of the amplitude characteristic for the dissipation. However, the whole system is reversible – there are no losses of energy due to the heat transfer or some other means of dissipation. Consequently, such solutions can be admissible solely under the condition that the energy of this mode of propagation must be transferred on some other modes – bulk or surface waves. If this is indeed the case we say that the wave loosing the energy is **leaky.**

We present briefly an example of such a wave considered by Viktorov [72]. It corresponds to one of the complex solutions of equation (133). Viktorov found analytically the complex solution of this equation under the assumption of small Poisson's ratio ν. He was able to show that this solution yields attenuation of the wave $\operatorname{Im}k\sim\nu^4$ and the velocity is of the order of the longitudinal wave c_L. The wave consists of the longitudinal and transversal parts. The amplitude of the longitudinal part decays exponentially with the depth (i.e. it is indeed a surface wave). The transversal part behaves like a bulk wave propagating from the boundary under the angle $\beta=\frac{\pi}{4}$ (see: Fig. 20). The energy of this wave is transferred on the transversal bulk wave.

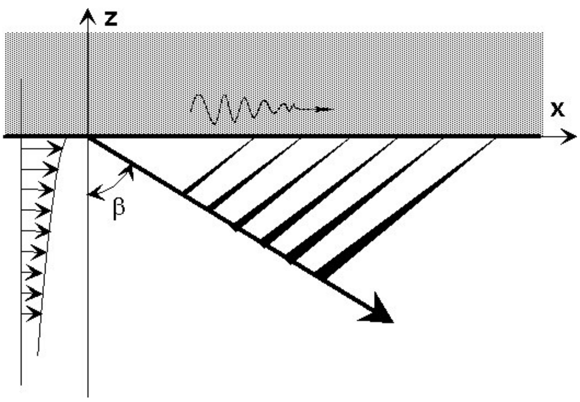


Fig. 20: *Scheme of propagation of the first leaky wave for the semiinfinite body with a layer of fluid.*

The existence of leaky waves has an important practical bearing. For instance, Norris [47] has presented a very neat model explaining a phenomenon of backscattering of ultrasound from a fluid-solid interface. It is shown that the energy radiated back into the fluid where the incident beam originates can be observed when the angle of incidence lies near the so-called leaky wave angle. Hence, it is proven that the backscattering is due to a leaky wave reflection zone.

Existence of leaky modes depends on the distribution of complex roots of the dispersion relation. Phinney [51] has designed a fairly general method (a generalization of the so-called Rosenbaum method) to find these roots by the analysis of a single integral expression. The method refers to properties of such an integral when the path of integration changes the Riemann surface.

5 Elastic two-component media

5.1 Biot's model and a simple mixture model of two-component poroelastic materials

5.1.1 Introduction

In contrast to single component elastic models, continuous modelling of porous materials requires the construction of a two-component model for fields describing the motion and deformation of the solid component (skeleton) and of the fluid component. In addition, it has been shown that modelling of porous media leads to the so-called **immiscible mixtures** which require an additional variable of the volume fraction of the fluid component (porosity). Porosity, in addition to the concentration of the fluid component in the mixture (i.e. the ratio of the partial mass density to the full mass density of the mixture), is the independent field in such a mixture theory.

We limit the attention to **linear poroelastic** models. Then the unknown fields are the following functions of spacial variable $\mathbf{x} \in \mathcal{B}$ and time $t \in \mathcal{T}$ ($\mathcal{B} \subset \Re^3$ is the current domain occupied simultaneously by both components, and $\mathcal{T} \subset \Re$ is the interval of time)

$$\left\{\rho^F, \mathbf{v}^F, \mathbf{v}^S, \mathbf{e}^S, n\right\}, \tag{134}$$

where ρ^F is the current mass density of the fluid component referred to the unit macroscopic volume of the mixture, $\mathbf{v}^F, \mathbf{v}^S$ are macroscopic velocity fields of the fluid and of the skeleton, respectively, $\mathbf{e}^S$ is the Almansi-Hamel tensor of small macroscopic deformations of the skeleton, and n is the current porosity. The current mass density of the skeleton ρ^S does not appear in this list because in the case of small deformations it is determined by the tensor of deformations

$$\rho^S = \rho_0^S \left(1 - \operatorname{tr} \mathbf{e}^S\right), \tag{135}$$

where ρ_0^S denotes the initial value of this mass density (in a chosen reference configuration for which $\mathbf{e}^S = 0$), and $\operatorname{tr} \mathbf{e}^S$ reflects small volume changes of the skeleton.

In soil mechanics and geotechnics usually the set of fields does not coincide with this presented above. Instead of the mass density of the fluid, often the concepts of the volume change of the fluid ε or the **increment of the fluid content** ζ are used. They are defined by the following relations

$$\varepsilon := \frac{\rho_0^F - \rho^F}{\rho_0^F}, \quad \zeta := n_0 \left(\operatorname{tr} \mathbf{e}^S - \varepsilon\right), \tag{136}$$

where ρ_0^F is the initial value of the fluid mass density (in the reference configuration $\mathbf{e}^S = 0$), and n_0 is the initial porosity in the reference configuration.

A lot of confusion arises also due to the fact that the mass densities are sometimes referred not to the common macroscopic unit volume but rather to partial volumes of components. These so-called **true mass densities** satisfy the following relations

$$\rho^{SR} = \frac{\rho^S}{1-n}, \quad \rho^{FR} = \frac{\rho^F}{n}, \tag{137}$$

where n is the current porosity.

In our presentation we either rely on the choice (134) or we replace ρ^F by ε.

Two-component models describing these fields are based on partial balance equations. In the linear model, they have the following form

1) mass balance equation of the fluid component

$$\frac{\partial \rho^F}{\partial t} + \rho_0^F \operatorname{div} \mathbf{v}^F = 0 \quad \Rightarrow \quad \frac{\partial \varepsilon}{\partial t} = \operatorname{div} \mathbf{v}^F, \tag{138}$$

2) partial momentum balance equations for both components

$$\rho_0^S \frac{\partial \mathbf{v}^S}{\partial t} = \operatorname{div} \mathbf{T}^S + \hat{\mathbf{p}} + \rho^S \mathbf{b}^S, \quad \rho_0^F \frac{\partial \mathbf{v}^F}{\partial t} = \operatorname{div} \mathbf{T}^F - \hat{\mathbf{p}} + \rho^F \mathbf{b}^F, \tag{139}$$

where $\mathbf{T}^S, \mathbf{T}^F$ are partial Cauchy stress tensors, $\hat{\mathbf{p}}$ denotes the source of momentum, and $\mathbf{b}^S, \mathbf{b}^F$ are external mass forces,

3) integrability condition for the deformation of the skeleton

$$\frac{\partial \mathbf{e}^S}{\partial t} = \operatorname{sym} \operatorname{grad} \mathbf{v}^S, \tag{140}$$

and this is equivalent to the existence of the displacement vector $\mathbf{u}^S$ for the skeleton.

We have to close this system by appropriate constitutive relations. Further in this article we discuss two models: the **Biot's model** and the **simple mixture model**.

The Biot's model is based on the following constitutive relations ([10], see also a full collection of Biot's works on porous materials [65])

$$\begin{aligned}
\mathbf{T}^S &= \mathbf{T}_0^S + \lambda^S \operatorname{tr} \mathbf{e}^S \mathbf{1} + 2\mu^S \mathbf{e}^S + Q\varepsilon \mathbf{1}, \\
\mathbf{T}^F &= -p^F \mathbf{1}, \quad p^F = p_0^F - \kappa \rho_0^F \varepsilon - Q \operatorname{tr} e^S, \\
\hat{\mathbf{p}} &= \pi \left(\mathbf{v}^F - \mathbf{v}^S \right) - \rho_{12} \left(\frac{\partial \mathbf{v}^F}{\partial t} - \frac{\partial \mathbf{v}^S}{\partial t} \right), \\
n &= n_0 \left(1 + \delta \operatorname{tr} \mathbf{e}^S + \gamma \left(\operatorname{tr} \mathbf{e}^S - \varepsilon \right) \right),
\end{aligned} \tag{141}$$

where $\mathbf{T}_0^S, p_0^F$ are the initial partial stress in the skeleton and the initial partial pressure in the fluid, respectively. λ^S, μ^S are effective (i.e. macroscopic, dependent on the initial porosity n_0) Lamé constants describing the skeleton, κ is the effective compressibility coefficient of the fluid component, Q is the **coupling parameter** between partial stresses, ρ_{12} is the **added mass parameter**, and δ, γ are parameters describing changes of porosity caused by volume changes of both components.

The above set of material parameters is not commonly accepted and the literature is full of its different variations. For instance, Stoll [61] relies on the following set

$$\begin{aligned}
K &= \lambda^S + \frac{2}{3}\mu^S + \rho_0^F \kappa + 2Q, \quad G = \mu^S, \\
C &= \frac{1}{n_0} \left(Q + \rho_0^F \kappa \right), \quad M = \frac{\rho_0^F \kappa}{n_0^2}.
\end{aligned} \tag{142}$$

In the standard reference book on linear acoustics of porous materials of Bourbie, Coussy, Zinszner [14], the following set of equations is applied

$$\rho \frac{\partial \mathbf{v}^S}{\partial t} + \rho_{uw} \frac{\partial^2 \mathbf{w}}{\partial t^2} = \operatorname{div} \mathbf{T}, \quad \mathbf{T} := \mathbf{T}^S + \mathbf{T}^F,$$

$$\rho_{uw} \frac{\partial \mathbf{v}^S}{\partial t} + \rho_w \frac{\partial^2 \mathbf{w}}{\partial t^2} = -\operatorname{grad} p - \frac{1}{\mathcal{K}} \mathbf{w}, \quad \mathbf{w} := n_0 \left(\mathbf{v}^F - \mathbf{v}^S \right), \tag{143}$$

$$\mathbf{T} = \lambda_f \operatorname{tr} \mathbf{e}^S \mathbf{1} + 2\mu \mathbf{e}^S - \beta M \zeta \mathbf{1}, \quad \mathbf{T}^F = -n_0 p \mathbf{1}, \quad p = M \left(-\beta \operatorname{tr} \mathbf{e}^S + \zeta \right),$$

where $\mathbf{w}$ corresponds to the so-called filter velocity and

$$\rho := \rho_0^S + \rho_0^F, \quad \rho_{uw} := \rho_0^{FR}, \quad \rho_w := \frac{1}{n_0^2} \left(\rho_0^F - \rho_{12} \right), \tag{144}$$

$$\lambda_f := \lambda^S + \kappa \rho_0^F + 2Q, \quad \mu := \mu^S, \quad \beta M := \frac{1}{n_0} \left(Q + \kappa \rho_0^F \right), \quad \frac{1}{\mathcal{K}} := \frac{\pi}{n_0^2}.$$

The quantity $p = \frac{1}{n_0} p^F$ is called the **fluid pore pressure** in contrast to p^F which is the partial pressure.

Certainly, all these relations are equivalent to the original Biot's equations.

The added mass is usually related to the **tortuosity** a by the relation

$$\rho_{12} = \rho_0^F (1 - a). \tag{145}$$

In addition to the porosity n, the tortuosity a describes the morphology of the porous material and, roughly speaking, it is the ratio of the average length of microchannels in the porous material to the average characteristic distance on the microlevel, i.e. for straight channels it is equal to one and otherwise bigger than one: $1 \leq a < \infty$. In the linear model, this quantity is constant.

It can be easily shown that the added mass effect incorporated in the Biot's model through the relative acceleration violates the **principle of material objectivity** [79] and the coupling effect described by the material parameter Q violates the **second law of thermodynamics** [82]. However, one can construct a **nonlinear model** [83], [84] which satisfies these two principles of continuum thermodynamics and, simultaneously, whose linearization leads to the Biot's model. In this sense the Biot's model can be considered as a thermodynamically sound way of a **linear description** of dynamics of saturated porous materials.

The second model which can be used to describe dynamics of linear poroelastic materials is the so-called **simple mixture model.** This model has been constructed by means of continuum thermodynamics for modelling **nonlinear** processes [76], [77]. For this reason changes of porosity are described by the balance equation rather then the relation $(141)_4$. Namely, in the linear model this equation is of the form

$$\frac{\partial (n - n_E)}{\partial t} + \Phi_0 \operatorname{div} \left(\mathbf{v}^F - \mathbf{v}^S \right) = -\frac{n - n_E}{\tau}, \quad n_E = n_0 \left(1 + \delta \operatorname{tr} \mathbf{e}^S \right), \tag{146}$$

where τ is the **relaxation time of porosity** and $\Phi_0 = n_0 \gamma$.

The coupling parameter Q is equal to zero in the simple mixture model and the tortuosity $a = 1$, i.e there is no influence of relative acceleration. Such a model satisfies both the principle of material objectivity and the second law of thermodynamics.

The questions arise if the simple mixture model reflects properly dynamical features of porous materials and what are the differences between the results obtained by the Biot's model and the simple mixture model. Before we proceed to a discussion of surface waves described by a two-component model we address these two questions.

5.1.2 Objective relative acceleration

In order to appreciate the problem of material objectivity, we consider some kinematic properties of the two-component system which we describe in two different reference systems specified by the following transformation of the point $\mathbf{x}$ in the space of motions (the so-called configuration space)

$$\mathbf{x}^{*} = \mathbf{O}\,(t)\,\mathbf{x} + \mathbf{c}\,(t)\,, \quad \mathbf{O}^{-1} = \mathbf{O}^{T}, \tag{147}$$

where the time dependent orthogonal matrix $\mathbf{O}$ describes the rotation of one reference system with respect to the other while the time dependent vector $\mathbf{c}$ describes the motion of the origin of one system with respect to the other.

This transformation (**isometry**) describes the most general change of reference which preserves the distance between two arbitrary points, say $\mathbf{x}_1, \mathbf{x}_2$:

$$|\mathbf{x}_1^{*} - \mathbf{x}_2^{*}|^2 = (\mathbf{x}_1 - \mathbf{x}_2) \cdot \mathbf{O}^{T}\mathbf{O}\,(\mathbf{x}_1 - \mathbf{x}_2) \equiv |\mathbf{x}_1 - \mathbf{x}_2|^2\,. \tag{148}$$

In particular, this transformation may describe the change of the inertial reference system into a noninertial one (e.g. the change from the reference with respect to fixed stars to the reference which rotates with a turntable in the laboratory).

It is shown in continuum mechanics that the above transformation yields certain rules of transformation for fields of mechanics. Any scalar quantity does not change under this transformation. We say for this reason that scalars are **objective**. Vectors and tensors may or may not be objective. If a vector $\mathbf{w}$ and a tensor of the second grade $\mathbf{e}$ transform according to the rules

$$\mathbf{w}^{*} = \mathbf{O}\mathbf{w}, \quad \mathbf{e}^{*} = \mathbf{O}\mathbf{e}\mathbf{O}^{T}, \tag{149}$$

we call them objective. For instance, the Almansi-Hamel deformation tensor is objective. Certainly, these rules reduce to simple transformations of coordinates if we refer the objects to their coordinates.

Time dependence of the orthogonal tensor $\mathbf{O}$ and of the vector $\mathbf{c}$ yield modifications of these transformation rules if they involve some time differentiation. In the case of a two-component mixture for the basic kinematic quantities we have the rules of the following form

1) partial velocities

$$\mathbf{v}^{S*} = \mathbf{O}\mathbf{v}^{S} + \dot{\mathbf{O}}\mathbf{x} + \dot{\mathbf{c}}, \quad \mathbf{v}^{F*} = \mathbf{O}\mathbf{v}^{F} + \dot{\mathbf{O}}\mathbf{x} + \dot{\mathbf{c}}, \tag{150}$$

2) partial velocity gradients

$$\begin{aligned} \mathbf{L}^{S*} &= \mathbf{O}\mathbf{L}^{S}\mathbf{O}^{T} + \dot{\mathbf{O}}\mathbf{O}^{T}, \quad \mathbf{L}^{F*} = \mathbf{O}\mathbf{L}^{F}\mathbf{O}^{T} + \dot{\mathbf{O}}\mathbf{O}^{T}, \\ \mathbf{L}^{S} &\equiv \operatorname{grad}\mathbf{v}^{S}, \quad \mathbf{L}^{F} \equiv \operatorname{grad}\mathbf{v}^{F}, \end{aligned} \tag{151}$$

3) accelerations

$$\mathbf{a}^{S*} = \mathbf{O}\mathbf{a}^{S} + 2\dot{\mathbf{O}}\mathbf{v}^{S} + \ddot{\mathbf{O}}\mathbf{x} + \ddot{\mathbf{c}}, \quad \mathbf{a}^{F*} = \mathbf{O}\mathbf{a}^{F} + 2\dot{\mathbf{O}}\mathbf{v}^{F} + \ddot{\mathbf{O}}\mathbf{x} + \ddot{\mathbf{c}}, \tag{152}$$

and the nonobjective contributions lead to the so-called Coriolis, centrifugal, Euler and translational accelerations (e.g. [40]). When multiplied by the mass densities they may be considered as **apparent mass forces**. It is known that such forces indeed appear in equations of motion in noninertial frames of reference.

It is obvious that the relative acceleration does not transform in an objective manner. We have

$$\mathbf{a}^{F*} - \mathbf{a}^{S*} = \mathbf{O}\left(\mathbf{a}^{F} - \mathbf{a}^{S}\right) + 2\dot{\mathbf{O}}\left(\mathbf{v}^{F} - \mathbf{v}^{S}\right). \tag{153}$$

Hence, the result depends on an arbitrary angular velocity $\dot{\mathbf{O}}$. As the relative acceleration enters the momentum balance equations with the constitutive (material) parameter ρ_{12} solutions of these equations depend on the choice of the reference system in a nontrivial manner. Hence, such contributions violate the principle of material objectivity and they are not admissible in continuum models.

Considering a suspension of bubbles in a fluid Drew, Cheng and Lahey, Jr. [22] have introduced a definition of the relative acceleration which transforms in an objective manner. Such a definition reminds definitions of various objective time derivatives appearing in continuum mechanics (such as Jaumann derivative, Oldroyd derivative, Truesdell derivative, etc.). It is rather easy to show that an **objective relative acceleration** can be also introduced in the nonlinear mechanics of porous materials [84]. The definition may have the following form

$$\begin{aligned} \mathbf{a}_r &= \frac{\partial}{\partial t}\left(\mathbf{v}^{F} - \mathbf{v}^{S}\right) + \left(\mathbf{L}^{F} - \mathbf{L}^{S}\right)\mathbf{v}^{S} - \left(1 - \mathfrak{z}\right)\mathbf{L}^{F}\left(\mathbf{v}^{F} - \mathbf{v}^{S}\right) - \mathfrak{z}\mathbf{L}^{S}\left(\mathbf{v}^{F} - \mathbf{v}^{S}\right), \\ \mathbf{a}_r^{*} &= \mathbf{O}\mathbf{a}_r, \end{aligned} \tag{154}$$

where $\mathfrak{z}$ is an arbitrary scalar. Some properties of a thermodynamical model constructed with a contribution of such a relative acceleration can be found elsewhere [84] but we do not need to go into any details in this work. It is sufficient to state that the thermodynamics admits a linear dependence of the momentum source $\hat{\mathbf{p}}$ on the objective relative acceleration, i.e the following relation for momentum source

$$\hat{\mathbf{p}} = \pi\left(\mathbf{v}^{F} - \mathbf{v}^{S}\right) - \rho_{12}\mathbf{a}_r, \tag{155}$$

satisfies the principle of material objectivity and the second law of thermodynamics.

It is clear that the Biot's contribution follows as a linearization of the above formula. One has $\rho_{12}\mathbf{a}_r \approx \rho_{12}\frac{\partial}{\partial t}\left(\mathbf{v}^{F} - \mathbf{v}^{S}\right)$ as required by the relations (141) of the Biot's model. After such a linearization the model is not objective anymore and, consequently, we cannot transform reference systems within the Biot's model. Experiments performed in static conditions cannot be compared with experiments performed in a centrifuge if we process the experimental data with the Biot's model.

We return to the problem of added mass further in order to estimate the order of its contributions to the propagation conditions of acoustic waves.

5.1.3 Gassmann relations

The problem of violation of the second law of thermodynamics by Biot's equations is related to the fact that thermodynamics of mixtures does not admit couplings between components if constitutive relations do not contain higher gradients. This property has been noticed by I. Müller in 1973 [44], [45] who has shown that partial quantities describing components of a mixture of fluids such as partial free energies or partial pressures may depend on partial mass densities of all components solely in the case when they also depend in a constitutive manner on gradients of mass densities. Otherwise the mixture becomes **simple**, i.e. partial free energies, partial pressures, etc. depend solely on their own partial mass densities.

The two-component model of porous materials possesses the same property. The partial stress tensor in the skeleton, say, may depend on volume changes of the fluid only if it depends also on some higher gradients such as the gradient of porosity. Otherwise the second law of thermodynamics yields $Q \equiv 0$.

A thermodynamical correction of the model through the extension of the set of constitutive variables has been introduced in the work [82]. It has been shown that a dependence on the gradient of porosity $\operatorname{grad} n$ leads to the following constitutive relations for partial stresses in the linear model

$$\begin{aligned} \mathbf{T}^S &= \mathbf{T}_0^S + \lambda^S \operatorname{tr} \mathbf{e}^S \mathbf{1} + 2\mu^S \mathbf{e}^S + Q\varepsilon \mathbf{1} + \beta (n - n_E) - N (n - n_0), \\ \mathbf{T}^F &= -p^F \mathbf{1}, \quad p^F = p_0^F - \kappa \rho_0^F \varepsilon - Q \operatorname{tr} e^S + \beta (n - n_E) - N (n - n_0), \end{aligned} \tag{156}$$

where the parameter β describes the influence of local nonequilibrium changes of porosity and the parameter N stems from the contribution to the source of momentum of the form $N \operatorname{grad} n$. It has been discussed in earlier works on acoustics of poroelastic materials (e.g. [78]) that the influence of β is negligible in processes with slow relative motion (small filtration velocity). This is the case in geomechanics and we neglect this contribution also in this article, i.e. we assume $\beta \equiv 0$.

However, the influence of the constant N cannot be evaluated so easily and we shall do it in this section by means of some Gedankenexperiments. Their role in geoacoustics is quite fundamental (e.g. [73]) and we present here the main results.

One of the main questions in applications of waves to nondestructive testing are relations between macroscopic quantities entering a model describing waves and microscopic quantities which we are trying to test. The typical example of such a quantity is the porosity of porous materials. We present here a brief review of results for granular materials which follow by means of certain static Gedankenexperiments for saturated materials. These experiments proposed by Gassmann [29] and incorporated into the Biot's model by Biot and Willis [11] describe the behavior of homogeneous samples of fully saturated porous materials in **jacketed drained and undrained** experiments as well as in an **unjacketed** experiment. In such Gedankenexperiments we know constitutive relations between microscopic pressures and microscopic volume changes of both components as well as corresponding macroscopic relations. Microscopic and macroscopic deformations must fulfil kinematic and dynamic compatibility relations which lead to a system of equations for unknown fields which is, in turn, overdetermined. Consequently, we obtain admissibility relations which connect microscopic and macroscopic material parameters.

We assume that the relaxation time of porosity τ is sufficiently large in comparison with other characteristic quantities (for example, $1/\omega_0$, where ω_0 is the characteristic frequency of monochromatic waves) in order to neglect the relaxation in the porosity balance equation (146). As the Biot's model does not describe the relaxation of porosity this assumption is necessary for the comparison of the two models.

We skip here the derivation of micro-macro relations referring the reader to the paper [83] and present the results. It is assumed that microscopic (true) compressibility properties of components are known. It means that we assume the measurability of the true compressibility modulus of the solid component K_s, the true compressibility modulus of the fluid component K_f, the drained modulus K_d (compare [39]) and the initial porosity n_0. Macroscopic parameters K, M, C, N are then given by four equations [83]

$$\mathcal{C}_1 := C + \frac{K_f (K - K_s) - N(K - K_V)}{n_0 (K_s - K_f)} = 0, \tag{157}$$

$$\mathcal{C}_2 := n_0 - \frac{C}{M} - \frac{K_b}{K_s} \frac{1 - (1 - n_0)\frac{K_s}{K_b}}{1 - \frac{1-n_0}{n_0}\frac{NC}{K_b M}} \left\{ 1 - \frac{N(K - n_0 C)}{n_0 M K_b} \right\} = 0, \tag{158}$$

$$\mathcal{C}_3 := K - K_d - C \frac{C - N}{M - \frac{N}{n_0}} = 0. \tag{159}$$

$$\mathcal{C}_4 := \left(1 - \frac{K}{K_W}\right) \left(M - C - N \frac{1 - n_0}{n_0} \frac{C}{K_s}\right) + \tag{160}$$
$$+ \left(1 - \frac{C}{K_W}\right) \left(K - C - N \frac{1 - n_0}{n_0} \left(1 - \frac{K}{K_s}\right)\right) = 0,$$

where

$$K_V = (1 - n_0) K_s + n_0 K_f, \quad K_b = K - \frac{C^2}{M}, \tag{161}$$
$$\frac{1}{K_W} = \frac{1 - n_0}{K_s} + \frac{n_0}{K_f}.$$

In general, this set cannot be solved analytically. However, one can show that it possesses two physically admissible solutions. One of them yields a nonzero value of the parameter N but this value is small when compared with other moduli. The other solution contains $N = 0$. Then the remaining parameters K, C, M can be found analytically. Their values almost coincide with these obtained in the case of $N \neq 0$. We conclude that the form of relations appearing in the Biot's model ($N = 0$) is thermodynamically acceptable within reasonable limits.

The analytical solution of the micro-macro transition problem for $N = 0$ was obtained for the first time by Gassmann [29] and it has the following form

$$K = \frac{(K_s - K_d)^2}{\frac{K_s^2}{K_W} - K_d} + K_d, \quad C = \frac{K_s (K_s - K_d)}{\frac{K_s^2}{K_W} - K_d}, \quad M = \frac{K_s^2}{\frac{K_s^2}{K_W} - K_d}. \tag{162}$$

In Figure 21, we present a numerical example for the above relations. It is clear that the coupling parameter Q is much smaller than the other moduli of the Biot's model. In the next Section we investigate its influence on the propagation conditions of bulk waves.

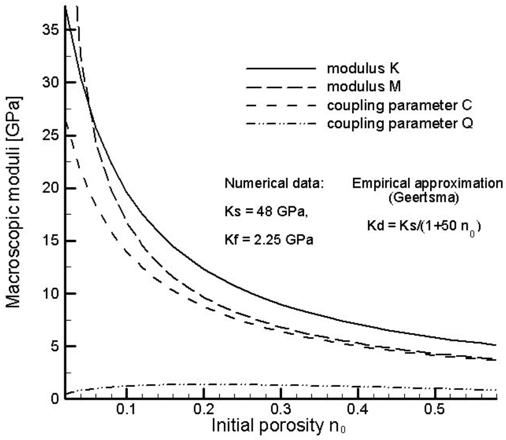


Fig.21: *Dependence of macroscopic moduli of Biot's model on the porosity* n_0 *following from Gassmann relations.*

The procedure of micro-macrotransition does not give any information on the shear modulus μ^S. This must be supplemented by some macroscopic measurements. A possible estimate based on values of the Poisson's ratio ν is given in [4]. This may have some practical bearing in soil mechanics because the Poisson's ratio seems to vary only a little for soils.

Finally let us mention a result obtained from some geometrical considerations by Berryman for the tortuosity parameter. He has shown [9] that for granular materials with moderate porosities it may be roughly estimated by the following simple relation

$$a = \frac{1}{2}\left(\frac{1}{n_0} + 1\right).^{5} \tag{163}$$

We use this relation in numerical examples presented further in this article.

5.2 Bulk monochromatic waves in two-component poroelastic materials

We proceed to analyze the propagation of monochromatic waves in two-component (saturated) poroelastic materials. There exists a vast literature on this subject. We mention here only two excellent books [14], [5] in which both the wave analysis for Biot's model as well as a comparison with experiments is presented.

The governing equations of the Biot's model have the form presented in the Box 7.

[5] Johnson, Plona, Scala, Pasierb and Kojima [33] suggest the relation $a = n_0^{-\beta}$, where $\beta = 2/3$ for random array of needles and $\beta = 1/2$ for random array of spheres. Certainly, none of these relations can be true in general as tortuosity is independent of porosity for arbitrary morphology. For instance, two classes of pipe-like channels differing in the area of cross-section on, say, factor two and yielding the same porosity give tortuosities differing also on factor two as the characteristic length in both cases is the same.

Box 7: *Biot's model*

$$\begin{aligned}\rho_{11}\frac{\partial \mathbf{v}^S}{\partial t}+\rho_{12}\frac{\partial \mathbf{v}^F}{\partial t} &= \lambda^S \operatorname{grad}\operatorname{tr}\mathbf{e}^S + 2\mu^S \operatorname{div}\mathbf{e}^S + Q\operatorname{grad}\varepsilon + \pi\left(\mathbf{v}^F-\mathbf{v}^S\right),\\ \rho_{12}\frac{\partial \mathbf{v}^S}{\partial t}+\rho_{22}\frac{\partial \mathbf{v}^F}{\partial t} &= \rho_0^F \kappa \operatorname{grad}\varepsilon + Q\operatorname{grad}\operatorname{tr}\mathbf{e}^S - \pi\left(\mathbf{v}^F-\mathbf{v}^S\right), \qquad (164)\end{aligned}$$

$$\frac{\partial \mathbf{e}^S}{\partial t} = \operatorname{sym}\operatorname{grad}\mathbf{v}^S, \quad \frac{\partial \varepsilon}{\partial t} = \operatorname{div}\mathbf{v}^F,$$

$$\begin{aligned}\rho_{11} &= \rho_0^S + \rho_{12} \equiv \rho_0^S\left[1-r\left(1-a\right)\right], \quad \rho_{22} = \rho_0^F + \rho_{12} \equiv ra\rho_0^S, \qquad (165)\\ \rho_{12} &= r\left(1-a\right)\rho_0^S, \quad r := \frac{\rho_0^F}{\rho_0^S},\end{aligned}$$

Material parameters: $\lambda^S, \mu^S, \kappa, \pi, Q, a$.

For the simple mixture model: $Q = 0, a = 1$.

The equation for ε follows easily from the mass balance for the fluid (see (138)). In the case of the simple mixture model, we have $Q = 0, \rho_{12} = 0$ (i.e. $a = 1$).

We seek solutions of equations (164) which have the form of the following monochromatic waves

$$\mathbf{v}^S = \mathbf{V}^S\mathcal{E}, \quad \mathbf{v}^F = \mathbf{V}^F\mathcal{E}, \quad \mathbf{e}^S = \mathbf{E}^S\mathcal{E}, \quad \varepsilon = E^F\mathcal{E}, \qquad (166)$$
$$\mathcal{E} := \exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right],$$

where $\mathbf{V}^S, \mathbf{V}^F, \mathbf{E}^S, E^F$ are constant amplitudes, $\mathbf{k}$ is the wave vector, ω real frequency.

By means of the last two equations (164) we can eliminate the amplitudes $\mathbf{E}^S, E^F$

$$\mathbf{E}^S = -\frac{1}{2\omega}\left(\mathbf{k}\otimes\mathbf{V}^S + \mathbf{V}^S\otimes\mathbf{k}\right), \quad E^F = -\frac{1}{\omega}\mathbf{V}^F\cdot\mathbf{k}. \qquad (167)$$

The remaining field equations yield the following compatibility conditions

$$\begin{aligned}&\left[\rho_{11}\omega^2\mathbf{1} - \lambda^S\mathbf{k}\otimes\mathbf{k} - \mu^S\left(k^2\mathbf{1} + \mathbf{k}\otimes\mathbf{k}\right) + i\pi\omega\mathbf{1}\right]\mathbf{V}^S +\\ &\quad + \left[\rho_{12}\omega^2\mathbf{1} - Q\mathbf{k}\otimes\mathbf{k} - i\pi\omega\mathbf{1}\right]\mathbf{V}^F = 0, \qquad (168)\\ &\left[\rho_{12}\omega^2\mathbf{1} - Q\mathbf{k}\otimes\mathbf{k} - i\pi\omega\mathbf{1}\right]\mathbf{V}^S + \left[\rho_{22}\omega^2\mathbf{1} - \kappa\rho_0^F\mathbf{k}\otimes\mathbf{k} + i\pi\omega\mathbf{1}\right]\mathbf{V}^F = 0.\end{aligned}$$

As usual, the problem of existence of such waves reduces to an eigenvalue problem with the eigenvector $\left[\mathbf{V}^S, \mathbf{V}^F\right]$. As in the case of classical one-component elasticity we split the problem into two parts: in the direction $\mathbf{k}_\perp = \mathbf{k} - \frac{\mathbf{k}\cdot\mathbf{k}}{k^2}\mathbf{k}$, perpendicular to $\mathbf{k}$ (transversal modes) and in the direction of the wave vector $\mathbf{k}$ (longitudinal modes). This yields two dispersion relations $k = k\left(\omega\right)$.

In contrast to cases which we have considered before for the elastic single component materials, the above dispersion problem possesses solely complex solutions. For a given real frequency ω, the imaginary part of k arises due to the **dissipation** of energy whose amount is controlled by the permeability coefficient π and it is physically caused by the **diffusion**. This energy is transferred as heat to the exterior and cannot be recovered by the system. This makes the physical interpretation of complex solutions of the dispersion relation different from this which we indicated in Section 4.4. on leaky waves. We see further that both leaky and dissipative waves appear in two-component poroelastic materials.

Simultaneously, inspection of the set (168) shows that the contribution of permeability π enters these equations exactly in the same way as the damping η in equation (26) – accelerations (second time derivatives) are related to ω^2 and the permeability, similarly to the first time derivative, to ω. Consequently, we could expect the existence of a critical wave number k if this was chosen as the independent variable rather than the frequency ω. This was indeed shown in the work of Edelman [23]. Even though it is an interesting mathematical problem, it does not seem to have much to do with the physical reality as, due to dispersion, one practically does not observe waves of a particular wavelength (i.e. of a particular value of k) but rather broad band packages of waves in which the influence of such a critical wave number cannot be spotted. For this reason we use in the analysis of diffusive systems solely the frequency ω as the independent variable in dispersion relations.

For transversal modes (monochromatic shear waves) we have

$$\begin{aligned} \left[\rho_{11}\omega^2 - \mu^S k^2 + i\pi\omega\right] V_\perp^S + \left[\rho_{12}\omega^2 - i\pi\omega\right] V_\perp^F &= 0, \quad k^2 = \mathbf{k}\cdot\mathbf{k}, \\ \left[\rho_{12}\omega^2 - i\pi\omega\right] V_\perp^S + \left[\rho_{22}\omega^2 + i\pi\omega\right] V_\perp^F &= 0, \end{aligned} \tag{169}$$

$$V_\perp^S = \mathbf{V}^S \cdot \mathbf{k}_\perp, \quad V_\perp^F = \mathbf{V}^F \cdot \mathbf{k}_\perp.$$

The dispersion relation can be written in this case in the following form

$$\omega\left\{\left(\rho_{11}\rho_{22} - \rho_{12}^2\right)\left(\frac{\omega}{k}\right)^2 - \mu^S\rho_{22}\right\} + $$
$$+ i\pi\left\{\left(\rho_{11} + \rho_{22} + 2\rho_{12}\right)\left(\frac{\omega}{k}\right)^2 - \mu^S\right\} = 0, \tag{170}$$

i.e.

$$\left(\frac{\omega}{k}\right)^2 = \frac{\omega r a + i\frac{\pi}{\rho_0^S}}{\omega r\left[a - r\left(1-a\right)\right] + i\frac{\pi}{\rho_0^S}\left(1+r\right)} c_S^2, \quad c_S^2 := \frac{\mu^S}{\rho_0^S}. \tag{171}$$

Consequently, neither the phase velocity $\omega/\operatorname{Re} k$ nor the attenuation $\operatorname{Im} k$ of monochromatic shear waves is dependent on the coupling coefficient Q.

In the two limits of frequencies we have then the following solutions

$$\omega \to 0: \quad \lim_{\omega\to 0}\left(\frac{\omega}{\operatorname{Re} k}\right)^2 = \frac{\mu^S}{\rho_0^S + \rho_0^F}, \quad \lim_{\omega\to 0}\left(\operatorname{Im} k\right) = 0,$$
$$\omega \to \infty: \quad \lim_{\omega\to \infty}\left(\frac{\omega}{\operatorname{Re} k}\right)^2 = \frac{\rho_{22}}{\rho_{11}\rho_{22} - \rho_{12}^2}\mu^S, \tag{172}$$

$$\lim_{\omega\to\infty}(\operatorname{Im}k)=\frac{\pi}{2\sqrt{\rho_0^S\mu^S}}\frac{1}{a^2}\sqrt{\frac{a}{a-r(1-a)}}.$$

The first result checks with the results of the classical one-component model commonly used in soil mechanics.

In the simple mixture model, the result for $\omega\to 0$ is, of course, the same. For $\omega\to\infty$ we get

$$\lim_{\omega\to\infty}\left(\frac{\omega}{\operatorname{Re}k}\right)^2=\frac{\mu^S}{\rho_0^S}\equiv c_S^2, \tag{173}$$

and this is identical with the result of the classical elasticity where we have used the notation c_T rather than c_S for this limit.

Let us notice that the attenuation in the limit $\omega\to\infty$ is finite and reaches there a maximum. Consequently, for $\pi=0$ the solution of the dispersion relation would be real. This means that indeed, for the above investigated waves, the attenuation is caused solely by the diffusion.

We demonstrate properties of monochromatic waves on a numerical example in the whole range of frequencies $\omega\in[0,\infty)$. Both for shear waves and for longitudinal waves we use the following numerical data

$$\begin{aligned}
c_{P1} &\equiv \sqrt{\frac{\lambda^S+2\mu^S}{\rho_0^S}}=2500\,\frac{\mathrm{m}}{\mathrm{s}},\quad c_{P2}\equiv\sqrt{\kappa}=1000\,\frac{\mathrm{m}}{\mathrm{s}},\quad c_S\equiv\sqrt{\frac{\mu^S}{\rho_0^S}}=1500\,\frac{\mathrm{m}}{\mathrm{s}},\\
\rho_0^S &= 2500\,\frac{\mathrm{kg}}{\mathrm{m}^3},\quad r=0.1,\quad \pi=10^8\,\frac{\mathrm{kg}}{\mathrm{m}^3\mathrm{s}},\\
Q &= 0.8\text{ GPa},\quad n_0=0.4,\quad a=1.75.
\end{aligned} \tag{174}$$

Velocities c_{P1},c_{P2},c_S, the mass density ρ_0^S (i.e $\rho_0^{SR}=4167\,\frac{\mathrm{kg}}{\mathrm{m}^3}$ for the porosity $n_0=0.4$) and the fraction $r=\rho_0^F/\rho_0^S$ possess values typical for many granular materials under a confining pressure of a few atmospheres and saturated by water. In units standard for soil mechanics the permeability π corresponds to app. 0.1 Darcy. The coupling coefficient Q has been estimated by means of the Gassmann relation discussed in Sec. 5.1.3. The tortuosity coefficient $a=1.75$ follows from Berryman formula (163) for the chosen value of porosity.

Transversal waves described by the relation (171) are characterized by the following distribution of velocities and attenuations in function of the frequency (Fig. 22). The solid lines correspond to the solution of Biot's model and the dashed lines to the solution of the simple mixture model.

It is clear that the qualitative behavior of the velocity of propagation is the same in both models. It is a few percent smaller in Biot's model than this in the simple mixture model in the range of high frequencies. A large quantitative difference between these models appears for the attenuation. In the range of higher frequencies it is much smaller in the Biot's model, i.e. tortuosity decreases the dissipation of shear waves.

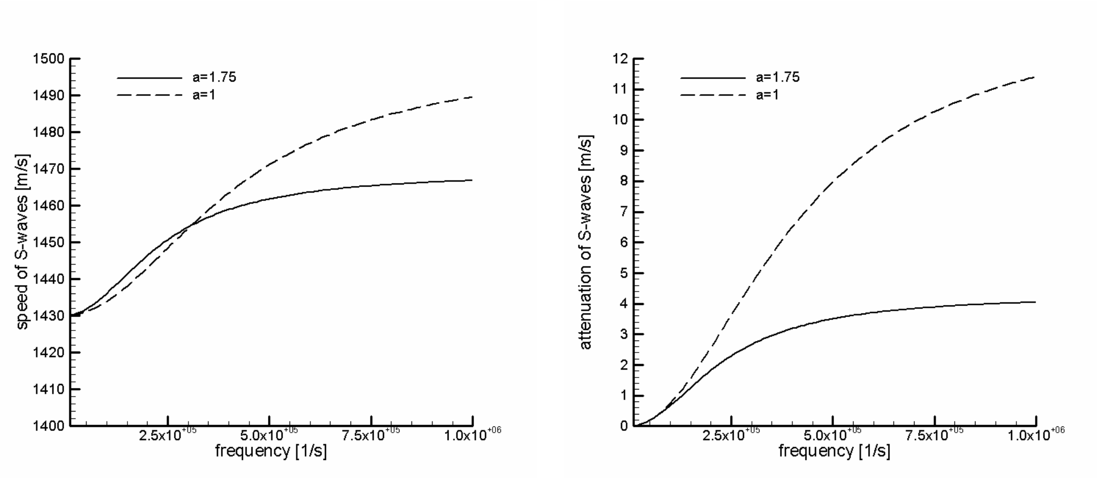


Figure 22: *Velocity of propagation and attenuation of monochromatic S-waves for two values of the tortuosity coefficient* $a:$ 1.75 *(Biot)*, 1.00 *(simple mixture).*

The latter property is illustrated in Fig. 23 where we plot the attenuation of the front of shear waves, i.e. $\lim_{\omega \to \infty} \operatorname{Im} k$, as a function of the tortuosity coefficient a. This behavior of attenuation indicates that damping of waves created by the tortuosity, which is connected in the macroscopic model to the relative velocity of components, is not related to scattering of waves on the microstructure. It is rather related to the decrease of the macroscopic diffusion velocity in comparison with the difference of velocities on the microscopic level due to the curvature of channels and volume averaging. Fluctuations are related solely to this averaging and not to temporal deviation from time averages (lack of ergodicity!). Whatever the argument may be the influence of tortuosity on the attenuation of waves seems to be much too strong.

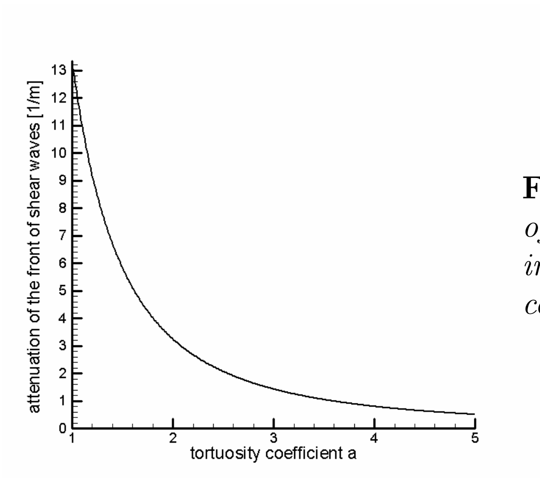


Figure 23: *Attenuation of the front of shear waves in function of the tortuosity coefficient a.*

Let us remark in passing that, in order to describe properly the influence of morphology (i.e. a random real geometry of channels, their volume contribution – porosity, their curvature – tortuosity, etc.) on the propagation of waves one would have to account for random scattering of waves on microscopic obstacles. This is, certainly, not done by the added mass coefficient ρ_{12} of Biot's model. However, one way of doing it is the overall interpretation of the permeability coefficient π. Usually in the literature on Biot's model, the permeability π is attributed to the viscosity of the true fluid component. This is, of course, an unnecessary restriction as curvature microeffects contribute to values of π as well and, consequently, a dissipative influence of tortuosity can be included in the model by an effective value of permeability coefficient π.

Let us return to the second part of the dispersion relation (168). For longitudinal waves we obtain

$$\begin{aligned}\left[\rho_{11}\omega^2-\left(\lambda^S+2\mu^S\right)k^2+i\pi\omega\right]\left[\rho_{22}\omega^2-\kappa\rho_0^F k^2+i\pi\omega\right]-\\ -\left(\rho_{12}\omega^2-Qk^2-i\pi\omega\right)^2=0,\end{aligned} \tag{175}$$

or, after easy manipulations,

$$\begin{aligned}&\omega\left\{\left[1-r\left(1-a\right)\right]\left(\frac{\omega}{k}\right)^2-c_{P1}^2\right\}\left\{a\left(\frac{\omega}{k}\right)^2-c_{P2}^2\right\}+\\ &+\frac{1}{r}i\frac{\pi}{\rho_0^S}\left(\frac{\omega}{k}\right)^2\left\{\left(1+r\right)\left(\frac{\omega}{k}\right)^2-rc_{P2}^2-c_{P1}^2-2\frac{Q}{\rho_0^S}\right\}-\\ &-\frac{1}{r}\omega\left\{r\left(1-a\right)\left(\frac{\omega}{k}\right)^2-\frac{Q}{\rho_0^S}\right\}^2=0,\quad c_{P1}^2=\frac{\lambda^S+2\mu^S}{\rho_0^S},\quad c_{P2}^2=\kappa.\end{aligned} \tag{176}$$

Again we check first solutions for two limits of frequencies: $\omega\to 0$, and $\omega\to\infty$.

Let us begin with the second limit which corresponds to the propagation of wave fronts. We have

$$\omega\to\infty:\quad c_\infty:=\lim_{\omega\to\infty}\left(\frac{\omega}{\operatorname{Re}k}\right),$$

$$r\left\{\left[1-r\left(1-a\right)\right]c_\infty^2-c_{P1}^2\right\}\left\{ac_\infty^2-c_{P2}^2\right\}-\left\{r\left(1-a\right)c_\infty^2-\frac{Q}{\rho_0^S}\right\}^2=0. \tag{177}$$

This is a biquadratic equation for c_∞ which yields two nontrivial solutions. These are called the P1-**wave** and the P2-**wave** (or Biot's wave). In contrast to the simple mixture model, in the case of the Biot's model these waves are not longitudinal even though they are customarily called so. It can be easily checked by the calculation of corresponding eigenvectors. We shall not do it in this article.

For the simple mixture model, the solution of (177) is immediate

$$c_\infty=\begin{cases}c_{P1},\\ c_{P2}.\end{cases} \tag{178}$$

In the classical elasticity, we have $c_{P1}=c_L$ and the second longitudinal wave does not exist.

The second, slow wave has been discovered in 1944 by Frenkel [28] and then rediscovered by Biot. However, this kind of waves is known since the discovery of the so-called **second sound** in liquid helium by Tisza in 1938. They appear in all hyperbolic models of multicomponent systems – for liquid helium these are the normal fluid and the superfluid. They are very difficult to verify experimentally due to a very high attenuation. However, at least for some porous materials (e.g. sintered glass spheres), they have been indeed observed (e.g. [52]).

Simultaneously, we obtain the following attenuation in the limit of infinite frequencies

$$\lim_{\omega\to\infty}\left(\operatorname{Im}k\right)=\frac{\pi\Gamma_1}{2\rho_0^S r\Gamma_2}, \tag{179}$$

$$\Gamma_1 = c_\infty \left[1 + r - \frac{1}{c_\infty^2}\left(c_{P1}^2 + rc_{P2}^2 + 2\frac{Q}{\rho_0^S}\right)\right],$$
$$\Gamma_2 = c_{P1}^2\left(a - \frac{c_{P2}^2}{c_\infty^2}\right) + c_{P2}^2\left(1 - r(1-a) - \frac{c_{P1}^2}{c_\infty^2}\right) + 2\frac{Q}{\rho_0^S}\left(1 - a - \frac{Q}{r\rho_0^S c_\infty^2}\right),$$

and, for the simple mixture model,

$$\lim_{\omega\to\infty}(\operatorname{Im} k) = \begin{cases} \dfrac{\pi}{2\rho_0^S c_{P1}} & \text{for P1-wave,} \\ \dfrac{\pi}{2\rho_0^F c_{P2}} & \text{for P2-wave.} \end{cases} \tag{180}$$

Hence both limits of attenuation for the P1-wave and P2-wave are finite and caused by the permeability π.

In the case of the low frequency limit we obtain

$$\omega \to 0: \quad c_0 := \lim_{\omega\to 0}\left(\frac{\omega}{\operatorname{Re} k}\right),$$
$$c_0^2\left\{(1+r)\,c_0^2 - rc_{P2}^2 - c_{P1}^2 + 2\frac{Q}{\rho_0^S}\right\} = 0, \quad \lim_{\omega\to 0}(\operatorname{Im} k) = 0. \tag{181}$$

Obviously, we obtain two real solutions of this equation

$$\lim_{\omega\to 0}\left(\frac{\omega}{\operatorname{Re} k}\right)^2\Bigg|_1 \; : = c_{oP1}^2 = \frac{c_{P1}^2 + rc_{P2}^2 + 2\frac{Q}{\rho_0^S}}{1+r} \equiv \frac{\lambda^S + 2\mu^S + \rho_0^F\kappa + 2\frac{Q}{\rho_0^S}}{\rho_0^S + \rho_0^F}, \tag{182}$$
$$\lim_{\omega\to 0}\left(\frac{\omega}{\operatorname{Re} k}\right)^2\Bigg|_2 \; : = c_{oP2}^2 = 0.$$

These are squares of velocities of propagation of two longitudinal modes in the limit of zero frequency. Clearly, the second mode, P2-wave, does not propagate in this limit. Both limits are independent of tortuosity. The result (182) checks with the relation for the velocity of longitudinal waves used in the classical one-component model of soil mechanics provided $Q = 0$.

For the data (174) we construct now the numerical example for the P1- and P2-waves discussed above. The solid lines in the following Figures correspond again to Biot's model, the dashed lines to the simple mixture model. In order to show separately the influence of tortuosity a and of the coupling Q we plot as well the solutions with $a = 1$ (dashed dotted lines) and the solutions with $Q = 0$ (dashed double dotted lines).

Even though similar, again, the quantitative differences are much more substantial for P1-waves (Fig. 24). This is primarily the influence of the coupling through partial stresses described by the parameter Q. The simple mixture model ($Q = 0, a = 1$) as well as Biot's model with $Q = 0$ yield velocities of these waves which differ only a few percent (lower curves in the left diagram). The coupling Q shifts the curves to higher values and reduces the difference caused by the tortuosity. This result does not seem to be very realistic because the real differences between low frequency and high frequency velocities were measured in soils to be rather as big as indicated by the simple mixture model. This may be an indication that Gassmann relations give much too big values of the coupling parameter Q with respect to those indeed appearing in real granular materials.

Both the tortuosity a and the coupling Q reduce the attenuation quite considerably as indicated in the right Figure.

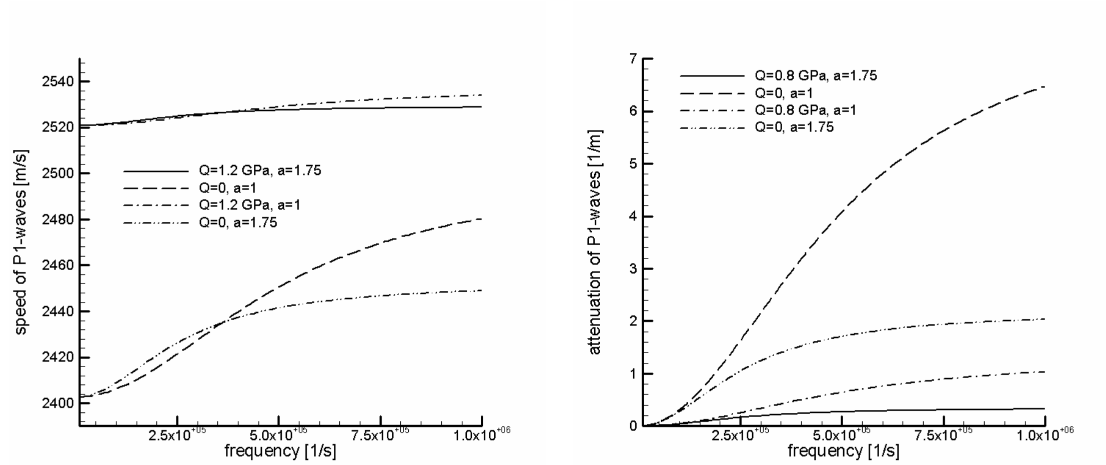


Figure 24: *Velocity of propagation and attenuation of monochromatic P1-waves for various coupling parameters Q and tortuosity coefficients a.*

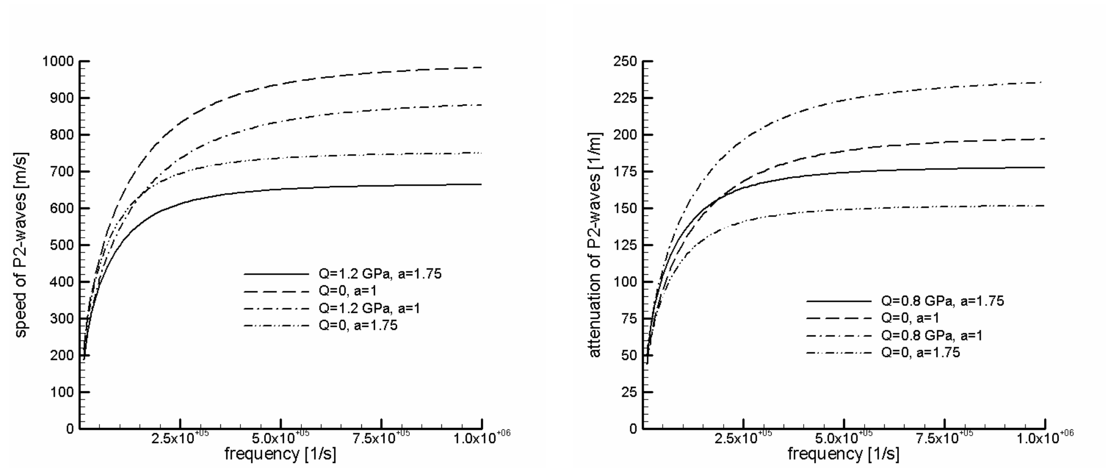


Fig. 25: *Velocity of propagation and attenuation of monochromatic P2-waves for various coupling parameters Q and tortuosity coefficients a.*

In spite of some claims in the literature the tortuosity a does not influence the existence of the slow (P2-) wave (Fig. 25). Velocities of this wave are again qualitatively similar in Biot's model and in the simple mixture model. The maximum differences appear in the range of high frequencies and reach some 35 percent. The same concerns the attenuation even though quantitative differences are not so big (app. 8 percent).

Let us notice that the lack of coupling through diffusion, $\pi \equiv 0$, yields for both shear waves (170) and longitudinal waves (176) dispersion relations whose solutions – phase velocities – are independent of frequencies. The waves are nondispersive. Hence, both dispersion and dissipation in the system are caused by the diffusive coupling.

5.3 Some remarks on the simple mixture model

It is evident from the above analysis that the simple mixture model yields very substantial technical simplifications in the analysis of wave propagation when compared with the full Biot's model. There appear quantitative differences between both models, in some cases substantial. The question arises if Biot's model leads to some qualitative effects which are not appearing in the simple mixture model. The claims that experiments confirm Biot's model are based on quantitative analysis of data rather than on any additional wave effects. Such an analysis may not be very reliable as it is based already on the *a priori* presumption that Biot's model is the only way to describe linear dynamics of porous saturated materials. However, some features of this model like the disastrous diagram in Figure 23 indicate that the model requires corrections.

Simultaneously, an extensive wave analysis for the simple mixture model (e.g. [76], [78], [80], [81], [84], [85], [4]) does not reveal any effects which would not appear as well in Biot's model and, *vice versa*, Biot's model does not indicate any additional effects either. Even such a sophisticated coupling of P1- and P2-waves which yields a local minimum in the velocity of Rayleigh waves (see: [14]) appears in both models. Similar effects are discussed in details in the contribution of B. Albers [3] to this volume.

As an important consequence of these remarks we proceed with our analysis of surface waves in linear poroelastic materials on the basis of the simple mixture model.

6 Surface waves in two-component media

6.1 Preliminaries

The theory of surface waves in two-component systems differs qualitatively from such a theory for one-component continua. Such waves are produced in linear models by a combination of bulk waves. In the case of a one-component continuum there are two bulk modes of propagation which yield a single Rayleigh wave. For two-component systems we have three bulk modes: P1-waves, P2-waves and S-waves which produce two surface modes in the case of impermeable boundary. For the permeable boundary, i.e. for the case of an additional system – a fluid in the exterior, there may exists three surface modes, etc. In addition, as all these waves are dissipative, there may exist additional leaky modes similar to these which we have mentioned in Sec. 4.4.

In this Section we consider surface waves in two-component homogeneous poroelastic materials with an impermeable boundary. However, we indicate as well some properties related to the permeable boundary condition. This condition has been proposed in 1962 by Deresiewicz [18], [21]. The analysis is based on the simple mixture model. We limit the attention solely to high and low frequency ranges. The presentation should be considered to be an introduction to the much more extensive article of B. Albers [3] contributed to this volume. That article contains also a numerical analysis of dispersion relations in the whole range of frequencies for three types of interfaces: sealed porous medium/vacuum, sealed porous medium/ideal fluid, unsealed porous medium/ideal fluid. We leave entirely open the problem of existence which is at least as complicated as in the case of single component materials.

To the end of this Section we quote a few results obtained within Biot's model. As this model is much more complicated than the simple mixture model results are limited only to some special cases.

6.2 Compatibility conditions and dispersion relation

As discussed in [81] we seek a solution of the set of fields equations which we obtain from Biot's equations (Box 7) by the substitution $Q = 0, a = 1$. It is convenient to introduce the displacement vector $\mathbf{u}^S$ for the skeleton, and, formally, the displacement vector $\mathbf{u}^F$ for the fluid. The latter is introduced solely for the technical symmetry of considerations and it does not have any physical bearing. Then

$$\begin{aligned}\mathbf{u}^S &= \operatorname{grad}\varphi^S + \operatorname{rot}\boldsymbol{\psi}^S, \quad \mathbf{v}^S = \frac{\partial \mathbf{u}^S}{\partial t}, \quad \mathbf{e}^S = \operatorname{sym}\operatorname{grad}\mathbf{u}^S, \\ \mathbf{u}^F &= \operatorname{grad}\varphi^F + \operatorname{rot}\boldsymbol{\psi}^F, \quad \mathbf{v}^F = \frac{\partial \mathbf{u}^F}{\partial t},\end{aligned} \tag{183}$$

where $\varphi^S, \boldsymbol{\psi}^S, \varphi^F, \boldsymbol{\psi}^F$ are two pairs of potentials analogous to those which we were using in the classical elasticity model.

We choose the axes with the downward orientation of the z-axis and the x-axis in the direction of propagation of the wave. As the problem is assumed to be two-dimensional we make the following ansatz for solutions

$$\begin{aligned}\varphi^S &= A^S(z)\exp[i(kx-\omega t)], \quad \varphi^F = A^F(z)\exp[i(kx-\omega t)], \\ \psi_y^S &= B^S(z)\exp[i(kx-\omega t)], \quad \psi_y^F = B^F(z)\exp[i(kx-\omega t)], \\ \psi_x^S &= \psi_z^S = \psi_x^F = \psi_z^F = 0,\end{aligned} \tag{184}$$

and

$$\begin{aligned}\rho^S - \rho_0^S &= A_\rho^S(z)\exp[i(kx-\omega t)], \quad \rho^F - \rho_0^F = A_\rho^F(z)\exp[i(kx-\omega t)], \\ n - n_0 &= A^\Delta \exp[i(kx-\omega t)].\end{aligned} \tag{185}$$

Substitution in field equations leads after straightforward calculations to the following compatibility conditions for $z \geq 0$

$$\begin{aligned}B^F &= \frac{i\pi}{\rho_0^F\omega + i\pi}B^S, \quad A^\Delta = -\frac{n_0\omega\tau}{i+\omega\tau}\left(\frac{d^2}{dz^2} - k^2\right)\left(A^F - A^S\right), \\ A_\rho^S &= -\rho_0^S\left(\frac{d^2}{dz^2} - k^2\right)A^S, \quad A_\rho^F = -\rho_0^F\left(\frac{d^2}{dz^2} - k^2\right)A^F,\end{aligned} \tag{186}$$

as well as

$$\left[\kappa\left(\frac{d^2}{dz^2} - k^2\right) + \omega^2\right]A^F + \left[\frac{n_0\beta\omega\tau}{\rho_0^F(i+\omega\tau)}\left(\frac{d^2}{dz^2} - k^2\right) + \frac{i\pi}{\rho_0^F}\omega\right]\left(A^F - A^S\right) = 0, \tag{187}$$

$$\left[\frac{\lambda^S + 2\mu^S}{\rho_0^S}\left(\frac{d^2}{dz^2} - k^2\right) + \omega^2\right]A^S - \left[\frac{n_0\beta\omega\tau}{\rho_0^S(i+\omega\tau)}\left(\frac{d^2}{dz^2} - k^2\right) + \frac{i\pi}{\rho_0^S}\omega\right]\left(A^F - A^S\right) = 0, \tag{188}$$

$$\left[\frac{\mu^S}{\rho_0^S}\left(\frac{d^2}{dz^2} - k^2\right) + \omega^2\right]B^S + \frac{i\pi\rho_0^F}{\rho_0^S\left(\rho_0^F\omega + i\pi\right)}\omega^2 B^S = 0. \tag{189}$$

It is convenient to introduce a dimensionless notation. In order to do so we define the following auxiliary quantities

$$\begin{aligned} c_s &= \frac{c_S}{c_{P1}} < 1, \quad c_f = \frac{c_{P2}}{c_{P1}}, \quad \pi' = \frac{\pi\tau}{\rho_0^S} > 0, \quad \beta' = \frac{n_0\beta}{\rho_0^S c_{P1}^2} > 0, \\ r &= \frac{\rho_0^F}{\rho_0^S} < 1, \qquad z' = \frac{z}{c_{P1}\tau}, \quad k' = kc_{P1}\tau, \quad \omega' = \omega\tau. \end{aligned} \tag{190}$$

where the velocities c_{P1}, c_{P2}, c_S are defined by (174) and, in the simple mixture model, they describe the velocities of fronts of bulk P1-, P2-, and S-wave, respectively. These are, of course, identical with the limits of bulk phase velocities for $\omega \to \infty$. As we neglect processes of relaxation of porosity, the reference time τ can be chosen arbitrarily.

As we have already mentioned, we neglect further the influence of the nonequilibrium changes of porosity, i.e. $\beta = 0$. In the compatibility relations derived in this Subsection we still keep it in the relations in order to show the way in which this influence enters the model if not ignored.

Further we omit the prime for typographical reasons. Substitution of (190) in equations (187), (188), (189) yields

$$\begin{aligned} \left[c_f^2\left(\frac{d^2}{dz^2} - k^2\right) + \omega^2\right] A^F + \left[\frac{\beta\omega}{r(i+\omega)}\left(\frac{d^2}{dz^2} - k^2\right) + i\frac{\pi}{r}\omega\right]\left(A^F - A^S\right) &= 0, \\ \left[\left(\frac{d^2}{dz^2} - k^2\right) + \omega^2\right] A^S - \left[\frac{\beta\omega}{i+\omega}\left(\frac{d^2}{dz^2} - k^2\right) + i\pi\omega\right]\left(A^F - A^S\right) &= 0, \\ \left[c_s^2\left(\frac{d^2}{dz^2} - k^2\right) + \omega^2 + \frac{i\pi\omega}{\omega + i\frac{\pi}{r}}\right] B^S &= 0. \end{aligned} \tag{191}$$

This differential eigenvalue problem can be easily solved because the matrix of coefficients for homogeneous materials is independent of z. This is different from the case of waves in heterogeneous materials (compare, for instance, the article of C. G. Lai [38]). Consequently, we seek solutions in the form

$$A^F = A_f^1 e^{\gamma_1 z} + A_f^2 e^{\gamma_2 z}, \quad A^S = A_s^1 e^{\gamma_1 z} + A_s^2 e^{\gamma_2 z}, \quad B^S = B_s e^{\zeta z}, \tag{192}$$

where, due to the chosen direction of the z-axis, the exponents $\gamma_1, \gamma_2, \zeta$ must possess negative real parts. This is the existence requirement for surface waves. Substitution in (191) yields them in the form

$$\left(\frac{\zeta}{k}\right)^2 = 1 - \frac{1}{c_s^2}\left(1 + \frac{i\pi}{\omega + i\frac{\pi}{r}}\right)\left(\frac{\omega}{k}\right)^2, \tag{193}$$

and

$$\begin{aligned} &\left[c_f^2 + \left(c_f^2 + \frac{1}{r}\right)\frac{\beta\omega}{i+\omega}\right]\left[\left(\frac{\gamma}{k}\right)^2 - 1\right]^2 + \left[1 + \left(1 + \frac{1}{r}\right)\frac{i\pi}{\omega}\right]\left(\frac{\omega}{k}\right)^4 \\ &+ \left[1 + c_f^2 + \left(1 + \frac{1}{r}\right)\frac{\beta\omega}{i+\omega} + \left(c_f^2 + \frac{1}{r}\right)\frac{i\pi}{\omega}\right]\left[\left(\frac{\gamma}{k}\right)^2 - 1\right]\left(\frac{\omega}{k}\right)^2 = 0. \end{aligned} \tag{194}$$

Simultaneously we obtain the following relations for eigenvectors

$$\mathbf{R}^1 = \left(B_s, A_s^1, A_f^1\right)^T, \quad \mathbf{R}^2 = \left(B_s, A_s^2, A_f^2\right)^T, \tag{195}$$

where

$$A_f^1 = \delta_f A_s^1, \quad A_s^2 = \delta_s A_f^2, \tag{196}$$

$$\delta_f := \frac{1}{r} \frac{\frac{\beta\omega}{i+\omega}\left[\left(\frac{\gamma_1}{k}\right)^2 - 1\right] + \frac{i\pi}{\omega}\frac{\omega^2}{k^2}}{\left(c_f^2 + \frac{1}{r}\frac{\beta\omega}{i+\omega}\right)\left[\left(\frac{\gamma_1}{k}\right)^2 - 1\right] + \left(\frac{\omega}{k}\right)^2 + \frac{i\pi}{\omega r}\frac{\omega^2}{k^2}}, \tag{197}$$

$$\delta_s := \frac{\frac{\beta\omega}{i+\omega}\left[\left(\frac{\gamma_2}{k}\right)^2 - 1\right] + \frac{i\pi}{\omega}\frac{\omega^2}{k^2}}{\left(1 + \frac{\beta\omega}{i+\omega}\right)\left[\left(\frac{\gamma_2}{k}\right)^2 - 1\right] + \left(\frac{\omega}{k}\right)^2 + \frac{i\pi}{\omega}\frac{\omega^2}{k^2}}. \tag{198}$$

The above solution for the exponents still leaves three unknown constants B_s, A_f^2, A_s^1 which must be specified from boundary conditions. This is the subject of the next Subsection.

For technical reasons, we limit the attention solely the limit problems in the range of high and low frequencies.

In the case of **high frequency approximation** we immediately obtain from relations (193) and (194)

$$\frac{1}{\omega} \ll 1: \qquad \left(\frac{\zeta}{k}\right)^2 = 1 - \frac{1}{c_s^2}\left(\frac{\omega}{k}\right)^2,$$
$$\left(\frac{\gamma_1}{k}\right)^2 = 1 - \left(\frac{\omega}{k}\right)^2, \quad \left(\frac{\gamma_2}{k}\right)^2 = 1 - \frac{1}{c_f^2}\left(\frac{\omega}{k}\right)^2, \tag{199}$$

and

$$\delta_f = \delta_s = 0 \quad \Rightarrow \quad \mathbf{R}^1 = \left(B_s, A_s^1, 0\right)^T, \quad \mathbf{R}^2 = \left(B_s, 0, A_f^2\right)^T. \tag{200}$$

For the case of **low frequency approximation** the equation (194) becomes singular. It can be written in the following form

$$\omega \ll 1: \qquad c_f^2\omega^2\left[\left(\frac{\gamma}{k}\right)^2 - 1\right]^2 + \omega\left[\omega + i\pi\left(1 + \frac{1}{r}\right)\right]\left(\frac{\omega}{k}\right)^4 +$$
$$+ \left[\omega\left(1 + c_f^2\right) + i\pi\left(c_f^2 + \frac{1}{r}\right)\right]\omega\left[\left(\frac{\gamma}{k}\right)^2 - 1\right]\left(\frac{\omega}{k}\right)^2 = 0. \tag{201}$$

Making the following substitution

$$W := \omega\left[\left(\frac{\gamma}{k}\right)^2 - 1\right] + \omega\left(\frac{\omega}{k}\right)^2\frac{1 + c_f^2}{2c_f^2}, \tag{202}$$

we obtain a quadratic equation for W

$$c_f^2 W^2 + i\pi\left(c_f^2 + \frac{1}{r}\right)\left(\frac{\omega}{k}\right)^2 W - \tag{203}$$
$$-\left\{i\pi\left[\frac{\left(c_f^2 + \frac{1}{r}\right)\left(1 + c_f^2\right)}{2c_f^2} - \left(1 + \frac{1}{r}\right)\right]\left(\frac{\omega}{k}\right)^4\right\}\omega + O\left(\omega^2\right) = 0,$$

which for small ω can be solved by the regular perturbation method

$$W = W_0 + \omega W_1 + O\left(\omega^2\right). \tag{204}$$

After easy calculations we obtain

$$W = \begin{cases} \left[\frac{1+c_f^2}{2c_f^2} - \frac{r+1}{rc_f^2+1}\right]\left(\frac{\omega}{k}\right)^2 \omega, \\ -\left[\frac{1+c_f^2}{2c_f^2} - \frac{r+1}{rc_f^2+1}\right]\left(\frac{\omega}{k}\right)^2 \omega - i\pi\frac{rc_f^2+1}{rc_f^2}\left(\frac{\omega}{k}\right)^2 \end{cases}. \tag{205}$$

Bearing the relation (202) in mind we arrive at the following results for the exponents

$$\omega \ll 1: \qquad \left(\frac{\zeta}{k}\right)^2 = 1 - \frac{r+1}{c_s^2}\left(\frac{\omega}{k}\right)^2,$$

$$\left(\frac{\gamma_1}{k}\right)^2 = 1 - \frac{r+1}{rc_f^2+1}\left(\frac{\omega}{k}\right)^2, \tag{206}$$

$$\left(\frac{\gamma_2}{k}\right)^2 = 1 - \frac{rc_f^4+1}{c_f^2\left(rc_f^2+1\right)}\left(\frac{\omega}{k}\right)^2 - \frac{i\pi}{\omega}\frac{rc_f^2+1}{rc_f^2}\left(\frac{\omega}{k}\right)^2,$$

and for the coefficients of amplitudes

$$\delta_f = 1 - \frac{\omega r}{i\pi}\frac{1-c_f^2}{1+rc_f^2}, \quad \delta_s = -rc_f^2\left(1 - \frac{\omega r}{i\pi}\frac{1-c_f^2}{1+rc_f^2}\right). \tag{207}$$

Obviously due to the singular character of the equation (201) the last contribution to $\frac{\gamma_2}{k}$ becomes singular for $\omega \to 0$.

6.3 Boundary value problems for surface waves

In order to determine surface waves in saturated poroelastic medium we need conditions for $z = 0$. We discuss in some details the problem in which this boundary is **impermeable**, and a poroelastic medium is in contact with **vacuum**. Boundary conditions have then the form

$$T_{13}\big|_{z=0} \equiv T_{13}^S\big|_{z=0} = \mu^S\left(\frac{\partial u_1^S}{\partial z} + \frac{\partial u_3^S}{\partial x}\right)\bigg|_{z=0} = 0, \tag{208}$$

$$T_{33}\big|_{z=0} \equiv \left(T_{33}^S - p^F\right)\big|_{z=0} =$$

$$= c_{P1}^2\rho_0^S\left(\frac{\partial u_1^S}{\partial x} + \frac{\partial u_3^S}{\partial z}\right) - 2c_S^2\rho_0^S\frac{\partial u_1^S}{\partial x} - c_{P2}^2\left(\rho^F - \rho_0^F\right)\bigg|_{z=0} = 0, \tag{209}$$

$$\frac{\partial}{\partial t}\left(u_3^F - u_3^S\right)\bigg|_{z=0} = 0, \tag{210}$$

where the first two conditions mean that the surface $z = 0$ is stress-free (far-field approximation), and the last condition means that there is no transport of fluid mass through this surface (impermeable boundary). u_1^S, u_3^S denote the components of the displacement

$\mathbf{u}^S$ in the direction of x-axis and z-axis, respectively, while u_3^F is the z-component of the displacement $\mathbf{u}^F$.

In the case of a permeable boundary neither the condition (208) nor the condition (210) would hold.

The first condition would have to possess the right-hand side reflecting the external pressure p_{ext} appearing in the fluid outside of the porous material. This change would appear as well in the case of impermeable boundary when we did not have the vacuum outside.

Condition (210) which reflects the fact that the impermeable boundary is **material** for both the solid and fluid component would have to describe mass transport through the surface specified by a relation to a driving force. According to the proposition of Deresiewicz and Skalak [21] such a driving force is proportional to the difference of pore pressures on both sides of the boundary

$$\left. \rho_0^F \frac{\partial}{\partial t} \left(u_3^F - u_3^S \right) - \alpha \left(p^F - n_0 p_{ext} \right) \right|_{z=0} = 0, \tag{211}$$

where α denotes a surface permeability coefficient and p_{ext} is an external pressure.

The coefficient α is an overall macroscopic description of a boundary layer which is created by the flow of the fluid component from conditions specified by the porous material (i.e. by the permeability π, porosity n, a geometry of the microscopic vicinity of the boundary such as a shape of openings of channels, their average orientation with respect to the surface normal, etc.) to the free space of a pure fluid. It is clear that the limit $\alpha \to 0$ corresponds to the impermeable (sealed) boundary, and the limit $\alpha \to \infty$ corresponds to the continuity of pressure in the fluid: $p^F = n_0 p_{ext}$. Such a boundary condition is used, for instance, in theories of porous materials with a rigid skeleton which are used in the description of various geotechnical diffusion and seepage processes.

In addition, for the permeable boundary we have to account for the continuity of the mass flux through the boundary. This additional boundary condition is necessary with respect to the existence of an additional constant in the solution for the exterior (in the range $z < 0$).

Substitution of results of the previous Subsection in boundary conditions (208)-(210) yields the following equations for three unknown constants B_s, A_f^2, A_s^1

$$\mathbf{AX} = \mathbf{0}, \tag{212}$$

where

$$\mathbf{A} := \begin{pmatrix} \left(\frac{\zeta}{k}\right)^2 + 1 & 2i\frac{\gamma_2}{k}\delta_s & 2i\frac{\gamma_1}{k} \\ -2ic_s^2\frac{\zeta}{k} & \begin{array}{c}\left[\left(\frac{\gamma_2}{k}\right)^2 - 1 + 2c_s^2\right]\delta_s + \\ +rc_f^2\left[\left(\frac{\gamma_2}{k}\right)^2 - 1\right]\end{array} & \begin{array}{c}\left(\frac{\gamma_1}{k}\right)^2 - 1 + 2c_s^2 + \\ +rc_f^2\left[\left(\frac{\gamma_1}{k}\right)^2 - 1\right]\delta_f\end{array} \\ i\frac{r\omega}{r\omega + i\pi} & -\left(\delta_s - 1\right)\frac{\gamma_2}{k} & \left(\delta_f - 1\right)\frac{\gamma_1}{k} \end{pmatrix}, \tag{213}$$

$$\mathbf{X} := \begin{pmatrix} B_s, & A_f^2, & A_s^1 \end{pmatrix}^T.$$

This homogeneous set yields the **dispersion relation:** $\det \mathbf{A} = 0$ determining the $\omega - k$ relation. We investigate separately solutions of this equation for high and low frequencies.

6.4 High frequency approximation

In the case of high frequencies $\frac{1}{\omega} \ll 1$ we have $\delta_s = \delta_f = 0$ and the dispersion relation follows in the form

$$\mathcal{P}_R \sqrt{1 - \frac{1}{c_f^2}\left(\frac{\omega}{k}\right)^2} + \frac{r}{c_s^4}\left(\frac{\omega}{k}\right)^4 \sqrt{1 - \left(\frac{\omega}{k}\right)^2} = 0, \tag{214}$$

where

$$\mathcal{P}_R := \left(2 - \frac{1}{c_s^2}\left(\frac{\omega}{k}\right)^2\right)^2 - 4\sqrt{1 - \left(\frac{\omega}{k}\right)^2}\sqrt{1 - \frac{1}{c_s^2}\left(\frac{\omega}{k}\right)^2}. \tag{215}$$

Hence for $r = 0$ the relation (214) reduces to $\mathcal{P}_R = 0$ which is the Rayleigh dispersion relation for single component continua. Otherwise we obtain the relation identical with this analyzed by I. Edelman and K. Wilmanski [24] in the limit of short waves (i.e. $\frac{1}{k} \ll 1$). Consequently, the conclusions for this case are the same as well. As shown in the paper [24] the equation (214) possesses two roots defining two surface waves: a true Stoneley wave which propagates with the finite attenuation and with the velocity a bit smaller than c_f as well as a generalized Rayleigh wave which is leaky (i.e. it radiates the energy to the P2-wave) and propagates with the velocity c_R: $c_f < c_R < c_s$. The Rayleigh wave is leaky because its attenuation is unbounded, i.e. such a wave cannot exist in the range of high frequencies. Immediately after the initiation, it transforms into bulk waves.

These results are not very surpprising because the dispersion relation (214) is identical with the dispersion relation in Box 6 for the Stoneley-Scholte wave. The only difference is that the real Stoneley-Scholte wave propagates on both sides of the interface and the above presented wave propagates solely below the boundary ($z > 0$) in the porous medium.

The detailed description of these waves can be found in [3].

6.5 Low frequency approximation

If we account for the relations (206) and (207) in the condition $\det \mathbf{A} = 0$ then we obtain the dispersion relation reflecting a dependence of $\frac{\omega}{k}$ on ω. The expansion with respect to $\sqrt{\omega}$ yields the identity in the zeroth order and the following relation for the higher order

$$\left(\frac{\omega}{k}\right)\left\{\left(2 - \frac{r+1}{c_s^2}\left(\frac{\omega}{k}\right)^2\right)^2 - 4\sqrt{1 - \frac{r+1}{c_s^2}\left(\frac{\omega}{k}\right)^2}\sqrt{1 - \frac{r+1}{rc_f^2+1}\left(\frac{\omega}{k}\right)^2}\right\} + \tag{216}$$

$$+ O\left(\sqrt{\omega}\right) = 0.$$

Clearly we obtain two solutions:

1. Rayleigh wave whose velocity is different from zero in the limit $\omega \to 0$ and whose attenuation is of the order $O\left(\sqrt{\omega}\right)$. The relation for the velocity reminds the relation (215) with the velocities of bulk waves replaced by the low frequency limits. Namely we have

$$\frac{r+1}{c_s^2} = c_{P1}^2 \frac{\rho_0^S + \rho_0^F}{\mu^S} \equiv \frac{c_{P1}^2}{c_{oS}^2}, \quad \frac{r+1}{rc_f^2+1} = c_{P1}^2 \frac{\rho_0^S + \rho_0^F}{\lambda^S + 2\mu^S + \rho_0^F \kappa} \equiv \frac{c_{P1}^2}{c_{oP1}^2}. \tag{217}$$

Consequently

$$\left(2-\frac{c_{P1}^2}{c_{oS}^2}\left(\frac{\omega}{k}\right)^2\right)^2-4\sqrt{1-\frac{c_{P1}^2}{c_{oS}^2}\left(\frac{\omega}{k}\right)^2}\sqrt{1-\frac{c_{P1}^2}{c_{oP1}^2}\left(\frac{\omega}{k}\right)^2}=0. \tag{218}$$

2. The Stoneley wave has the velocity of propagation of the order $O\left(\sqrt{\omega}\right)$. Hence, it goes to zero in the same way as the velocity of propagation of the P2-wave.

Box 8: *Surface waves on an impermeable boundary of porous materials – dispersion relations*

High frequency approximation: $\omega \to \infty$

$$\mathcal{P}_R\sqrt{1-\frac{1}{c_{P2}^2}\left(\frac{\omega}{k}\right)^2}+\frac{r}{c_S^4}\left(\frac{\omega}{k}\right)^4\sqrt{1-\frac{1}{c_{P1}^2}\left(\frac{\omega}{k}\right)^2}=0,$$

$$\mathcal{P}_R:=\left(2-\frac{1}{c_S^2}\left(\frac{\omega}{k}\right)^2\right)^2-4\sqrt{1-\frac{1}{c_{P1}^2}\left(\frac{\omega}{k}\right)^2}\sqrt{1-\frac{1}{c_S^2}\left(\frac{\omega}{k}\right)^2},$$

Low frequency approximation: $\omega \to 0$

$$\left(2-\frac{1}{c_{oS}^2}\left(\frac{\omega}{k}\right)^2\right)^2-4\sqrt{1-\frac{1}{c_{oS}^2}\left(\frac{\omega}{k}\right)^2}\sqrt{1-\frac{1}{c_{oP1}^2}\left(\frac{\omega}{k}\right)^2}=0,$$

where

$$\begin{aligned} c_{P1} &= \sqrt{\frac{\lambda^S+2\mu^S}{\rho_0^S}}, \quad c_{P2}=\sqrt{\kappa}, \quad c_S=\sqrt{\frac{\mu^S}{\rho_0^S}}, \\ c_{oP1} &= \sqrt{\frac{\lambda^S+2\mu^S+\rho_0^F\kappa}{\rho_0^S+\rho_0^F}}, \quad c_{oS}=\sqrt{\frac{\mu^S}{\rho_0^S+\rho_0^F}}, \quad r=\frac{\rho_0^F}{\rho_0^S}. \end{aligned}$$

6.6 Remarks on modelling surface waves by Biot's model and the simple mixture model

The results for a two-component model of porous solid-fluid mixtures presented in this Section should be compared with those obtained by means of the Biot's model and with experimental observations. We shall not go into details of such a comparison in this work. However, as mentioned before, there is a very good qualitative agreement of both models as far as propagation of acoustic waves is concerned. Velocities of bulk waves are influenced by the coupling parameter Q but this influence reflected in high frequency – low frequency relations seems to be too strong for values of this parameter following from the classical Gassmann relations. The influence of tortuosity a on the velocities of propagation is rather small and, simultaneously, essential changes of attenuation do not correspond to the physical inside, particularly to mechanisms of scattering of waves, of the morphology of porous materials in spite of some claims in the literature (e.g. compare [33]).

Neither bulk waves nor surface waves reveal any qualitative differences between Biot's model and the simple mixture model. It can be expected that ranges of existence of different surface modes are different for both models but results of analysis of this problem are not yet available.

It should be mentioned that results for surface waves within the Biot's model are often limited to high frequency limits [26]. Some early results of Deresiewicz for low frequencies [17], [18], [19], [20] do not depart from those obtained within the simple mixture model. There exists a number of works in which solutions similar to those of Deresiewicz are obtained for more complex systems (e.g. [35], [36], [59], [60]). For instance, in the work [36] a porous semiinfinite medium described by the Biot's model is considered to be in contact with a layer of an ideal homogeneous fluid on which there is another layer of ideal inhomogeneous fluid with a linear dependence of material parameters on the z-coordinate. Surface waves are sougth under the following simplifying assumptions. In the Deresiewicz-Skalak boundary condition (211), it is assumed that the boundary permeability $\alpha \to \infty$. The second assumption concerns the diffusion. Authors assume lack of dissipation which means that the partial velocities of components are equal. This happens approximately indeed in the range of low frequencies where the components move almost synchronized but not in the range of high frequencies. However, both limits are investigated numerically in the work. In these calculations, the data are chosen in such a way that the tortuosity $a = 1.01$ which is, of course almost eliminating this effect from Biot's model. It is rather natural that results practically do not differ from those for single component materials.

The most interesting work with a very extensive comparison of different models has been done by Norris [48] for the propagation of Stoneley waves on the permeable boundary of the porous material. He investigates two problems. A low frequency regime is first described by the model of pore-pressure diffusion in a compressible frame. In this range, the Stoneley wave in a borehole is called a **tube wave** and Norris presents a comparison of various so-called quasi-static models. He is using the full Deresiewicz-Skalak boundary condition. The most extensive part of the work concerns the application of Biot's model for which the frequency variations of the velocity and attenuation of the Stoneley wave are discussed in details for the range app. 0-50 kHz. For the lack of space in this article we have to refer to the original work for details. It would be, certainly, interesting to compare these results based on Biot's model with those which follow from the simple mixture model.

Further details concerning the comparison of both models and a full account of numerical results for surface waves described by the simple mixture model can be found in the article of B. Albers [3]. Certainly, it should be born in mind that the simple mixture model must be quantitatively a worse approximation than Biot's model because it does not contain natural physical couplings. However, it is known from the theory of mixture of fluids that many results of the simple mixture theory are good enough for some practical purposes. This seems to be also the case for porous materials as far as a qualitative analysis of acoustic waves is concerned.

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