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A general asymptotic model for Lipschitzian curved rods

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Abstract

In this paper we show that the asymptotic methods provide an advantageous approach to obtain models of thin elastic bodies under minimal regularity assumptions on the geometry. Our investigation is devoted to clamped curved rods with a nonsmooth line of centroids and the obtained model is a generalization of results already available in the literature.

1 Introduction

The asymptotic methods in modelling thin elastic bodies like plates, beams, arches or shells have a long history. The recent treatise of Ciarlet [4] together with the previously published volumes provide a unitary and modern mathematical treatment of the contemporary research in elasticity. In the case of elastic curved rods, the book of Trabucho and Viaño [16] and the articles of Jurak and Tambača [10] and [11] demonstrate the application of asymptotic approaches in a general geometric setting.

In the papers of Blouza and Le Dret [2], Ignat, Sprekels and Tiba [6], [7], the possibility of relaxing the regularity assumptions on the shape of the elastic body is examined for polynomial-type models. It turns out that the most advantageous way in reaching minimal regularity hypotheses on the geometry is the asymptotic modelling method.

We discuss this for curved elastic rods, but many of the ideas seem possible to be further applied to shells. As announced in the title, we study (for the line of centroids) unit speed curves in $\mathbb{R}^3$ which admit Lipschitz parametrization. However, by the standard reparametrization with respect to the arc length, one may consider general absolutely continuous regular parametrization (i.e. with nonzero tangent vector a.e.).

The basic idea is rather simple and natural. If we denote by $\epsilon > 0$ the "thickness" parameter specific to asymptotic methods, we also introduce another small parameter $\delta = \epsilon^r$ $(0 < r < \frac{1}{3})$ associated to a regularization procedure applied to the nonsmooth line of centroids. A careful examination of the convergence properties of the arising smooth coefficients and sharp estimates in the corresponding variational formulation of the linear elasticity system (after scaling) allows to pass to the limit $\epsilon \to 0$ and to obtain the asymptotic model. In the smooth case, this is similar to the model of Jurak and Tambača [10] and [11].

An important ingredient in our argument is the construction of a local frame (different than the classical Frenêt basis) applicable for Lipschitzian parametrizations. This is a generalization of the ideas developed in Ignat, Sprekels and Tiba [7]. Let us also mention other related works discussing asymptotic models: Aganovič and Tutek [1] (for beams), Nazarov and Slutskij [12], Jamal [8], Jamal and Sanchez-Palencia [9] (for curved rods). The very recent work of Tambača [15] discusses a regularization procedure for piecewise C^1 parametrizations and in the absence of surface tractions, directly in the setting of ordinary differential equations (with different boundary conditions) obtained as the asymptotic model in the smooth case. Our work is more general from these points of view and our arguments are constructive and certainly different.

Finally, we mention a brief outline of the paper. In Section 2, we introduce the basic notations and notions that will be further needed. Section 3 deals with the construction of a local frame for piecewise C^1 parametrizations and its regularization. This will be later refined (in the Appendix) to the case of Lipschitzian parametrizations. In Section 4 we study the functional space which plays the essential role in our approach. Section 5 contains several auxiliary results, some of them collected from the existing literature. Section 6 is devoted to the formulation of the linear elasticity equations and their transformation. Section 7 gives the basic estimates including a Korn-type inequality with explicit constants with respect to the thickness $\epsilon > 0$. In Section 8 the passage to the limit $\epsilon \to 0$ is performed and the main existence and uniqueness result is proved. Section 9 provides a short comparison with other results available in the literature.

2 Basic notation

We denote by $\mathbb{R}^3$ the usual three dimensional Euclidean space with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. By "$\cdot \times \cdot$" we shall denote the vector product in $\mathbb{R}^3$ and by $\langle\cdot,\cdot\rangle$ any ordered pair. In the text the symbol "$\times$" is also used for the Cartesian product of two spaces and $|A|$ will also denote the Lebesgue measure of some measurable set A, without danger of confusion. The summation convention with respect to repeated indices will be also used, if not otherwise explicitly stated.

Let $S \subset \mathbb{R}^2$ be a bounded simply connected domain of class C^1 satisfying the "symmetry" condition

$$\int_S x_2 \; dx_2 dx_3 = \int_S x_3 \; dx_2 dx_3 = \int_S x_2 x_3 \; dx_2 dx_3 = 0. \tag{2.1}$$

We denote by $\Omega = (0,l) \times S$, $\Omega_\epsilon = (0,l) \times \epsilon S$ open "cylinders" in $\mathbb{R}^3$, where $l > 0$ and $\epsilon > 0$ "small", are given.

Let $\mathcal{C}$ be a unit speed curve of length l in $\mathbb{R}^3$ defined by its parametrization $\mathbf{\Phi} : [0,l] \to \mathbb{R}^3$, and let $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ denote its tangent, normal and binormal vectors. As we shall assume less regularity for $\mathbf{\Phi}$ as for instance in [3], the local frame $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ is not necessarily the Frenêt one. Alternative ways to construct local frames under low regularity assumptions may be found in [7] and in Appendix. Let $\mathbf{\Phi}_\epsilon : [0,l] \to \mathbb{R}^3$ be a smoothing of $\mathbf{\Phi}$ such that it remains a unit speed curve (i.e. $|\mathbf{\Phi}'_\epsilon(y_1)| = 1$, $\forall y_1 \in [0,l]$) and $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ be the associated local frame. The regularization parameter will be of the form ϵ^r, $r \in (0, \frac{1}{3})$, and we just write $\mathbf{\Phi}_\epsilon$, $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ to simplify notation. More details on the construction of $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ and their regularizations are given in Section 3 and in Appendix.

Further, we define the auxiliary functions α_ϵ, β_ϵ, γ_ϵ (corresponding to the usual notions of curvature and torsion) by

$$\alpha_\epsilon = (\mathbf{t}'_\epsilon, \mathbf{b}_\epsilon),\ \beta_\epsilon = (\mathbf{t}'_\epsilon, \mathbf{n}_\epsilon),\ \gamma_\epsilon = (\mathbf{b}'_\epsilon, \mathbf{n}_\epsilon),$$

where $\mathbf{t}'_\epsilon$ is the derivative of $\mathbf{t}_\epsilon$ with respect to x_1, etc. To obtain this relations, we use the assumed orthonormality of the local basis $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ which gives the orthogonality properties $(\mathbf{t}_\epsilon, \mathbf{t}'_\epsilon) = 0$, $(\mathbf{n}_\epsilon, \mathbf{n}'_\epsilon) = 0$, $(\mathbf{b}_\epsilon, \mathbf{b}'_\epsilon) = 0$, that is $\mathbf{t}'_\epsilon$ may be expressed via $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ and so on. We obtain the "laws of motion" of the local frame

$$\begin{aligned} \mathbf{t}'_\epsilon &= \alpha_\epsilon \mathbf{b}_\epsilon + \beta_\epsilon \mathbf{n}_\epsilon, \\ \mathbf{n}'_\epsilon &= -\beta_\epsilon \mathbf{t}_\epsilon - \gamma_\epsilon \mathbf{b}_\epsilon, \\ \mathbf{b}'_\epsilon &= -\alpha_\epsilon \mathbf{t}_\epsilon + \gamma_\epsilon \mathbf{n}_\epsilon. \end{aligned} \tag{2.2}$$

We introduce the mappings $\mathbf{R}_\epsilon$

$$\mathbf{R}_\epsilon : \Omega \to \Omega_\epsilon,\ \mathbf{R}_\epsilon(x_1, x_2, x_3) = (x_1, \epsilon x_2, \epsilon x_3), \tag{2.3}$$

and $\bar{\mathbf{P}}_\epsilon$

$$\bar{\mathbf{P}}_\epsilon : \Omega_\epsilon \to \mathbb{R}^3,\ \bar{\mathbf{P}}_\epsilon(y) = \mathbf{\Phi}_\epsilon(y_1) + y_2 \mathbf{n}_\epsilon(y_1) + y_3 \mathbf{b}_\epsilon(y_1), \tag{2.4}$$

$(y_1, y_2, y_3) \in (0, l) \times \epsilon S$, which gives the parametrization of the curved rod $\widetilde{\Omega}_\epsilon = \bar{\mathbf{P}}_\epsilon(\Omega_\epsilon)$. Further,

$$\bar{d}_\epsilon(y) = \det(\bar{\nabla}\bar{\mathbf{P}}_\epsilon(y)) = 1 - \beta_\epsilon(y_1)y_2 - \alpha_\epsilon(y_1)y_3 \text{ for all } y \in \overline{\Omega}_\epsilon. \tag{2.5}$$

We can suppose that $\bar{d}_\epsilon(y) \neq 0$ for all $y \in \overline{\Omega}_\epsilon$ (see (2.10) and Corollary 3.3) and for ϵ "small". Then $\bar{\mathbf{P}}_\epsilon : \Omega_\epsilon \to \widetilde{\Omega}_\epsilon$ is a C^1-diffeomorphism, Ciarlet [4], Theorem 3.1-1. In the sequel, we shall write $\widetilde{\partial}_i = \frac{\partial}{\partial \widetilde{y}_i}$, where $\widetilde{y} = (\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3) \in \widetilde{\Omega}_\epsilon$, $\bar{\partial}_i = \frac{\partial}{\partial y_i}$, for $y = (y_1, y_2, y_3) \in \Omega_\epsilon$, and $\partial_i = \frac{\partial}{\partial x_i}$, where $x = (x_1, x_2, x_3) \in \Omega$. Thus, in (2.5) $\bar{\nabla} = (\bar{\partial}_1, \bar{\partial}_2, \bar{\partial}_3)$. In an analogous way, we denote by $\widetilde{V}$ a function defined on $\widetilde{\Omega}_\epsilon$, $\bar{V}$ a function defined on Ω_ϵ and V a function defined on Ω. We suppose throughout this subsection that all derivatives, that we need, exist, which will follow from Section 3 later.

The covariant basis at point $\bar{\mathbf{P}}_\epsilon(y)$, $y \in \Omega_\epsilon$, of the curved rod is defined by $\bar{\mathbf{g}}_{i,\epsilon}(y) = \bar{\partial}_i \bar{\mathbf{P}}_\epsilon(y)$ and (using (2.2)) these vectors are given by

$$\begin{aligned} \bar{\mathbf{g}}_{1,\epsilon}(y) &= (1 - y_2\beta_\epsilon(y_1) - y_3\alpha_\epsilon(y_1))\mathbf{t}_\epsilon(y_1) + y_3\gamma_\epsilon(y_1)\mathbf{n}_\epsilon(y_1) - y_2\gamma_\epsilon(y_1)\mathbf{b}_\epsilon(y_1), \\ \bar{\mathbf{g}}_{2,\epsilon}(y) &= \mathbf{n}_\epsilon(y_1),\ \bar{\mathbf{g}}_{3,\epsilon}(y) = \mathbf{b}_\epsilon(y_1). \end{aligned} \tag{2.6}$$

The vectors $\bar{\mathbf{g}}^{j,\epsilon}$ defined by the relations $(\bar{\mathbf{g}}_{i,\epsilon}, \bar{\mathbf{g}}^{j,\epsilon}) = \delta^{ij}$, constitute the contravariant basis of the curved rod at the point $\bar{\mathbf{P}}_\epsilon(y)$. They have the form

$$\bar{\mathbf{g}}^{1,\epsilon}(y) = \frac{\mathbf{t}_\epsilon(y_1)}{\bar{d}_\epsilon(y)},\ \bar{\mathbf{g}}^{2,\epsilon}(y) = \frac{-y_3\gamma_\epsilon(y_1)\mathbf{t}_\epsilon(y_1)}{\bar{d}_\epsilon(y)} + \mathbf{n}_\epsilon(y_1),$$

$$\bar{\mathbf{g}}^{3,\epsilon}(y) = \frac{y_2\gamma_\epsilon(y_1)\mathbf{t}_\epsilon(y_1)}{\bar{d}_\epsilon(y)} + \mathbf{b}_\epsilon(y_1). \tag{2.7}$$

Further, we define the covariant and contravariant metric tensors $(\bar{g}_{ij,\epsilon})_{i,j=1}^3$ and $(\bar{g}^{ij,\epsilon})_{i,j=1}^3$, where

$$\bar{g}_{ij,\epsilon} = (\bar{\mathbf{g}}_{i,\epsilon}, \bar{\mathbf{g}}_{j,\epsilon}), \; \bar{g}^{ij,\epsilon} = (\bar{\mathbf{g}}^{i,\epsilon}, \bar{\mathbf{g}}^{j,\epsilon}). \tag{2.8}$$

After substitution $y = \mathbf{R}_\epsilon(x)$, we adopt the notation

$$g^{ij,\epsilon}(x) = \bar{g}^{ij,\epsilon}(\mathbf{R}_\epsilon(x)), \; g_{ij,\epsilon}(x) = \bar{g}_{ij,\epsilon}(\mathbf{R}_\epsilon(x)), \; \mathbf{g}_{i,\epsilon}(x) = \bar{\mathbf{g}}_{i,\epsilon}(\mathbf{R}_\epsilon(x)), \tag{2.9}$$

$$\mathbf{g}^{j,\epsilon}(x) = \bar{\mathbf{g}}^{j,\epsilon}(\mathbf{R}_\epsilon(x)), \; d_\epsilon(x) = \bar{d}_\epsilon(\mathbf{R}_\epsilon(x)), \; A_\epsilon^{ijkl}(x) = \bar{A}_\epsilon^{ijkl}(\mathbf{R}_\epsilon(x)), \tag{2.10}$$

where $x \in \Omega$ and $(\bar{A}_\epsilon^{ijkl}(y))_{i,j,k,l=1}^3$ is a fourth-order tensor to be defined later in (6.2).

In an analogous way, we can derive the covariant basis at the point $(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)(x)$, $x \in \Omega$. Thus $\mathbf{o}_{i,\epsilon}(x) = \partial_i(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)(x)$ and these vectors are given by

$$\mathbf{o}_{1,\epsilon}(x) = (1 - \epsilon x_2\beta_\epsilon(x_1) - \epsilon x_3\alpha_\epsilon(x_1))\mathbf{t}_\epsilon(x_1) + \epsilon x_3\gamma_\epsilon(x_1)\mathbf{n}_\epsilon(x_1) - \epsilon x_2\gamma_\epsilon(x_1)\mathbf{b}_\epsilon(x_1),$$

$$\mathbf{o}_{2,\epsilon}(x) = \epsilon\mathbf{n}_\epsilon(x_1), \; \mathbf{o}_{3,\epsilon}(x) = \epsilon\mathbf{b}_\epsilon(x_1). \tag{2.11}$$

The vectors $\mathbf{o}^{j,\epsilon}$ defined by the relations $(\mathbf{o}_{i,\epsilon}, \mathbf{o}^{j,\epsilon}) = \delta^{ij}$, constitute the contravariant basis at the point $(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)(x)$, $x \in \Omega$. They have the form

$$\mathbf{o}^{1,\epsilon}(x) = \frac{\mathbf{t}_\epsilon(x_1)}{d_\epsilon(x)}, \; \mathbf{o}^{2,\epsilon}(x) = \frac{-x_3\gamma_\epsilon(x_1)\mathbf{t}_\epsilon(x_1)}{d_\epsilon(x)} + \frac{\mathbf{n}_\epsilon(x_1)}{\epsilon},$$

$$\bar{\mathbf{o}}^{3,\epsilon}(x) = \frac{x_2\gamma_\epsilon(x_1)\mathbf{t}_\epsilon(x_1)}{d_\epsilon(x)} + \frac{\mathbf{b}_\epsilon(x_1)}{\epsilon}. \tag{2.12}$$

We can define the covariant and contravariant metric tensors $(o_{ij,\epsilon})_{i,j=1}^3$ and $(o^{ij,\epsilon})_{i,j=1}^3$, where

$$o_{ij,\epsilon} = (\mathbf{o}_{i,\epsilon}, \mathbf{o}_{j,\epsilon}), \; o^{ij,\epsilon} = (\mathbf{o}^{i,\epsilon}, \mathbf{o}^{j,\epsilon}). \tag{2.13}$$

These tensors have the form

$$(o_{ij,\epsilon})_{i,j=1}^3 = \begin{pmatrix} d_\epsilon^2 + \epsilon^2 x_3^2\gamma_\epsilon^2 + \epsilon^2 x_2^2\gamma_\epsilon^2 & \epsilon^2 x_3\gamma_\epsilon & -\epsilon^2 x_2\gamma_\epsilon \\ \epsilon^2 x_3\gamma_\epsilon & \epsilon^2 & 0 \\ -\epsilon^2 x_2\gamma_\epsilon & 0 & \epsilon^2 \end{pmatrix} \tag{2.14}$$

and

$$(o^{ij,\epsilon})_{i,j=1}^3 = \begin{pmatrix} \frac{1}{d_\epsilon^2} & \frac{-x_3\gamma_\epsilon}{d_\epsilon^2} & \frac{x_2\gamma_\epsilon}{d_\epsilon^2} \\ \frac{-x_3\gamma_\epsilon}{d_\epsilon^2} & \frac{1}{\epsilon^2} + \frac{x_3^2\gamma_\epsilon^2}{d_\epsilon^2} & \frac{-x_2x_3\gamma_\epsilon^2}{d_\epsilon^2} \\ \frac{x_2\gamma_\epsilon}{d_\epsilon^2} & \frac{-x_2x_3\gamma_\epsilon^2}{d_\epsilon^2} & \frac{1}{\epsilon^2} + \frac{x_2^2\gamma_\epsilon^2}{d_\epsilon^2} \end{pmatrix}. \tag{2.15}$$

Now, we can compute

$$o_\epsilon(x) = \sqrt{\det(o_{ij,\epsilon}(x))_{i,j=1}^3} = \epsilon^2 d_\epsilon(x). \tag{2.16}$$

We use for constants the symbols C or C_i, for $i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Constant vectors will be denoted by $\mathbf{C}$ or $\mathbf{C}_i$ for $i \in \mathbb{N}_0$.

The symbols $H^1(\Omega)$, $H_0^1(\Omega)$ and $L^p(\Omega)$, respectively, denote (for $p \in [1, \infty]$) the standard Sobolev and Lebesgue spaces endowed with the norms $\|\cdot\|_{1,2}$ or $\|\cdot\|_p$. We will use the same notation of the norms also for vector or tensor functions in the case that all their components belong to above mentioned Sobolev or Lebesgue spaces. $H^{-1}(\Omega)$ stands for the dual space to $H_0^1(\Omega)$. The notation $C^m(\overline{\Omega})$, with $m \in \mathbb{N}_0$, means the usual spaces of continuous functions whose derivatives up to the order m are continuous in $\overline{\Omega}$. The symbols $L^p(0, l; X)$, $p \in [1, \infty)$, and $C([0, l]; X)$, where X is a Banach space, stand for the Bochner spaces endowed with the norms

$$\|v\|_{L^p(0,l;X)} = \left(\int_0^l \|v(x_1)\|_X^p \, dx_1 \right)^{1/p} \quad \text{and} \quad \|v\|_{C([0,l];X)} = \max_{x_1 \in [0,l]} \|v(x_1)\|_X.$$

The definitions of the domains $\widetilde{\Omega}_\epsilon$, Ω_ϵ and Ω enable us to introduce the following notation:

$$V(\widetilde{\Omega}_\epsilon) = \{\widetilde{\mathbf{V}} \in H^1(\widetilde{\Omega}_\epsilon)^3 : \widetilde{\mathbf{V}}|_{\bar{\mathbf{P}}_\epsilon(\{0\}\times\epsilon S)} = \widetilde{\mathbf{V}}|_{\bar{\mathbf{P}}_\epsilon(\{l\}\times\epsilon S)} = 0\},$$

$$V(\Omega_\epsilon) = \{\bar{\mathbf{V}} \in H^1(\Omega_\epsilon)^3 : \bar{\mathbf{V}}|_{(\{0\}\times\epsilon S)} = \bar{\mathbf{V}}|_{(\{l\}\times\epsilon S)} = 0\},$$
$$V(\Omega) = \{\mathbf{V} \in H^1(\Omega)^3 : \mathbf{V}|_{(\{0\}\times S)} = \mathbf{V}|_{(\{l\}\times S)} = 0\}$$

and further we introduce the space

$$\begin{aligned} \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0, l) = \{\langle \mathbf{V}, \psi\rangle \in H_0^1(0, l)^3 \times L^2(0, l) :\ (\mathbf{V}', \mathbf{t}) = 0 \\ \text{and } \mathbf{V}_* = -\psi\mathbf{t} + (\mathbf{V}', \mathbf{b})\mathbf{n} - (\mathbf{V}', \mathbf{n})\mathbf{b} \in H_0^1(0, l)^3\}. \end{aligned} \tag{2.17}$$

The properties of the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0, l)$ will be studied in Section 4.

3 Construction of the local frame for the unit speed curve $\mathcal{C}$ and its regularization

We start by recalling the result established in [7]:

Proposition 3.1 *If $\mathbf{\Psi} \in C^1([0, l])^3$ then the tangent vector $\mathbf{t}_\mathbf{\Psi} \in C([0, l])^3$ is defined by $\mathbf{t}_\mathbf{\Psi} = \mathbf{\Psi}'$ and there exists a normal vector $\mathbf{n}_\mathbf{\Psi} \in C([0, l])^3$ such that $|\mathbf{n}_\mathbf{\Psi}(x_1)| = 1$, $(\mathbf{n}_\mathbf{\Psi}(x_1), \mathbf{t}_\mathbf{\Psi}(x_1)) = 0$, $x_1 \in [0, l]$. The vector $\mathbf{b}_\mathbf{\Psi} = \mathbf{t}_\mathbf{\Psi} \times \mathbf{n}_\mathbf{\Psi}$ has the same regularity properties and completes the local frame.*

In this section, we extend and complete Proposition 3.1 as follows

Proposition 3.2 *Let the function $\mathbf{\Phi} \in C([0,l])^3$ be such that its tangent vector $\mathbf{t} = \mathbf{\Phi}'$ is a piecewise continuous function with a finite set D of points of discontinuity. Then there exists the functions $\mathbf{n}$ and $\mathbf{b}$ piecewise continuous such that*

$$|\mathbf{t}| = |\mathbf{n}| = |\mathbf{b}| = 1,\ \mathbf{t} \perp \mathbf{n} \perp \mathbf{b} \ \text{in } [0,l] \setminus D. \tag{3.1}$$

In addition, there exist the functions

$$\{\mathbf{\Phi}_\epsilon\}_{\epsilon \in (0,1)},\ \{\mathbf{t}_\epsilon\}_{\epsilon \in (0,1)},\ \{\mathbf{n}_\epsilon\}_{\epsilon \in (0,1)},\ \{\mathbf{b}_\epsilon\}_{\epsilon \in (0,1)} \subset C^\infty([0,l])^3$$

such that

$$\mathbf{\Phi}'_\epsilon = \mathbf{t}_\epsilon,\ |\mathbf{t}_\epsilon| = |\mathbf{n}_\epsilon| = |\mathbf{b}_\epsilon| = 1,\ \mathbf{t}_\epsilon \perp \mathbf{n}_\epsilon \perp \mathbf{b}_\epsilon \ \text{on } [0,l] \tag{3.2}$$

$$\mathbf{t}_\epsilon \to \mathbf{t},\ \mathbf{n}_\epsilon \to \mathbf{n},\ \mathbf{b}_\epsilon \to \mathbf{b} \ \text{pointwisely in } [0,l] \setminus D, \tag{3.3}$$

$$\|\mathbf{t}'_\epsilon\|_\infty,\ \|\mathbf{n}'_\epsilon\|_\infty,\ \|\mathbf{b}'_\epsilon\|_\infty \sim O(\frac{1}{\epsilon^r}) \tag{3.4}$$

and

$$\|\mathbf{t}''_\epsilon\|_\infty,\ \|\mathbf{n}''_\epsilon\|_\infty,\ \|\mathbf{b}''_\epsilon\|_\infty \sim O(\frac{1}{\epsilon^{2r}}) \tag{3.5}$$

for $r \in (0, \frac{1}{3})$.

P r o o f: First, we denote by D_{t_i}, $i = 1,2,3$, the sets given by the points of discontinuity for the functions t_i, $i = 1,2,3$. Further, we have $D = \bigcup_{i=1}^3 D_{t_i}$. The existence of the functions $\mathbf{n}$ and $\mathbf{b}$ satisfying (3.1) immediately follows from Proposition 3.1 after application on the intervals forming the set $[0,l] \setminus D$.

In the first step, we construct the continuous approximation of the functions $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$. As D has a finite number of elements, its points are isolated and we denote them in increasing order by $\widehat{x}_j$, $j = 1, \dots, k$. The approximation will be demonstrated on some points $\widehat{x}_j < \widehat{x}_{j+1} < \widehat{x}_{j+2} \in D$. We define the functions

$$\widehat{t}_{i,\sigma} = \begin{cases} t_i & \text{on } [\widehat{x}_j + \sigma, \widehat{x}_{j+1} - \sigma] \\ l_{i,\sigma} & \text{on } [\widehat{x}_{j+1} - \sigma, \widehat{x}_{j+1} + \sigma] \\ t_i & \text{on } [\widehat{x}_{j+1} + \sigma, \widehat{x}_{j+2} - \sigma], \end{cases} \tag{3.6}$$

where $\sigma > 0$ is arbitrary and sufficiently small and the functions $l_{i,\sigma}$ are continuous and such that

$$l_{i,\sigma}(\widehat{x}_{j+1} - \sigma) = t_i(\widehat{x}_{j+1} - \sigma), \tag{3.7}$$

$$l_{i,\sigma}(\widehat{x}_{j+1} + \sigma) = t_i(\widehat{x}_{j+1} + \sigma) \tag{3.8}$$

and there is no point $z \in [\widehat{x}_{j+1} - \sigma, \widehat{x}_{j+1} + \sigma]$ such that simultaneously

$$l_{i,\sigma}(z) = 0,\ \forall i = 1,2,3. \tag{3.9}$$

Thus we have constructed the continuous vectorial function $\widehat{\mathbf{t}}_\sigma$ such that

$$\widehat{\mathbf{t}}_\sigma \to \widehat{\mathbf{t}} \ \text{pointwisely in } [0,l] \setminus D \tag{3.10}$$

for $\sigma \to 0$ and

$$|\widehat{\mathbf{t}}_\sigma| > 0 \text{ on } [0, l] \tag{3.11}$$

for all σ sufficiently small. Then the functions

$$\mathbf{t}_\sigma = \frac{\widehat{\mathbf{t}}_\sigma}{|\widehat{\mathbf{t}}_\sigma|} \tag{3.12}$$

are continuous and satisfy the convergence

$$\mathbf{t}_\sigma \to \mathbf{t} \text{ pointwisely in } [0, l] \setminus D \tag{3.13}$$

for $\sigma \to 0$.

Now, using Proposition 3.1 we can construct the continuous normal and binormal vector functions $\check{\mathbf{n}}_\sigma$ and $\check{\mathbf{b}}_\sigma$ to the curve $\mathbf{\Phi}_\sigma$, where we put

$$\mathbf{\Phi}_\sigma(x_1) = \int_0^{x_1} \mathbf{t}_\sigma(z_1)\, dz_1 + \mathbf{\Phi}(0),\ x_1 \in [0, l].$$

Let us define the function $\widehat{\mathbf{n}}_\sigma$ in this way

$$\widehat{n}_{i,\sigma} = \begin{cases} n_i & \text{on } [\widehat{x}_j + 2\sigma, \widehat{x}_{j+1} - 2\sigma] \\ (1 - \widehat{l}_{i,1,\sigma})n_i + \widehat{l}_{i,1,\sigma}\check{n}_{i,\sigma} & \text{on } [\widehat{x}_{j+1} - 2\sigma, \widehat{x}_{j+1} - \sigma] \\ \check{n}_{i,\sigma} & \text{on } [\widehat{x}_{j+1} - \sigma, \widehat{x}_{j+1} + \sigma] \\ (1 - \widehat{l}_{i,2,\sigma})\check{n}_{i,\sigma} + \widehat{l}_{i,2,\sigma}n_i & \text{on } [\widehat{x}_{j+1} + \sigma, \widehat{x}_{j+1} + 2\sigma] \\ n_i & \text{on } [\widehat{x}_{j+1} + 2\sigma, \widehat{x}_{j+2} - 2\sigma], \end{cases} \tag{3.14}$$

where $\widehat{l}_{i,m,\sigma}$, $i = 1, 2, 3$ and $m = 1, 2$, are linear functions such that

$$\widehat{l}_{i,1,\sigma}(\widehat{x}_{j+1} - 2\sigma) = 0,\ \widehat{l}_{i,1,\sigma}(\widehat{x}_{j+1} - \sigma) = 1,$$

$$\widehat{l}_{i,2,\sigma}(\widehat{x}_{j+1} + \sigma) = 0,\ \widehat{l}_{i,2,\sigma}(\widehat{x}_{j+1} + 2\sigma) = 1.$$

It is easy to see from (3.6) and (3.14) that the vector $\widehat{\mathbf{n}}_\sigma$ is orthogonal to $\mathbf{t}_\sigma$ and all functions $\widehat{n}_{i,\sigma}$, $i = 1, 2, 3$, cannot be equal to zero at the same point, if

$$n_i(\widehat{x}_{j+1} - \frac{3}{2}\sigma) \neq -\check{n}_{i,\sigma}(\widehat{x}_{j+1} - \frac{3}{2}\sigma) \tag{3.15}$$

and

$$n_i(\widehat{x}_{j+1} + \frac{3}{2}\sigma) \neq -\check{n}_{i,\sigma}(\widehat{x}_{j+1} + \frac{3}{2}\sigma) \tag{3.16}$$

for some i, $i = 1, 2, 3$.

Now, we show how to modify the definition (3.14) of the function $\widehat{\mathbf{n}}_\sigma$ in the case that (3.15) or (3.16) do not hold. We suppose for instance that

$$n_i(\widehat{x}_{j+1} - \frac{3}{2}\sigma) = -\check{n}_{i,\sigma}(\widehat{x}_{j+1} - \frac{3}{2}\sigma) \tag{3.17}$$

for all i, $i = 1, 2, 3$, and

$$n_i(\widehat{x}_{j+1} + \frac{3}{2}\sigma) \neq -\check{n}_{i,\sigma}(\widehat{x}_{j+1} + \frac{3}{2}\sigma) \tag{3.18}$$

for some i, $i = 1, 2, 3$. In this case, we define the functions $\widehat{n}_{i,\sigma}$, $i = 1, 2, 3$, by

$$\widehat{n}_{i,\sigma} = \begin{cases} n_i & \text{on } [\widehat{x}_j + 2\sigma, \widehat{x}_{j+1} - 2\sigma] \\ (1 - \widehat{l}_{i,1,\sigma})n_i - \widehat{l}_{i,1,\sigma}\check{n}_{i,\sigma} & \text{on } [\widehat{x}_{j+1} - 2\sigma, \widehat{x}_{j+1} - \sigma] \\ -(1 - \widehat{l}_{i,2,\sigma})\check{n}_{i,\sigma} + \widehat{l}_{i,2,\sigma}\check{b}_{i,\sigma} & \text{on } [\widehat{x}_{j+1} - \sigma, \widehat{x}_{j+1}] \\ (1 - \widehat{l}_{i,3,\sigma})\check{b}_{i,\sigma} + \widehat{l}_{i,3,\sigma}\check{n}_{i,\sigma} & \text{on } [\widehat{x}_{j+1}, \widehat{x}_{j+1} + \sigma] \\ (1 - \widehat{l}_{i,4,\sigma})\check{n}_{i,\sigma} + \widehat{l}_{i,4,\sigma}n_i & \text{on } [\widehat{x}_{j+1} + \sigma, \widehat{x}_{j+1} + 2\sigma] \\ n_i & \text{on } [\widehat{x}_{j+1} + 2\sigma, \widehat{x}_{j+2} - 2\sigma], \end{cases} \tag{3.19}$$

where $\check{b}_{i,\sigma}$ are the components of the binormal $\check{\mathbf{b}}_\sigma$ and $\widehat{l}_{i,m,\sigma}$, for $i = 1, 2, 3$ and $m = 1, 2, 3, 4$, are linear functions such that

$$\widehat{l}_{i,1,\sigma}(\widehat{x}_{j+1} - 2\sigma) = 0,\ \widehat{l}_{i,1,\sigma}(\widehat{x}_{j+1} - \sigma) = 1,\ \widehat{l}_{i,2,\sigma}(\widehat{x}_{j+1} - \sigma) = 0,\ \widehat{l}_{i,2,\sigma}(\widehat{x}_{j+1}) = 1,$$

$$\widehat{l}_{i,3,\sigma}(\widehat{x}_{j+1}) = 0,\ \widehat{l}_{i,3,\sigma}(\widehat{x}_{j+1} + \sigma) = 1,\ \widehat{l}_{i,4,\sigma}(\widehat{x}_{j+1} + \sigma) = 0,\ \widehat{l}_{i,4,\sigma}(\widehat{x}_{j+1} + 2\sigma) = 1.$$

We can derive from (3.6) and (3.19) that

$$\mathbf{t}_\sigma \perp \widehat{\mathbf{n}}_\sigma,\ \ |\widehat{\mathbf{n}}_\sigma| > 0 \text{ on } [0, l] \tag{3.20}$$

for all σ sufficiently small and

$$\widehat{\mathbf{n}}_\sigma \to \mathbf{n} \text{ pointwisely in } [0, l] \setminus D \tag{3.21}$$

for $\sigma \to 0$. We can easily check using (3.19) and (3.20) that the functions

$$\mathbf{n}_\sigma = \frac{\widehat{\mathbf{n}}_\sigma}{|\widehat{\mathbf{n}}_\sigma|} \tag{3.22}$$

are continuous, orthogonal to $\mathbf{t}_\sigma$ on $[0, l]$ for all $\sigma \in (0, 1)$ and

$$\mathbf{n}_\sigma \to \mathbf{n} \text{ pointwisely in } [0, l] \setminus D \tag{3.23}$$

for $\sigma \to 0$.

Defining the function $\mathbf{b}_\sigma$ by

$$\mathbf{b}_\sigma = \mathbf{t}_\sigma \times \mathbf{n}_\sigma \tag{3.24}$$

we can complete the local frame. It is easy to verify that $|\mathbf{b}_\sigma| = 1$ and

$$\mathbf{b}_\sigma \to \mathbf{b} \text{ pointwisely in } [0, l] \setminus D \tag{3.25}$$

for $\sigma \to 0$.

Now, we construct C^∞-approximation of the local frame. Let us define the functions

$$\mathbf{t}_{\sigma,\delta} = \frac{\mathbf{t}_\sigma * \vartheta_\delta}{|\mathbf{t}_\sigma * \vartheta_\delta|}, \tag{3.26}$$

$$\mathbf{n}_{\sigma,\delta} = \frac{\mathbf{n}_\sigma * \vartheta_\delta - (\mathbf{n}_\sigma * \vartheta_\delta, \mathbf{t}_{\sigma,\delta})\mathbf{t}_{\sigma,\delta}}{|\mathbf{n}_\sigma * \vartheta_\delta - (\mathbf{n}_\sigma * \vartheta_\delta, \mathbf{t}_{\sigma,\delta})\mathbf{t}_{\sigma,\delta}|}, \tag{3.27}$$

where $\vartheta \in C_0^\infty(-1,1)$, $\int_{-1}^1 \vartheta(x_1)\, dx_1 = 1$, $0 \leq \vartheta \leq 1$ and $\vartheta_\delta(x_1) = \frac{1}{\delta}\vartheta(\frac{x_1}{\delta})$. From the convergence

$$\mathbf{t}_\sigma * \vartheta_\delta \to \mathbf{t}_\sigma \text{ in } C([0,l])^3 \tag{3.28}$$

for $\delta \to 0$, which imply that

$$|\mathbf{t}_\sigma * \vartheta_\delta(x_1)| \geq C_1(\sigma,\delta),\ \forall x_1 \in [0,l], \tag{3.29}$$

where $C_1(\sigma,\delta) \to 1$ for $\delta \to 0$, we see that (3.26) makes sense and

$$\mathbf{t}_{\sigma,\delta} \to \mathbf{t}_\sigma \text{ in } C([0,l])^3 \tag{3.30}$$

for fixed σ and $\delta \to 0$. Similarly, we deduce that

$$\mathbf{n}_{\sigma,\delta} \to \mathbf{n}_\sigma \text{ in } C([0,l])^3 \tag{3.31}$$

for fixed σ and $\delta \to 0$. Defining, now, the function $\mathbf{b}_\delta$ by

$$\mathbf{b}_{\sigma,\delta} = \mathbf{t}_{\sigma,\delta} \times \mathbf{n}_{\sigma,\delta} \tag{3.32}$$

completes the approximating local frame and the convergences (3.30)–(3.31) imply that

$$\mathbf{b}_{\sigma,\delta} \to \mathbf{b}_\sigma \text{ in } C([0,l])^3 \tag{3.33}$$

for fixed σ and $\delta \to 0$. The fact that $\mathbf{t}_{\sigma,\delta}$, $\mathbf{n}_{\sigma,\delta}$ and $\mathbf{b}_{\sigma,\delta} \in C^\infty([0,l])^3$ follows from (3.26)–(3.27), (3.29) and from the definition of mollifiers.

From (3.6), (3.12), (3.19), (3.22) and (3.24), it follows that

$$\mathbf{t}_\sigma = \mathbf{t} \text{ on } [\widehat{x}_j + \sigma, \widehat{x}_{j+1} - \sigma],\ \mathbf{n}_\sigma = \mathbf{n},\ \mathbf{b}_\sigma = \mathbf{b} \text{ on } [\widehat{x}_j + 2\sigma, \widehat{x}_{j+1} - 2\sigma], \tag{3.34}$$

which implies that

$$\Big| x_1 \in [\widehat{x}_j, \widehat{x}_{j+1}] :\ |\mathbf{t}_\sigma(x_1) - \mathbf{t}(x_1)| + |\mathbf{n}_\sigma(x_1) - \mathbf{n}(x_1)|$$

$$+ |\mathbf{b}_\sigma(x_1) - \mathbf{b}(x_1)| \geq \epsilon_1 \Big| = 4\sigma, \tag{3.35}$$

where σ is independent of ϵ_1. Further we deduce from (3.30), (3.31) and (3.33) that $\forall \sigma \in (0,1)\ \forall \epsilon_2 > 0\ \exists \delta_0(\sigma,\epsilon_2) : \forall \delta \in (0, \delta_0(\sigma,\epsilon_2))$

$$||\mathbf{t}_{\sigma,\delta} - \mathbf{t}_\sigma||_{C([\widehat{x}_j,\widehat{x}_{j+1}])} + ||\mathbf{n}_{\sigma,\delta} - \mathbf{n}_\sigma||_{C([\widehat{x}_j,\widehat{x}_{j+1}])} + ||\mathbf{b}_{\sigma,\delta} - \mathbf{b}_\sigma||_{C([\widehat{x}_j,\widehat{x}_{j+1}])} < \epsilon_2. \tag{3.36}$$

Let now $\epsilon_2 = \epsilon_2(\sigma)$, where $\epsilon_2(\sigma) \to 0$ for $\sigma \to 0$, then we conclude from (3.36) that $\exists \delta_0(\sigma) : \forall \delta \in (0, \delta_0(\sigma))$

$$\|\mathbf{t}_{\sigma,\delta} - \mathbf{t}_\sigma\|_{C([\widehat{x}_j,\widehat{x}_{j+1}])} + \|\mathbf{n}_{\sigma,\delta} - \mathbf{n}_\sigma\|_{C([\widehat{x}_j,\widehat{x}_{j+1}])} + \|\mathbf{b}_{\sigma,\delta} - \mathbf{b}_\sigma\|_{C([\widehat{x}_j,\widehat{x}_{j+1}])} < \epsilon_2(\sigma). \quad (3.37)$$

From (3.34) and (3.37) it follows that

$$\|\mathbf{t}_{\sigma,\delta} - \mathbf{t}\|_{C([\widehat{x}_j+2\sigma,\widehat{x}_{j+1}-2\sigma])} + \|\mathbf{n}_{\sigma,\delta} - \mathbf{n}\|_{C([\widehat{x}_j+2\sigma,\widehat{x}_{j+1}-2\sigma])}$$
$$+\|\mathbf{b}_{\sigma,\delta} - \mathbf{b}\|_{C([\widehat{x}_j+2\sigma,\widehat{x}_{j+1}-2\sigma])} < \epsilon_2(\sigma) \quad (3.38)$$

and

$$\Big| x_1 \in [\widehat{x}_j, \widehat{x}_{j+1}] : \ |\mathbf{t}_{\sigma,\delta}(x_1) - \mathbf{t}(x_1)| + |\mathbf{n}_{\sigma,\delta}(x_1) - \mathbf{n}(x_1)|$$
$$+|\mathbf{b}_{\sigma,\delta}(x_1) - \mathbf{b}(x_1)| \geq \epsilon_2(\sigma)\Big| = 4\sigma \quad (3.39)$$

for all $\delta \in (0, \delta_0(\sigma))$. It is clear that the choice of δ depends on σ. Now, we reformulate the argument in such way that σ depends on δ. We suppose for simplicity that we have two decreasing sequences $\{\sigma_n\}_{n=1}^{\infty}$ and $\{\delta(\sigma_n)\}_{n=1}^{\infty}$, where $\delta(\sigma_n) \to 0$ for $\sigma_n \to 0$, such that

$$\|\mathbf{t}_{\sigma_n,\delta(\sigma_n)} - \mathbf{t}\|_{C([\widehat{x}_j+2\sigma_n,\widehat{x}_{j+1}-2\sigma_n])} + \|\mathbf{n}_{\sigma_n,\delta(\sigma_n)} - \mathbf{n}\|_{C([\widehat{x}_j+2\sigma_n,\widehat{x}_{j+1}-2\sigma_n])} +$$
$$\|\mathbf{b}_{\sigma_n,\delta(\sigma_n)} - \mathbf{b}\|_{C([\widehat{x}_j+2\sigma_n,\widehat{x}_{j+1}-2\sigma_n])} < \epsilon_2(\sigma_n) \quad (3.40)$$

and

$$\|\mathbf{t}_{\sigma_n,\delta} - \mathbf{t}\|_{C([\widehat{x}_j+2\sigma_n,\widehat{x}_{j+1}-2\sigma_n])} + \|\mathbf{n}_{\sigma_n,\delta} - \mathbf{n}\|_{C([\widehat{x}_j+2\sigma_n,\widehat{x}_{j+1}-2\sigma_n])}$$
$$+\|\mathbf{b}_{\sigma_n,\delta} - \mathbf{b}\|_{C([\widehat{x}_j+2\sigma_n,\widehat{x}_{j+1}-2\sigma_n])} < \epsilon_2(\sigma_n) \quad (3.41)$$

for all $\delta \in (\delta(\sigma_{n+1}), \delta(\sigma_n))$ see (3.38). Now, we can put $\delta_n = \delta(\sigma_n)$ and we can define the function

$$\sigma(\delta) = \sigma_n \text{ for } \delta \in (\delta_{n+1}, \delta_n], \ n = 1, 2, \ldots. \quad (3.42)$$

This enables us rewrite (3.40) using (3.41) and (3.42) as

$$\|\mathbf{t}_{\sigma(\delta),\delta} - \mathbf{t}\|_{C([\widehat{x}_j+2\sigma(\delta),\widehat{x}_{j+1}-2\sigma(\delta)])} + \|\mathbf{n}_{\sigma(\delta),\delta} - \mathbf{n}\|_{C([\widehat{x}_j+2\sigma(\delta),\widehat{x}_{j+1}-2\sigma(\delta)])}$$
$$+\|\mathbf{b}_{\sigma(\delta),\delta} - \mathbf{b}\|_{C([\widehat{x}_j+2\sigma(\delta),\widehat{x}_{j+1}-2\sigma(\delta)])} \to 0 \quad (3.43)$$

for $\delta \to 0$. We denote

$$\mathbf{t}_\delta = \mathbf{t}_{\sigma(\delta),\delta}, \ \mathbf{n}_\delta = \mathbf{n}_{\sigma(\delta),\delta}, \ \mathbf{b}_\delta = \mathbf{b}_{\sigma(\delta),\delta}$$

and

$$\mathbf{\Phi}_\delta(x_1) = \int_0^{x_1} \mathbf{t}_\delta(z_1)\, dz_1 + \mathbf{\Phi}(0), \ x_1 \in [0, l].$$

In the sequel, we fix $\delta = \epsilon^r$, $r \in (0, \frac{1}{3})$, and redenote simply $\mathbf{\Phi}_\delta$, $\mathbf{t}_\delta$, $\mathbf{n}_\delta$, $\mathbf{b}_\delta$ by $\mathbf{\Phi}_\epsilon$, $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$. Analogously as we have obtained (3.43), we get from (3.37) using the above mentioned notation

$$\|\mathbf{t}_\epsilon - \mathbf{t}_{\sigma(\epsilon^r)}\|_{C([\widehat{x}_j,\widehat{x}_{j+1}])} + \|\mathbf{n}_\epsilon - \mathbf{n}_{\sigma(\epsilon^r)}\|_{C([\widehat{x}_j,\widehat{x}_{j+1}])} + \|\mathbf{b}_\epsilon - \mathbf{b}_{\sigma(\epsilon^r)}\|_{C([\widehat{x}_j,\widehat{x}_{j+1}])} < \epsilon_2(\sigma(\epsilon^r)),$$

which together with the definition of mollifiers and (3.26), (3.27) and (3.32) give (3.4) and (3.5). $\square$

Corollary 3.3 *Let the function $\mathbf{\Phi}_\epsilon$, $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$ and $\mathbf{b}_\epsilon$ have the properties given by Proposition 3.2. Then the functions α_ϵ, β_ϵ, γ_ϵ defined by (2.2) belong to $C^\infty([0,l])$, have the following behaviour*

$$\|\alpha_\epsilon\|_\infty,\ \|\beta_\epsilon\|_\infty,\ \|\gamma_\epsilon\|_\infty \sim O\left(\frac{1}{\epsilon^r}\right), \tag{3.44}$$

$$\|\alpha'_\epsilon\|_\infty,\ \|\beta'_\epsilon\|_\infty,\ \|\gamma'_\epsilon\|_\infty \sim O\left(\frac{1}{\epsilon^{2r}}\right), \tag{3.45}$$

for $r \in (0, \frac{1}{3})$. In addition,

$$\sup_{y_1\in[0,l]}\left(\sup_{(y_2,y_3)\in\epsilon S}|\beta_\epsilon(y_1)y_2 + \alpha_\epsilon(y_1)y_3|\right) < 1 \tag{3.46}$$

for ϵ sufficiently small and thus the mapping $\bar{\mathbf{P}}_\epsilon$ defined by (2.4) is injective and there exist constants C_j, $j = 0, 1$, independent of ϵ and x such that

$$0 < C_0 \le d_\epsilon(x) \le C_1,\ \forall\epsilon \in (0,1) \text{ and } \forall x \in \overline{\Omega}. \tag{3.47}$$

P r o o f: From (2.2), it follows that

$$\alpha_\epsilon = (\mathbf{t}'_\epsilon, \mathbf{b}_\epsilon),\ \alpha'_\epsilon = (\mathbf{t}''_\epsilon, \mathbf{b}_\epsilon) + (\mathbf{t}'_\epsilon, \mathbf{b}'_\epsilon),$$
$$\beta_\epsilon = (\mathbf{t}'_\epsilon, \mathbf{n}_\epsilon),\ \beta'_\epsilon = (\mathbf{t}''_\epsilon, \mathbf{n}_\epsilon) + (\mathbf{t}'_\epsilon, \mathbf{n}'_\epsilon),$$
$$\gamma_\epsilon = (\mathbf{b}'_\epsilon, \mathbf{n}_\epsilon),\ \gamma'_\epsilon = (\mathbf{b}''_\epsilon, \mathbf{n}_\epsilon) + (\mathbf{b}'_\epsilon, \mathbf{n}'_\epsilon).$$

Hence and from Proposition 3.2, we get (3.44) and (3.45).

(3.44) yields (3.46) for ϵ sufficiently small, because $y_2 = \epsilon x_2$ and $y_3 = \epsilon x_3$ for $(x_2, x_3) \in S$ (see (2.3)). (3.46) together with (2.5) imply that the mapping $\bar{\mathbf{P}}_\epsilon$ defined by (2.4) is injective and that (3.47) holds. □

Remark 3.4 Without loss of generality, by rescaling the domain S, we can suppose that (3.46) remains valid for $\epsilon \in (0,1)$.

4 Properties of the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$

Proposition 4.1 *Let the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ be defined by (2.17). Then*

$$\psi = -(\mathbf{V}_*, \mathbf{t}) \text{ and } \mathbf{V}(x_1) = \int_0^{x_1} [-(\mathbf{V}_*, \mathbf{b})\mathbf{n} + (\mathbf{V}_*, \mathbf{n})\mathbf{b}]\, dz_1 \tag{4.1}$$

for $x_1 \in [0, l]$, where ψ is a piecewise continuous function, and

$$\mathbf{V}(l) = \int_0^l [-(\mathbf{V}_*, \mathbf{b})\mathbf{n} + (\mathbf{V}_*, \mathbf{n})\mathbf{b}]\, dx_1 = 0. \tag{4.2}$$

$\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ is a nontrivial Hilbert space endowed with the norm

$$\|\langle\mathbf{V}, \psi\rangle\|^2 = \|\mathbf{V}\|^2_{1,2} + \|\psi\|^2_2 + \|\mathbf{V}_*\|^2_{1,2}. \tag{4.3}$$

P r o o f: The relations in (4.1) follow from (2.17), since $(\mathbf{V}', \mathbf{b}) = (\mathbf{V}_*, \mathbf{n})$ and $-(\mathbf{V}', \mathbf{n}) = (\mathbf{V}_*, \mathbf{b})$. Relation (4.2) is a consequence of the assumed boundary conditions for the function $\mathbf{V}$.

Using the embbeding theorem, we obtain from the definition of the functions $\mathbf{V}_*$, $\mathbf{t}$ (see (2.17) and Proposition 3.2) and from (4.1) that ψ is piecewise continuous.

It is obvious that the set $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ is linear and the norm (4.3) is induced by the scalar product

$$((\langle \mathbf{V}, \psi\rangle, \langle \widehat{\mathbf{V}}, \widehat{\psi}\rangle)) = \int_0^l [(\mathbf{V}, \widehat{\mathbf{V}}) + (\mathbf{V}', \widehat{\mathbf{V}}')]\, dx_1$$

$$+ \int_0^l \psi\widehat{\psi}\, dx_1 + \int_0^l [(\mathbf{V}_*, \widehat{\mathbf{V}}_*) + (\mathbf{V}_*', \widehat{\mathbf{V}}_*')]\, dx_1 \tag{4.4}$$

for arbitrary couples $\langle \mathbf{V}, \psi\rangle, \langle \widehat{\mathbf{V}}, \widehat{\psi}\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$. It remains to show that the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ is complete in the norm introduced in (4.3). Using completeness of the spaces $H_0^1(0,l)^3$ and $L^2(0,l)$ and taking a Cauchy sequence $\{\langle \mathbf{V}_n, \psi_n\rangle\}_{n=1}^\infty$ in $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$, we can find such functions $\mathbf{V}$, $\mathbf{V}_* \in H_0^1(0,l)^3$ and $\psi \in L^2(0,l)$ that

$$\mathbf{V}_n \to \mathbf{V},\ \mathbf{V}_{*,n} \to \mathbf{V}_* \text{ in } H_0^1(0,l)^3 \tag{4.5}$$

and

$$\psi_n \to \psi \text{ in } L^2(0,l), \tag{4.6}$$

where $\{\langle \mathbf{V}_n, \psi_n\rangle\}_{n=1}^\infty \subset \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$. One can pass to the limit in the definition of $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ and the completeness is proved.

Now, we want to show that the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ also contains nontrivial couples. To prove this we take an arbitrary function $\widehat{\mathbf{V}}_* \in H_0^1(0,l)^3$ such that the functions $\widehat{V}_{*,i}$, $i = 1,2,3$, are not identically zero. Then the function $\widehat{\mathbf{V}}$ defined by

$$\widehat{\mathbf{V}}(x_1) = \int_0^{x_1} [-(\widehat{\mathbf{V}}_*, \mathbf{b})\mathbf{n} + (\widehat{\mathbf{V}}_*, \mathbf{n})\mathbf{b}]\, dz_1,\ x_1 \in [0,l],$$

satisfies

$$\widehat{\mathbf{V}}(l) = \int_0^l [-(\widehat{\mathbf{V}}_*, \mathbf{b})\mathbf{n} + (\widehat{\mathbf{V}}_*, \mathbf{n})\mathbf{b}]\, dx_1 = \mathbf{C}_1$$

for some constant vector $\mathbf{C}_1$. Now, we take another function $\mathbf{h} \in H_0^1(0,l)^3$, which is not proportional with $\widehat{\mathbf{V}}_*$ and whose components are not identically zero, such that

$$\int_0^l [-(\mathbf{h}, \mathbf{b})\mathbf{n} + (\mathbf{h}, \mathbf{n})\mathbf{b}]\, dx_1 = \mathbf{C}_2,$$

where $\mathbf{C}_2 = (C_{2,1}, C_{2,2}, C_{2,3})$, $C_{2,i} \neq 0$ for $i = 1,2,3$. We define the function $\mathbf{V}_*$ by (we do not use the summation convention here)

$$V_{*,i}(x_1) = \widehat{V}_{*,i}(x_1) - \frac{C_{1,i}}{C_{2,i}} h_i(x_1),\ x_1 \in [0,l]. \tag{4.7}$$

Then $\mathbf{V}_* \in H_0^1(0,l)^3$, the functions $V_{*,i}$, $i = 1,2,3$, are not identically zero and

$$\mathbf{V}(l) = \int_0^l [-(\mathbf{V}_*, \mathbf{b})\mathbf{n} + (\mathbf{V}_*, \mathbf{n})\mathbf{b}] \, dx_1 = 0.$$

This implies that the function $\mathbf{V}$ defined by

$$\mathbf{V}(x_1) = \int_0^{x_1} [-(\mathbf{V}_*, \mathbf{b})\mathbf{n} + (\mathbf{V}_*, \mathbf{n})\mathbf{b}] \, dz_1, \; x_1 \in [0,l], \tag{4.8}$$

belongs to $H_0^1(0,l)^3$, $(\mathbf{V}', \mathbf{t}) = 0$, $\psi = -(\mathbf{V}_*, \mathbf{t})$ is piecewise continuous and thus the nontrivial couple $\langle \mathbf{V}, \psi \rangle$ belongs to $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$. □

Now, we construct "approximating spaces" to the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$.

Proposition 4.2 *Let $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$ and $\mathbf{b}_\epsilon$ be the functions from Proposition* 3.2 *and let the space $\mathcal{V}_0^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0,l)$ be defined by* (2.17) *using the functions $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ instead of $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$. Let, further, $\langle \mathbf{V}, \psi \rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$. Then there exist couples $\langle \mathbf{V}_\epsilon, \psi_\epsilon \rangle \in \mathcal{V}_0^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0,l)$ generating the functions $\mathbf{V}_{*,\epsilon}$ such that*

$$\{\mathbf{V}_\epsilon\}_{\epsilon \in (0,1)} \subset C_0^\infty(0,l)^3, \; \{\psi_\epsilon\}_{\epsilon \in (0,1)} \subset C_0^\infty(0,l), \; \{\mathbf{V}_{*,\epsilon}\}_{\epsilon \in (0,1)} \subset C_0^\infty(0,l)^3,$$

$$\mathbf{V}_\epsilon \to \mathbf{V}, \; \mathbf{V}_{*,\epsilon} \to \mathbf{V}_* \textit{ in } H_0^1(0,l)^3, \tag{4.9}$$

$$\psi_\epsilon \to \psi \textit{ pointwisely in } [0,l] \setminus D \textit{ and in } L^p(0,l), \; \forall p \in (1,\infty), \tag{4.10}$$

for $\epsilon \to 0$ and

$$\|\mathbf{V}_\epsilon''\|_2 \sim O(\frac{1}{\epsilon^r}), \; \|\psi_\epsilon'\|_2 \sim O(\frac{1}{\epsilon^r}), \; r \in (0, \frac{1}{3}). \tag{4.11}$$

P r o o f: In the definition (2.17) of the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$, we have defined the function $\mathbf{V}_*$ with the help of the function $\mathbf{V}$. But we can use the inverse procedure as in Proposition 4.1. We can easily construct by regularization the set of functions $\{\widehat{\mathbf{V}}_{*,\epsilon}\}_{\epsilon \in (0,1)} \subset C_0^\infty(0,l)^3$ such that

$$\widehat{\mathbf{V}}_{*,\epsilon} \to \mathbf{V}_* \text{ in } H_0^1(0,l)^3 \tag{4.12}$$

for $\epsilon \to 0$. We know from Proposition 3.2 that $\mathbf{t}_\epsilon \to \mathbf{t}$, $\mathbf{n}_\epsilon \to \mathbf{n}$ and $\mathbf{b}_\epsilon \to \mathbf{b}$ pointwisely in $[0,l] \setminus D$ and strongly in $L^p(0,l)^3$, $p \in (1,\infty)$, and thus, using the Lebesgue theorem,

$$\int_0^l [-(\widehat{\mathbf{V}}_{*,\epsilon}, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon + (\widehat{\mathbf{V}}_{*,\epsilon}, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon] \, dx_1 = \mathbf{C}_3(\epsilon) \to 0, \tag{4.13}$$

for $\epsilon \to 0$. Let $\mathbf{h}$ be some vector function from $C_0^\infty(0,l)^3$, which is not proportional with $\widehat{\mathbf{V}}_*$ and whose components are not identically zero, such that

$$\int_0^l [-(\mathbf{h}, \mathbf{b})\mathbf{n} + (\mathbf{h}, \mathbf{n})\mathbf{b}] \, dx_1 = \mathbf{C}_4 = (C_{4,1}, C_{4,2}, C_{4,3}),$$

with $C_{4,i} \neq 0,\ i = 1, 2, 3$. Then

$$\int_0^l [-(\mathbf{h}, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon + (\mathbf{h}, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon]\ dx_1 = \mathbf{C}_4 + \mathbf{C}_5(\epsilon), \tag{4.14}$$

where $\mathbf{C}_5(\epsilon) \to 0$ for $\epsilon \to 0$. Now, we define the function $\mathbf{V}_{*,\epsilon}$ by (we do not use the summation convention here)

$$V_{*,\epsilon,i}(x_1) = \widehat{V}_{*,\epsilon,i}(x_1) - \frac{C_{3,i}(\epsilon)}{C_{4,i} + C_{5,i}(\epsilon)} h_i(x_1),\ \ x_1 \in [0, l],\ \ i = 1, 2, 3. \tag{4.15}$$

Then $\mathbf{V}_{*,\epsilon} \in C_0^\infty(0, l)^3$, the functions $V_{*,\epsilon,i}$, $i = 1, 2, 3$, are not identically zero and

$$\int_0^l [-(\mathbf{V}_{*,\epsilon}, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon + (\mathbf{V}_{*,\epsilon}, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon]\ dx_1 = 0. \tag{4.16}$$

Then analogously as in Proposition 4.1 we define the functions

$$\mathbf{V}_\epsilon(x_1) = \int_0^{x_1} [-(\mathbf{V}_{*,\epsilon}, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon + (\mathbf{V}_{*,\epsilon}, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon]\ dz_1, \tag{4.17}$$

$$\psi_\epsilon = -(\mathbf{V}_{*,\epsilon}, \mathbf{t}_\epsilon) \tag{4.18}$$

and thus $\langle \mathbf{V}_\epsilon, \psi_\epsilon \rangle \in \mathcal{V}_0^{\mathbf{t}_\epsilon, \mathbf{n}_\epsilon, \mathbf{b}_\epsilon}(0, l)$ for $\epsilon \in (0, 1)$. Since $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$ and $\mathbf{b}_\epsilon \in C^\infty([0, l])^3$, we get easily from (4.15) and from the properties of the functions $\widehat{\mathbf{V}}_{*,\epsilon}$ and $\mathbf{h}$ that $\mathbf{V}_{*,\epsilon} \in C_0^\infty(0, l)^3$ and thus using (4.17)–(4.18), $\mathbf{V}_\epsilon \in C_0^\infty(0, l)^3$ and $\psi_\epsilon \in C_0^\infty(0, l)$ for all $\epsilon \in (0, 1)$.

The verification of (4.9) and (4.10) is left to the reader. From (4.17), it follows the estimate

$$\|\mathbf{V}_\epsilon''\|_2 = \| - (\mathbf{V}_{*,\epsilon}', \mathbf{b}_\epsilon)\mathbf{n}_\epsilon - (\mathbf{V}_{*,\epsilon}, \mathbf{b}_\epsilon')\mathbf{n}_\epsilon - (\mathbf{V}_{*,\epsilon}, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon' + (\mathbf{V}_{*,\epsilon}', \mathbf{n}_\epsilon)\mathbf{b}_\epsilon$$

$$+(\mathbf{V}_{*,\epsilon}, \mathbf{n}_\epsilon')\mathbf{b}_\epsilon + (\mathbf{V}_{*,\epsilon}, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon'\|_2 \leq 2\Big(\|\mathbf{V}_{*,\epsilon}'\|_2 + \|\mathbf{V}_{*,\epsilon}\|_2(\|\mathbf{b}_\epsilon'\|_\infty + \|\mathbf{n}_\epsilon'\|_\infty)\Big),$$

which together with (3.4) yields the first relation in (4.11). The second relation in (4.11) easily follows from the fact that $\psi_\epsilon = -(\mathbf{V}_{*,\epsilon}, \mathbf{t}_\epsilon)$ and from (3.4). □

5 Auxiliary propositions

Proposition 5.1 [11] *Let $w \in H^1(\Omega)$. Then $\partial_i \partial_j w \in L^2(0, l; H^{-1}(S))$ for i, $j = 1, 2, 3$ except for $i = j = 1$. If, in addition, $w|_{x_1=0} = w|_{x_1=l} = 0$, then $\partial_j w|_{x_1=0} = \partial_j w|_{x_1=l} = 0$, for $j = 2, 3$, in the sense of the space $C([0, l]; H^{-1}(S))$. Furthermore, if $v \in L^2(0, l; L^2(S))$, $\partial_1 v \in L^2(0, l; H^{-1}(S))$ and $v|_{x_1=0} = v|_{x_1=l} = 0$ in the sense of the space $C([0, l]; H^{-1}(S))$, then there is a constant C independent of v such that*

$$\|v\|_{L^2(0,l;L^2(S))} \leq C\|\nabla v\|_{L^2(0,l;H^{-1}(S))}. \tag{5.1}$$

Proposition 5.2 [11] *Let* $\{v_n\}_{n=1}^{\infty} \subset L^2(0,l;L^2(S))$, $\{\partial_1 v_n\}_{n=1}^{\infty} \subset L^2(0,l;H^{-1}(S))$ *and let* $v_n|_{x_1=0} = v_n|_{x_1=l} = 0$, *for all* $n \in \mathbb{N}$, *in the sense of the space* $C([0,l];H^{-1}(S))$. *Assume, in addition, that this sequence satisfies*

$$\partial_1 v_n \rightharpoonup \xi, \ \partial_j v_n \rightharpoonup 0 \ in \ L^2(0,l;H^{-1}(S)), \ j=2,3, \tag{5.2}$$

where $\xi \in L^2(0,l;H^{-1}(S))$. *Then* $\xi \in L^2(0,l)$ *and there exists a unique function* $v \in H_0^1(0,l)$ *such that* $v' = \xi$ *and*

$$v_n \rightharpoonup v \ in \ L^2(0,l;L^2(S)), \tag{5.3}$$

$$v_n \to v \ in \ C([0,l];H^{-1}(S)). \tag{5.4}$$

If the convergences in (5.2) *are strong then the convergence* (5.3) *is also strong.*

Every function $\mathbf{V} \in H^1(\Omega)^3$ may be represented in the local frame generated by the vectors $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$:

$$\mathbf{V}(x) = v_{1,\epsilon}(x)\mathbf{t}_\epsilon(x_1) + v_{2,\epsilon}(x)\mathbf{n}_\epsilon(x_1) + v_{3,\epsilon}(x)\mathbf{b}_\epsilon(x_1), \tag{5.5}$$

where the components of the vector $\mathbf{v}_\epsilon = (v_{1,\epsilon}, v_{2,\epsilon}, v_{3,\epsilon}) \in H^1(\Omega)^3$ are defined by

$$(\mathbf{V},\mathbf{t}_\epsilon) = v_{1,\epsilon}, \ (\mathbf{V},\mathbf{n}_\epsilon) = v_{2,\epsilon}, \ (\mathbf{V},\mathbf{b}_\epsilon) = v_{3,\epsilon}. \tag{5.6}$$

Using (2.2) together with (5.5) we get similar relations for the derivative ∂_1 of $\mathbf{V}$

$$(\partial_1\mathbf{V}(x),\mathbf{t}_\epsilon(x_1)) = \partial_1 v_{1,\epsilon}(x) - \alpha_\epsilon(x_1)v_{3,\epsilon}(x) - \beta_\epsilon(x_1)v_{2,\epsilon}(x), \tag{5.7}$$

$$(\partial_1\mathbf{V}(x),\mathbf{n}_\epsilon(x_1)) = \partial_1 v_{2,\epsilon}(x) + \beta_\epsilon(x_1)v_{1,\epsilon}(x) + \gamma_\epsilon(x_1)v_{3,\epsilon}(x), \tag{5.8}$$

$$(\partial_1\mathbf{V}(x),\mathbf{b}_\epsilon(x_1)) = \partial_1 v_{3,\epsilon}(x) + \alpha_\epsilon(x_1)v_{1,\epsilon}(x) - \gamma_\epsilon(x_1)v_{2,\epsilon}(x) \tag{5.9}$$

for a.a. $x \in \Omega$. The following proposition shows that the relations (5.7)–(5.9) remain valid under weaker assumptions on the function $\mathbf{V}$.

Proposition 5.3 *Let* $\mathbf{V} \in L^2(\Omega)^3$ *and let the vector function* $\mathbf{v}_\epsilon = (v_{1,\epsilon}, v_{2,\epsilon}, v_{3,\epsilon})$ *from* (5.6) *be such that* $\partial_1\mathbf{v}_\epsilon \in L^2(0,l;H^{-1}(S)^3)$. *Then the function* $\mathbf{V}$ *of the form* (5.5) *is such that* $\partial_1\mathbf{V} \in L^2(0,l;H^{-1}(S)^3)$ *and fulfills the relations* (5.7)–(5.9) *in the sense of the space* $L^2(0,l;H^{-1}(S))$ *for all* $\epsilon \in (0,1)$.

P r o o f: We must find the functions $m_i \in L^2(0,l;H^{-1}(S))$, $i=1,2,3$, such that

$$\lim_{h\to 0}\left\|\frac{V_i(\cdot+h,\cdot,\cdot)-V_i(\cdot,\cdot,\cdot)}{h} - m_i\right\|_{L^2(o,l-o;H^{-1}(S))} \to 0, \ i=1,2,3, \tag{5.10}$$

where o is arbitrary small and h satisfies $|h| < o$. We can substitute the expression (5.5) in the fraction in (5.10), which leads to the expression

$$\frac{V_i(x_1+h,x_2,x_3)-V_i(x_1,x_2,x_3)}{h}$$

$$= \frac{v_{1,\epsilon}(x_1+h,x_2,x_3)t_{i,\epsilon}(x_1+h) - v_{1,\epsilon}(x_1,x_2,x_3)t_{i,\epsilon}(x_1)}{h}$$
$$+\frac{v_{2,\epsilon}(x_1+h,x_2,x_3)n_{i,\epsilon}(x_1+h) - v_{2,\epsilon}(x_1,x_2,x_3)n_{i,\epsilon}(x_1)}{h}$$
$$+\frac{v_{3,\epsilon}(x_1+h,x_2,x_3)b_{i,\epsilon}(x_1+h) - v_{3,\epsilon}(x_1,x_2,x_3)b_{i,\epsilon}(x_1)}{h}, \ i=1,2,3.$$

The first term may be rewritten as

$$\frac{\Big(v_{1,\epsilon}(x_1+h,x_2,x_3) - v_{1,\epsilon}(x_1,x_2,x_3)\Big)t_{i,\epsilon}(x_1+h)}{h}$$
$$+\frac{\Big(t_{i,\epsilon}(x_1+h) - t_{i,\epsilon}(x_1)\Big)v_{1,\epsilon}(x_1,x_2,x_3)}{h}$$

for $i=1,2,3$ and a.a. $x \in \Omega$ and similarly the last two terms. Using (2.2), the fact that the functions $t_{i,\epsilon}$, $n_{i,\epsilon}$, $b_{i,\epsilon}$, $i=1,2,3$, belong to $C^\infty([0,l])$ and that $\partial_1 \mathbf{v} \in L^2(0,l;H^{-1}(S)^3)$, we can deduce that the relation (5.10) is fulfilled for the vector function $\mathbf{m}=(m_1,m_2,m_3) \in L^2(0,l;H^{-1}(S)^3)$ defined by

$$\mathbf{m} = \partial_1 v_{1,\epsilon}\mathbf{t}_\epsilon + v_{1,\epsilon}(\alpha_\epsilon \mathbf{b}_\epsilon + \beta_\epsilon \mathbf{n}_\epsilon) + \partial_1 v_{2,\epsilon}\mathbf{n}_\epsilon + v_{2,\epsilon}(-\beta_\epsilon \mathbf{t}_\epsilon - \gamma_\epsilon \mathbf{b}_\epsilon)$$
$$+\partial_1 v_{3,\epsilon}\mathbf{b}_\epsilon + v_{3,\epsilon}(-\alpha_\epsilon \mathbf{t}_\epsilon + \gamma_\epsilon \mathbf{n}_\epsilon). \qquad (5.11)$$

Since the function $\mathbf{V}$ does not depend on ϵ, then the function $\mathbf{m}$ is an independent function of ϵ as well. Since o is arbitrary, using the relations (5.10), (5.11), we can conclude that $\partial_1 \mathbf{V} = \mathbf{m}$ in the sense of the space $L^2(0,l;H^{-1}(S)^3)$. We get from (5.11) that

$$\int_S (\mathbf{m},\mathbf{t}_\epsilon)\varphi \, dx_2dx_3 = \int_S (\partial_1 v_{1,\epsilon} - \beta_\epsilon v_{2,\epsilon} - \alpha_\epsilon v_{3,\epsilon})\varphi \, dx_2dx_3$$

for arbitrary $\varphi \in H_0^1(S)$, which implies that

$$(\mathbf{m},\mathbf{t}_\epsilon) = \partial_1 v_{1,\epsilon} - \beta_\epsilon v_{2,\epsilon} - \alpha_\epsilon v_{3,\epsilon} \qquad (5.12)$$

in the sense of the space $L^2(0,l;H^{-1}(S))$. Since $\partial_1 \mathbf{V} = \mathbf{m}$ in the sense of the space $L^2(0,l;H^{-1}(S)^3)$ and thus $(\partial_1 \mathbf{V},\mathbf{t}_\epsilon) = (\mathbf{m},\mathbf{t}_\epsilon)$ in the sense of the space $L^2(0,l;H^{-1}(S))$, we obtain from (5.12) the identity (5.7). The identities (5.8)–(5.9) can be derived analogously. □

6 Variational equations for the curved rods and their transformation

We consider $\widetilde{\Omega}_\epsilon$ defined by the mapping $\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon$ (see (2.3)–(2.4)) for $\epsilon \in (0,1)$ arbitrary but fixed as a three-dimensional homogeneous and isotropic elastic body with the Lamé constants $\lambda \geq 0$ and $\mu > 0$. Let $\widetilde{\mathbf{F}}_\epsilon$ be the body force and $\widetilde{\mathbf{G}}_\epsilon$ the surface traction acting on the curved rod $\widetilde{\Omega}_\epsilon$ such that $\widetilde{\mathbf{F}}_\epsilon \in L^2(\widetilde{\Omega}_\epsilon)^3$ and $\widetilde{\mathbf{G}}_\epsilon \in L^2((\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)((0,l) \times \partial S))^3$, for $\epsilon \in (0,1)$. Let $\widetilde{\Omega}_\epsilon$ be clamped on both bases $\bar{\mathbf{P}}_\epsilon(\{0\} \times \epsilon S)$ and $\bar{\mathbf{P}}_\epsilon(\{l\} \times \epsilon S)$. The equilibrium displacement $\widetilde{\mathbf{U}}_\epsilon$ is the solution of the variational equation

$$\int_{\widetilde{\Omega}_\epsilon} \widetilde{A}^{ijkl} e_{kl}(\widetilde{\mathbf{U}}_\epsilon) e_{ij}(\widetilde{\mathbf{V}})\, d\widetilde{y} = \int_{\widetilde{\Omega}_\epsilon} (\widetilde{\mathbf{F}}_\epsilon, \widetilde{\mathbf{V}})\, d\widetilde{y}$$

$$+ \int_{(\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)((0,l)\times\partial S)} (\widetilde{\mathbf{G}}_\epsilon, \widetilde{\mathbf{V}})\, d\widetilde{S}_\epsilon d\widetilde{y}_1, \ \forall \widetilde{\mathbf{V}} \in V(\widetilde{\Omega}_\epsilon), \tag{6.1}$$

where $\widetilde{S}_\epsilon = (\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon)((0,l) \times \partial S)$, $\widetilde{A}^{ijkl} = \lambda \delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk})$ and $(e_{ij}(\widetilde{\mathbf{V}}))_{i,j=1}^3$ stands for the symmetric part of the gradient of the function $\widetilde{\mathbf{V}}$.

From (2.3)–(2.4) and from the regularization of the local frame constructed in Section 3, it follows that the mapping $\bar{\mathbf{P}}_\epsilon \circ \mathbf{R}_\epsilon$ is the parametrization of the smooth three-dimensional curved rod.

We transform the equation (6.1) to an equation on the domain Ω. We combine the standard transformation from [4, pp. 27–32] and [10] with the idea from [2], which enables us to exclude the Christoffel symbols.

Transformation: Since the detailed derivation of the following expressions can be found in [4, pp. 30–31], we mentioned only the basic identities here. Let $\widetilde{\mathbf{V}} \in H^1(\widetilde{\Omega}_\epsilon)^3$, the mapping $\bar{\mathbf{P}}_\epsilon$ be given by (2.4), $\bar{\mathbf{g}}^{k,\epsilon} = ([\bar{\mathbf{g}}^{k,\epsilon}]_1, [\bar{\mathbf{g}}^{k,\epsilon}]_2, [\bar{\mathbf{g}}^{k,\epsilon}]_3)$ be given by (2.7) and $\bar{\mathbf{W}}_\epsilon = (\bar{W}_{1,\epsilon}, \bar{W}_{2,\epsilon}, \bar{W}_{3,\epsilon}) = \bar{\nabla}\mathbf{P}_\epsilon(\widetilde{\mathbf{V}} \circ \mathbf{P}_\epsilon)$ be such that

$$\widetilde{V}_i \circ \bar{\mathbf{P}}_\epsilon = \bar{W}_{k,\epsilon}[\bar{\mathbf{g}}^{k,\epsilon}]_i, \ i = 1,2,3.$$

Then

$$(\widetilde{\partial}_j \widetilde{V}_i) \circ \bar{\mathbf{P}}_\epsilon = (\bar{\partial}_l \bar{W}_{k,\epsilon} - \bar{W}_{q,\epsilon}\Gamma^q_{lk,\epsilon})[\bar{\mathbf{g}}^{k,\epsilon}]_i[\bar{\mathbf{g}}^{l,\epsilon}]_j,$$

where the Christoffel symbols $\Gamma^i_{jk,\epsilon}$ are defined by

$$\Gamma^i_{jk,\epsilon} = (\bar{\mathbf{g}}^{i,\epsilon}, \bar{\partial}_j \bar{\mathbf{g}}_{k,\epsilon}), \ i,j,k = 1,2,3.$$

Using the notation

$$e_{i||j}(\bar{\mathbf{W}}_\epsilon) = \frac{1}{2}(\bar{\partial}_i \bar{W}_{j,\epsilon} + \bar{\partial}_j \bar{W}_{i,\epsilon}) - \bar{W}_{p,\epsilon}\Gamma^p_{ij,\epsilon},$$

we obtain

$$e_{ij}(\widetilde{\mathbf{V}}) \circ \bar{\mathbf{P}}_\epsilon = e_{k||l}(\bar{\mathbf{W}}_\epsilon)[\bar{\mathbf{g}}^{k,\epsilon}]_i[\bar{\mathbf{g}}^{l,\epsilon}]_j, \ i,j = 1,2,3.$$

Now, we define the vector function $\bar{\mathbf{V}}_\epsilon$ by

$$\bar{\mathbf{V}}_\epsilon = \bar{W}_{i,\epsilon}\bar{\mathbf{g}}^{i,\epsilon} (= \widetilde{\mathbf{V}} \circ \bar{\mathbf{P}}_\epsilon).$$

Then

$$\frac{1}{2}(\bar{\partial}_i\bar{W}_{j,\epsilon}+\bar{\partial}_j\bar{W}_{i,\epsilon}) = \frac{1}{2}\Big(\bar{\partial}_i(\bar{W}_{k,\epsilon}\bar{\mathbf{g}}^{k,\epsilon},\bar{\mathbf{g}}_{j,\epsilon})+\bar{\partial}_j(\bar{W}_{l,\epsilon}\bar{\mathbf{g}}^{l,\epsilon},\bar{\mathbf{g}}_{i,\epsilon})\Big)$$
$$= \frac{1}{2}\Big(\bar{\partial}_i(\bar{\mathbf{V}}_\epsilon,\bar{\mathbf{g}}_{j,\epsilon})+\bar{\partial}_j(\bar{\mathbf{V}}_\epsilon,\bar{\mathbf{g}}_{i,\epsilon})\Big).$$

Since the Christoffel symbols are symmetric in the indices i, j, we get, using the identities

$$\bar{W}_{k,\epsilon}\Gamma^k_{ij,\epsilon} = \bar{W}_{k,\epsilon}(\bar{\mathbf{g}}^{k,\epsilon},\bar{\partial}_i\bar{\mathbf{g}}_{j,\epsilon}) = \bar{\partial}_i(\bar{\mathbf{V}}_\epsilon,\bar{\mathbf{g}}_{j,\epsilon})-(\bar{\partial}_i\bar{\mathbf{V}}_\epsilon,\bar{\mathbf{g}}_{j,\epsilon}),$$
$$\bar{W}_{k,\epsilon}\Gamma^k_{ij,\epsilon} = \bar{\partial}_j(\bar{\mathbf{V}}_\epsilon,\bar{\mathbf{g}}_{i,\epsilon})-(\bar{\partial}_j\bar{\mathbf{V}}_\epsilon,\bar{\mathbf{g}}_{i,\epsilon})$$

and the notation

$$\bar{\omega}^\epsilon_{ij}(\bar{\mathbf{V}}_\epsilon) = \frac{1}{2}\left((\bar{\partial}_i\bar{\mathbf{V}}_\epsilon,\bar{\mathbf{g}}_{j,\epsilon})+(\bar{\partial}_j\bar{\mathbf{V}}_\epsilon,\bar{\mathbf{g}}_{i,\epsilon})\right),$$

that

$$e_{i||j}(\bar{\mathbf{W}}_\epsilon) = \bar{\omega}^\epsilon_{ij}(\bar{\mathbf{V}}_\epsilon).$$

Hence

$$e_{ij}(\widetilde{\mathbf{V}})\circ\bar{\mathbf{P}}_\epsilon = \bar{\omega}^\epsilon_{ij}(\bar{\mathbf{V}}_\epsilon)[\bar{\mathbf{g}}^{k,\epsilon}]_i[\bar{\mathbf{g}}^{l,\epsilon}]_j.$$

•

Using the above transformation, we can denote (see also [4, p.31])

$$\bar{A}^{ijkl}_\epsilon = \lambda\bar{g}^{ij,\epsilon}\bar{g}^{kl,\epsilon}+\mu(\bar{g}^{ik,\epsilon}\bar{g}^{jl,\epsilon}+\bar{g}^{il,\epsilon}\bar{g}^{jk,\epsilon}) \tag{6.2}$$

and thus we can transform the left-hand side of the equation (6.1) as

$$\int_{\Omega_\epsilon}\bar{A}^{ijkl}_\epsilon\bar{\omega}^\epsilon_{kl}(\bar{\mathbf{U}}_\epsilon)\bar{\omega}^\epsilon_{ij}(\bar{\mathbf{V}})\bar{d}_\epsilon(y)\ dy,\ \forall\bar{\mathbf{V}}\in V(\Omega_\epsilon). \tag{6.3}$$

After the substitution $y=\mathbf{R}_\epsilon(x)$ in (6.3) and using the transformation of the right-hand side of (6.1), which is given by the mapping $\bar{\mathbf{P}}_\epsilon\circ\mathbf{R}_\epsilon$ and by the relation

$$d\widetilde{S}_\epsilon d\widetilde{y}_1 = o_\epsilon\sqrt{\nu_i o^{ij,\epsilon}\nu_j}dSdx_1 = \epsilon^2 d_\epsilon\sqrt{\nu_i o^{ij,\epsilon}\nu_j}dSdx_1$$

see [4, p.19], and (2.13)–(2.16), we get

$$\int_\Omega A^{ijkl}_\epsilon\omega^\epsilon_{kl}(\mathbf{U}_\epsilon)\omega^\epsilon_{ij}(\mathbf{V})d_\epsilon\ dx = \int_\Omega(\mathbf{F}_\epsilon,\mathbf{V})d_\epsilon\ dx$$
$$+\int_{(0,l)\times\partial S}(\mathbf{G}_\epsilon,\mathbf{V})d_\epsilon\sqrt{\nu_i o^{ij,\epsilon}\nu_j}\ dSdx_1,\ \forall\mathbf{V}\in V(\Omega), \tag{6.4}$$

where ν_i, $i=1,2,3$, are the components of the unit outward normal to $(0,l)\times\partial S$.

The symmetric tensor $\omega^\epsilon(\mathbf{V})$, obtained after composition with $\mathbf{R}_\epsilon$ has the form

$$\omega^\epsilon(\mathbf{V}) = \frac{1}{\epsilon}\theta^\epsilon(\mathbf{V})+\kappa^\epsilon(\mathbf{V}), \tag{6.5}$$

where the individual nonzero components of the symmetric tensors θ^ϵ and κ^ϵ are defined by

$$\theta^\epsilon_{12}(\mathbf{V}) = \frac{1}{2}(\partial_2 \mathbf{V}, \mathbf{g}_{1,\epsilon}),\ \theta^\epsilon_{22}(\mathbf{V}) = (\partial_2 \mathbf{V}, \mathbf{n}_\epsilon),\ \theta^\epsilon_{33}(\mathbf{V}) = (\partial_3 \mathbf{V}, \mathbf{b}_\epsilon), \tag{6.6}$$

$$\theta^\epsilon_{13}(\mathbf{V}) = \frac{1}{2}(\partial_3 \mathbf{V}, \mathbf{g}_{1,\epsilon}),\ \theta^\epsilon_{23}(\mathbf{V}) = \frac{1}{2}\Big((\partial_2 \mathbf{V}, \mathbf{b}_\epsilon) + (\partial_3 \mathbf{V}, \mathbf{n}_\epsilon)\Big), \tag{6.7}$$

$$\kappa^\epsilon_{11}(\mathbf{V}) = (\partial_1 \mathbf{V}, \mathbf{g}_{1,\epsilon}),\ \kappa^\epsilon_{12}(\mathbf{V}) = \frac{1}{2}(\partial_1 \mathbf{V}, \mathbf{n}_\epsilon),\ \kappa^\epsilon_{13}(\mathbf{V}) = \frac{1}{2}(\partial_1 \mathbf{V}, \mathbf{b}_\epsilon) \tag{6.8}$$

(compare with [2]), where $\mathbf{g}_{1,\epsilon} \to \mathbf{t}$, $\mathbf{n}_\epsilon \to \mathbf{n}$, $\mathbf{b}_\epsilon \to \mathbf{b}$ pointwisely in $\Omega \setminus (S \times D)$ or $[0,l] \setminus D$ for $\epsilon \to 0$. The other components of θ^ϵ and κ^ϵ are equal to zero.

Now, we check that the inequality

$$\frac{1}{\epsilon^2}\|\omega^\epsilon(\mathbf{U}_\epsilon)\|_2^2 \le \frac{C}{\epsilon^2}\int_\Omega A^{ijkl}_\epsilon \omega^\epsilon_{kl}(\mathbf{U}_\epsilon)\omega^\epsilon_{ij}(\mathbf{U}_\epsilon) d_\epsilon \, dx, \tag{6.9}$$

with the constant C independent of ϵ, holds. This inequality together with the Korn inequality derived in the next section enable us not only to prove the existence of a unique solution $\mathbf{U}_\epsilon$ for the equation (6.4) and to study the behaviour of the functions $\mathbf{U}_\epsilon$ and $\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon)$ for $\epsilon \to 0$. In Corollary 3.3, we have proved that $d_\epsilon(x) \ge C_0 > 0$, for all $x \in \overline{\Omega}$ and $\epsilon \in (0,1)$ and without loss of generality we can suppose that it holds for all $\epsilon \in (0,1]$. Then the estimate (6.9) is a consequence of the proposition:

Proposition 6.1 *Let $\lambda \ge 0$, $\mu > 0$ and*

$$A^{ijkl}_\epsilon = \lambda g^{ij,\epsilon} g^{kl,\epsilon} + \mu(g^{ik,\epsilon} g^{jl,\epsilon} + g^{il,\epsilon} g^{jk,\epsilon}).$$

Then there exists a constant $C > 0$ such that the estimate

$$\sum_{i,j=1}^{3} |t_{ij}|^2 \le C A^{ijkl}_\epsilon(x) t_{kl} t_{ij} \tag{6.10}$$

holds for all $x \in \overline{\Omega}$, all $\epsilon \in [0,1]$ and all symmetric matrices $(t_{ij})_{i,j=1}^3$, with the constant C being independent of ϵ and x.

P r o o f: First, we verify that

$$g^{ik,\epsilon}(x) g^{jl,\epsilon}(x) t_{kl} t_{ij} > 0 \text{ if } t_{ij} \ne 0$$

for all $\epsilon \in [0,1]$ and $x \in \overline{\Omega}$. In case $\epsilon \in (0,1]$ the proof proceeds in the same way as in [4] Theorem 1.8-1. The case $\epsilon = 0$ is an obvious consequence of the fact that $\mathbf{g}^{1,\epsilon} \to \mathbf{t}$ pointwisely in $\Omega \setminus (S \times D)$ for $\epsilon \to 0$ (see (2.5), (2.7), Proposition 3.2, (3.44)). The mapping

$$(\epsilon, x, (t_{ij})) \in \mathbf{K} = [0,1] \times \overline{\Omega} \times \{t_{ij};\ \sum_{i,j=1}^{3} |t_{ij}|^2 = 1\} \to g^{ik,\epsilon}(x) g^{jl,\epsilon}(x) t_{kl} t_{ij} \tag{6.11}$$

is continuous. The only difficulty could appear for $\epsilon \to 0$. We argue, for instance, for the term

$$g^{12,\epsilon}(x) = -\frac{\epsilon x_3 \gamma_\epsilon(x_1)}{d_\epsilon^2(x)}, \ \forall x \in \overline{\Omega},$$

(see (2.7)–(2.9)), which converge to zero in $C(\overline{\Omega})$ for $\epsilon \to 0$ because of Corollary 3.3. Since, in addition, the domain in (6.11) is compact, we infer

$$C = \inf_{(\epsilon, x, t_{ij}) \in \mathbf{K}} g^{ik,\epsilon}(x) g^{jl,\epsilon}(x) t_{kl} t_{ij} > 0.$$

Hence, we get the assertion of the proposition. □

7 Korn's inequality for the curved rod

In this section, we derive a special form of the Korn inequality and we study the dependence of the constant from this inequality on ϵ. A similar problem was discussed by Nazarov and Slutskij in [13]. The main result of this section can be summarized into the following theorem.

Theorem 7.1 *There exist constant $C > 0$, independent of ϵ, such that*

$$\|\mathbf{V}\|_{1,2} \le \frac{C}{\epsilon} \|\omega^\epsilon(\mathbf{V})\|_2, \ \forall \mathbf{V} \in V(\Omega) \ \textit{and} \ \forall \epsilon \in (0,1). \tag{7.1}$$

The proof of Theorem 7.1 is based on the following proposition.

Proposition 7.2 *Suppose that $\{\epsilon_n\}_{n=1}^\infty \subset (0,1)$ and $\epsilon_n \to 0$. Let, in addition, a sequence $\{\mathbf{U}_{\epsilon_n}\}_{n=1}^\infty \subset V(\Omega)$ be such that*

$$\mathbf{U}_{\epsilon_n} \rightharpoonup \mathbf{U} \ \textit{in} \ H^1(\Omega)^3, \tag{7.2}$$

$$\frac{1}{\epsilon_n} \omega^{\epsilon_n}(\mathbf{U}_{\epsilon_n}) \rightharpoonup \zeta \ \textit{in} \ L^2(\Omega)^9 \tag{7.3}$$

for $\epsilon_n \to 0$. Then the couple $\langle \mathbf{U}, \phi \rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ (in the sense $\partial_j \mathbf{U} = 0$, $j = 2,3$), where the function ϕ is such that

$$\frac{1}{2\epsilon_n}\Big((\partial_2 \mathbf{U}_{\epsilon_n}, \mathbf{b}_{\epsilon_n}) - (\partial_3 \mathbf{U}_{\epsilon_n}, \mathbf{n}_{\epsilon_n})\Big) \rightharpoonup \phi$$

in $L^2(\Omega)$ for $\epsilon_n \to 0$. In addition, the couple $\langle \mathbf{U}, \phi \rangle$ generates the function $\mathbf{U}_* \in H_0^1(0,l)^3$ which together with the function $\mathbf{U}$ satisfy the relations

$$(\mathbf{U}', \mathbf{t}) = 0 \ \textit{a.e. on} \ [0,l], \tag{7.4}$$

$$(\mathbf{U}'_*, \mathbf{t}) = \partial_3 \zeta_{12} - \partial_2 \zeta_{13} \ \textit{in} \ L^2(0,l; H^{-1}(S)), \tag{7.5}$$

$$(\mathbf{U}'_*, \mathbf{n}) = -\partial_3 \zeta_{11} \ \textit{a.e. on} \ [0,l], \tag{7.6}$$

$$(\mathbf{U}'_*, \mathbf{b}) = \partial_2 \zeta_{11} \ \textit{a.e. on} \ [0,l]. \tag{7.7}$$

If the sequence $\{\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{U}_{\epsilon_n})\}_{n=1}^\infty$ converges strongly in $L^2(\Omega)^9$, then the convergence in (7.2) is strong as well.

P r o o f of Theorem 7.1: Assume the contrary, i.e., there exist ϵ_n, $\epsilon_n \in (0, 1/n)$, and $\mathbf{V}_{\epsilon_n}$, $\|\mathbf{V}_{\epsilon_n}\|_{1,2} = 1$, such that

$$\frac{1}{\epsilon_n}\|\omega^{\epsilon_n}(\mathbf{V}_{\epsilon_n})\|_2 \leq \frac{1}{n}.$$

Hence (passing to a subsequence if it is necessary),

$$\mathbf{V}_{\epsilon_n} \rightharpoonup \mathbf{V} \text{ in } H^1(\Omega)^3 \text{ and } \frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{V}_{\epsilon_n}) \to 0 \text{ in } L^2(\Omega)^9.$$

Proposition 7.2 implies that the sequence $\mathbf{V}_{\epsilon_n} \to \mathbf{V}$ strongly in $H^1(\Omega)^3$ and

$$(\mathbf{V}', \mathbf{t}) = 0, \ (\mathbf{V}'_*, \mathbf{t}) = 0, \ (\mathbf{V}'_*, \mathbf{n}) = 0, \ (\mathbf{V}'_*, \mathbf{b}) = 0. \tag{7.8}$$

Further, from Proposition 7.2 and from the definition (2.17) of the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0, l)$, it follows that the couple $\langle \mathbf{V}, \psi \rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0, l)$ and thus $\mathbf{V} \in H_0^1(0, l)^3$ and $\mathbf{V}_* \in H_0^1(0, l)^3$. Hence we conclude using (7.8) that $\mathbf{V}_* = \mathbf{0}$ and thus $\mathbf{V} = \mathbf{0}$, a contradiction. □

P r o o f of Proposition 7.2: The proof of Proposition 7.2 follows from Lemma 7.5, 7.10 and Corollary 7.9 and 7.11. □

We will use ϵ instead of ϵ_n to simplify the notation in these lemmas.

Lemma 7.3 *Under the assumptions in Proposition* 7.2 *the following convergences*

$$\frac{1}{\epsilon^q}\theta^\epsilon(\mathbf{U}_\epsilon) \to 0 \ in \ L^2(\Omega)^9, \ q \in [0, 1), \tag{7.9}$$

$$\left(\frac{1}{\epsilon^2}\theta^\epsilon(\mathbf{U}_\epsilon) + \frac{1}{\epsilon}\kappa^\epsilon(\mathbf{U}_\epsilon)\right) \rightharpoonup \zeta \ in \ L^2(\Omega)^9 \tag{7.10}$$

hold.

P r o o f: We can observe that the weak convergences (7.2) and (7.3) together with (6.5)–(6.8) imply the boundedness of the set of the tensors $\{\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon)\}_{\epsilon \in (0,1)}$ and $\{\kappa^\epsilon(\mathbf{U}_\epsilon)\}_{\epsilon \in (0,1)}$ in $L^2(\Omega)^9$. Using these facts, we can easily deduce (7.9). Relation (7.10) immediately follows from (7.9) and (6.5). □

Corollary 7.4 *Under hypotheses* (7.2)–(7.3) *we have:*

$$\frac{1}{\epsilon^q}(\partial_2 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) \to 0, \ (\partial_2 \mathbf{U}, \mathbf{t}) = 0, \tag{7.11}$$

$$\frac{1}{\epsilon^q}(\partial_3 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) \to 0, \ (\partial_3 \mathbf{U}, \mathbf{t}) = 0, \tag{7.12}$$

$$\frac{1}{\epsilon^q}(\partial_1 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) \to 0, \ (\partial_1 \mathbf{U}, \mathbf{t}) = 0, \tag{7.13}$$

$$\frac{1}{\epsilon^q}\left(\frac{1}{\epsilon}(\partial_2\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})+(\partial_1\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\right)\to 0, \tag{7.14}$$

$$\frac{1}{\epsilon^q}\left(\frac{1}{\epsilon}(\partial_3\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})+(\partial_1\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)\right)\to 0 \tag{7.15}$$

in $L^2(\Omega)$ *for* $q\in[0,1)$ *and* $\epsilon\to 0$,

$$\partial_j\frac{1}{\epsilon^q}\left(\frac{1}{\epsilon}(\partial_2\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})+(\partial_1\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\right)\to 0,\ \ j=2,3, \tag{7.16}$$

$$\partial_j\frac{1}{\epsilon^q}\left(\frac{1}{\epsilon}(\partial_3\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})+(\partial_1\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)\right)\to 0,\ \ j=2,3, \tag{7.17}$$

in $L^2(0,l;H^{-1}(S))$ *for* $\epsilon\to 0$ *and* $q\in[0,1)$,

$$\frac{1}{\epsilon^{q_1}}(\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\to 0,\ \ (\partial_2\mathbf{U},\mathbf{n})=0, \tag{7.18}$$

$$\frac{1}{\epsilon^{q_1}}(\partial_3\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)\to 0,\ \ (\partial_3\mathbf{U},\mathbf{b})=0 \tag{7.19}$$

$$\frac{1}{\epsilon^{q_1}}\Big((\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)+(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\Big)\to 0,\ \ (\partial_2\mathbf{U},\mathbf{b})+(\partial_3\mathbf{U},\mathbf{n})=0 \tag{7.20}$$

in $L^2(\Omega)$ *for* $q_1\in[0,2)$ *and* $\epsilon\to 0$, *and*

$$\frac{1}{\epsilon^{q_2}}(\partial_j\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)\to 0\ \textit{in}\ L^2(\Omega),\ \ j=2,3, \tag{7.21}$$

$$\frac{1}{\epsilon^{q_2}}(\partial_1\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)\to 0\ \textit{in}\ L^2(\Omega) \tag{7.22}$$

for $q_2\in[0,1-r)$, $r\in(0,\frac{1}{3})$, *and* $\epsilon\to 0$.

P r o o f: We can easily derive from (7.9)–(7.10) and (6.5)–(6.8) the convergences (7.11)–(7.15) and (7.18)–(7.20). It remains to prove the associated equalities. For instance, that $(\partial_2\mathbf{U},\mathbf{t})=0$. The proof for the other functions proceeds in almost the same way. Since $\mathbf{t}\in L^\infty(0,l)^3$ and $\mathbf{g}_{1,\epsilon}\to\mathbf{t}$ pointwisely in $\Omega\setminus(S\times D)$ (see (2.6), Proposition 3.2, (3.44)), we can easily derive from (7.2) that

$$(\partial_2\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})\rightharpoonup(\partial_2\mathbf{U},\mathbf{t})\text{ in }L^{q_3}(\Omega),\ \ q_3\in(1,2).$$

In addition, $\|\mathbf{g}_{1,\epsilon}\|_\infty\le C$, where C is independent of ϵ, and thus

$$(\partial_2\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})\rightharpoonup(\partial_2\mathbf{U},\mathbf{t})\text{ in }L^2(\Omega).$$

Since $(\partial_2\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})\to 0$ in $L^2(\Omega)$, $(\partial_2\mathbf{U},\mathbf{t})=0$ a.e. in Ω.

The convergences (7.16)–(7.17) easily follow from (7.14)–(7.15) (see also Proposition 5.1).

Further, we can derive from (2.6) that

$$(\partial_j \mathbf{U}_\epsilon, \mathbf{t}_\epsilon) = (\partial_j \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) + \epsilon\beta_\epsilon x_2(\partial_j \mathbf{U}_\epsilon, \mathbf{t}_\epsilon) + \epsilon\alpha_\epsilon x_3(\partial_j \mathbf{U}_\epsilon, \mathbf{t}_\epsilon)$$

$$-\epsilon\gamma_\epsilon x_3(\partial_j \mathbf{U}_\epsilon, \mathbf{n}_\epsilon) + \epsilon\gamma_\epsilon x_2(\partial_j \mathbf{U}_\epsilon, \mathbf{b}_\epsilon), \ j = 2, 3, \quad \text{in } \Omega.$$

Hence and from (3.44), we get the estimate

$$(1 - C\epsilon^{1-r})\|(\partial_j \mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\|_2 \le \|(\partial_j \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})\|_2$$

$$+C\epsilon^{1-r}(\|(\partial_j \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\|_2 + \|(\partial_j \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\|_2),$$

which together with (7.2), (7.11)–(7.12) and the fact that $r \in (0, \frac{1}{3})$ lead to (7.21). The convergence (7.22) can be proved analogously and we omit its proof. □

Lemma 7.5 *Under the assumptions of Proposition* 7.2, *we have* $\mathbf{U} \in H_0^1(0,l)^3$ (*in the sense* $\partial_j \mathbf{U} = 0$, $j = 2, 3$) *and satisfies the relation* (7.4).

P r o o f: Since $\mathbf{U}_\epsilon \in V(\Omega)$, $\forall \epsilon \in (0,1)$, the convergence (7.2) implies that the function $\mathbf{U} \in V(\Omega)$ as well. Hence we can see that it is enough to show that the function $\mathbf{U}$ depends only on x_1. But the identity

$$\mathbf{U} = (\mathbf{U}, \mathbf{t})\mathbf{t} + (\mathbf{U}, \mathbf{n})\mathbf{n} + (\mathbf{U}, \mathbf{b})\mathbf{b}$$

enables us to reduce this problem to the problem to check the dependence on x_1 only for the terms $(\mathbf{U}, \mathbf{t})$, $(\mathbf{U}, \mathbf{n})$ and $(\mathbf{U}, \mathbf{b})$.

The equalities (7.11)–(7.12) enable us to conclude that $(\mathbf{U}, \mathbf{t})$ depends only on x_1. Using (7.18)–(7.19), we can assert that

$$(\mathbf{U}, \mathbf{n})(x_1, x_2, x_3) = \widehat{\xi}_1(x_1, x_3) \text{ and } (\mathbf{U}, \mathbf{b})(x_1, x_2, x_3) = \widehat{\xi}_2(x_1, x_2),$$

where $\widehat{\xi}_i \in L^\infty(0, l; L^2(S)) \cap L^2(0, l; H^1(S))$, $i = 1, 2$. Let the point $\langle x_2^0, x_3^0 \rangle \in S$. Since S is open, there exists a square $S_0 \subset S$ such that the point $\langle x_2^0, x_3^0 \rangle$ is the corner of this square satisfying $x_2^0 \le x_2$ and $x_3^0 \le x_3$ for $\langle x_2, x_3 \rangle \in S_0$. Integrating the equality in (7.20) on the set $[x_2^0, x_2] \times [x_3^0, x_3]$ yields the identity

$$\Big(\widehat{\xi}_1(x_1, x_3) - \widehat{\xi}_1(x_1, x_3^0)\Big)x_2 = -\Big(\widehat{\xi}_2(x_1, x_2) - \widehat{\xi}_2(x_1, x_2^0)\Big)x_3.$$

If we fix x_2 and then x_3, we obtain a linear dependence of $\widehat{\xi}_1$ on x_3 and of $\widehat{\xi}_2$ on x_2. Hence we get that $\widehat{\xi}_1(x_1, x_3) = \xi_1^0(x_1)x_3 + \xi_2^0(x_1)$ and $\widehat{\xi}_2(x_1, x_2) = -\xi_1^0(x_1)x_2 + \xi_3^0(x_1)$ on $S_0 \times (0, l)$. Take the point $\langle x_2^1, x_3^1 \rangle \in S$, which is the corner of the square S_1, $x_2^1 \le x_2$ and $x_3^1 \le x_3$ for $\langle x_2, x_3 \rangle \in S_1$ and $|S_0 \cap S_1| \ne 0$. Analogously as above we can derive the functions $\xi_i^1(x_1)$, $i = 1, 2, 3$, such that $\widehat{\xi}_1(x_1, x_3) = \xi_1^1(x_1)x_3 + \xi_2^1(x_1)$ and $\widehat{\xi}_2(x_1, x_2) = -\xi_1^1(x_1)x_2 + \xi_3^1(x_1)$ on $S_1 \times (0, l)$. In addition, we obtain that

$$\xi_1^0(x_1)x_3 + \xi_2^0(x_1) = \xi_1^1(x_1)x_3 + \xi_2^1(x_1), \ -\xi_1^0(x_1)x_2 + \xi_3^0(x_1) = -\xi_1^1(x_1)x_2 + \xi_3^1(x_1)$$

a.e. on $(S_0 \cap S_1) \times (0,l)$, which implies that $\xi_i^0(x_1) = \xi_i^1(x_1)$, $i = 1,2,3$, a.e. on $(0,l)$. Since S is a connected domain and thus any two points from S can be connected with some curve lying in S, we can easily obtain that the functions ξ_i, $i = 1,2,3$, satisfy $\widehat{\xi_1}(x_1,x_3) = \xi_1(x_1)x_3 + \xi_2(x_1)$ and $\widehat{\xi_2}(x_1,x_2) = -\xi_1(x_1)x_2 + \xi_3(x_1)$ in Ω and hence

$$(\mathbf{U},\mathbf{n}) = \xi_1(x_1)x_3 + \xi_2(x_1), \ (\mathbf{U},\mathbf{b}) = -\xi_1(x_1)x_2 + \xi_3(x_1) \text{ in } \Omega. \tag{7.23}$$

Since, in addition, the functions $\mathbf{n}$ and $\mathbf{b} \in L^\infty(0,l)^3$, the functions $\xi_i \in L^2(0,l)$, $i = 1,2,3$.

Now, we prove that the functions $(\mathbf{U},\mathbf{n})$ and $(\mathbf{U},\mathbf{b})$ depend only on x_1, which together with the fact, that $(\mathbf{U},\mathbf{t})$ depends on x_1, imply that the function $\mathbf{U}$ depends only on x_1.

Taking into account the definition (2.6) of the function $\mathbf{g}_{1,\epsilon}$ and changing the position of the derivative ∂_3 with ∂_2 in the first term, we find

$$\begin{aligned}
&\frac{1}{\epsilon}\Big(\partial_3(\partial_2\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon}) - \partial_2(\partial_3\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})\Big) \\
&= \frac{1}{\epsilon}\Big(\partial_3\partial_2(\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon}) - \partial_3(\mathbf{U}_\epsilon,\partial_2\mathbf{g}_{1,\epsilon}) - \partial_2(\partial_3\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})\Big) \\
&= \frac{1}{\epsilon}\Big(\partial_3\partial_2(\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon}) - \partial_3(\mathbf{U}_\epsilon,-\epsilon\beta_\epsilon\mathbf{t}_\epsilon - \epsilon\gamma_\epsilon\mathbf{b}_\epsilon) - \partial_2(\partial_3\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})\Big) \\
&= \beta_\epsilon(\partial_3\mathbf{U}_\epsilon,\mathbf{t}_\epsilon) + \gamma_\epsilon(\partial_3\mathbf{U}_\epsilon,\mathbf{b}_\epsilon) + \frac{1}{\epsilon}\Big(\partial_2(\partial_3\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon}) \\
&\quad +\partial_2(\mathbf{U}_\epsilon,\partial_3\mathbf{g}_{1,\epsilon}) - \partial_2(\partial_3\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon}\Big) \\
&= \beta_\epsilon(\partial_3\mathbf{U}_\epsilon,\mathbf{t}_\epsilon) - \alpha_\epsilon(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon) + \gamma_\epsilon\Big((\partial_3\mathbf{U}_\epsilon,\mathbf{b}_\epsilon) + (\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\Big)
\end{aligned} \tag{7.24}$$

in $L^2(0,l;H^{-1}(S))$. Further, from the identities (2.2), we can derive ("changing the position of the derivatives ∂_j, $j = 2,3$, with ∂_1") that

$$\begin{aligned}
&\partial_3(\partial_1\mathbf{U}_\epsilon,\mathbf{n}_\epsilon) - \partial_2(\partial_1\mathbf{U}_\epsilon,\mathbf{b}_\epsilon) = \partial_3\partial_1(\mathbf{U}_\epsilon,\mathbf{n}_\epsilon) - (\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon') \\
&-\partial_2\partial_1(\mathbf{U}_\epsilon,\mathbf{b}_\epsilon) + (\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon') = \Big(\partial_1(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon) - \partial_1(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)\Big) \\
&+\beta_\epsilon(\partial_3\mathbf{U}_\epsilon,\mathbf{t}_\epsilon) - \alpha_\epsilon(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon) + \gamma_\epsilon\Big((\partial_3\mathbf{U}_\epsilon,\mathbf{b}_\epsilon) + (\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\Big)
\end{aligned} \tag{7.25}$$

in $H^{-1}(\Omega)$. Now, we add (7.25) to (7.24) and from (3.44), (7.16)–(7.19), (7.21), it follows that the functions $\partial_1(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon) - \partial_1(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)$ converge to zero strongly in $L^2(0,l;H^{-1}(S))$. In an analogous way as in the verification of (7.11), we can check that $(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon) - (\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)$ converge to $(\partial_3\mathbf{U},\mathbf{n}) - (\partial_2\mathbf{U},\mathbf{b})$ weakly in $L^2(\Omega)$ and thus

$$\partial_1(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon) - \partial_1(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon) \rightharpoonup \partial_1(\partial_3\mathbf{U},\mathbf{n}) - \partial_1(\partial_2\mathbf{U},\mathbf{b}) \text{ in } H^{-1}(\Omega),$$

which yields $\partial_1(\partial_3\mathbf{U},\mathbf{n})-\partial_1(\partial_2\mathbf{U},\mathbf{b})=0$ in the space $L^2(0,l;H^{-1}(S))$. Substituting (7.23) to the term $\partial_1(\partial_3\mathbf{U},\mathbf{n})-\partial_1(\partial_2\mathbf{U},\mathbf{b})$ yields

$$\partial_1(\partial_3\mathbf{U},\mathbf{n})-\partial_1(\partial_2\mathbf{U},\mathbf{b})=2\xi_1'=0.$$

We have proved that $\xi_1\in L^2(0,l)$ and $\xi_1'=0$ in the sense of distributions, which implies that ξ_1 is a constant. Thus we will write ξ_1 instead of $\xi_1(x_1)$. Now, we want to prove that $\xi_1=0$. After substitution (7.23) to the identity

$$\mathbf{U}=(\mathbf{U},\mathbf{t})\mathbf{t}+(\mathbf{U},\mathbf{n})\mathbf{n}+(\mathbf{U},\mathbf{b})\mathbf{b},$$

we get

$$\mathbf{U}(x_1,x_2,x_3)=\Big((\mathbf{U},\mathbf{t})\mathbf{t}\Big)(x_1)+(\xi_1x_3+\xi_2(x_1))\mathbf{n}(x_1)+(-\xi_1x_2+\xi_3(x_1))\mathbf{b}(x_1). \quad (7.26)$$

Since $\mathbf{U}\in V(\Omega)$, Proposition 5.1 implies that $\partial_j\mathbf{U}\in C_0(0,l;H^{-1}(S)^3)$, $j=2,3$. Taking $\varphi\in H_0^1(S)$ such that $\int_S\varphi\ dx_2dx_3=1$, we get from (7.26)

$$\int_S\partial_2\mathbf{U}(x_1)\varphi\ dx_2dx_3=-\xi_1\mathbf{b}(x_1),\ x_1\in[0,l].$$

Since the function $\int_S\partial_2\mathbf{U}(x_1)\varphi\ dx_2dx_3$ belongs to $C_0(0,l)$, the function $-\xi_1\mathbf{b}\in C_0(0,l)$ as well, which implies that the function $\mathbf{b}$ must be continuous for $\xi_1\neq 0$. If not, then ξ_1 must be equal to zero. Let us suppose that $\mathbf{b}$ is a continuous function. We know that

$$0=\lim_{x_1\to 0}\int_S\partial_2\mathbf{U}(x_1)\varphi\ dx_2dx_3=-\xi_1\mathbf{b}(0),$$

because $\partial_2\mathbf{U}\in C_0(0,l;H^{-1}(S)^3)$. Since $|\mathbf{b}(0)|=1$, $\xi_1=0$. Therefore $(\mathbf{U},\mathbf{n})$ and $(\mathbf{U},\mathbf{b})$ depend only on x_1. The equation (7.4) follows immediately from the equality in (7.13). □

In the following lemmas and corollaries, we construct the function ϕ from Proposition 7.2, we show that $\langle\mathbf{U},\phi\rangle\in\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ and we derive the equations (7.5)–(7.7). But first we introduce the following notation. Let the functions $\mathbf{U}_\epsilon\in V(\Omega)$, $\epsilon\in(0,1)$, be the functions from Proposition 7.2. We define auxiliary functions ϕ_ϵ, $\epsilon\in(0,1)$, by the relation

$$\phi_\epsilon=\frac{1}{2\epsilon}\Big((\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)-(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\Big). \quad (7.27)$$

Further, we define the vector functions $\mathbf{u}_{*,\epsilon}=(u^\epsilon_{*,1},u^\epsilon_{*,2},u^\epsilon_{*,3})$ by

$$u^\epsilon_{*,1}=-\phi_\epsilon,\ u^\epsilon_{*,2}=-\frac{1}{\epsilon}(\partial_3\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon}),\ u^\epsilon_{*,3}=\frac{1}{\epsilon}(\partial_2\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon}) \quad (7.28)$$

and the vector functions $\mathbf{U}_{*,\epsilon}$, $\epsilon\in(0,1)$, by

$$\mathbf{U}_{*,\epsilon}=-\phi_\epsilon\mathbf{t}_\epsilon-\frac{1}{\epsilon}(\partial_3\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})\mathbf{n}_\epsilon+\frac{1}{\epsilon}(\partial_2\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})\mathbf{b}_\epsilon. \quad (7.29)$$

Lemma 7.6 *We have*

$$\partial_j \phi_\epsilon \to 0 \ \textit{in} \ L^2(0,l;H^{-1}(S)), \ j=2,3, \tag{7.30}$$

for $\epsilon \to 0$ *and* $\phi_\epsilon|_{x_1=0} = \phi_\epsilon|_{x_1=l} = 0$ *for all* $\epsilon \in (0,1)$ *in the sense of the space* $C([0,l];H^{-1}(S))$.

P r o o f: Since $\mathbf{U}_\epsilon \in V(\Omega)$, then Proposition 5.1 together with the fact that $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon \in C^\infty([0,l])^3$ imply that $\phi_\epsilon|_{x_1=0} = \phi_\epsilon|_{x_1=l} = 0$ for all $\epsilon \in (0,1)$ in the sense of the space $C([0,l];H^{-1}(S))$.

Further, we can express the functions $\partial_2\phi_\epsilon$ in this way

$$\partial_2\phi_\epsilon = \frac{1}{2\epsilon}\Big(\partial_2(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon) - \partial_2(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\Big)$$

$$= \frac{1}{2\epsilon}\Big(\partial_2(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon) + \partial_2(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\Big) - \frac{1}{\epsilon}\partial_3(\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)$$

in $L^2(0,l;H^{-1}(S))$ (see Proposition 5.1). Applying now (7.18)–(7.20) for $q_1 = 1$, we obtain the convergence (7.30) for $j=2$. The proof of the convergence (7.30) for $j=3$ proceeds in almost the same way. □

Lemma 7.7 *Let the assumptions of Proposition* 7.2 *be fulfilled. Then*

$$(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{t}_\epsilon) \rightharpoonup \partial_3\zeta_{12} - \partial_2\zeta_{13} \ \textit{in} \ L^2(0,l;H^{-1}(S)), \tag{7.31}$$

$$(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{b}_\epsilon) \rightharpoonup \partial_2\zeta_{11} \ \textit{in} \ L^2(0,l;H^{-1}(S)), \tag{7.32}$$

$$(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{n}_\epsilon) \rightharpoonup -\partial_3\zeta_{11} \ \textit{in} \ L^2(0,l;H^{-1}(S)) \tag{7.33}$$

and thus

$$\partial_1\mathbf{U}_{*,\epsilon} \rightharpoonup (\partial_3\zeta_{12} - \partial_2\zeta_{13})\mathbf{t} - \partial_3\zeta_{11}\mathbf{n} + \partial_2\zeta_{11}\mathbf{b} \tag{7.34}$$

in $L^2(0,l;H^{-1}(S)^3)$ *for* $\epsilon \to 0$.

P r o o f: From (7.10) and (6.5)–(6.8), it follows that

$$\frac{1}{\epsilon^2}\partial_3\theta^\epsilon_{12}(\mathbf{U}_\epsilon) + \frac{1}{\epsilon}\partial_3\kappa^\epsilon_{12}(\mathbf{U}_\epsilon) - \frac{1}{\epsilon^2}\partial_2\theta^\epsilon_{13}(\mathbf{U}_\epsilon) - \frac{1}{\epsilon}\partial_2\kappa^\epsilon_{13}(\mathbf{U}_\epsilon) \rightharpoonup \partial_3\zeta_{12} - \partial_2\zeta_{13} \tag{7.35}$$

and

$$\frac{\partial_j\kappa^\epsilon_{11}(\mathbf{U}_\epsilon)}{\epsilon} \rightharpoonup \partial_j\zeta_{11}, \ j=2,3, \tag{7.36}$$

in $L^2(0,l;H^{-1}(S))$ for $\epsilon \to 0$. Thus to prove (7.31)–(7.34) it is enough to check that

$$(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{t}_\epsilon) - \Big(\frac{1}{\epsilon^2}\partial_3\theta^\epsilon_{12}(\mathbf{U}_\epsilon) + \frac{1}{\epsilon}\partial_3\kappa^\epsilon_{12}(\mathbf{U}_\epsilon)$$

$$-\frac{1}{\epsilon^2}\partial_2\theta^\epsilon_{13}(\mathbf{U}_\epsilon) - \frac{1}{\epsilon}\partial_2\kappa^\epsilon_{13}(\mathbf{U}_\epsilon)\Big) \to 0 \text{ in } L^2(\Omega), \tag{7.37}$$

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{b}_\epsilon) - \frac{\partial_2 \kappa_{11}^\epsilon(\mathbf{U}_\epsilon)}{\epsilon} \to 0 \text{ in } L^2(\Omega), \tag{7.38}$$

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{n}_\epsilon) + \frac{\partial_3 \kappa_{11}^\epsilon(\mathbf{U}_\epsilon)}{\epsilon} \to 0 \text{ in } L^2(\Omega). \tag{7.39}$$

First, we find the expressions for the terms $(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon)$, $(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{n}_\epsilon)$ and $(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{b}_\epsilon)$. Using the definition (6.5)–(6.8) of the tensors θ^ϵ and κ^ϵ, it is easy to see that it is enough to add (7.24) to (7.25) and to multiply this sum with $\frac{1}{2\epsilon}$ to obtain

$$\frac{1}{\epsilon^2}\partial_3\theta_{12}^\epsilon(\mathbf{U}_\epsilon) + \frac{1}{\epsilon}\partial_3\kappa_{12}^\epsilon(\mathbf{U}_\epsilon) - \frac{1}{\epsilon^2}\partial_2\theta_{13}^\epsilon(\mathbf{U}_\epsilon) - \frac{1}{\epsilon}\partial_2\kappa_{13}^\epsilon(\mathbf{U}_\epsilon)$$
$$= \frac{1}{2\epsilon}\Big(\partial_1(\partial_3 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon) - \partial_1(\partial_2 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\Big)$$
$$+\frac{1}{\epsilon}\Bigg(\beta_\epsilon(\partial_3 \mathbf{U}_\epsilon, \mathbf{t}_\epsilon) - \alpha_\epsilon(\partial_2 \mathbf{U}_\epsilon, \mathbf{t}_\epsilon) + \gamma_\epsilon\Big((\partial_3 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon) + (\partial_2 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\Big)\Bigg).$$

By rewriting the above mentioned expression in such a way that it involves the terms $\frac{1}{\epsilon}\beta_\epsilon(\partial_3 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})$ and $\frac{1}{\epsilon}\alpha_\epsilon(\partial_2 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})$ instead of $\frac{1}{\epsilon}\beta_\epsilon(\partial_3 \mathbf{U}_\epsilon, \mathbf{t}_\epsilon)$ and $\frac{1}{\epsilon}\alpha_\epsilon(\partial_2 \mathbf{U}_\epsilon, \mathbf{t}_\epsilon)$, we conclude that

$$\frac{1}{\epsilon^2}\partial_3\theta_{12}^\epsilon(\mathbf{U}_\epsilon) + \frac{1}{\epsilon}\partial_3\kappa_{12}^\epsilon(\mathbf{U}_\epsilon) - \frac{1}{\epsilon^2}\partial_2\theta_{13}^\epsilon(\mathbf{U}_\epsilon) - \frac{1}{\epsilon}\partial_2\kappa_{13}^\epsilon(\mathbf{U}_\epsilon)$$
$$= \Big(-\partial_1\phi_\epsilon + \frac{1}{\epsilon}\beta_\epsilon(\partial_3 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) - \frac{1}{\epsilon}\alpha_\epsilon(\partial_2 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})\Big)$$
$$+\Big((\beta_\epsilon^2 x_2 + \alpha_\epsilon\beta_\epsilon x_3)(\partial_3 \mathbf{U}_\epsilon, \mathbf{t}_\epsilon) - (\alpha_\epsilon\beta_\epsilon x_2 + \alpha_\epsilon^2 x_3)(\partial_2 \mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\Big)$$
$$+\Big((\beta_\epsilon\gamma_\epsilon x_2 + \frac{\gamma_\epsilon}{\epsilon})(\partial_3 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon) + (\alpha_\epsilon\gamma_\epsilon x_3 + \frac{\gamma_\epsilon}{\epsilon})(\partial_2 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\Big)$$
$$-\Big(\beta_\epsilon\gamma_\epsilon x_3(\partial_3 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon) + \alpha_\epsilon\gamma_\epsilon x_2(\partial_2 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\Big). \tag{7.40}$$

in $H^{-1}(\Omega)$. In addition, since all terms except $\partial_1\phi_\epsilon$ belong to $L^2(0,l;H^{-1}(S))$ then $\partial_1\phi_\epsilon \in L^2(0,l;H^{-1}(S))$ as well. From (7.29), (7.40), it follows that

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon) = \Big(\frac{1}{\epsilon^2}\partial_3\theta_{12}^\epsilon(\mathbf{U}_\epsilon) + \frac{1}{\epsilon}\partial_3\kappa_{12}^\epsilon(\mathbf{U}_\epsilon) - \frac{1}{\epsilon^2}\partial_2\theta_{13}^\epsilon(\mathbf{U}_\epsilon) - \frac{1}{\epsilon}\partial_2\kappa_{13}^\epsilon(\mathbf{U}_\epsilon)\Big)$$
$$-\Big((\beta_\epsilon^2 x_2 + \alpha_\epsilon\beta_\epsilon x_3)(\partial_3 \mathbf{U}_\epsilon, \mathbf{t}_\epsilon) - (\alpha_\epsilon\beta_\epsilon x_2 + \alpha_\epsilon^2 x_3)(\partial_2 \mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\Big)$$
$$-\Big((\beta_\epsilon\gamma_\epsilon x_2 + \frac{\gamma_\epsilon}{\epsilon})(\partial_3 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon) + (\alpha_\epsilon\gamma_\epsilon x_3 + \frac{\gamma_\epsilon}{\epsilon})(\partial_2 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\Big)$$
$$+\Big(\beta_\epsilon\gamma_\epsilon x_3(\partial_3 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon) + \alpha_\epsilon\gamma_\epsilon x_2(\partial_2 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\Big) \tag{7.41}$$

in $L^2(0,l;H^{-1}(S))$.

Further, using (2.6) and (6.8), we get (in the sense of $L^2(0,l;H^{-1}(S))$)

$$\frac{\partial_2\kappa_{11}^{\epsilon}(\mathbf{U}_\epsilon)}{\epsilon} = \frac{1}{\epsilon}\partial_2(\partial_1\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) + \partial_2(\partial_1\mathbf{U}_\epsilon, -x_2\beta_\epsilon\mathbf{t}_\epsilon) + \partial_2(\partial_1\mathbf{U}_\epsilon, -x_3\alpha_\epsilon\mathbf{t}_\epsilon)$$

$$+\partial_2(\partial_1\mathbf{U}_\epsilon, x_3\gamma_\epsilon\mathbf{n}_\epsilon) + \partial_2(\partial_1\mathbf{U}_\epsilon, -x_2\gamma_\epsilon\mathbf{b}_\epsilon) = \sum_{j=1}^{5} I_j.$$

Now, we express the terms I_i, $i = 1, \ldots, 5$, individually. Changing the position of the derivatives ∂_2 with ∂_1 in the terms above and using (2.2) lead (in the sense of the space $H^{-1}(\Omega)$) to

$$\begin{aligned} I_1 &= \frac{1}{\epsilon}\partial_2\partial_1(\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) - \frac{1}{\epsilon}(\partial_2\mathbf{U}_\epsilon, \mathbf{t}'_\epsilon) = \frac{1}{\epsilon}\partial_1(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) - \frac{1}{\epsilon}(\partial_2\mathbf{U}_\epsilon, \alpha_\epsilon\mathbf{b}_\epsilon + \beta_\epsilon\mathbf{n}_\epsilon) \\ &= \frac{1}{\epsilon}\partial_1(\partial_2\mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) - \frac{\alpha_\epsilon}{\epsilon}(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon) - \frac{\beta_\epsilon}{\epsilon}(\partial_2\mathbf{U}_\epsilon, \mathbf{n}_\epsilon) \\ &+ \partial_1\Big(x_2\beta_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) + x_3\alpha_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) - x_3\gamma_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{n}_\epsilon) \\ &+ x_2\gamma_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\Big) = \frac{1}{\epsilon}\partial_1(\partial_2\mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) - \frac{\alpha_\epsilon}{\epsilon}(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon) - \frac{\beta_\epsilon}{\epsilon}(\partial_2\mathbf{U}_\epsilon, \mathbf{n}_\epsilon) \\ &+ x_2\beta'_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) + x_2\beta_\epsilon\partial_1(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) + x_3\alpha'_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) \\ &+ x_3\alpha_\epsilon\partial_1(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) - x_3\gamma'_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{n}_\epsilon) + x_3\gamma_\epsilon\partial_1(\partial_2\mathbf{U}_\epsilon, \mathbf{n}_\epsilon) \\ &+ x_2\gamma'_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon) + x_2\gamma_\epsilon\partial_1(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon) \end{aligned}$$

$$\begin{aligned} I_2 &= -\beta_\epsilon(\partial_1\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) - x_2\beta_\epsilon\partial_2(\partial_1\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) = -\beta_\epsilon(\partial_1\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) \\ &- x_2\beta_\epsilon\partial_2\partial_1(\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) + x_2\beta_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{t}'_\epsilon) = -\beta_\epsilon(\partial_1\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) \\ &- x_2\beta_\epsilon\partial_1(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) + x_2\alpha_\epsilon\beta_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon) + x_2\beta_\epsilon^2(\partial_2\mathbf{U}_\epsilon, \mathbf{n}_\epsilon), \end{aligned}$$

$$\begin{aligned} I_3 &= -x_3\alpha_\epsilon\partial_2\partial_1(\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) + x_3\alpha_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{t}'_\epsilon) \\ &= -x_3\alpha_\epsilon\partial_1(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) + x_3\alpha_\epsilon^2(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon) + x_3\alpha_\epsilon\beta_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{n}_\epsilon), \end{aligned}$$

$$\begin{aligned} I_4 &= x_3\gamma_\epsilon\partial_2\partial_1(\mathbf{U}_\epsilon, \mathbf{n}_\epsilon) - x_3\gamma_\epsilon\partial_2(\mathbf{U}_\epsilon, \mathbf{n}'_\epsilon) \\ &= x_3\gamma_\epsilon\partial_1(\partial_2\mathbf{U}_\epsilon, \mathbf{n}_\epsilon) + x_3\beta_\epsilon\gamma_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) + x_3\gamma_\epsilon^2(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon), \end{aligned}$$

$$\begin{aligned} I_5 &= -\gamma_\epsilon(\partial_1\mathbf{U}_\epsilon, \mathbf{b}_\epsilon) - x_2\gamma_\epsilon\partial_2\partial_1(\mathbf{U}_\epsilon, \mathbf{b}_\epsilon) + x_2\gamma_\epsilon\partial_2(\mathbf{U}_\epsilon, \mathbf{b}'_\epsilon) \\ &= -\gamma_\epsilon(\partial_1\mathbf{U}_\epsilon, \mathbf{b}_\epsilon) - x_2\gamma_\epsilon\partial_1(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon) - x_2\alpha_\epsilon\gamma_\epsilon(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) \\ &+ x_2\gamma_\epsilon^2(\partial_2\mathbf{U}_\epsilon, \mathbf{n}_\epsilon) \end{aligned}$$

Then we get

$$\frac{\partial_2\kappa_{11}^{\epsilon}(\mathbf{U}_\epsilon)}{\epsilon} = \sum_{j=1}^{5} I_j = \frac{1}{\epsilon}\partial_1(\partial_2\mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) - \frac{\alpha_\epsilon}{\epsilon}(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon) - \frac{\beta_\epsilon}{\epsilon}(\partial_2\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)$$

$$+x_2\beta_\epsilon'(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)+x_2\beta_\epsilon\partial_1(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)+x_3\alpha_\epsilon'(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)+x_3\alpha_\epsilon\partial_1(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)$$
$$-x_3\gamma_\epsilon'(\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)-x_3\gamma_\epsilon\partial_1(\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)+x_2\gamma_\epsilon'(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)+x_2\gamma_\epsilon\partial_1(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)$$
$$-\beta_\epsilon(\partial_1\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)-x_2\beta_\epsilon\partial_1(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)+x_2\alpha_\epsilon\beta_\epsilon(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)+x_2\beta_\epsilon^2(\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)$$
$$-x_3\alpha_\epsilon\partial_1(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)+x_3\alpha_\epsilon^2(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)+x_3\alpha_\epsilon\beta_\epsilon(\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)+x_3\gamma_\epsilon\partial_1(\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)$$
$$+x_3\beta_\epsilon\gamma_\epsilon(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)+x_3\gamma_\epsilon^2(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)-\gamma_\epsilon(\partial_1\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)-x_2\gamma_\epsilon\partial_1(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)$$
$$-x_2\alpha_\epsilon\gamma_\epsilon(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)+x_2\gamma_\epsilon^2(\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)=\frac{1}{\epsilon}\partial_1(\partial_2\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})-\frac{\alpha_\epsilon}{\epsilon}(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)$$
$$-\frac{\beta_\epsilon}{\epsilon}(\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)+x_2\beta_\epsilon'(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)+x_3\alpha_\epsilon'(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)-x_3\gamma_\epsilon'(\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)$$
$$+x_2\gamma_\epsilon'(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)-\beta_\epsilon(\partial_1\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)+x_2\alpha_\epsilon\beta_\epsilon(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)+x_2\beta_\epsilon^2(\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)$$
$$+x_3\alpha_\epsilon^2(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)+x_3\alpha_\epsilon\beta_\epsilon(\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)+x_3\beta_\epsilon\gamma_\epsilon(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)+x_3\gamma_\epsilon^2(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)$$
$$-\gamma_\epsilon(\partial_1\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)-x_2\alpha_\epsilon\gamma_\epsilon(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)+x_2\gamma_\epsilon^2(\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)$$

in $H^{-1}(\Omega)$. Using (5.9) and (7.28)–(7.29) yield (after rearrangement)

$$\begin{aligned}
(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{b}_\epsilon)&=\partial_1\left(\frac{1}{\epsilon}(\partial_2\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})\right)-\alpha_\epsilon\phi_\epsilon+\gamma_\epsilon\frac{1}{\epsilon}(\partial_3\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})\\
&=\left(\frac{\partial_2\kappa_{11}^\epsilon(\mathbf{U}_\epsilon)}{\epsilon}+\frac{\alpha_\epsilon}{\epsilon}\left(\frac{(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)+(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)}{2}\right)\right)\\
&\quad+\gamma_\epsilon\left(\frac{1}{\epsilon}(\partial_3\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})+(\partial_1\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)\right)\\
&\quad-\left((-\frac{\beta_\epsilon}{\epsilon}+\beta_\epsilon^2x_2+\alpha_\epsilon\beta_\epsilon x_3-\gamma_\epsilon'x_3+\gamma_\epsilon^2x_2)(\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\right)\\
&\quad-\Big((\beta_\epsilon'x_2+\alpha_\epsilon'x_3+\beta_\epsilon\gamma_\epsilon x_3-\alpha_\epsilon\gamma_\epsilon x_2)(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)\Big)\\
&\quad-\Big((\alpha_\epsilon\beta_\epsilon x_2+\alpha_\epsilon^2x_3+\gamma_\epsilon^2x_3+\gamma_\epsilon'x_2)(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)-\beta_\epsilon(\partial_1\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)\Big)
\end{aligned}\tag{7.42}$$

in $L^2(0,l;H^{-1}(S))$. In an analogous way applied to $\frac{\partial_3\kappa_{11}^\epsilon(\mathbf{U}_\epsilon)}{\epsilon}$, we can derive that

$$\begin{aligned}
(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{n}_\epsilon)&=\left(-\frac{\partial_3\kappa_{11}^\epsilon(\mathbf{U}_\epsilon)}{\epsilon}+\frac{\beta_\epsilon}{\epsilon}\left(\frac{(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)+(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)}{2}\right)\right)\\
&\quad+\gamma_\epsilon\left(\frac{1}{\epsilon}(\partial_2\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})+(\partial_1\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\right)\\
&\quad+\left((-\frac{\alpha_\epsilon}{\epsilon}+\alpha_\epsilon^2x_3+\alpha_\epsilon\beta_\epsilon x_2+\gamma_\epsilon'x_2+\gamma_\epsilon^2x_3)(\partial_3\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)\right)\\
&\quad+\Big((\beta_\epsilon'x_2+\alpha_\epsilon'x_3+\beta_\epsilon\gamma_\epsilon x_3-\alpha_\epsilon\gamma_\epsilon x_2)(\partial_3\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)\Big)\\
&\quad+\Big((\alpha_\epsilon\beta_\epsilon x_3+\beta_\epsilon^2x_2+\gamma_\epsilon^2x_2-\gamma_\epsilon'x_3)(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)-\alpha_\epsilon(\partial_1\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)\Big)
\end{aligned}\tag{7.43}$$

in $L^2(0,l;H^{-1}(S))$.

Now, we check the convergence (7.37). The convergences (7.38)–(7.39) can be proved analogously. From (7.41) and the facts that $\mathbf{U}_\epsilon \in V(\Omega)$, $\alpha_\epsilon,\ \beta_\epsilon,\ \gamma_\epsilon \in C^\infty([0,l])$, $\mathbf{g}_{1,\epsilon} \in C^\infty(\overline{\Omega})^3$, $\mathbf{t}_\epsilon,\ \mathbf{n}_\epsilon,\ \mathbf{b}_\epsilon \in C^\infty([0,l])^3$, it follows that the difference

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon) - \Big(\frac{1}{\epsilon^2}\partial_3\theta_{12}^\epsilon(\mathbf{U}_\epsilon) + \frac{1}{\epsilon}\partial_3\kappa_{12}^\epsilon(\mathbf{U}_\epsilon) - \frac{1}{\epsilon^2}\partial_2\theta_{13}^\epsilon(\mathbf{U}_\epsilon) - \frac{1}{\epsilon}\partial_2\kappa_{13}^\epsilon(\mathbf{U}_\epsilon)\Big)$$

is well-defined in $L^2(\Omega)$ for all $\epsilon \in (0,1)$ and satisfies for $r \in (0,\frac{1}{3})$ the estimate

$$\begin{aligned}
&\|(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon) - (\frac{1}{\epsilon^2}\partial_3\theta_{12}^\epsilon(\mathbf{U}_\epsilon) + \frac{1}{\epsilon}\partial_3\kappa_{12}^\epsilon(\mathbf{U}_\epsilon) - \frac{1}{\epsilon^2}\partial_2\theta_{13}^\epsilon(\mathbf{U}_\epsilon) - \frac{1}{\epsilon}\partial_2\kappa_{13}^\epsilon(\mathbf{U}_\epsilon))\|_2 \\
&\overset{(7.41)}{\leq} \|(\beta_\epsilon^2 x_2 + \alpha_\epsilon\beta_\epsilon x_3)(\partial_3\mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\|_2 + \|(\alpha_\epsilon\beta_\epsilon x_2 + \alpha_\epsilon^2 x_3)(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\|_2 \\
&\quad + \|(\beta_\epsilon\gamma_\epsilon x_2 + \frac{\gamma_\epsilon}{\epsilon})(\partial_3\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\|_2 + \|(\alpha_\epsilon\gamma_\epsilon x_3 + \frac{\gamma_\epsilon}{\epsilon})(\partial_2\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\|_2 \\
&\quad + \|\beta_\epsilon\gamma_\epsilon x_3(\partial_3\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\|_2 + \|\alpha_\epsilon\gamma_\epsilon x_2(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\|_2 \\
&\overset{(3.44)}{\leq} C\Big(\frac{1}{\epsilon^{2r}}\|(\partial_3\mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\|_2 + \frac{1}{\epsilon^{2r}}\|(\partial_2\mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\|_2 + \frac{1}{\epsilon^{1+r}}\|(\partial_3\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\|_2 \\
&\quad + \frac{1}{\epsilon^{1+r}}\|(\partial_2\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\|_2 + \frac{1}{\epsilon^{2r}}\|(\partial_3\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\|_2 + \frac{1}{\epsilon^{2r}}\|(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\|_2\Big) \\
&= C(\epsilon) + \frac{1}{\epsilon^{2r}}\Big(\|(\partial_3\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\|_2 + \|(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\|_2\Big),
\end{aligned} \tag{7.44}$$

where $C(\epsilon) \to 0$ for $\epsilon \to 0$ as a consequence of (7.18)–(7.19), (7.21). It remains to study the behaviour of the terms

$$\frac{1}{\epsilon^{2r}}\|(\partial_3\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\|_2,\ \frac{1}{\epsilon^{2r}}\|(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\|_2.$$

The estimate

$$\begin{aligned}
&\|\frac{1}{\epsilon^{2r}}(\partial_3\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\|_2 + \|\frac{1}{\epsilon^{2r}}(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\|_2 \leq \|\frac{1}{\epsilon^{2r}}((\partial_3\mathbf{U}_\epsilon, \mathbf{n}_\epsilon) + (\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon))\|_2 \\
&\quad + \|\frac{1}{\epsilon^{2r}}((\partial_3\mathbf{U}_\epsilon, \mathbf{n}_\epsilon) - (\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon))\|_2 \overset{(7.27)}{=} C_1(\epsilon) + 2\epsilon^{1-2r}\|\phi_\epsilon\|_2 \\
&\overset{(5.1),\ \text{Lemma 7.6}}{\leq} C_1(\epsilon) + C\epsilon^{1-2r}\sum_{j=1}^{3}\|\partial_j\phi_\epsilon\|_{L^2(0,l;H^{-1}(S))} \\
&\overset{(7.30)}{=} C_1(\epsilon) + C_2(\epsilon) + C\epsilon^{1-2r}\|\partial_1\phi_\epsilon\|_{L^2(0,l;H^{-1}(S))} \\
&\overset{(7.40),(3.44)}{\leq} C_1(\epsilon) + C_2(\epsilon) + \frac{C}{\epsilon^{2r}}\Big(\|\frac{1}{\epsilon}\partial_3\theta_{12}^\epsilon(\mathbf{U}_\epsilon) + \partial_3\kappa_{12}^\epsilon(\mathbf{U}_\epsilon)\|_{L^2(0,l;H^{-1}(S))} \\
&\quad + \|\frac{1}{\epsilon}\partial_2\theta_{13}^\epsilon(\mathbf{U}_\epsilon) + \partial_2\kappa_{13}^\epsilon(\mathbf{U}_\epsilon)\|_{L^2(0,l;H^{-1}(S))}\Big)
\end{aligned}$$

$$+\frac{C}{\epsilon^{3r}}\Big(\|(\partial_3\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})\|_2+\|(\partial_2\mathbf{U}_\epsilon,\mathbf{g}_{1,\epsilon})\|_2\Big)$$
$$+C\epsilon^{1-4r}\Big(\|(\partial_3\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)\|_2+\|(\partial_2\mathbf{U}_\epsilon,\mathbf{t}_\epsilon)\|_2\Big)+\frac{C}{\epsilon^{3r}}\Big(\|(\partial_3\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)\|_2+\|(\partial_2\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\|_2\Big)$$
$$+C\epsilon^{1-2r}\Big(\|\frac{1}{\epsilon^{2r}}(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)\|_2+\|\frac{1}{\epsilon^{2r}}(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\|_2\Big)$$
$$=\sum_{j=1}^{6}C_j(\epsilon)+C\epsilon^{1-2r}\Big(\|\frac{1}{\epsilon^{2r}}(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)\|_2+\|\frac{1}{\epsilon^{2r}}(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\|_2\Big),$$

leads to the estimate

$$\|\frac{1}{\epsilon^{2r}}(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\|_2+\|\frac{1}{\epsilon^{2r}}(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)\|_2\le C\sum_{j=1}^{6}C_j(\epsilon)$$

for $\epsilon\in(0,1)$, where $C_1(\epsilon)\to 0$ see (7.20), $C_2(\epsilon)\to 0$ as a consequence of (7.30), $C_3(\epsilon)\to 0$ see (7.16)–(7.17), because $r\in(0,\frac{1}{3})$, $C_4(\epsilon)\to 0$ and $C_6(\epsilon)\to 0$ as a result of (7.11)–(7.12), (7.18)–(7.19) and the fact that $r\in(0,\frac{1}{3})$, $C_5(\epsilon)\to 0$ as a consequence of (7.21), because $4r-1<1-r$ for $r\in(0,\frac{1}{3})$. Hence, we can conclude that

$$\frac{1}{\epsilon^{2r}}\Big(\|(\partial_3\mathbf{U}_\epsilon,\mathbf{n}_\epsilon)\|_2+\|(\partial_2\mathbf{U}_\epsilon,\mathbf{b}_\epsilon)\|_2\Big)\to 0 \tag{7.45}$$

for $r\in(0,\frac{1}{3})$, which together with (7.44) imply (7.37) and thus (using (7.35)) (7.31).

Now, it remains to prove (7.34). Since

$$\partial_1\mathbf{U}_{*,\epsilon}=(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{t}_\epsilon)\mathbf{t}_\epsilon+(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{n}_\epsilon)\mathbf{n}_\epsilon+(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{b}_\epsilon)\mathbf{b}_\epsilon,$$

it is enough to show that

$$(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{t}_\epsilon)\mathbf{t}_\epsilon\rightharpoonup(\partial_3\zeta_{12}-\partial_2\zeta_{13})\mathbf{t}\text{ in }L^2(0,l;H^{-1}(S)^3), \tag{7.46}$$
$$(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{n}_\epsilon)\mathbf{n}_\epsilon\rightharpoonup-\partial_3\zeta_{11}\mathbf{n}\text{ in }L^2(0,l;H^{-1}(S)^3), \tag{7.47}$$
$$(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{b}_\epsilon)\mathbf{b}_\epsilon\rightharpoonup\partial_2\zeta_{11}\mathbf{n}\text{ in }L^2(0,l;H^{-1}(S)^3) \tag{7.48}$$

for $\epsilon\to 0$. We check only (7.46). The convergences (7.47) and (7.48) can be proved in almost the same way. Since $\mathbf{t}$ is a bounded function depending only on x_1, then (7.31) yields

$$(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{t}_\epsilon)\mathbf{t}\rightharpoonup(\partial_3\zeta_{12}-\partial_2\zeta_{13})\mathbf{t}\text{ in }L^2(0,l;H^{-1}(S)^3).$$

It remains to show that

$$(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{t}_\epsilon)\mathbf{t}_\epsilon-(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{t}_\epsilon)\mathbf{t}\rightharpoonup 0\text{ in }L^2(0,l;H^{-1}(S)^3)$$

for $\epsilon\to 0$, which follows from the estimate

$$\left|\int_\Omega(\partial_1\mathbf{U}_{*,\epsilon},\mathbf{t}_\epsilon)(\mathbf{t}_\epsilon-\mathbf{t})\varphi\,dx\right|$$

$$\leq C \left(\int_0^l |\mathbf{t}_\epsilon(x_1) - \mathbf{t}(x_1)|^2 \|\varphi(x_1)\|^2_{1,2,S} \, dx_1 \right)^{\frac{1}{2}} \to 0, \tag{7.49}$$

for $\epsilon \to 0$ and for arbitrary but fixed function $\varphi \in L^2(0,l;H^1_0(S))$, because $|\mathbf{t}_\epsilon| = |\mathbf{t}| = 1$, $\forall \epsilon \in (0,1)$, $\mathbf{t}_\epsilon \to \mathbf{t}$ pointwisely in $[0,l] \setminus D$ and thus we can use the Lebesgue theorem. □

To derive the equations (7.5)–(7.7), we must describe more precisely the limit state of the functions $\mathbf{U}_{*,\epsilon}$ for $\epsilon \to 0$. This will be done in the following lemma and corollary.

Lemma 7.8 *Let the assumptions of Proposition* 7.2 *be fulfilled. Then*

$$\partial_j \mathbf{U}_{*,\epsilon} \rightharpoonup \mathbf{0} \ \textit{in} \ L^2(0,l;H^{-1}(S)^3), \ j = 2,3, \tag{7.50}$$

and $\mathbf{U}_{*,\epsilon}|_{x_1=0} = \mathbf{U}_{*,\epsilon}|_{x_1=l} = 0$ *in the sense of the space* $C([0,l];H^{-1}(S)^3)$.

P r o o f: Since $\phi_\epsilon|_{x_1=0} = \phi_\epsilon|_{x_1=l} = 0$ for all $\epsilon \in (0,1)$ in the sense of the space $C([0,l];H^{-1}(S)^3)$ (see Lemma 7.6), $\mathbf{U}_\epsilon \in V(\Omega)$ and since the functions $\mathbf{g}_{1,\epsilon}$, $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ belong to $C^\infty(\overline{\Omega})^3$, we can use the definition (7.29) of the function $\mathbf{U}_{*,\epsilon}$ and applying Proposition 5.1, we get that $\mathbf{U}_{*,\epsilon}|_{x_1=0} = \mathbf{U}_{*,\epsilon}|_{x_1=l} = 0$ in the sense of the space $C([0,l];H^{-1}(S)^3)$.

It remains to show (7.50). Using the definition (7.29) of the function $\mathbf{U}_{*,\epsilon}$, we obtain the identity

$$\begin{aligned}
\partial_j \mathbf{U}_{*,\epsilon} &= -\partial_j \phi_\epsilon \mathbf{t}_\epsilon - \partial_j \frac{1}{\epsilon}(\partial_3 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})\mathbf{n}_\epsilon + \partial_j \frac{1}{\epsilon}(\partial_2 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})\mathbf{b}_\epsilon \\
&= -\partial_j \phi_\epsilon \mathbf{t}_\epsilon + \partial_j(\partial_1 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon - \partial_j(\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon \\
-\partial_j \Big(\frac{1}{\epsilon}(\partial_3 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) &+ (\partial_1 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\Big)\mathbf{n}_\epsilon + \partial_j \Big(\frac{1}{\epsilon}(\partial_2 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) + (\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\Big)\mathbf{b}_\epsilon
\end{aligned} \tag{7.51}$$

in $L^2(0,l;H^{-1}(S)^3)$, $j = 2,3$. From (7.16), (7.17), (7.30) and from the fact that the functions $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$ are bounded in $L^\infty(0,l)^3$, it follows that

$$\partial_j \phi_\epsilon \mathbf{t}_\epsilon \to \mathbf{0} \text{ in } L^2(0,l;H^{-1}(S)^3), \ j = 2,3, \tag{7.52}$$

$$\partial_j \Big(\frac{1}{\epsilon}(\partial_3 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) + (\partial_1 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\Big)\mathbf{n}_\epsilon \to \mathbf{0} \text{ in } L^2(0,l;H^{-1}(S)^3), \ j = 2,3, \tag{7.53}$$

$$\partial_j \Big(\frac{1}{\epsilon}(\partial_2 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) + (\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\Big)\mathbf{b}_\epsilon \to \mathbf{0} \text{ in } L^2(0,l;H^{-1}(S)^3), \ j = 2,3, \tag{7.54}$$

for $\epsilon \to 0$. We can see from (7.51) that it remains to prove that

$$\partial_j(\partial_1 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon \rightharpoonup \mathbf{0} \text{ in } L^2(0,l;H^{-1}(S)^3), \ j = 2,3, \tag{7.55}$$

$$\partial_j(\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon \rightharpoonup \mathbf{0} \text{ in } L^2(0,l;H^{-1}(S)^3), \ j = 2,3 \tag{7.56}$$

for $\epsilon \to 0$. From (7.2), it follows that $(\partial_1 \mathbf{U}_\epsilon, \mathbf{n}) \rightharpoonup (\partial_1 \mathbf{U}, \mathbf{n})$ in $L^2(\Omega)$, because $\mathbf{n}$ is a bounded function. Further, we have the estimate

$$\left| \int_\Omega (\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon - \mathbf{n}) \varphi \, dx \right| \le C \left(\int_0^l |\mathbf{n}_\epsilon(x_1) - \mathbf{n}(x_1)|^2 \|\varphi(x_1)\|_{2,S}^2 \, dx_1 \right)^{\frac{1}{2}} \to 0,$$

where $\varphi \in L^2(\Omega)$ is arbitrary but fixed, $\mathbf{n}_\epsilon \to \mathbf{n}$ pointwisely in $[0,l] \setminus D$ for $\epsilon \to 0$ and thus we can use the Lebesgue theorem. Hence we can deduce that

$$(\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon) \rightharpoonup (\partial_1 \mathbf{U}, \mathbf{n}) \text{ in } L^2(\Omega).$$

The proof that

$$(\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon \rightharpoonup (\partial_1 \mathbf{U}, \mathbf{n})\mathbf{b} \text{ in } L^2(\Omega)^3$$

is almost the same as the proof that

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon)\mathbf{t}_\epsilon \rightharpoonup (\partial_3 \zeta_{12} - \partial_2 \zeta_{13})\mathbf{t} \text{ in } L^2(0,l; H^{-1}(S)^3),$$

because we take only $\varphi \in L^2(\Omega)$ instead of $\varphi \in L^2(0,l; H^{-1}(S))$ in the estimate (7.49) modified for the functions $(\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon$. The analogous result can be obtained for $(\partial_1 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon$. Hence we get that

$$\partial_j(\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon \rightharpoonup \partial_j(\partial_1 \mathbf{U}, \mathbf{n})\mathbf{b} \text{ in } L^2(0,l; H^{-1}(S)^3),\ j = 2,3,$$

$$\partial_j(\partial_1 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon \rightharpoonup \partial_j(\partial_1 \mathbf{U}, \mathbf{b})\mathbf{n} \text{ in } L^2(0,l; H^{-1}(S)^3),\ j = 2,3.$$

In Lemma 7.5 we have proved that the function $\mathbf{U}$ depends only on x_1 and hence

$$\partial_j(\partial_1 \mathbf{U}, \mathbf{n})\mathbf{b} = \mathbf{0},\ \partial_j(\partial_1 \mathbf{U}, \mathbf{b})\mathbf{n} = \mathbf{0},\ j = 2,3.$$

Thus we have proved (7.55) and (7.56). □

Corollary 7.9 *Let the assumptions of Proposition 7.2 be fulfilled. Then*

$$\partial_i \mathbf{U}_{*,\epsilon} \rightharpoonup \partial_i \mathbf{U}_* \text{ in } L^2(0,l; H^{-1}(S)^3),\ i = 1,2,3, \tag{7.57}$$

$$\mathbf{U}_{*,\epsilon} \rightharpoonup \mathbf{U}_* \text{ in } L^2(\Omega)^3, \tag{7.58}$$

$$\mathbf{U}_{*,\epsilon} \to \mathbf{U}_* \text{ in } C_0(0,l; H^{-1}(S)^3) \tag{7.59}$$

for $\epsilon \to 0$, and $\mathbf{U}_* \in H_0^1(0,l)^3$, where

$$\mathbf{U}_*(x_1) = \int_0^{x_1} [(\partial_3 \zeta_{12}(z_1,x_2,x_3) - \partial_2 \zeta_{13}(z_1,x_2,x_3))\mathbf{t}(z_1)$$

$$-\partial_3 \zeta_{11}(z_1,x_2,x_3)\mathbf{n}(z_1) + \partial_2 \zeta_{11}(z_1,x_2,x_3)\mathbf{b}(z_1)] \, dz_1 \tag{7.60}$$

for $(x_1,x_2,x_3) \in (0,l) \times S$. In addition,

$$\phi_\epsilon \rightharpoonup \phi = (\mathbf{U}_*, \mathbf{t}) \text{ in } L^2(\Omega) \tag{7.61}$$

for $\epsilon \to 0$ and ϕ is piecewise continuous.

P r o o f: Lemma 7.7 and 7.8 enable us to use Proposition 5.1 and 5.2 to prove (7.57)–(7.60) and $\mathbf{U}_* \in H_0^1(0,l)^3$. From (7.29), it follows that $\phi_\epsilon = -(\mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon)$. Then (7.61) easily follows from (7.58) using the pointwise convergence on $[0,l] \setminus D$ of the functions $\mathbf{t}_\epsilon$. □

Lemma 7.10 *Let the assumptions of Proposition* 7.2 *be fulfilled. Let the function* $\mathbf{U}$ *be determined by* (7.2) *and the function* ϕ *by* (7.61). *Then the couple* $\langle \mathbf{U}, \phi \rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$.

P r o o f: To prove that $\langle \mathbf{U}, \phi \rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$, it is enough to check that $\mathbf{U} = \widehat{\mathbf{U}}$, where

$$\widehat{\mathbf{U}}(x_1) = \int_0^{x_1} [-(\mathbf{U}_*, \mathbf{b})\mathbf{n} + (\mathbf{U}_*, \mathbf{n})\mathbf{b}]\, dz_1, \;\; x_1 \in [0,l]$$

(see (2.17) and Proposition 4.1). We define the function $\widehat{\mathbf{U}}_\epsilon$ by

$$\widehat{\mathbf{U}}_\epsilon(x_1, x_2, x_3) = \int_0^{x_1} [-(\mathbf{U}_{*,\epsilon}(z_1, x_2, x_3), \mathbf{b}_\epsilon(z_1))\mathbf{n}_\epsilon(z_1)$$

$$+ (\mathbf{U}_{*,\epsilon}(z_1, x_2, x_3), \mathbf{n}_\epsilon(z_1))\mathbf{b}_\epsilon(z_1)]\, dz_1, \;\; (x_1, x_2, x_3) \in [0,l] \times S. \tag{7.62}$$

The definition (7.29) of the function $\mathbf{U}_{*,\epsilon}$ together with (7.62) enable us to express the function $\widehat{\mathbf{U}}_\epsilon$ by

$$\widehat{\mathbf{U}}_\epsilon = -\int_0^{x_1} \left[\frac{1}{\epsilon}(\partial_2 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})\mathbf{n}_\epsilon + \frac{1}{\epsilon}(\partial_3 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon})\mathbf{b}_\epsilon\right]\, dz_1, \tag{7.63}$$

where we omit to write the points (z_1, x_2, x_3) and (z_1) in the right-hand side to simplify the notation. Using (7.63), we can deduce that

$$\mathbf{U}_\epsilon = \int_0^{x_1} \partial_1 \mathbf{U}_\epsilon\, dz_1 = \int_0^{x_1} [(\partial_1 \mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\mathbf{t}_\epsilon + (\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{n}_\epsilon + (\partial_1 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{b}_\epsilon]\, dz_1$$

$$= \widehat{\mathbf{U}}_\epsilon + \int_0^{x_1} \left[(\partial_1 \mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\mathbf{t}_\epsilon + \Big(\frac{1}{\epsilon}(\partial_2 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) + (\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\Big)\mathbf{n}_\epsilon\right.$$

$$\left. + \Big(\frac{1}{\epsilon}(\partial_3 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) + (\partial_1 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\Big)\mathbf{b}_\epsilon\right]\, dz_1. \tag{7.64}$$

As a result of (7.64) and (7.14)–(7.15), (7.22), we get

$$\partial_1 \widehat{\mathbf{U}}_\epsilon - \partial_1 \mathbf{U}_\epsilon \to 0 \text{ in } L^2(\Omega)^3$$

and

$$\widehat{\mathbf{U}}_\epsilon - \mathbf{U}_\epsilon \to 0 \text{ in } C([0,l]; L^2(S)^3)$$

for $\epsilon \to 0$. Since, in addition, $\mathbf{U}_\epsilon \rightharpoonup \mathbf{U}$ in $H^1(\Omega)^3$ and $\mathbf{U} \in H_0^1(0,l)^3$, we can conclude that $\mathbf{U} = \widehat{\mathbf{U}}$ a.e. in $[0,l]$ and thus

$$\mathbf{U}(x_1) = \int_0^{x_1} [-(\mathbf{U}_*, \mathbf{b})\mathbf{n} + (\mathbf{U}_*, \mathbf{n})\mathbf{b}]\, dz_1, \;\; x_1 \in [0,l],$$

and

$$\mathbf{U}(l) = \int_0^l [-(\mathbf{U}_*, \mathbf{b})\mathbf{n} + (\mathbf{U}_*, \mathbf{n})\mathbf{b}]\, dx_1 = 0.$$

Hence, from (2.17) and Proposition 4.1, we get that $\langle \mathbf{U}, \phi\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$. □

Corollary 7.11 *Let the function* $\mathbf{U}_*$ *be defined by* (7.60). *Then the function* $\mathbf{U}_*$ *satisfies the equations* (7.5)–(7.7).

P r o o f: The proof immediately follows from (7.60). □

Lemma 7.12 *Let the assumptions of Proposition* 7.2 *be fulfilled. Let, in addition,* $\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon) \to \zeta$ *in* $L^2(\Omega)^9$. *Then*

$$\mathbf{U}_\epsilon \to \mathbf{U} \text{ in } H^1(\Omega) \tag{7.65}$$

for $\epsilon \to 0$.

P r o o f: From (7.18) and (7.19) for $q_1 = 0$ and from (7.21) for $q_2 = 0$, it follows

$$(\partial_j \mathbf{U}_\epsilon, \mathbf{t}_\epsilon) \to 0,\ j = 2,3,\ (\partial_2 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon) \to 0 \text{ and } (\partial_3 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon) \to 0 \tag{7.66}$$

in $L^2(\Omega)$. To prove that $\partial_2\mathbf{U}_\epsilon$ and $\partial_3\mathbf{U}_\epsilon$ converge strongly in $L^2(\Omega)^3$, we must verify the strong convergence of the functions $(\partial_2\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)$ and $(\partial_3\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)$ to zero in $L^2(\Omega)$, which follows from (7.45). The rest of the proof is a consequence of the identity

$$\partial_j\mathbf{U}_\epsilon = (\partial_j\mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\mathbf{t}_\epsilon + (\partial_j\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{n}_\epsilon + (\partial_j\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{b}_\epsilon,\ j = 2,3,$$

because $|\mathbf{t}_\epsilon| = |\mathbf{n}_\epsilon| = |\mathbf{b}_\epsilon| = 1$ for all $\epsilon \in (0,1)$.

It remains to investigate the functions $\partial_1\mathbf{U}_\epsilon$. We have proved in (7.22) for $q_2 = 0$ that

$$(\partial_1\mathbf{U}_\epsilon, \mathbf{t}_\epsilon) \to 0 \text{ in } L^2(\Omega).$$

Since

$$\partial_1\mathbf{U}_\epsilon = (\partial_1\mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\mathbf{t}_\epsilon + (\partial_1\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{n}_\epsilon + (\partial_1\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{b}_\epsilon,$$

it remains to study the strong convergences

$$(\partial_1\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{n}_\epsilon \to (\partial_1\mathbf{U}, \mathbf{n})\mathbf{n} \text{ and } (\partial_1\mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{b}_\epsilon \to (\partial_1\mathbf{U}, \mathbf{b})\mathbf{b}$$

in $L^2(\Omega)^3$. Let us suppose first that we know that

$$(\partial_1\mathbf{U}_\epsilon, \mathbf{n}_\epsilon) \to (\partial_1\mathbf{U}, \mathbf{n}),\ (\partial_1\mathbf{U}_\epsilon, \mathbf{b}_\epsilon) \to (\partial_1\mathbf{U}, \mathbf{b}) \text{ in } L^2(\Omega). \tag{7.67}$$

Now, we get

$$\|(\partial_1\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{n}_\epsilon - (\partial_1\mathbf{U}, \mathbf{n})\mathbf{n}\|_2 \le \left(\int_\Omega \left|\Big((\partial_1\mathbf{U}_\epsilon, \mathbf{n}_\epsilon) - (\partial_1\mathbf{U}, \mathbf{n})\Big)\mathbf{n}_\epsilon\right|^2 dx\right)^{\frac{1}{2}}$$

$$+\left(\int_\Omega |(\partial_1 \mathbf{U}, \mathbf{n})(\mathbf{n}_\epsilon - \mathbf{n})|^2\, dx\right)^{\frac{1}{2}} \le \|(\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon) - (\partial_1 \mathbf{U}, \mathbf{n})\|_2$$

$$+\left(\int_0^l |\mathbf{n}_\epsilon - \mathbf{n}|^2 \|(\partial_1 \mathbf{U}, \mathbf{n})(x_1)\|_{2,S}^2\, dx_1\right)^{\frac{1}{2}} \to 0$$

for $\epsilon \to 0$ using (7.67), the facts that $\mathbf{n}_\epsilon \to \mathbf{n}$ pointwisely in $[0,l] \setminus D$ and $|\mathbf{n}_\epsilon| = 1$, which enables us to use the Lebesgue theorem. Hence, we can see that the convergence of the terms $(\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{n}_\epsilon$ and $(\partial_1 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{b}_\epsilon$ can be replaced with the problem to check (7.67). Further, from (7.14) and (7.15) for $q = 0$, it follows that this problem is equivalent to the problem to show that

$$\frac{1}{\epsilon}(\partial_2 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) \to -(\partial_1 \mathbf{U}, \mathbf{n}) \text{ in } L^2(\Omega), \tag{7.68}$$

$$\frac{1}{\epsilon}(\partial_3 \mathbf{U}_\epsilon, \mathbf{g}_{1,\epsilon}) \to -(\partial_1 \mathbf{U}, \mathbf{b}) \text{ in } L^2(\Omega), \tag{7.69}$$

and these convergences are equivalent (using the definition (7.29) of the functions $\mathbf{U}_{*,\epsilon}$ and the fact that $\langle \mathbf{U}, \phi\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$, i.e. the definition of the function $\mathbf{U}_*$ in (2.17)) to the problem to verify that

$$(\mathbf{U}_{*,\epsilon}, \mathbf{b}_\epsilon) \to (\mathbf{U}_*, \mathbf{b}) \text{ in } L^2(\Omega), \tag{7.70}$$

$$(\mathbf{U}_{*,\epsilon}, \mathbf{n}_\epsilon) \to (\mathbf{U}_*, \mathbf{n}) \text{ in } L^2(\Omega). \tag{7.71}$$

The estimate

$$\|(\mathbf{U}_{*,\epsilon}, \mathbf{n}_\epsilon) - (\mathbf{U}_*, \mathbf{n})\|_2 \le \left(\int_\Omega |(\mathbf{U}_{*,\epsilon} - \mathbf{U}_*, \mathbf{n}_\epsilon)|^2\, dx\right)^{\frac{1}{2}}$$

$$+\left(\int_\Omega |(\mathbf{U}_*, \mathbf{n}_\epsilon - \mathbf{n})|^2\, dx\right)^{\frac{1}{2}} \le \|\mathbf{U}_{*,\epsilon} - \mathbf{U}\|_2$$

$$+\left(\int_0^l |\mathbf{n}_\epsilon - \mathbf{n}|^2 \|\mathbf{U}_*(x_1)\|_{2,S}^2\, dx_1\right)^{\frac{1}{2}} \tag{7.72}$$

and the similar arguments as in (7.49) enable us to assert that if we prove that

$$\mathbf{U}_{*,\epsilon} \to \mathbf{U}_* \text{ in } L^2(\Omega)^3, \tag{7.73}$$

then the convergences in (7.67) immediately follow from (7.68)–(7.72). To check (7.73), we use the inequality (C is independent of v)

$$\|v\|_2 \le C(\|v\|_{-1} + \|\nabla v\|_{-1}),\ \forall v \in L^2(\Omega) \tag{7.74}$$

(see [14, p. 189]). In the first step, we show that

$$\nabla \mathbf{U}_{*,\epsilon} \to \nabla \mathbf{U}_* \text{ in } H^{-1}(\Omega)^9. \tag{7.75}$$

Since we suppose that $\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon) \to \zeta$ in $L^2(\Omega)^9$, $\partial_j \frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon) \to \partial_j \zeta$ for $\epsilon \to 0$ in the space $L^2(0,l;H^{-1}(S)^9)$, $j=2,3$, and using (6.5)–(6.8) together with (7.37)–(7.39) and (7.60), we can deduce that

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon) \to (\partial_1 \mathbf{U}_*, \mathbf{t}) \text{ in } L^2(0,l;H^{-1}(S)), \tag{7.76}$$

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{n}_\epsilon) \to (\partial_1 \mathbf{U}_*, \mathbf{n}) \text{ in } L^2(0,l;H^{-1}(S)), \tag{7.77}$$

$$(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{b}_\epsilon) \to (\partial_1 \mathbf{U}_*, \mathbf{b}) \text{ in } L^2(0,l;H^{-1}(S)). \tag{7.78}$$

Since

$$\partial_1 \mathbf{U}_{*,\epsilon} = (\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon)\mathbf{t}_\epsilon + (\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{n}_\epsilon)\mathbf{n}_\epsilon + (\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{b}_\epsilon)\mathbf{b}_\epsilon$$

and since

$$\begin{aligned} &\|(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon)\mathbf{t}_\epsilon - (\partial_1 \mathbf{U}_*, \mathbf{t})\mathbf{t}\|_{L^2(0,l;H^{-1}(S))} \\ &\leq \|((\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{t}_\epsilon) - (\partial_1 \mathbf{U}_{*,}, \mathbf{t}))\mathbf{t}_\epsilon\|_{L^2(0,l;H^{-1}(S))} \\ &+ \left(\int_0^l |\mathbf{t}_\epsilon - \mathbf{t}|^2 \|(\partial_1 \mathbf{U}_*, \mathbf{t})(x_1)\|^2_{H^{-1}(S)}\, dx_1\right)^{\frac{1}{2}} \to 0 \end{aligned} \tag{7.79}$$

for $\epsilon \to 0$ as a consequence of (7.76) and the fact that $\mathbf{t}_\epsilon \to \mathbf{t}$ pointwisely in $[0,l]\setminus D$, and since we can easily modify the estimate (7.79) for the functions $(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{n}_\epsilon)\mathbf{n}_\epsilon$ and $(\partial_1 \mathbf{U}_{*,\epsilon}, \mathbf{b}_\epsilon)\mathbf{b}_\epsilon$, then we can conclude that

$$\partial_1 \mathbf{U}_{*,\epsilon} \to \partial_1 \mathbf{U}_* \text{ in } L^2(0,l;H^{-1}(S)^3) \tag{7.80}$$

and thus strongly in $H^{-1}(\Omega)^3$.

Further, we want to show that

$$\partial_j \mathbf{U}_{*,\epsilon} \to 0 \text{ in } H^{-1}(\Omega)^3,\ j=2,3, \tag{7.81}$$

for $\epsilon \to 0$. From (7.51)–(7.54), it follows that to prove (7.81) it remains to show that

$$\partial_j(\partial_1 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon \to 0 \text{ in } H^{-1}(\Omega)^3,\ j=2,3, \tag{7.82}$$

$$\partial_j(\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon \to 0 \text{ in } H^{-1}(\Omega)^3,\ j=2,3, \tag{7.83}$$

for $\epsilon \to 0$. The relations in (2.2) provide

$$\begin{aligned} \partial_j(\partial_1 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon &= \partial_j\partial_1(\mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon - \partial_j(\mathbf{U}_\epsilon, \mathbf{n}'_\epsilon)\mathbf{b}_\epsilon \\ &= \partial_1(\partial_j \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon + \beta_\epsilon(\partial_j \mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\mathbf{b}_\epsilon + \gamma_\epsilon(\partial_j \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{b}_\epsilon \end{aligned} \tag{7.84}$$

and analogously

$$\partial_j(\partial_1 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon = \partial_1(\partial_j \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon + \alpha_\epsilon(\partial_j \mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\mathbf{n}_\epsilon - \gamma_\epsilon(\partial_j \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{n}_\epsilon, \tag{7.85}$$

$j=2,3$. We get from the convergences (7.18)–(7.19), (7.21) and (7.45) together with (3.44) and with the fact that $|\mathbf{n}_\epsilon| = |\mathbf{b}_\epsilon| = 1$, that

$$\beta_\epsilon(\partial_j \mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\mathbf{b}_\epsilon \to 0,\ \gamma_\epsilon(\partial_j \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{b}_\epsilon \to 0 \text{ in } L^2(\Omega)^3,$$

$$\alpha_\epsilon(\partial_j \mathbf{U}_\epsilon, \mathbf{t}_\epsilon)\mathbf{n}_\epsilon \to 0,\ \gamma_\epsilon(\partial_j \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{n}_\epsilon \to 0 \text{ in } L^2(\Omega)^3.$$

Now, we prove for instance that

$$\partial_1(\partial_j \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon \to 0 \text{ in } H^{-1}(\Omega)^3,\ j = 2, 3. \tag{7.86}$$

Then the proof of the convergence

$$\partial_1(\partial_j \mathbf{U}_\epsilon, \mathbf{b}_\epsilon)\mathbf{n}_\epsilon \to 0 \text{ in } H^{-1}(\Omega)^3,\ j = 2, 3, \tag{7.87}$$

proceeds analogously.

Let $\varphi \in H_0^1(\Omega)$ be an arbitrary function. Then (since $\mathbf{b}_\epsilon \in C^\infty([0, l])^3$) using (3.44) and (2.2), we deduce the estimate

$$\left|\int_\Omega (\partial_j \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\partial_1(\mathbf{b}_\epsilon \varphi)\ dx\right| \le \left|\int_\Omega (\partial_j \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}'_\epsilon \varphi\ dx\right|$$

$$+\left|\int_\Omega (\partial_j \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\mathbf{b}_\epsilon \partial_1 \varphi\ dx\right| \le \frac{C}{\epsilon^r}\|(\partial_j \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\|_2 \|\varphi\|_2 + \|(\partial_j \mathbf{U}_\epsilon, \mathbf{n}_\epsilon)\|_2 \|\partial_1 \varphi\|_2$$

for $\epsilon \in (0, 1)$, $r \in (0, \frac{1}{3})$, $j = 2, 3$. Then the convergence (7.86) easily follows from (7.18) for $j = 2$ and from (7.45) for $j = 3$. Hence we get (7.81), which together with (7.80) yields (7.75). The convergences (7.58) and (7.75) together with (7.74) lead to the strong convergence

$$\mathbf{U}_{*,\epsilon} \to \mathbf{U}_* \text{ in } L^2(\Omega)^3.$$

□

8 The main result

In this section, we pass from the three-dimensional model to the asymptotic one-dimensional model and our main result is stated and proved.

We suppose that $\mathbf{F}_\epsilon = \epsilon^2 \mathbf{F}$, $\mathbf{F} \in L^2(\Omega)^3$, and $\mathbf{G}_\epsilon = \epsilon^3 \mathbf{G}$, $\mathbf{G} \in L^2(0, l; L^2(\partial S)^3)$, for $\epsilon \in (0, 1)$ in the scaled equation (6.4). Using (2.15) and (3.44), we deduce the convergence

$$\epsilon\sqrt{\nu_i o^{ij,\epsilon}\nu_j} \to \sqrt{\nu_2^2 + \nu_3^2} = 1 \text{ in } C(\overline{\Omega}), \tag{8.1}$$

because $\nu_1 = 0$ (the domain is $[0, l] \times S$). Here we "use" one power of ϵ from the above assumption on the function $\mathbf{G}_\epsilon$. Dividing by ϵ^2 in (6.4) after substitution of the assumptions $\mathbf{F}_\epsilon = \epsilon^2 \mathbf{F}$ and $\mathbf{G}_\epsilon = \epsilon^3 \mathbf{G}$, (3.47), (6.4), (6.9), (8.1) and Theorem 7.1, we obtain the estimate

$$\|\mathbf{U}_\epsilon\|_{1,2}^2 \le \frac{C^2}{\epsilon^2}\|\omega^\epsilon(\mathbf{U}_\epsilon)\|_2^2 \le \frac{C^2}{C_0 \epsilon^2}\int_\Omega A_\epsilon^{ijkl}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon)\omega_{ij}^\epsilon(\mathbf{U}_\epsilon) d_\epsilon\ dx$$

$$= \frac{C^2}{C_0}\left(\int_\Omega (\mathbf{F}, \mathbf{U}_\epsilon) d_\epsilon\ dx + \int_0^l \int_{\partial S} (\mathbf{G}, \mathbf{U}_\epsilon) d_\epsilon \epsilon \sqrt{\nu_j o^{ij,\epsilon}\nu_j}\ dS dx_1\right)$$

$$\leq \frac{C^2 C_1}{C_0}\Big(\|\mathbf{F}\|_2\|\mathbf{U}_\epsilon\|_{1,2} + \|\mathbf{G}\|_{2,[0,l]\times\partial S}\|\mathbf{U}_\epsilon\|_{L^2(0,l;L^2(\partial S)^3)}\Big) \leq C_3\|\mathbf{U}_\epsilon\|_{1,2},$$

for all $\epsilon \in (0,1)$, because $\mathbf{U}_\epsilon \in V(\Omega)$ and thus $\mathbf{U}_\epsilon \in L^2(0,l;L^2(\partial S)^3)$ in the sense of the trace. By the above relations (passing to a subsequence), we have that

$$\mathbf{U}_{\epsilon_n} \rightharpoonup \mathbf{U} \text{ in } H^1(\Omega)^3, \tag{8.2}$$

$$\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{U}_{\epsilon_n}) \rightharpoonup \zeta \text{ in } L^2(\Omega)^9 \tag{8.3}$$

for $\epsilon_n \to 0$, where $\mathbf{U} \in H_0^1(0,l)^3$ according to Proposition 7.2.

To find the form of the tensor ζ, we must obtain the corresponding equations for its components.

Proposition 8.1 *Let the tensor ζ be the limit determined by* (8.3). *Then it satisfies the equation*

$$\int_\Omega A_0^{ijkl}\zeta_{kl}\theta^0_{ij}(\mathbf{V})\ dx = 0,\ \forall \mathbf{V} \in L^2(0,l;H^1(S)^3), \tag{8.4}$$

where the tensor $\theta^0(\mathbf{V})$ is defined by

$$\theta^0(\mathbf{V}) = \begin{pmatrix} 0 & \frac{(\partial_2\mathbf{V},\mathbf{t})}{2} & \frac{(\partial_3\mathbf{V},\mathbf{t})}{2} \\ \frac{(\partial_2\mathbf{V},\mathbf{t})}{2} & (\partial_2\mathbf{V},\mathbf{n}) & \frac{(\partial_2\mathbf{V},\mathbf{b})+(\partial_3\mathbf{V},\mathbf{n})}{2} \\ \frac{(\partial_3\mathbf{V},\mathbf{t})}{2} & \frac{(\partial_2\mathbf{V},\mathbf{b})+(\partial_3\mathbf{V},\mathbf{n})}{2} & (\partial_3\mathbf{V},\mathbf{b}). \end{pmatrix}. \tag{8.5}$$

P r o o f: In the proof, we will use ϵ instead of ϵ_n to simplify the notation. Letting $\epsilon \to 0$, we want to pass from the equation

$$\int_\Omega A_\epsilon^{ijkl}\frac{1}{\epsilon}\omega^\epsilon_{kl}(\mathbf{U}_\epsilon)\epsilon\omega^\epsilon_{ij}(\mathbf{V})d_\epsilon\ dx = \epsilon^2\int_\Omega(\mathbf{F},\mathbf{V})d_\epsilon\ dx$$

$$+\epsilon^2\int_0^l\int_{\partial S}(\mathbf{G},\mathbf{V})d_\epsilon\epsilon\sqrt{\nu_j o^{ij,\epsilon}\nu_j}\ dSdx_1,\ \forall\mathbf{V} \in V(\Omega),$$

to the equation

$$\int_\Omega A_0^{ijkl}\zeta_{kl}\theta^0_{ij}(\mathbf{V})\ dx = 0,\ \forall\mathbf{V} \in V(\Omega), \tag{8.6}$$

where the tensor $\theta^0(\mathbf{V})$ is defined by (8.5). We show that the tensor $\theta^0(\mathbf{V})$ is the limit state of the tensors $\theta^\epsilon(\mathbf{V}) + \epsilon\kappa^\epsilon(\mathbf{V})$ for $\epsilon \to 0$ (see (6.5)–(6.8)). Since the functions $\mathbf{g}_{1,\epsilon}$, $\mathbf{n}_\epsilon$ and $\mathbf{b}_\epsilon$ are bounded in $L^\infty(\Omega)^3$ or $L^\infty(0,l)^3$, it is easily seen that $\epsilon\kappa^\epsilon(\mathbf{V}) \to 0$ in $L^2(\Omega)^9$ (see (6.8)). Thus it remains to show that $\theta^\epsilon(\mathbf{V}) \to \theta^0(\mathbf{V})$ in $L^2(\Omega)^9$ for $\epsilon \to 0$. Since we know that $\mathbf{g}_{1,\epsilon} \to \mathbf{t}$ and $\mathbf{n}_\epsilon \to \mathbf{n}$, $\mathbf{b}_\epsilon \to \mathbf{b}$ pointwisely in $\Omega \setminus (S \times D)$ or in $[0,l] \setminus D$, respectively, and are bounded in $L^\infty(\Omega)^3$ or $L^\infty(0,l)^3$, respectively, we can combine (6.6)–(6.7) with the technique we have used in (7.72) to prove the above mentioned strong convergence and thus we omit the detailed proof.

Using the definition (see (2.10) and (6.2)) of the tensor $(A_\epsilon^{ijkl})_{i,j,k,l=1}^3$, we can easily check by (2.6)–(2.12) that

$$A_\epsilon^{ijkl} \to A_0^{ijkl} \text{ in } C(\overline{\Omega}), \text{ where } A_0^{ijkl} = \lambda\delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \tag{8.7}$$

for $i, j, k, l = 1, 2, 3$. The rest of the proof follows from density of the space $V(\Omega)$ in $L^2(0, l; H^1(S)^3)$ and from (8.5) and (8.6). □

Now, we introduce the following notation:

$$\zeta_{22}^H = \zeta_{22} + \frac{1}{2}\frac{\lambda}{\lambda+\mu}\zeta_{11}, \ \zeta_{33}^H = \zeta_{33} + \frac{1}{2}\frac{\lambda}{\lambda+\mu}\zeta_{11}, \ \zeta_{23}^H = \zeta_{23}. \tag{8.8}$$

Corollary 8.2 *We have*

$$\int_S \zeta_{12} = \int_S \zeta_{13} = \int_S \zeta_{12}x_2 = \int_S \zeta_{13}x_3 = \int_S [\zeta_{12}x_3 + \zeta_{13}x_2] = 0, \tag{8.9}$$

$$\int_S \zeta_{23}^H = \int_S \zeta_{23}^H x_2 = \int_S \zeta_{23}^H x_3 = 0 \tag{8.10}$$

and

$$\int_S (\zeta_{22}^H + \zeta_{33}^H) = \int_S (\zeta_{22}^H + \zeta_{33}^H)x_2 = \int_S (\zeta_{22}^H + \zeta_{33}^H)x_3 = 0. \tag{8.11}$$

P r o o f: Let $v \in L^2(0, l)$ be arbitrary but fixed function and $\mathbf{V} = v\mathbf{t}$. Testing equation (8.4) with functions $\mathbf{V}x_2$, $\mathbf{V}x_3$, $\mathbf{V}x_2^2/2$, $\mathbf{V}x_3^2/2$ and $\mathbf{V}x_2x_3$, we can derive (8.9).

Let us take now some arbitrary function $\mathbf{V} \in L^2(0, l; H^1(S)^3)$ such that $(\mathbf{V}, \mathbf{t}) = (\mathbf{V}, \mathbf{b}) = 0$. Then we can derive from (8.4) and (8.5) that

$$\int_\Omega [(\lambda(\zeta_{11} + \zeta_{22} + \zeta_{33}) + 2\mu\zeta_{22})(\partial_2\mathbf{V}, \mathbf{n}) + 2\mu\zeta_{23}(\partial_3\mathbf{V}, \mathbf{n})] \, dx = 0. \tag{8.12}$$

Analogously we deduce for arbitrary functions $\mathbf{V} \in L^2(0, l; H^1(S)^3)$, which satisfy $(\mathbf{V}, \mathbf{t}) = (\mathbf{V}, \mathbf{n}) = 0$, that

$$\int_\Omega [(\lambda(\zeta_{11} + \zeta_{22} + \zeta_{33}) + 2\mu\zeta_{33})(\partial_3\mathbf{V}, \mathbf{b}) + 2\mu\zeta_{23}(\partial_2\mathbf{V}, \mathbf{b})] \, dx = 0. \tag{8.13}$$

After substitution of (8.8) we can transform (8.12) and (8.13) as

$$\int_\Omega [(\lambda(\zeta_{22}^H + \zeta_{33}^H) + 2\mu\zeta_{22}^H)(\partial_2\mathbf{V}, \mathbf{n}) + 2\mu\zeta_{23}^H(\partial_3\mathbf{V}, \mathbf{n})] \, dx = 0 \tag{8.14}$$

and

$$\int_\Omega [(\lambda(\zeta_{22}^H + \zeta_{33}^H) + 2\mu\zeta_{33}^H)(\partial_3\mathbf{V}, \mathbf{b}) + 2\mu\zeta_{23}^H(\partial_2\mathbf{V}, \mathbf{b})] \, dx = 0, \tag{8.15}$$

respectively. Taking $\mathbf{V}x_3$, $\mathbf{V}x_3^2/2$ and $\mathbf{V}x_2^2/2$, where $\mathbf{V} = v\mathbf{n}$ or $\mathbf{V} = v\mathbf{b}$, as test functions in (8.14) and (8.15), respectively, yields (8.10). In an analogous way, we substitute the functions $\mathbf{V}x_2$, $\mathbf{V}x_3$, $\mathbf{V}x_2^2/2$, $\mathbf{V}x_2x_3$ and $\mathbf{V}x_2x_3$, $\mathbf{V}x_3^2/2$, where $\mathbf{V} = v\mathbf{n}$ or $\mathbf{V} = v\mathbf{b}$, to (8.14) and (8.15), respectively, to derive (8.11). $\square$

If we define the vector $\boldsymbol{\eta} \in L^2(\Omega)^2$ by $\boldsymbol{\eta} = \langle \zeta_{12}, \zeta_{13} \rangle$, then the equations (8.4) after putting $\mathbf{V} = \varphi\mathbf{t}$, $\varphi \in L^2(0,l; H^1(S))$, and (7.5) can be rewritten in the form

$$\int_\Omega (\boldsymbol{\eta}, \nabla_{23}\varphi)_2 \; dx = 0, \;\; \forall \varphi \in L^2(0,l; H^1(S)), \tag{8.16}$$

$$\int_\Omega (\boldsymbol{\eta}, \mathrm{rot}_{23}\psi)_2 \; dx = \int_\Omega (\mathbf{U}'_*, \mathbf{t})\psi \; dx, \;\; \forall \psi \in H^1_0(\Omega), \tag{8.17}$$

where we have denoted $\nabla_{23}\varphi = \langle \partial_2\varphi, \partial_3\varphi \rangle$, $\mathrm{rot}_{23}\psi = \langle -\partial_3\psi, \partial_2\psi \rangle$ and $(\cdot,\cdot)_2$ means the scalar product in the usual two dimensional Euclidean space $\mathbb{R}^2$.

Lemma 8.3 *Let S be a simply connected domain and let $\partial S \in C^1$. The system* (8.16), (8.17) *has unique solution in $L^2(\Omega)^2$, given by*

$$\boldsymbol{\eta} = \langle \zeta_{12}, \zeta_{13} \rangle = -\frac{1}{2}(\mathbf{U}'_*, \mathbf{t})\langle \partial_2 p - x_3, \partial_3 p + x_2 \rangle \tag{8.18}$$

where the function $p \in H^1(S)$ is the unique solution to the Neumann problem

$$\int_S [(\partial_2 p - x_3)\partial_2 r + (\partial_3 p + x_2)\partial_3 r] \; dx_2dx_3 = 0, \;\; \int_S p \; dx_2dx_3 = 0, \tag{8.19}$$

for all $r \in H^1(S)$.

P r o o f: After substitution of (8.18) to (8.16) and (8.17), we obtain using (8.19) that

$$\int_\Omega (\boldsymbol{\eta}, \nabla_{23}\varphi)_2 \; dx = -\frac{1}{2}\int_\Omega (\mathbf{U}'_*, \mathbf{t})(\partial_2 p - x_3)\partial_2\varphi \; dx - \frac{1}{2}\int_\Omega (\mathbf{U}'_*, \mathbf{t})(\partial_3 p + x_2)\partial_3\varphi \; dx$$

$$= -\frac{1}{2}\int_0^l (\mathbf{U}'_*, \mathbf{t}) \int_S [(\partial_2 p - x_3)\partial_2\varphi + (\partial_3 p + x_2)\partial_3\varphi] \; dx_2dx_3dx_1 \overset{(8.19)}{=} 0$$

and

$$\int_\Omega (\boldsymbol{\eta}, \mathrm{rot}_{23}\psi)_2 \; dx = \frac{1}{2}\int_\Omega (\mathbf{U}'_*, \mathbf{t})(\partial_2 p - x_3)\partial_3\psi \; dx - \frac{1}{2}\int_\Omega (\mathbf{U}'_*, \mathbf{t})(\partial_3 p + x_2)\partial_2\psi \; dx$$

$$= -\frac{1}{2}\int_0^l (\mathbf{U}'_*, \mathbf{t})\left[\int_S \partial_3 p\partial_2\psi - \partial_2 p\partial_3\psi \; dx_2dx_3 + \int_S x_3\partial_3\psi + x_2\partial_2\psi \; dx_2dx_3\right] dx_1$$

$$= \int_\Omega (\mathbf{U}'_*, \mathbf{t})\psi \; dx,$$

for all $\psi \in C_0^\infty(\Omega)$, which implies that $\psi(x_1,\cdot,\cdot) \in C_0^\infty(S)$ for all $x_1 \in (0,l)$. Thus by density the relation remains valid for all $\psi \in H_0^1(\Omega)$.

To prove uniqueness, we assume that there exist two solutions $\boldsymbol{\eta}_i \in L^2(\Omega)^2$, $i = 1,2$. Taking $\varphi = s\widehat{\varphi}$ in (8.16) and $\psi = s\widehat{\psi}$ in (8.17) for all $s \in C_0^\infty(0,l)$, $\widehat{\varphi} \in H^1(S)$ and $\widehat{\psi} \in H_0^1(S)$, it is easy to verify that the function $\boldsymbol{\eta}_s = \langle \eta_{1,s}, \eta_{2,s}\rangle = \int_0^l s\boldsymbol{\eta}\ dx_1$, where $\boldsymbol{\eta} = \boldsymbol{\eta}_1 - \boldsymbol{\eta}_2$, satisfies the equations

$$\int_S (\boldsymbol{\eta}_s, \nabla_{23}\widehat{\varphi})_2\ dx_2dx_3 = 0 \text{ and } \int_S (\boldsymbol{\eta}_s, \mathrm{rot}_{23}\widehat{\psi})_2\ dx_2dx_3 = 0. \tag{8.20}$$

Let us define the vector functions $\widehat{\boldsymbol{\eta}}_s = (0, \eta_{1,s}, \eta_{2,s})$ and $\widehat{\boldsymbol{\psi}} = (-\check{\psi}, \psi_1, \psi_2)$, where the functions $\check{\psi}$, ψ_1, $\psi_2 \in C_0^\infty(\Omega)$ are arbitrary. Since the function $\widehat{\boldsymbol{\eta}}_s$ is defined only on S, we can deduce from (8.20) that

$$\int_0^l \int_S (\widehat{\boldsymbol{\eta}}_s, \mathrm{rot}\,\widehat{\boldsymbol{\psi}})\ dx = \int_0^l \int_S (\boldsymbol{\eta}_s, \mathrm{rot}_{23}\check{\psi}(x_1))_2\ dx_2dx_3dx_1 = 0.$$

Hence, we can easily derive that $\mathrm{rot}\,\widehat{\boldsymbol{\eta}}_s = 0$ in $\mathcal{D}'(\Omega)$. Since S is simply connected, then $\Omega = [0,l]\times S$ is simply connected as well and there exists a function $h_s \in H^1(\Omega)$, unique up to a constant, such that $\widehat{\boldsymbol{\eta}}_s = \nabla h_s$ (see [5]), which means

$$\partial_1 h_s = 0,\ \partial_2 h_s = \eta_{1,s},\ \partial_3 h_s = \eta_{2,s},$$

and hence we get that $h_s \in H^1(S)$ and $\boldsymbol{\eta}_s = \nabla_{23}h_s$. After substitution $\widehat{\varphi} = h_s$ to (8.20), it follows that $\|\nabla_{23}h_s\|_2 = 0$. Hence $\boldsymbol{\eta}_s = \mathbf{0}$ for all $s \in L^2(0,l)$ which implies $\boldsymbol{\eta} = \mathbf{0}$. $\square$

Now, we derive the asymptotic one-dimensional model. First we introduce some constants:

$$I_{x_2^2} = \int_S x_2^2\ dx_2dx_3,\ I_{x_3^2} = \int_S x_3^2\ dx_2dx_3, \tag{8.21}$$

$$E = \mu\frac{3\lambda + 2\mu}{\lambda + \mu},\ K = \int_S [(\partial_2 p - x_3)^2 + (\partial_3 p + x_2)^2]\ dx_2dx_3, \tag{8.22}$$

where $p \in H^1(S)$ is the unique solution of the Neumann problem (8.19).

Lemma 8.4 *Let $\{\mathbf{U}_{\epsilon_n}\}_{n=1}^\infty$, $\epsilon_n \to 0$, be a subsequence of the solutions of the problem* (6.4) *with $\mathbf{F}_{\epsilon_n} = \epsilon_n^2\mathbf{F}$, $\mathbf{G}_{\epsilon_n} = \epsilon_n^3\mathbf{G}$, satisfying* (8.2) *and* (8.3)*. Then the limit $\langle \mathbf{U}, \phi\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ obtained in Proposition* 7.2 *generates the function $\mathbf{U}_*$, which satisfies the equation*

$$\int_0^l E[I_{x_2^2}(\mathbf{U}_*', \mathbf{b})(\mathbf{V}_*', \mathbf{b}) + I_{x_3^2}(\mathbf{U}_*', \mathbf{n})(\mathbf{V}_*', \mathbf{n})]\ dx_1$$

$$+ \int_0^l \mu K(\mathbf{U}_*', \mathbf{t})(\mathbf{V}_*', \mathbf{t})\ dx_1 = \int_0^l (\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}, \mathbf{V})\ dx_1 \tag{8.23}$$

for all functions $\mathbf{V}_* \in H_0^1(0,l)^3$ generated by any arbitrary couple $\langle \mathbf{V}, \psi\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ (see (2.17)*), where $\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}(x_1) = \int_S \mathbf{F}(x_1)\ dx_2dx_3 + \int_{\partial S} \mathbf{G}(x_1)\ dS$, $x_1 \in [0,l]$.*

P r o o f: In the proof, we will use ϵ instead of ϵ_n to simplify the notation. Let $\langle \mathbf{V}, \psi\rangle$ be an arbitrary couple from the space $\mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ and let the function $\mathbf{W} \in L^2(\Omega)^3$ be defined by

$$\mathbf{W}(x_1,x_2,x_3) = -\Big((\mathbf{V}'(x_1),\mathbf{n}(x_1))x_2 + (\mathbf{V}'(x_1),\mathbf{b}(x_1))x_3\Big)\mathbf{t}(x_1)$$

$$-x_3\psi(x_1)\mathbf{n}(x_1) + x_2\psi(x_1)\mathbf{b}(x_1) \text{ for } (x_1,x_2,x_3)\in\Omega. \tag{8.24}$$

Proposition 4.2 enables us to approximate the couple $\langle \mathbf{V}, \psi\rangle$ with couples $\langle \mathbf{V}_\epsilon, \psi_\epsilon\rangle \in \mathcal{V}_0^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0,l)$ satisfying $\mathbf{V}_\epsilon \in C_0^\infty(0,l)^3$ and $\psi_\epsilon \in C_0^\infty(0,l)$. In an analogous way as in (8.24), we define the functions $\mathbf{W}_\epsilon \in C^\infty(\overline{\Omega})^3$ by

$$\mathbf{W}_\epsilon(x_1,x_2,x_3) = -\Big((\mathbf{V}_\epsilon'(x_1),\mathbf{n}_\epsilon(x_1))x_2 + (\mathbf{V}_\epsilon'(x_1),\mathbf{b}_\epsilon(x_1))x_3\Big)\mathbf{t}_\epsilon(x_1)$$

$$-x_3\psi_\epsilon(x_1)\mathbf{n}_\epsilon(x_1) + x_2\psi_\epsilon(x_1)\mathbf{b}_\epsilon(x_1) \tag{8.25}$$

for $(x_1,x_2,x_3)\in\Omega$.

Let us define the function $\widehat{\mathbf{V}}_\epsilon$ by

$$\widehat{\mathbf{V}}_\epsilon = \mathbf{V}_\epsilon + \epsilon\mathbf{W}_\epsilon \in C^\infty(\overline{\Omega})^3 \cap V(\Omega).$$

After substitution to (6.5)–(6.8) we get by using $(\mathbf{V}_\epsilon', \mathbf{t}_\epsilon) = 0$ (see (2.17))

$$\omega_{11}^\epsilon(\widehat{\mathbf{V}}_\epsilon) = \kappa_{11}^\epsilon(\mathbf{V}_\epsilon) + \epsilon\kappa_{11}^\epsilon(\mathbf{W}_\epsilon)$$

$$= \Big((1-\epsilon x_2\beta_\epsilon - \epsilon x_3\alpha_\epsilon)(\mathbf{V}_\epsilon',\mathbf{t}_\epsilon) + \epsilon x_3\gamma_\epsilon(\mathbf{V}_\epsilon',\mathbf{n}_\epsilon) - \epsilon x_2\gamma_\epsilon(\mathbf{V}_\epsilon',\mathbf{b}_\epsilon)\Big)$$

$$+\epsilon\Big((1-\epsilon x_2\beta_\epsilon - \epsilon x_3\alpha_\epsilon)(\partial_1\mathbf{W}_\epsilon,\mathbf{t}_\epsilon) + \epsilon x_3\gamma_\epsilon(\partial_1\mathbf{W}_\epsilon,\mathbf{n}_\epsilon) - \epsilon x_2\gamma_\epsilon(\partial_1\mathbf{W}_\epsilon,\mathbf{b}_\epsilon)\Big)$$

$$= \Big(\epsilon x_3\gamma_\epsilon(\mathbf{V}_\epsilon',\mathbf{n}_\epsilon) - \epsilon x_2\gamma_\epsilon(\mathbf{V}_\epsilon',\mathbf{b}_\epsilon)\Big)$$

$$+\epsilon\Big((1-\epsilon x_2\beta_\epsilon - \epsilon x_3\alpha_\epsilon)(\partial_1\mathbf{W}_\epsilon,\mathbf{t}_\epsilon) + \epsilon x_3\gamma_\epsilon(\partial_1\mathbf{W}_\epsilon,\mathbf{n}_\epsilon) - \epsilon x_2\gamma_\epsilon(\partial_1\mathbf{W}_\epsilon,\mathbf{b}_\epsilon)\Big) = Z$$

(new notation). Since

$$\epsilon(1-\epsilon x_2\beta_\epsilon - \epsilon x_3\alpha_\epsilon)(\partial_1\mathbf{W}_\epsilon,\mathbf{t}_\epsilon) = -\epsilon(1-\epsilon x_2\beta_\epsilon - \epsilon x_3\alpha_\epsilon)(x_2(\mathbf{V}_\epsilon',\mathbf{n}_\epsilon)' + x_3(\mathbf{V}_\epsilon',\mathbf{b}_\epsilon)')$$

$$-\epsilon(1-\epsilon x_2\beta_\epsilon - \epsilon x_3\alpha_\epsilon)x_3\psi_\epsilon(\mathbf{n}_\epsilon',\mathbf{t}_\epsilon) + \epsilon(1-\epsilon x_2\beta_\epsilon - \epsilon x_3\alpha_\epsilon)x_2\psi_\epsilon(\mathbf{b}_\epsilon',\mathbf{t}_\epsilon)$$

$$\stackrel{(2.2)}{=} -\epsilon(1-\epsilon x_2\beta_\epsilon - \epsilon x_3\alpha_\epsilon)(x_2(\mathbf{V}_\epsilon',\mathbf{n}_\epsilon)' + x_3(\mathbf{V}_\epsilon',\mathbf{b}_\epsilon)')$$

$$+\epsilon(1-\epsilon x_2\beta_\epsilon - \epsilon x_3\alpha_\epsilon)\beta_\epsilon x_3\psi_\epsilon - \epsilon(1-\epsilon x_2\beta_\epsilon - \epsilon x_3\alpha_\epsilon)\alpha_\epsilon x_2\psi_\epsilon$$

$$= \epsilon\Big(-x_2(\mathbf{V}_\epsilon',\mathbf{n}_\epsilon)' - x_3(\mathbf{V}_\epsilon',\mathbf{b}_\epsilon)' + \beta_\epsilon x_3\psi_\epsilon - \alpha_\epsilon x_2\psi_\epsilon\Big)$$

$$+\epsilon^2(\beta_\epsilon x_2 + \alpha_\epsilon x_3)(x_2(\mathbf{V}_\epsilon',\mathbf{n}_\epsilon)' + x_3(\mathbf{V}_\epsilon',\mathbf{b}_\epsilon)' - \beta_\epsilon x_3\psi_\epsilon + \alpha_\epsilon x_2\psi_\epsilon).$$

Then

$$Z = \epsilon\Big(\gamma_\epsilon x_3(\mathbf{V}_\epsilon',\mathbf{n}_\epsilon) - \gamma_\epsilon x_2(\mathbf{V}_\epsilon',\mathbf{b}_\epsilon)\Big) + \epsilon\Big(-x_2(\mathbf{V}_\epsilon',\mathbf{n}_\epsilon)' - x_3(\mathbf{V}_\epsilon',\mathbf{b}_\epsilon)' + \beta_\epsilon x_3\psi_\epsilon$$

$$-\alpha_\epsilon x_2\psi_\epsilon\Big) + \epsilon^2\Big((\beta_\epsilon x_2 + \alpha_\epsilon x_3)(x_2(\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon)' + x_3(\mathbf{V}'_\epsilon, \mathbf{b}_\epsilon)' - \beta_\epsilon x_3\psi_\epsilon + \alpha_\epsilon x_2\psi_\epsilon)$$

$$+\gamma_\epsilon x_3(\partial_1\mathbf{W}_\epsilon, \mathbf{n}_\epsilon) - \gamma_\epsilon x_2(\partial_1\mathbf{W}_\epsilon, \mathbf{b}_\epsilon)\Big) = \epsilon x_2\Big(-(\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon)' - \alpha_\epsilon\psi_\epsilon - \gamma_\epsilon(\mathbf{V}'_\epsilon, \mathbf{b}_\epsilon)\Big)$$

$$+\epsilon x_3\Big(-(\mathbf{V}'_\epsilon, \mathbf{b}_\epsilon)' + \beta_\epsilon\psi_\epsilon + \gamma_\epsilon(\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon)\Big) + B^{11}_\epsilon,$$

where

$$B^{11}_\epsilon = \epsilon^2\Big((\beta_\epsilon x_2 + \alpha_\epsilon x_3)(x_2(\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon)' + x_3(\mathbf{V}'_\epsilon, \mathbf{b}_\epsilon)' - \beta_\epsilon x_3\psi_\epsilon + \alpha_\epsilon x_2\psi_\epsilon)$$

$$+\gamma_\epsilon x_3(\partial_1\mathbf{W}_\epsilon, \mathbf{n}_\epsilon) - \gamma_\epsilon x_2(\partial_1\mathbf{W}_\epsilon, \mathbf{b}_\epsilon)\Big).$$

Since $\langle \mathbf{V}_\epsilon, \psi_\epsilon\rangle \in \mathcal{V}_0^{\mathbf{t}_\epsilon,\mathbf{n}_\epsilon,\mathbf{b}_\epsilon}(0,l)$, (2.17) and (5.5)–(5.6) enable us to introduce the notation

$$v^\epsilon_{*,1} = (\mathbf{V}_{*,\epsilon}, \mathbf{t}_\epsilon) = -\psi_\epsilon,\ \ v^\epsilon_{*,2} = (\mathbf{V}_{*,\epsilon}, \mathbf{n}_\epsilon) = (\mathbf{V}'_\epsilon, \mathbf{b}_\epsilon),$$
$$v^\epsilon_{*,3} = (\mathbf{V}_{*,\epsilon}, \mathbf{b}_\epsilon) = -(\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon).$$

Then

$$Z \overset{(5.8),(5.9)}{=} \epsilon(x_2(\mathbf{V}'_{*,\epsilon}, \mathbf{b}_\epsilon) - x_3(\mathbf{V}'_{*,\epsilon}, \mathbf{n}_\epsilon)) + B^{11}_\epsilon.$$

Since $\mathbf{V}_\epsilon \in C_0^\infty(0,l)$, then from (6.6) it follows that $\frac{1}{\epsilon}\theta^\epsilon_{12}(\mathbf{V}_\epsilon) = 0$ and thus

$$\omega^\epsilon_{12}(\widehat{\mathbf{V}}_\epsilon) \overset{(6.5)}{=} \kappa^\epsilon_{12}(\mathbf{V}_\epsilon) + \theta^\epsilon_{12}(\mathbf{W}_\epsilon) + \epsilon\kappa^\epsilon_{12}(\mathbf{W}_\epsilon)$$

$$\overset{(6.6)-(6.8)}{=} \frac{1}{2}\Big((\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon) + (1 - \epsilon x_2\beta_\epsilon - \epsilon x_3\alpha_\epsilon)(\partial_2\mathbf{W}_\epsilon, \mathbf{t}_\epsilon) + \epsilon x_3\gamma_\epsilon(\partial_2\mathbf{W}_\epsilon, \mathbf{n}_\epsilon)$$

$$-\epsilon x_2\gamma_\epsilon(\partial_2\mathbf{W}_\epsilon, \mathbf{b}_\epsilon) + \epsilon(\partial_1\mathbf{W}_\epsilon, \mathbf{n}_\epsilon)\Big) = \widehat{Z}$$

(new notation). We compute each term

$$(1 - \epsilon x_2\beta_\epsilon - \epsilon x_3\alpha_\epsilon)(\partial_2\mathbf{W}_\epsilon, \mathbf{t}_\epsilon) = -(1 - \epsilon x_2\beta_\epsilon - \epsilon x_3\alpha_\epsilon)(\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon),$$

$$\epsilon x_3\gamma_\epsilon(\partial_2\mathbf{W}_\epsilon, \mathbf{n}_\epsilon) = 0,\ \ -\epsilon x_2\gamma_\epsilon(\partial_2\mathbf{W}_\epsilon, \mathbf{b}_\epsilon) = -\epsilon x_2\gamma_\epsilon\psi_\epsilon,$$

$$\epsilon(\partial_1\mathbf{W}_\epsilon, \mathbf{n}_\epsilon) = -\epsilon\Bigg(\Big(x_2(\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon) + x_3(\mathbf{V}'_\epsilon, \mathbf{b}_\epsilon)\Big)(\mathbf{t}'_\epsilon, \mathbf{n}_\epsilon)$$

$$-x_3\psi'_\epsilon + x_2\psi_\epsilon(\mathbf{b}'_\epsilon, \mathbf{n}_\epsilon)\Bigg) = -\epsilon\beta_\epsilon\Big(x_2(\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon) + x_3(\mathbf{V}'_\epsilon, \mathbf{b}_\epsilon)\Big) - \epsilon x_3\psi'_\epsilon + \epsilon x_2\gamma_\epsilon\psi_\epsilon,$$

and we can conclude that

$$\widehat{Z} = \frac{1}{2}\Big((\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon) + -(\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon) + \epsilon x_2\beta_\epsilon(\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon) + \epsilon x_3\alpha_\epsilon(\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon)$$

$$-\epsilon x_2\gamma_\epsilon\psi_\epsilon - \epsilon x_2\beta_\epsilon(\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon) - \epsilon x_3\beta_\epsilon(\mathbf{V}'_\epsilon, \mathbf{b}_\epsilon) - \epsilon x_3\psi'_\epsilon + \epsilon x_2\gamma_\epsilon\psi_\epsilon\Big)$$

$$= \frac{\epsilon x_3}{2}\Big(-\psi'_\epsilon + \alpha_\epsilon(\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon) - \beta_\epsilon(\mathbf{V}'_\epsilon, \mathbf{b}_\epsilon)\Big).$$

Using the functions $v^\epsilon_{*,i}$, $i=1,2,3$, defined above, we get from (5.7)

$$\widehat{Z} = \frac{\epsilon x_3}{2}(\mathbf{V}'_{*,\epsilon}, \mathbf{t}_\epsilon).$$

Analogously we can derive that $\frac{1}{\epsilon}\theta^\epsilon_{13}(\mathbf{V}_\epsilon) = 0$ and thus

$$\omega^\epsilon_{13}(\widehat{\mathbf{V}}_\epsilon) = \kappa^\epsilon_{13}(\mathbf{V}_\epsilon) + \theta^\epsilon_{13}(\mathbf{W}_\epsilon) + \epsilon\kappa^\epsilon_{13}(\mathbf{W}_\epsilon)$$

$$= \frac{1}{2}\Bigg((\mathbf{V}'_\epsilon, \mathbf{b}_\epsilon) + \Big(-(\mathbf{V}'_\epsilon, \mathbf{b}_\epsilon) + \epsilon x_2 \beta_\epsilon(\mathbf{V}'_\epsilon, \mathbf{b}_\epsilon) + \epsilon x_3 \alpha_\epsilon(\mathbf{V}'_\epsilon, \mathbf{b}_\epsilon)$$

$$-\epsilon x_3 \gamma_\epsilon \psi_\epsilon\Big) - \Big(\epsilon x_2 \alpha_\epsilon(\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon) + \epsilon x_3 \alpha_\epsilon(\mathbf{V}'_\epsilon, \mathbf{b}_\epsilon)\Big) + \epsilon x_2 \psi'_\epsilon + \epsilon x_3 \gamma_\epsilon \psi_\epsilon\Bigg)$$

$$= \frac{\epsilon x_2}{2}\Big(\psi'_\epsilon - \alpha_\epsilon(\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon) + \beta_\epsilon(\mathbf{V}'_\epsilon, \mathbf{b}_\epsilon)\Big) \overset{(5.7)}{=} -\frac{\epsilon x_2}{2}(\mathbf{V}'_{*,\epsilon}, \mathbf{t}_\epsilon).$$

We leave to the reader the verification of

$$\omega^\epsilon_{i,j}(\widehat{\mathbf{V}}_\epsilon) = 0, \;\; i,j = 2,3.$$

Denoting $B_\epsilon = (B^{ij}_\epsilon)^3_{i,j=1}$, where $B^{ij}_\epsilon = 0$ except for $i=j=1$ and

$$B^{11}_\epsilon = \epsilon^2\Big((\beta_\epsilon x_2 + \alpha_\epsilon x_3)(x_2(\mathbf{V}'_\epsilon, \mathbf{n}_\epsilon)' + x_3(\mathbf{V}'_\epsilon, \mathbf{b}_\epsilon)' - \beta_\epsilon x_3 \psi_\epsilon + \alpha_\epsilon x_2 \psi_\epsilon)$$

$$+\gamma_\epsilon x_3(\partial_1 \mathbf{W}_\epsilon, \mathbf{n}_\epsilon) - \gamma_\epsilon x_2(\partial_1 \mathbf{W}_\epsilon, \mathbf{b}_\epsilon)\Big),$$

we can write

$$\omega^\epsilon(\widehat{\mathbf{V}}_\epsilon) = \epsilon\Upsilon(\mathbf{V}_{*,\epsilon}) + B_\epsilon, \tag{8.26}$$

where

$$\Upsilon_{11}(\mathbf{V}_{*,\epsilon}) = -(\mathbf{V}'_{*,\epsilon}, \mathbf{n}_\epsilon)x_3 + (\mathbf{V}'_{*,\epsilon}, \mathbf{b}_\epsilon)x_2, \tag{8.27}$$

$$\Upsilon_{12}(\mathbf{V}_{*,\epsilon}) = \Upsilon_{21}(\mathbf{V}_{*,\epsilon}) = \frac{x_3}{2}(\mathbf{V}'_{*,\epsilon}, \mathbf{t}_\epsilon), \tag{8.28}$$

$$\Upsilon_{13}(\mathbf{V}_{*,\epsilon}) = \Upsilon_{31}(\mathbf{V}_{*,\epsilon}) = -\frac{x_2}{2}(\mathbf{V}'_{*,\epsilon}, \mathbf{t}_\epsilon) \tag{8.29}$$

and

$$\Upsilon_{ij}(\mathbf{V}_{*,\epsilon}) = 0, \;\; i,j = 2,3. \tag{8.30}$$

Since we know that $\mathbf{t}_\epsilon \to \mathbf{t}$, $\mathbf{n}_\epsilon \to \mathbf{n}$, $\mathbf{b}_\epsilon \to \mathbf{b}$ pointwisely in $[0,l] \setminus D$, we can use (4.9) and the technique we have used in (7.72) to prove that

$$\Upsilon_{ij}(\mathbf{V}_{*,\epsilon}) \to \Upsilon_{ij}(\mathbf{V}_*) \text{ in } L^2(\Omega), \;\; i,j = 1,2,3.$$

Moreover, using (3.4), (4.11) and (8.25) we can easily check that

$$\|B_\epsilon\|_2 = \|B^{11}_\epsilon\|_2 \le C\epsilon^{2(1-r)}, \;\; r \in (0, \frac{1}{3}).$$

These convergences and estimates together with (8.1) and (8.3), (8.7) enable us to pass to the limit in the equation (since $\widehat{\mathbf{V}}_\epsilon \in C^\infty(\Omega)^3 \cap V(\Omega)$)

$$\int_\Omega A_\epsilon^{ijkl} \frac{1}{\epsilon} \omega_{kl}^\epsilon(\mathbf{U}_\epsilon) \frac{1}{\epsilon} \omega_{ij}^\epsilon(\widehat{\mathbf{V}}_\epsilon) d_\epsilon \, dx = \int_\Omega (\mathbf{F}, \widehat{\mathbf{V}}_\epsilon) d_\epsilon \, dx$$

$$+ \int_0^l \int_{\partial S} (\mathbf{G}, \widehat{\mathbf{V}}_\epsilon) \epsilon d_\epsilon \sqrt{\nu_j o^{ij,\epsilon} \nu_j} \, dS dx_1$$

and to establish

$$\int_\Omega A_0^{ijkl} \zeta_{kl} \Upsilon_{ij}(\mathbf{V}_*) \, dx = \int_\Omega (\mathbf{F}, \mathbf{V}) \, dx + \int_0^l \int_{\partial S} (\mathbf{G}, \mathbf{V}) \, dS dx_1 \tag{8.31}$$

for all $\langle \mathbf{V}, \psi \rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$, which generate the functions $\mathbf{V}_*$ (see (2.17)).

Let the point $\langle x_2^0, x_3^0 \rangle \in S$. Since S is open, there exists a square $S_0 \subset S$ such that the point $\langle x_2^0, x_3^0 \rangle$ is the corner of this square satisfying $x_2^0 \leq x_2$ and $x_3^0 \leq x_3$ for $\langle x_2, x_3 \rangle \in S_0$. Integrating the equality (7.6) on the interval $[x_3^0, x_3]$ we get

$$\zeta_{11}(x_1, x_2, x_3) = -(\mathbf{U}_*'(x_1), \mathbf{n}(x_1)) x_3 + (\mathbf{U}_*'(x_1), \mathbf{n}(x_1)) x_3^0 + \zeta_{11}(x_1, x_2, x_3^0)$$

for arbitrary but fixed $x_1 \in (0, l)$ and $\langle x_2, x_3 \rangle \in S_0$. After derivation according to the second variable we find from (7.7) that

$$(\partial_2 \zeta_{11}(x_1, x_2, x_3) =)(\mathbf{U}_*'(x_1), \mathbf{b}(x_1)) = \partial_2 \zeta_{11}(x_1, x_2, x_3^0).$$

Integrating on the interval $[x_2^0, x_2]$ we get

$$\zeta_{11}(x_1, x_2, x_3^0) = (\mathbf{U}_*'(x_1), \mathbf{b}(x_1)) x_2 - (\mathbf{U}_*'(x_1), \mathbf{b}(x_1)) x_2^0 + \zeta_{11}(x_1, x_2^0, x_3^0).$$

We denote

$$Q_0(x_1) = \zeta_{11}(x_1, x_2^0, x_3^0) - (\mathbf{U}_*'(x_1), \mathbf{b}(x_1)) x_2^0 + (\mathbf{U}_*'(x_1), \mathbf{n}(x_1)) x_3^0 \in L^2(0, l).$$

Analogously as in the derivation of (7.23) we can prove that Q_0 does not depend on the choice of the point from S and thus

$$\zeta_{11} = Q_0 + (\mathbf{U}_*', \mathbf{b}) x_2 - (\mathbf{U}_*', \mathbf{n}) x_3 \text{ in } \Omega. \tag{8.32}$$

By the form of the tensor $(A_0^{ijkl})_{i,j,k,l=1}^3$ (see (8.7)), we have after the substitution (8.27)–(8.30) to (8.31)

$$\int_\Omega A_0^{ijkl} \zeta_{kl} \Upsilon_{ij}(\mathbf{V}_*) \, dx = \int_\Omega [\lambda(\zeta_{11} + \zeta_{22} + \zeta_{33})(\Upsilon_{11}(\mathbf{V}_*) + \Upsilon_{22}(\mathbf{V}_*) + \Upsilon_{33}(\mathbf{V}_*))$$

$$+2\mu(\zeta_{11}\Upsilon_{11}(\mathbf{V}_*) + \zeta_{22}\Upsilon_{22}(\mathbf{V}_*) + \zeta_{33}\Upsilon_{33}(\mathbf{V}_*)$$

$$+2\zeta_{12}\Upsilon_{12}(\mathbf{V}_*) + 2\zeta_{13}\Upsilon_{13}(\mathbf{V}_*) + 2\zeta_{23}\Upsilon_{23}(\mathbf{V}_*))] \, dx$$

$$= \int_\Omega [\lambda(\zeta_{11}+\zeta_{22}+\zeta_{33})+2\mu\zeta_{11}]\Upsilon_{11}(\mathbf{V}_*)\ dx + \int_\Omega [4\mu(\zeta_{12}\Upsilon_{12}(\mathbf{V}_*)+\zeta_{13}\Upsilon_{13}(\mathbf{V}_*))]\ dx.$$

Hence using (8.27)–(8.29), we can rewrite (8.31) as

$$\mathcal{I}_1 + \mathcal{I}_2 = \int_\Omega (\mathbf{F},\mathbf{V})\ dx + \int_0^l \int_{\partial S} (\mathbf{G},\mathbf{V})\ dS dx_1, \tag{8.33}$$

where

$$\mathcal{I}_1 = \int_\Omega [\lambda(\zeta_{11}+\zeta_{22}+\zeta_{33})+2\mu\zeta_{11}][(\mathbf{V}'_*,\mathbf{b})x_2 - (\mathbf{V}'_*,\mathbf{n})x_3]\ dx,$$

$$\mathcal{I}_2 = 2\mu \int_\Omega [\zeta_{12}(\mathbf{V}'_*,\mathbf{t})x_3 - \zeta_{13}(\mathbf{V}'_*,\mathbf{t})x_2]\ dx.$$

Using (8.8), we find that

$$\lambda(\zeta_{11}+\zeta_{22}+\zeta_{33})+2\mu\zeta_{11} = (\lambda+2\mu)\zeta_{11} + \lambda\zeta_{22}^H + \lambda\zeta_{33}^H - \frac{\lambda^2}{\lambda+\mu}\zeta_{11}$$

$$= (\lambda+2\mu - \frac{\lambda^2}{\lambda+\mu})\zeta_{11} + \lambda(\zeta_{22}^H+\zeta_{33}^H).$$

Hence using (8.22) we can rewrite the integral $\mathcal{I}_1$ to the form

$$\mathcal{I}_1 = \int_\Omega [E\zeta_{11} + \lambda(\zeta_{22}^H+\zeta_{33}^H)][(\mathbf{V}'_*,\mathbf{b})x_2 - (\mathbf{V}'_*,\mathbf{n})x_3]\ dx. \tag{8.34}$$

The terms involving function $\zeta_{22}^H+\zeta_{33}^H$ disappear from (8.34) because of (8.11) and the dependence of the terms $(\mathbf{V}'_*,\mathbf{b})$ and $(\mathbf{V}'_*,\mathbf{n})$ only on x_1. After the substitution (8.32) to (8.34), we can conclude using (2.1) and (8.21)–(8.22) that

$$\mathcal{I}_1 = \int_0^l E[I_{x_2^2}(\mathbf{U}'_*,\mathbf{b})(\mathbf{V}'_*,\mathbf{b}) + I_{x_3^2}(\mathbf{U}'_*,\mathbf{n})(\mathbf{V}'_*,\mathbf{n})]\ dx_1. \tag{8.35}$$

After the substitution $\boldsymbol{\eta} = \langle\zeta_{12},\zeta_{13}\rangle$ from (8.18) to $\mathcal{I}_2$, we obtain

$$\mathcal{I}_2 = \int_\Omega \mu\left(-(\partial_2 p - x_3)x_3 + (\partial_3 p + x_2)x_2\right)(\mathbf{U}'_*,\mathbf{t})(\mathbf{V}'_*,\mathbf{t})\ dx, \tag{8.36}$$

where p is the unique solution to the Neumann problem (8.19) and it is easy to verify from (8.36) (using (8.22) and (8.19) with the test function $r=p$) that

$$\mathcal{I}_2 \overset{(8.19)}{=} \int_\Omega \mu(-\partial_2 p x_3 + x_3^2 + \partial_3 p x_2 + x_2^2)(\mathbf{U}'_*,\mathbf{t})(\mathbf{V}'_*,\mathbf{t})\ dx$$

$$+ \int_0^l (\mathbf{U}'_*,\mathbf{t})(\mathbf{V}'_*,\mathbf{t}) \int_S \mu[(\partial_2 p)^2 - \partial_2 p x_3 + (\partial_3 p)^2 + \partial_3 p x_2]\ dx_2 dx_3 dx_1$$

$$= \int_0^l \mu K(\mathbf{U}'_*,\mathbf{t})(\mathbf{V}'_*,\mathbf{t})\ dx_1. \tag{8.37}$$

Thus after adding (8.35) to (8.37) we obtain (8.23). □

Lemma 8.5 *The sequence* $\{\frac{1}{\epsilon_n}\omega^{\epsilon_n}(\mathbf{U}_{\epsilon_n})\}_{n=1}^{\infty}$ *converges strongly to* ζ *in* $L^2(\Omega)^9$ *for* $\epsilon_n \to 0$.

P r o o f: In the proof, we will write ϵ instead of ϵ_n to simplify the notation. Let us define

$$\Lambda_\epsilon = \int_\Omega A_\epsilon^{ijkl}\left(\frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon) - \zeta_{kl}\right)\left(\frac{1}{\epsilon}\omega_{ij}^\epsilon(\mathbf{U}_\epsilon) - \zeta_{ij}\right) d_\epsilon\, dx.$$

According to Proposition 6.1, there exists a constant $C > 0$ independent of ϵ such that

$$\left\|\frac{1}{\epsilon}\omega^\epsilon(\mathbf{U}_\epsilon) - \zeta\right\|_2^2 \leq C\Lambda_\epsilon. \tag{8.38}$$

Equation (6.4) implies for $\mathbf{F}_\epsilon = \epsilon^2\mathbf{F}$ and $\mathbf{G}_\epsilon = \epsilon^3\mathbf{G}$ that

$$\begin{aligned}\Lambda_\epsilon = &\int_\Omega (\mathbf{F}, \mathbf{U}_\epsilon) d_\epsilon\, dx + \int_0^l \int_{\partial S} (\mathbf{G}, \mathbf{U}_\epsilon) d_\epsilon \epsilon \sqrt{\nu_i o^{ij,\epsilon}\nu_j}\; dS dx_1\\
&+ \int_\Omega A_\epsilon^{ijkl}\left(\left(\zeta_{kl} - \frac{1}{\epsilon}\omega_{kl}^\epsilon(\mathbf{U}_\epsilon)\right)\zeta_{ij} - \zeta_{kl}\frac{1}{\epsilon}\omega_{ij}^\epsilon(\mathbf{U}_\epsilon)\right) d_\epsilon\, dx.\end{aligned}$$

As a result of (8.1)–(8.3) and (8.7), we obtain the convergence of the sequence Λ_ϵ, i.e.

$$\Lambda = \lim_{\epsilon \to 0} \Lambda_\epsilon = \int_0^l (\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}, \mathbf{U})\; dx_1 - \int_\Omega A_0^{ijkl}\zeta_{kl}\zeta_{ij}\; dx. \tag{8.39}$$

Using (8.7) leads after substitution of (8.8) to the identity

$$\int_\Omega A_0^{ijkl}\zeta_{kl}\zeta_{ij}\; dx = \int_\Omega [\lambda(\zeta_{11} + \zeta_{22} + \zeta_{33})^2 + 2\mu \sum_{i,j=1}^{3} \zeta_{ij}^2]\; dx$$

$$\begin{aligned}
\overset{(8.8)}{=} &\int_\Omega \Bigg[\lambda\left(\zeta_{11} + \zeta_{22}^H - \frac{\lambda}{\lambda+\mu}\zeta_{11} + \zeta_{33}^H\right)^2 + 2\mu\zeta_{11}^2 + 4\mu\left(\zeta_{12}^2 + \zeta_{13}^2 + (\zeta_{23}^H)^2\right)\\
&+2\mu\left(\zeta_{22}^H - \frac{1}{2}\frac{\lambda}{\lambda+\mu}\zeta_{11}\right)^2 + 2\mu\left(\zeta_{33}^H - \frac{1}{2}\frac{\lambda}{\lambda+\mu}\zeta_{11}\right)^2\Bigg]\; dx\\
= &\int_\Omega \Bigg[(\lambda + 2\mu)\zeta_{11}^2 + 4\mu(\zeta_{12}^2 + \zeta_{13}^2) + 4\mu(\zeta_{23}^H)^2 + 2\lambda\zeta_{11}\zeta_{22}^H - \frac{\lambda^2}{\lambda+\mu}\zeta_{11}^2\\
&+2\lambda\zeta_{11}\zeta_{33}^H - \frac{\lambda^2}{\lambda+\mu}\zeta_{11}^2 + 2\lambda\zeta_{22}^H\zeta_{33}^H - \frac{\lambda^2}{\lambda+\mu}\zeta_{11}\zeta_{22}^H - \frac{\lambda^2}{\lambda+\mu}\zeta_{11}\zeta_{33}^H + \frac{1}{2}\frac{\lambda^3}{(\lambda+\mu)^2}\zeta_{11}^2\\
&+\lambda(\zeta_{22}^H)^2 - \frac{\lambda^2}{\lambda+\mu}\zeta_{22}^H\zeta_{11} + \frac{1}{4}\frac{\lambda^3}{(\lambda+\mu)^2}\zeta_{11}^2 + \lambda(\zeta_{33}^H)^2 - \frac{\lambda^2}{\lambda+\mu}\zeta_{33}^H\zeta_{11}\\
&+\frac{1}{4}\frac{\lambda^3}{(\lambda+\mu)^2}\zeta_{11}^2 + 2\mu(\zeta_{22}^H)^2 - \frac{2\mu\lambda}{\lambda+\mu}\zeta_{22}^H\zeta_{11} + \frac{1}{2}\frac{\mu\lambda^2}{(\lambda+\mu)^2}\zeta_{11}^2
\end{aligned}$$

$$+2\mu(\zeta_{33}^H)^2 - \frac{2\mu\lambda}{\lambda+\mu}\zeta_{33}^H\zeta_{11} + \frac{1}{2}\frac{\mu\lambda^2}{(\lambda+\mu)^2}\zeta_{11}^2\Bigg]\ dx$$

$$= \int_\Omega \Bigg[(\lambda + 2\mu - \frac{2\lambda^2}{\lambda+\mu} + \frac{\lambda^3}{(\lambda+\mu)^2} + \frac{\mu\lambda^2}{(\lambda+\mu)^2})\zeta_{11}^2 + 4\mu(\zeta_{12}^2+\zeta_{13}^2)$$

$$+\lambda(\zeta_{22}^H+\zeta_{33}^H)^2 + 2\mu((\zeta_{22}^H)^2 + (\zeta_{33}^H)^2 + 2(\zeta_{23}^H)^2)$$

$$+(2\lambda - \frac{2\lambda^2}{\lambda+\mu} - \frac{2\mu\lambda}{\lambda+\mu})\zeta_{11}\zeta_{22}^H + (2\lambda - \frac{2\lambda^2}{\lambda+\mu} - \frac{2\mu\lambda}{\lambda+\mu})\zeta_{11}\zeta_{33}^H\Bigg]\ dx$$

$$\overset{(8.22)}{=} \int_\Omega [E\zeta_{11}^2 + 4\mu(\zeta_{12}^2+\zeta_{13}^2) + \lambda(\zeta_{22}^H+\zeta_{33}^H)^2 + 2\mu((\zeta_{22}^H)^2+(\zeta_{33}^H)^2+2(\zeta_{23}^H)^2)]\ dx. \quad (8.40)$$

The expressions for ζ_{11}, ζ_{12} and ζ_{13}, i.e (8.32) and (8.18), imply (together with (8.23) and (2.1)) after substitution to (8.40) that

$$\int_\Omega A_0^{ijkl}\zeta_{kl}\zeta_{ij}\ dx = \int_\Omega \Bigg[E\zeta_{11}^2 + 4\mu(\zeta_{12}^2+\zeta_{13}^2) + \lambda(\zeta_{22}^H+\zeta_{33}^H)^2$$

$$+2\mu((\zeta_{22}^H)^2 + (\zeta_{33}^H)^2 + 2(\zeta_{23}^H)^2)\Bigg]\ dx = \int_\Omega \Bigg[E\Big(Q_0 + (\mathbf{U}_*', \mathbf{b})x_2 - (\mathbf{U}_*', \mathbf{n})x_3\Big)^2$$

$$+4\mu\Big(-\frac{1}{2}(\mathbf{U}_*', \mathbf{t})(\partial_2 p - x_3)\Big)^2 + 4\mu\Big(-\frac{1}{2}(\mathbf{U}_*', \mathbf{t})(\partial_3 p + x_2)\Big)^2 + \lambda(\zeta_{22}^H+\zeta_{33}^H)^2$$

$$+2\mu((\zeta_{22}^H)^2 + (\zeta_{33}^H)^2 + 2(\zeta_{23}^H)^2)\Bigg]\ dx \overset{(8.23),(2.1)}{=} \int_0^l [(\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}, \mathbf{U}) + EQ_0^2]\ dx_1$$

$$+\int_\Omega \left[\lambda(\zeta_{22}^H+\zeta_{33}^H)^2 + 2\mu\left((\zeta_{22}^H)^2 + (\zeta_{33}^H)^2 + 2(\zeta_{23}^H)^2\right)\right]\ dx,$$

and substituting to (8.39) leads to

$$\Lambda = -\int_\Omega \left[\frac{EQ_0^2}{|S|} + \lambda(\zeta_{22}^H+\zeta_{33}^H)^2 + 2\mu\left((\zeta_{22}^H)^2 + (\zeta_{33}^H)^2 + 2(\zeta_{23}^H)^2\right)\right]\ dx.$$

The sequence Λ_ϵ consists of non-negative numbers by (8.38) and so $\Lambda = 0$. Hence

$$Q_0 = \zeta_{22}^H = \zeta_{23}^H = \zeta_{33}^H = 0.$$

In addition, the estimate (8.38) yields the strong convergence in (8.3). Hence by Proposition 7.2, the convergence (8.2) is also strong. □

Since we have proved that $Q_0 = \zeta_{22}^H = \zeta_{23}^H = \zeta_{33}^H = 0$, and we have denoted $\boldsymbol{\eta} = \langle\zeta_{12}, \zeta_{13}\rangle$, we obtain

$$\zeta_{11} \overset{(8.32)}{=} (\mathbf{U}_*', \mathbf{b})x_2 - (\mathbf{U}_*', \mathbf{n})x_3,$$

$$\zeta_{12} \overset{(8.18)}{=} \zeta_{21} = -\frac{1}{2}(\mathbf{U}_*', \mathbf{t})(\partial_2 p - x_3),$$

$$\zeta_{13} \overset{(8.18)}{=} \zeta_{31} = -\frac{1}{2}(\mathbf{U}'_*, \mathbf{t})(\partial_3 p + x_2), \tag{8.41}$$

$$\zeta_{22} \overset{(8.8)}{=} -\frac{1}{2}\frac{\lambda}{\lambda+\mu}\Big((\mathbf{U}'_*, \mathbf{b})x_2 - (\mathbf{U}'_*, \mathbf{n})x_3\Big),$$

$$\zeta_{23} = \zeta_{32} = 0,$$

$$\zeta_{33} \overset{(8.8)}{=} -\frac{1}{2}\frac{\lambda}{\lambda+\mu}\Big((\mathbf{U}'_*, \mathbf{b})x_2 - (\mathbf{U}'_*, \mathbf{n})x_3\Big).$$

We have proved that the asymptotic one-dimensional model for the curved rods has the form

$$a(\langle \mathbf{U}, \phi\rangle, \langle \mathbf{V}, \psi\rangle) = \mathcal{F}(\mathbf{V}) \tag{8.42}$$

for all $\langle \mathbf{V}, \psi\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0, l)$, where

$$a(\langle \mathbf{U}, \phi\rangle, \langle \mathbf{V}, \psi\rangle) = \int_0^l E[I_{x_2^2}(\mathbf{U}'_*, \mathbf{b})(\mathbf{V}'_*, \mathbf{b}) + I_{x_3^2}(\mathbf{U}'_*, \mathbf{n})(\mathbf{V}'_*, \mathbf{n})$$

$$+\mu K(\mathbf{U}'_*, \mathbf{t})(\mathbf{V}'_*, \mathbf{t})]\; dx_1$$

$$= \int_0^l \Big[EI_{x_2^2}\Big((-\phi\mathbf{t} + (\mathbf{U}', \mathbf{b})\mathbf{n} - (\mathbf{U}', \mathbf{n})\mathbf{b})', \mathbf{b}\Big)\Big((-\psi\mathbf{t} + (\mathbf{V}', \mathbf{b})\mathbf{n} - (\mathbf{V}', \mathbf{n})\mathbf{b})', \mathbf{b}\Big)$$

$$+EI_{x_3^2}\Big((-\phi\mathbf{t} + (\mathbf{U}', \mathbf{b})\mathbf{n} - (\mathbf{U}', \mathbf{n})\mathbf{b})', \mathbf{n}\Big)\Big((-\psi\mathbf{t} + (\mathbf{V}', \mathbf{b})\mathbf{n} - (\mathbf{V}', \mathbf{n})\mathbf{b})', \mathbf{n}\Big)$$

$$+\mu K\Big((-\phi\mathbf{t}+(\mathbf{U}', \mathbf{b})\mathbf{n}-(\mathbf{U}', \mathbf{n})\mathbf{b})', \mathbf{t}\Big)\Big((-\psi\mathbf{t}+(\mathbf{V}', \mathbf{b})\mathbf{n}-(\mathbf{V}', \mathbf{n})\mathbf{b})', \mathbf{t}\Big)\Big]\; dx_1 \tag{8.43}$$

and

$$\mathcal{F}(\mathbf{V}) = \int_0^l (\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}, \mathbf{V})\; dx_1 \tag{8.44}$$

is defined in Lemma 8.4. The convergences (8.2)–(8.3) ensure the existence of the solution to (8.42).

Remark 8.6 The existence of the strong limit $\mathbf{U}_\epsilon \to \mathbf{U}$ in $H^1(\Omega)$ for $\epsilon \to 0$ (see Proposition 7.2 and Lemma 8.5 for the strong convergence on some subsequence) is equivalent to the existence of the unique solution to the equation (8.23), which will be studied in the following proposition under more general assumptions.

Proposition 8.7 *Let* $\mathcal{F} \in H^{-1}(0, l)$ *and* $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b} \in L^\infty(0, l)^3$. *Then the solution* $\langle \mathbf{U}, \phi\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0, l)$ *to the equation* (8.42) *is unique.*

P r o o f: Let us suppose that there exist two solutions $\langle \mathbf{U}_i, \phi_i\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0, l)$, $i = 1, 2$ of the equation (8.42). Then the couple of functions $\langle \widehat{\mathbf{V}}, \widehat{\phi}\rangle$ defined by $\widehat{\mathbf{V}} = \mathbf{U}_1 - \mathbf{U}_2$ and $\widehat{\phi} = \phi_1 - \phi_2$ is the solution of the equation

$$a(\langle \widehat{\mathbf{V}}, \widehat{\phi}\rangle, \langle \mathbf{V}, \psi\rangle) = 0,\ \forall \langle \mathbf{V}, \psi\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0, l).$$

The uniqueness follows from the next estimate (using zero boundary conditions of the functions $\widehat{\mathbf{V}}$ and $\widehat{\mathbf{V}}_*$, (2.17), (5.5)–(5.6) for the functions $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ instead of $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$)

$$a(\langle \widehat{\mathbf{V}}, \widehat{\phi}\rangle, \langle \widehat{\mathbf{V}}, \widehat{\phi}\rangle) = \int_0^l \left[EI_{x_2^2}(\widehat{\mathbf{V}}'_*, \mathbf{b})^2 + EI_{x_3^2}(\widehat{\mathbf{V}}'_*, \mathbf{n})^2 + \mu K(\widehat{\mathbf{V}}'_*, \mathbf{t})^2 \right] dx_1$$

$$\geq \min\{EI_{x_2^2}, EI_{x_3^2}, \mu K\} \int_0^l \left[(\widehat{\mathbf{V}}'_*, \mathbf{t})^2 + (\widehat{\mathbf{V}}'_*, \mathbf{n})^2 + (\widehat{\mathbf{V}}'_*, \mathbf{b})^2 \right] dx_1$$

$$= C_2 \int_0^l |\widehat{\mathbf{V}}'_*|^2 \, dx_1 \geq C_3 \int_0^l |\widehat{\mathbf{V}}_*|^2 \, dx_1 = C_3 \int_0^l \left[(\widehat{\mathbf{V}}_*, \mathbf{t})^2 + (\widehat{\mathbf{V}}_*, \mathbf{n})^2 + (\widehat{\mathbf{V}}_*, \mathbf{b})^2 \right] dx_1$$

$$= C_3 \int_0^l \left[\widehat{\phi}^2 + (\widehat{\mathbf{V}}', \mathbf{n})^2 + (\widehat{\mathbf{V}}', \mathbf{b})^2 \right] dx_1 = C_3 \int_0^l \left[\widehat{\phi}^2 + (\widehat{\mathbf{V}}', \mathbf{t})^2 \right.$$

$$\left. + (\widehat{\mathbf{V}}', \mathbf{n})^2 + (\widehat{\mathbf{V}}', \mathbf{b})^2 \right] dx_1 = C_3 \int_0^l \left[\widehat{\phi}^2 + |\widehat{\mathbf{V}}'|^2 \right] dx_1 \geq C_4 \int_0^l \left[\widehat{\phi}^2 + |\widehat{\mathbf{V}}|^2 \right] dx_1.$$

□

The proof of the main theorem of this article is now complete and we can state it:

Theorem 8.8 *Let the function $\mathbf{\Phi}$ be the parametrization of a unit speed curve such that $\mathbf{\Phi} \in C([0,l])^3$ and $\mathbf{\Phi}'$ is piecewise continuous. Let, further, $\mathbf{F} \in L^2(\Omega)^3$, $\mathbf{G} \in L^2(0,l;L^2(\partial S)^3)$ and $\check{\mathbf{F}}_{\mathbf{F}+\mathbf{G}}$ be defined as in Lemma 8.4. Then, there is a unique pair $\langle \mathbf{U}, \phi\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ and satisfying the boundary value problem (8.42) with $a(\cdot,\cdot)$ given by (8.43). Moreover, the constant extension to $\Omega = (0,l) \times S$ of $\langle \mathbf{U}, \phi\rangle$ may be approximated by the solutions $\mathbf{U}_\epsilon \in V(\Omega)$ of the equation (6.4) as follows*

$$\mathbf{U} = \lim_{\epsilon \to 0} \mathbf{U}_\epsilon \text{ strongly in } H^1(\Omega)^3,$$

$$\phi = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \Big((\partial_2 \mathbf{U}_\epsilon, \mathbf{b}_\epsilon) - (\partial_3 \mathbf{U}_\epsilon, \mathbf{n}_\epsilon) \Big) \text{ weakly in } L^2(\Omega).$$

9 Applications and examples

We suppose now that the couples of functions $\langle \mathbf{U}, \phi\rangle$, $\langle \mathbf{V}, \psi\rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ and the functions $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ are smooth enough such that the following transformations have sense. First, we introduce the notation

$$v_1 = (\mathbf{V}, \mathbf{t}), \; v_2 = (\mathbf{V}, \mathbf{n}), \; v_3 = (\mathbf{V}, \mathbf{b}). \tag{9.1}$$

From (2.2), (9.1) with the functions $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$, α, β, γ instead of $\mathbf{t}_\epsilon$, $\mathbf{n}_\epsilon$, $\mathbf{b}_\epsilon$, α_ϵ, β_ϵ, γ_ϵ, and from (2.17), it follows that

$$\mathbf{V}'_* = (-\psi \mathbf{t} + (\mathbf{V}', \mathbf{b})\mathbf{n} - (\mathbf{V}', \mathbf{n})\mathbf{b})' =$$

$$-\psi'\mathbf{t}-\psi(\alpha\mathbf{b}+\beta\mathbf{n})+(\mathbf{V}',\mathbf{b})'\mathbf{n}+(\mathbf{V}',\mathbf{b})(-\beta\mathbf{t}-\gamma\mathbf{b})-(\mathbf{V}',\mathbf{n})'\mathbf{b}-(\mathbf{V}',\mathbf{n})(-\alpha\mathbf{t}+\gamma\mathbf{n})$$
$$=-\psi'\mathbf{t}-\alpha\psi\mathbf{b}-\beta\psi\mathbf{n}+v_3''\mathbf{n}-(\mathbf{V},-\alpha\mathbf{t}+\gamma\mathbf{n})'\mathbf{n}+(v_3'-(\mathbf{V},-\alpha\mathbf{t}+\gamma\mathbf{n}))(-\beta\mathbf{t}-\gamma\mathbf{b})$$
$$-v_2''\mathbf{b}+(\mathbf{V},-\beta\mathbf{t}-\gamma\mathbf{b})'\mathbf{b}+(-v_2'+(\mathbf{V},-\beta\mathbf{t}-\gamma\mathbf{b}))(-\alpha\mathbf{t}+\gamma\mathbf{n})$$
$$=-\psi'\mathbf{t}-\alpha\psi\mathbf{b}-\beta\psi\mathbf{n}+(v_3'+\alpha v_1-\gamma v_2)'\mathbf{n}-\beta(v_3'+\alpha v_1-\gamma v_2)\mathbf{t}-\gamma(v_3'+\alpha v_1-\gamma v_2)\mathbf{b}$$
$$-(v_2'+\beta v_1+\gamma v_3)'\mathbf{b}+\alpha(v_2'+\beta v_1+\gamma v_3)\mathbf{t}-\gamma(v_2'+\beta v_1+\gamma v_3)\mathbf{n}. \tag{9.2}$$

Using (2.2), (2.17) and (9.2) we get that

$$(\mathbf{V}',\mathbf{t})=v_1'-\alpha v_3-\beta v_2=0, \tag{9.3}$$

$$(\mathbf{V}_*',\mathbf{b})=-\alpha\psi-\gamma(v_3'+\alpha v_1-\gamma v_2)-(v_2'+\beta v_1+\gamma v_3)', \tag{9.4}$$
$$(\mathbf{V}_*',\mathbf{n})=-\beta\psi+(v_3'+\alpha v_1-\gamma v_2)'-\gamma(v_2'+\beta v_1+\gamma v_3), \tag{9.5}$$
$$(\mathbf{V}_*',\mathbf{t})=-\psi'-\beta(v_3'+\alpha v_1-\gamma v_2)+\alpha(v_2'+\beta v_1+\gamma v_3). \tag{9.6}$$

Let us suppose that the vectors $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ are obtained as the Frenêt basis, i.e.

$$\mathbf{\Phi}''(x_1)\neq\mathbf{0}, x_1\in[0,l],\ \mathbf{t}=\mathbf{\Phi}',\ \mathbf{n}=\frac{\mathbf{\Phi}''}{|\mathbf{\Phi}''|},\ \mathbf{b}=\mathbf{t}\times\mathbf{n}. \tag{9.7}$$

Let, further, $\kappa=|\mathbf{\Phi}''|$ be the curvature and $\tau=\frac{1}{\kappa^2}(\mathbf{\Phi}',\mathbf{\Phi}''\times\mathbf{\Phi}''')$ be the torsion. From (2.2), it follows that

$$\alpha=(\mathbf{t}',\mathbf{b})=(\mathbf{\Phi}'',\mathbf{\Phi}'\times\frac{\mathbf{\Phi}''}{|\mathbf{\Phi}''|})=0, \tag{9.8}$$

$$\beta=(\mathbf{t}',\mathbf{n})=(\mathbf{\Phi}'',\frac{\mathbf{\Phi}''}{|\mathbf{\Phi}''|})=\kappa, \tag{9.9}$$

$$\gamma=(\mathbf{b}',\mathbf{n})=(\mathbf{\Phi}''\times\frac{\mathbf{\Phi}''}{|\mathbf{\Phi}''|},\frac{\mathbf{\Phi}''}{|\mathbf{\Phi}''|})+(\mathbf{\Phi}'\times\frac{\mathbf{\Phi}'''|\mathbf{\Phi}''|-(\mathbf{\Phi}'',\mathbf{\Phi}''')\frac{\mathbf{\Phi}''}{|\mathbf{\Phi}''|}}{|\mathbf{\Phi}''|^2},\frac{\mathbf{\Phi}''}{|\mathbf{\Phi}''|})$$
$$=\frac{1}{|\mathbf{\Phi}''|^2}(\mathbf{\Phi}'\times\mathbf{\Phi}''',\mathbf{\Phi}'')=-\tau. \tag{9.10}$$

We denote after substitution of (9.8)–(9.10) to (9.3)–(9.6)

$$Q_1(\mathbf{v})=v_1'-\kappa v_2=0, \tag{9.11}$$

$$Q_2(\mathbf{v})=-(v_2'+\kappa v_1-\tau v_3)'+\tau(v_3'-\tau v_2), \tag{9.12}$$
$$Q_3(\mathbf{v},\psi)=(v_3'+\tau v_2)'+\tau(v_2'+\kappa v_1-\tau v_3)-\kappa\psi, \tag{9.13}$$
$$Q_4(\mathbf{v},\psi)=-\psi'-\kappa(v_3'+\tau v_2), \tag{9.14}$$

where $\mathbf{v}=(v_1,v_2,v_3)$. Then (8.42) can be transformed for $\mathbf{G}=0$ as

$$\int_0^l[EI_{x_2^2}Q_2(\mathbf{u})Q_2(\mathbf{v})+EI_{x_3^2}Q_3(\mathbf{u},\phi)Q_3(\mathbf{v},\psi)+\mu KQ_4(\mathbf{u},\phi)Q_4(\mathbf{v},\psi)]\,dx_1$$

$$= \int_0^l (\check{\mathbf{f}}, \mathbf{v}) \, dx_1, \tag{9.15}$$

where $\check{\mathbf{f}} = (\check{f}_1, \check{f}_2, \check{f}_3)$ and

$$\check{f}_1 = (\check{\mathbf{F}}, \mathbf{t}), \;\; \check{f}_2 = (\check{\mathbf{F}}, \mathbf{n}), \;\; \check{f}_3 = (\check{\mathbf{F}}, \mathbf{b}).$$

Let us notice that from $\langle \mathbf{U}, \phi \rangle \in \mathcal{V}_0^{\mathbf{t},\mathbf{n},\mathbf{b}}(0,l)$ and from the regularity of $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$, we see that $\langle \mathbf{u}, \phi \rangle \in \{\langle \mathbf{w}, \widehat{\psi} \rangle \in H_0^1(0,l) \times H_0^2(0,l) \times H_0^2(0,l) \times H_0^1(0,l) : \; w_1' - \kappa w_2 = 0\}$, where the function $\mathbf{u} = (u_1, u_2, u_3)$ is defined as in (9.1). For instance, the relation $w_1' - \kappa w_2 = 0$ is fulfilled by (9.3) and from (2.17) it follows that

$$u_2' = (\mathbf{U}, \mathbf{n})' = (\mathbf{U}', \mathbf{n}) + (\mathbf{U}, \mathbf{n}') = -(\mathbf{U}_*, \mathbf{b}) + (\mathbf{U}, \mathbf{n}') \in H_0^1(0,l)$$

and thus $u_2 \in H_0^2(0,l)$, because $\mathbf{U}_* \in H_0^1(0,l)^3$, $\phi = -(\mathbf{U}_*, \mathbf{t}) \in H_0^1(0,l)$, etc.

The equation (9.15) is nothing but the asymptotic one dimensional equation for curved rods, as derived in [10].

Consider now the case of smooth arches $\widehat{\mathbf{\Phi}} : [0,1] \to \mathbb{R}^2$ and let

$$c = \theta' = (\arctan(\frac{\widehat{\Phi}_2'}{\widehat{\Phi}_1'}))' = \frac{\widehat{\Phi}_2''\widehat{\Phi}_1' - \widehat{\Phi}_1''\widehat{\Phi}_2'}{|\widehat{\mathbf{\Phi}}'|^2} \tag{9.16}$$

denote its curvature. Then under the conditions that $l = 1$, $EI_{x_2^2} = 1$, $EI_{x_3^2} = 1$, $\mu K = 1$, $\check{f}_3 = 0$, $\phi = 0$, $\psi = 0$, $u_3 = 0$, $v_3 = 0$, $\tau = 0$ and $\kappa = c$, we get from (9.11)–(9.15)

$$\int_0^1 (u_2' + cu_1)'(v_2' + cv_1)' \, dx_1 = \int_0^1 (\check{\mathbf{f}}, \mathbf{v})_2 \, dx_1, \tag{9.17}$$

for all $\mathbf{v} \in \{\mathbf{w} \in H_0^1(0,1) \times H_0^2(0,1) : \; w_1' - cw_2 = 0\}$, which is nothing but the asymptotic one dimensional "flexural" model for arches from [6].

A Appendix

In this section, we construct a local frame for the unit speed curve $\mathcal{C}$ generated by a Lipschitz function, and its regularization. Finally, we show that slight modifications of the previous arguments enable us to derive the same asymptotic one dimensional model as in Theorem 8.8, when $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b} \in L^\infty(0,l)^3$.

Proposition A.1 *Let $\mathbf{\Phi} \in W^{1,\infty}(0,l)^3$ be a parametrization of a unit speed curve. Then there exist the tangent vector $\mathbf{t}$, the normal vector $\mathbf{n}$, the binormal vector $\mathbf{b}$, which belong to $L^\infty(0,l)^3$, satisfying*

$$|\mathbf{t}| = |\mathbf{n}| = |\mathbf{b}| = 1, \;\; \mathbf{t} \perp \mathbf{n} \perp \mathbf{b} \; \textit{a.e. on } (0,l). \tag{A.1}$$

Further, there exist functions $\mathbf{\Phi}_\sigma$, $\mathbf{t}_\sigma$, $\mathbf{n}_\sigma$ and $\mathbf{b}_\sigma$, $\sigma \in (0,1)$, such that

$$\mathbf{\Phi}_\sigma' = \mathbf{t}_\sigma, \;\; \mathbf{t}_\sigma \to \mathbf{t}, \;\; \mathbf{n}_\sigma \to \mathbf{n}, \;\; \mathbf{b}_\sigma \to \mathbf{b} \; \textit{in measure on } (0,l) \tag{A.2}$$

for $\sigma \to 0$,

$$|\mathbf{t}_\sigma| = |\mathbf{n}_\sigma| = |\mathbf{b}_\sigma| = 1,\ \mathbf{t}_\sigma \perp \mathbf{n}_\sigma \perp \mathbf{b}_\sigma \text{ in } [0,l] \setminus D_\sigma \tag{A.3}$$

for all $\sigma \in (0,1)$, $\mathbf{\Phi}_\sigma \in W^{1,\infty}(0,l)^3$ *and* $\mathbf{\Phi}'_\sigma$, $\mathbf{t}_\sigma$, $\mathbf{n}_\sigma$, $\mathbf{b}_\sigma$ *are piecewise continuous functions with a finite set* D_σ *of points of discontinuity.*

The proof of the proposition is a consequence of Lemmas A.2–A.6.

Lemma A.2 *Let* $\mathbf{\Phi} \in W^{1,\infty}(0,l)^3$. *Then there exist the tangent vector* $\mathbf{t}$, *the normal vector* $\mathbf{n}$ *and the binormal vector* $\mathbf{b}$, *which belong to* $L^\infty(0,l)^3$, *and satisfy* (A.1).

P r o o f: This construction is based on the solvability with respect to $\widehat{\mathbf{n}} = (\widehat{n}_1, \widehat{n}_2, \widehat{n}_3)$ of the equation

$$\widehat{n}_1 t_1 + \widehat{n}_2 t_2 + \widehat{n}_3 t_3 = 0, \tag{A.4}$$

where the tangent vector $\mathbf{t}$ is defined by $\mathbf{t} = \mathbf{\Phi}'$. Let us denote I_{t_i}, $i = 1,2,3$, the Lebesgue measurable sets such that

$$|t_i| \geq \sqrt{\frac{1}{3}} \text{ a.e. on } I_{t_i},\ i = 1,2,3. \tag{A.5}$$

Since $|\mathbf{t}| = 1$ then at least one of these sets must be nonempty and in addition $|[0,l] \setminus \bigcup_{i=1}^3 I_{t_i}| = 0$. Using (A.4) we can define the functions $\widehat{n}_i$, $i = 1,2,3$, in this way

$$\widehat{n}_2 = 1,\ \widehat{n}_3 = 1,\ \widehat{n}_1 = \frac{-t_2 - t_3}{t_1} \text{ a.e. on } I_{t_1}, \tag{A.6}$$

$$\widehat{n}_1 = 1,\ \widehat{n}_3 = 1,\ \widehat{n}_2 = \frac{-t_1 - t_3}{t_2} \text{ a.e. on } I_{t_2} \setminus (I_{t_2} \cap I_{t_1}), \tag{A.7}$$

$$\widehat{n}_1 = 1,\ \widehat{n}_2 = 1,\ \widehat{n}_3 = \frac{-t_1 - t_2}{t_3} \text{ a.e. on } I_{t_3} \setminus [(I_{t_3} \cap I_{t_1}) \cup (I_{t_3} \cap I_{t_2})]. \tag{A.8}$$

Hence we get that $|\widehat{\mathbf{n}}| \geq 1$ a.e. in $[0,l]$ and from (A.5), it follows that $\widehat{\mathbf{n}} \in L^\infty(0,l)^3$. Thus we can put

$$\mathbf{n} = \frac{\widehat{\mathbf{n}}}{|\widehat{\mathbf{n}}|} \tag{A.9}$$

and

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}, \tag{A.10}$$

which completes the definition of the local frame in $L^\infty(0,l)^3$. □

Lemma A.3 *Let* $I \subset (0,l)$ *be a Lebesgue measurable set and let* $\{I_{\sigma_1}\}_{\sigma_1 \in (0,1)}$ *be an arbitrary family of measurable sets such that* $I_{\sigma_1} \subset I$ *for all* $\sigma_1 \in (0,1)$, *and*

$$|I \setminus I_{\sigma_1}| \to 0 \text{ for } \sigma_1 \to 0. \tag{A.11}$$

Then there exist the open intervals $J_{I_{\sigma_1},k} \subset (0,l)$ *and* $m(\sigma_1) \in \mathbb{N}$, $m(\sigma_1) \to \infty$ *for* $\sigma_1 \to 0$, *such that*

$$I_{\sigma_1} \subset \bigcup_{k=1}^{\infty} J_{I_{\sigma_1},k},\ \forall \sigma_1 \in (0,1),\ |(I_{\sigma_1} \setminus \bigcup_{k=1}^{m(\sigma_1)} J_{I_{\sigma_1},k}) \cup (\bigcup_{k=1}^{m(\sigma_1)} J_{I_{\sigma_1},k} \setminus I_{\sigma_1})| \to 0 \tag{A.12}$$

and

$$|(I \setminus \bigcup_{k=1}^{m(\sigma_1)} J_{I_{\sigma_1},k}) \cup (\bigcup_{k=1}^{m(\sigma_1)} J_{I_{\sigma_1},k} \setminus I)| \to 0 \tag{A.13}$$

for $\sigma_1 \to 0$.

P r o o f: From the definition of the (outer) Lebesgue measure, it follows that

$$|I_{\sigma_1}| = \inf_{P, I_{\sigma_1} \subset P} |P|, \ P \text{ bounded open sets}, \ \sigma_1 \in (0,1).$$

Hence we can construct for arbitrary but fixed $\sigma_1 \in (0,1)$ the sequence $\{P_{n_{\sigma_1}}\}_{n_{\sigma_1}=1}^{\infty}$ consisting of bounded open sets such that $I_{\sigma_1} \subset P_{n_{\sigma_1}}$, $n_{\sigma_1} = 1, 2, \ldots$, and

$$\lim_{n_{\sigma_1} \to \infty} |P_{n_{\sigma_1}} \setminus I_{\sigma_1}| = 0$$

or equivalently $\forall \widehat{\epsilon}_1 \ \exists n_0(\sigma_1, \widehat{\epsilon}_1) \in \mathbb{N}_0 : \ \forall n_{\sigma_1} \geq n_0(\sigma_1, \widehat{\epsilon}_1)$

$$|P_{n_{\sigma_1}} \setminus I_{\sigma_1}| < \widehat{\epsilon}_1. \tag{A.14}$$

Since $P_{n_{\sigma_1}}$ are open bounded sets,

$$P_{n_{\sigma_1}} = \bigcup_{k=1}^{\infty} J_{P_{n_{\sigma_1}},k}, \ n_{\sigma_1} = 1, 2, \ldots,$$

where $J_{P_{n_{\sigma_1}},k}$ are bounded open intervals. We obtain

$$\lim_{m_{\sigma_1} \to \infty} |P_{n_{\sigma_1}} \setminus \bigcup_{k=1}^{m_{\sigma_1}} J_{P_{n_{\sigma_1}},k}| = 0$$

or equivalently $\forall \widehat{\epsilon}_2 \ \exists m_0(\widehat{\epsilon}_2, n_{\sigma_1}) \in \mathbb{N}_0 : \ \forall m_{\sigma_1} \geq m_0(\widehat{\epsilon}_2, n_{\sigma_1})$

$$|P_{n_{\sigma_1}} \setminus \bigcup_{k=1}^{m_{\sigma_1}} J_{P_{n_{\sigma_1}},k}| < \widehat{\epsilon}_2 \tag{A.15}$$

for $\sigma_1 \in (0,1)$ arbitrary but fixed. We notice that the sets of indices $\{\widehat{\epsilon}_j(\sigma_1)\}_{\sigma_1 \in (0,1)}$, $j = 1, 2$, may be chosen such that $\widehat{\epsilon}_j(\sigma_1) \to 0$ for $\sigma_1 \to 0$ and $j = 1, 2$. From (A.14), it follows the existence of $n(\sigma_1) = n_0(\sigma_1, \widehat{\epsilon}_1(\sigma_1)) \in \mathbb{N}_0$ such that for all $n_{\sigma_1} \geq n(\sigma_1)$ and $\sigma_1 \in (0,1)$

$$|P_{n_{\sigma_1}} \setminus I_{\sigma_1}| < \widehat{\epsilon}_1(\sigma_1). \tag{A.16}$$

We can suppose without loss of generality that $n(\sigma_1) \to \infty$ for $\sigma_1 \to 0$. From (A.15), it follows the existence of $m(\sigma_1) = m_0(\widehat{\epsilon}_2(\sigma_1), n(\sigma_1)) \in \mathbb{N}_0$ such that for all $m_{\sigma_1} \geq m(\sigma_1)$ and $\sigma_1 \in (0,1)$

$$|P_{n(\sigma_1)} \setminus \bigcup_{k=1}^{m_{\sigma_1}} J_{P_{n(\sigma_1)},k}| < \widehat{\epsilon}_2(\sigma_1). \tag{A.17}$$

We can suppose without loss of generality that $m(\sigma_1) \to \infty$ for $\sigma_1 \to 0$. Combining (A.16) and (A.17) leads to

$$|P_{n(\sigma_1)} \setminus I_{\sigma_1}| \to 0 \text{ and } |P_{n(\sigma_1)} \setminus \bigcup_{k=1}^{m(\sigma_1)} J_{P_{n(\sigma_1)},k}| \to 0,$$

which together with (A.11) imply

$$|(I_{\sigma_1} \setminus \bigcup_{k=1}^{m(\sigma_1)} J_{P_{n(\sigma_1)},k}) \cup (\bigcup_{k=1}^{m(\sigma_1)} J_{P_{n(\sigma_1)},k} \setminus I_{\sigma_1})| \to 0 \tag{A.18}$$

and

$$|(I \setminus \bigcup_{k=1}^{m(\sigma_1)} J_{P_{n(\sigma_1)},k}) \cup (\bigcup_{k=1}^{m(\sigma_1)} J_{P_{n(\sigma_1)},k} \setminus I)| \to 0 \tag{A.19}$$

for $\sigma_1 \to 0$. Since we suppose that $I_{\sigma_1} \subset I \subset (0,l)$ for all $\sigma_1 \in (0,1)$, we can define $J_{I_{\sigma_1},k} = J_{P_{n(\sigma_1)},k} \cap (0,l)$ and now it is easy to verify (A.12) and (A.13). □

Lemma A.4 *Let $I \subset (0,l)$ be a Lebesgue measurable set and let $\{I_{\sigma_1}\}_{\sigma_1 \in (0,1)}$ be an arbitrary family of measurable sets such that $I \subset I_{\sigma_1} \subset (0,l)$ for all $\sigma_1 \in (0,1)$, and*

$$|I_{\sigma_1} \setminus I| \to 0 \textit{ for } \sigma_1 \to 0. \tag{A.20}$$

Then there exist the open intervals $J_{I_{\sigma_1},k} \subset (0,l)$ and $m(\sigma_1) \in \mathbb{N}$, $m(\sigma_1) \to \infty$ for $\sigma_1 \to 0$, such that

$$I_{\sigma_1} \subset \sum_{k=1}^{\infty} J_{I_{\sigma_1},k}, \ \forall \sigma_1 \in (0,1), \ |(I_{\sigma_1} \setminus \bigcup_{k=1}^{m(\sigma_1)} J_{I_{\sigma_1},k}) \cup (\bigcup_{k=1}^{m(\sigma_1)} J_{I_{\sigma_1},k} \setminus I_{\sigma_1})| \to 0 \tag{A.21}$$

and

$$|(I \setminus \bigcup_{k=1}^{m(\sigma_1)} J_{I_{\sigma_1},k}) \cup (\bigcup_{k=1}^{m(\sigma)} J_{I_{\sigma_1},k} \setminus I)| \to 0 \tag{A.22}$$

for $\sigma_1 \to 0$.

P r o o f: Since the proof is analogous to the proof of Lemma A.3, we omit it. □

Lemma A.5 *Let the functions $\mathbf{t}$, $\mathbf{n} \in L^\infty(0,l)^3$ be as constructed in the proof of Lemma A.2. Then for any $\sigma > 0$ there exist piecewise continuous functions $\mathbf{t}_\sigma$, $\mathbf{n}_\sigma$, $\sigma \in (0,1)$, with finite sets $\widehat{D}_\sigma$ of points of discontinuity such that*

$$|\mathbf{t}_\sigma| = |\mathbf{n}_\sigma| = 1 \textit{ in } [0,l] \setminus \widehat{D}_\sigma \tag{A.23}$$

and

$$\mathbf{t}_\sigma \to \mathbf{t} \textit{ and } \mathbf{n}_\sigma \to \mathbf{n} \textit{ in measure on } (0,l) \tag{A.24}$$

for $\sigma \to 0$.

P r o o f: Let the sets $I^+_{t_i,\sigma_1}$ and $I^-_{t_i,\sigma_1}$ be determined by

$$x_1 \in I^+_{t_i,\sigma_1} \Leftrightarrow t_i(x_1) \geq \sigma_1 \text{ and } x_1 \in I^-_{t_i,\sigma_1} \Leftrightarrow t_i(x_1) \leq -\sigma_1,\ i = 1,2,3, \tag{A.25}$$

for $\sigma_1 \in [0,1]$. It is easy to see that

$$(0,l) = I^+_{t_i,0} \cup I^-_{t_i,0} \tag{A.26}$$

and

$$I^+_{t_i,\sigma_1} \subset I^+_{t_i,0},\ \forall \sigma_1 \in (0,1),\ |I^+_{t_i,0} \setminus I^+_{t_i,\sigma_1}| \to 0, \tag{A.27}$$

$$I^-_{t_i,\sigma_1} \subset I^-_{t_i,0},\ \forall \sigma_1 \in (0,1),\ |I^-_{t_i,0} \setminus I^-_{t_i,\sigma_1}| \to 0 \tag{A.28}$$

for $\sigma_1 \to 0$ and $i = 1,2,3$. From Lemma A.3, it follows the existence of the open intervals $J_{I^+_{t_i,\sigma_1},k}$ and $m_i(\sigma_1)$, $i = 1,2,3$, such that $m(\sigma_1) = \max_{i=1,2,3}\{m_i(\sigma_1)\} \to \infty$ for $\sigma_1 \to 0$, and

$$\bigcup_{k=1}^{m(\sigma_1)} J_{I^+_{t_i,\sigma_1},k} \subset (0,l),\ m(\sigma_1) = 1,2,\ldots, \tag{A.29}$$

$$|(I^+_{t_i,\sigma_1} \setminus \bigcup_{k=1}^{m(\sigma_1)} J_{I^+_{t_i,\sigma_1},k}) \cup (\bigcup_{k=1}^{m(\sigma_1)} J_{I^+_{t_i,\sigma_1},k} \setminus I^+_{t_i,\sigma_1})| \to 0 \tag{A.30}$$

and

$$|(I^+_{t_i,0} \setminus \bigcup_{k=1}^{m(\sigma_1)} J_{I^+_{t_i,\sigma_1},k}) \cup (\bigcup_{k=1}^{m(\sigma_1)} J_{I^+_{t_i,\sigma_1},k} \setminus I^+_{t_i,0})| \to 0 \tag{A.31}$$

for $\sigma_1 \to 0$ and $i = 1,2,3$.

The functions t_i, $i = 1,2,3$, are defined for almost all $x_1 \in (0,l)$ and we extend these functions by zero outside of $(0,l)$. This enables us to define the functions

$$t^+_{i,\sigma_1}(x_1) = \max\{\sigma_1, t_i(x_1)\} \text{ and } t^-_{i,\sigma_1}(x_1) = -\max\{\sigma_1, -t_i(x_1)\},\ x_1 \in \mathbb{R}^1. \tag{A.32}$$

We can deduce from (A.26)–(A.28) and (A.32) that

$$t^+_{i,\sigma_1} = t_i \text{ on } I^+_{t_i,\sigma_1} \text{ and } t^-_{i,\sigma_1} = t_i \text{ on } I^-_{t_i,\sigma_1} \tag{A.33}$$

and

$$t^+_{i,\sigma_1} \to t_i \text{ on } I^+_{t_i,0} \text{ and } t^-_{i,\sigma_1} \to t_i \text{ in measure on } (0,l) \setminus I^+_{t_i,0}, \tag{A.34}$$

for $\sigma_1 \to 0$ and $i = 1,2,3$. Further, we can easily see that

$$t^+_{i,\sigma_1} * \vartheta_{\sigma_2} \to t^+_{i,\sigma_1} \text{ on } \bigcup_{k=1}^{m(\sigma_1)} J_{I^+_{t_i,\sigma_1},k} \text{ and } t^-_{i,\sigma_1} * \vartheta_{\sigma_2} \to t^-_{i,\sigma_1} \text{ on } (0,l) \setminus \bigcup_{k=1}^{m(\sigma_1)} J_{I^+_{t_i,\sigma_1},k} \tag{A.35}$$

in measure for fixed σ_1, $i = 1,2,3$ and for $\sigma_2 \to 0$ (it follows from the convergence of mollifiers in L^p-spaces, $p \in [1,\infty)$), where $\vartheta \in C_0^\infty(-1,1)$, $0 \leq \vartheta \leq 1$, $\int_{-1}^1 \vartheta(x_1)\,dx_1 = 1$ and $\vartheta_{\sigma_2}(x_1) = \frac{1}{\sigma_2}\vartheta(\frac{x_1}{\sigma_2})$. Now, we define the functions

$$\widehat{t}_{i,\sigma_1} = \begin{cases} t^+_{i,\sigma_1} & \text{on } \bigcup_{k=1}^{m(\sigma_1)} J_{I^+_{t_i,\sigma_1},k} \\ t^-_{i,\sigma_1} & \text{on } (0,l) \setminus \bigcup_{k=1}^{m(\sigma_1)} J_{I^+_{t_i,\sigma_1},k} \end{cases} \tag{A.36}$$

and

$$\widehat{t}_{i,\sigma_1,\sigma_2} = \begin{cases} t^+_{i,\sigma_1} * \vartheta_{\sigma_2} & \text{on } \bigcup_{k=1}^{m(\sigma_1)} J_{I^+_{t_{i,\sigma_1}},k} \\ t^-_{i,\sigma_1} * \vartheta_{\sigma_2} & \text{on } [0,l] \setminus \bigcup_{k=1}^{m(\sigma_1)} J_{I^+_{t_{i,\sigma_1}},k}. \end{cases} \tag{A.37}$$

We get from (A.35) that

$$\widehat{t}_{i,\sigma_1,\sigma_2} \to \widehat{t}_{i,\sigma_1} \text{ in measure on } (0,l) \tag{A.38}$$

for fixed σ_1, $i = 1,2,3$ and for $\sigma_2 \to 0$, and from (A.26), (A.31), (A.34), (A.36) that

$$\widehat{t}_{i,\sigma_1} \to \widehat{t}_i \text{ in measure on } (0,l) \text{ for } \sigma_1 \to 0 \text{ and } i = 1,2,3. \tag{A.39}$$

In addition, we can deduce from (A.1), (A.32) and (A.36)–(A.37) that

$$1 \geq |\widehat{t}_{i,\sigma_1,\sigma_2}| \geq \sigma_1 \text{ and } 1 \geq |\widehat{t}_{i,\sigma_1}| \geq \sigma_1 \text{ a.e. on } [0,l] \tag{A.40}$$

for all $\sigma_2 \in (0,1)$.

Let us take arbitrary but fixed $\widehat{\epsilon}_1 > 0$. Let, further, $E_{\widehat{\epsilon}_1,\sigma_1} = \bigcup_{i=1}^3 E_{\widehat{\epsilon}_1,\sigma_1,i}$ and $E_{\widehat{\epsilon}_1,\sigma_1,\sigma_2} = \bigcup_{i=1}^3 E_{\widehat{\epsilon}_1,\sigma_1,\sigma_2,i}$, where $|t_{i,\sigma_1} - t_i| \geq \widehat{\epsilon}_1$ on $E_{\widehat{\epsilon}_1,\sigma_1,i}$ and $|t_{i,\sigma_1,\sigma_2} - t_{i,\sigma_1}| \geq \widehat{\epsilon}_1$ on $E_{\widehat{\epsilon}_1,\sigma_1,\sigma_2,i}$, $\sigma_j \in (0,1)$, $j = 1,2$ and $i = 1,2,3$. Then we conclude from (A.38)–(A.39) that

$$|E_{\widehat{\epsilon}_1,\sigma_1,i}| \to 0 \text{ for } \sigma_1 \to 0,\ |E_{\widehat{\epsilon}_1,\sigma_1,\sigma_2,i}| \to 0$$

for $\sigma_2 \to 0$ and σ_1 arbitrary but fixed, or equivalently $\forall \widehat{\epsilon}_2\ \exists \sigma_1^0(\widehat{\epsilon}_1, \widehat{\epsilon}_2, i) > 0 :\ \forall \sigma_1 \in (0, \sigma_1^0(\widehat{\epsilon}_1, \widehat{\epsilon}_2, i))$

$$|E_{\widehat{\epsilon}_1,\sigma_1,i}| < \widehat{\epsilon}_2 \tag{A.41}$$

and $\forall \widehat{\epsilon}_3\ \exists \sigma_2^0(\widehat{\epsilon}_1, \widehat{\epsilon}_3, \sigma_1, i) > 0 :\ \forall \sigma_2 \in (0, \sigma_2^0(\widehat{\epsilon}_1, \widehat{\epsilon}_3, \sigma_1, i))$

$$|E_{\widehat{\epsilon}_1,\sigma_1,\sigma_2,i}| < \widehat{\epsilon}_3 \tag{A.42}$$

for $i = 1,2,3$. Now, we can take the family of indices $\{\widehat{\epsilon}_j(\sigma)\}_{\sigma \in (0,1)}$ such that $\widehat{\epsilon}_j(\sigma) \to 0$ for $\sigma \to 0$ and $j = 1,2,3$. Then there exist $\sigma_1^0(\widehat{\epsilon}_1(\sigma), \widehat{\epsilon}_2(\sigma), i) > 0 :\ \forall \sigma_1 \in (0, \sigma_1^0(\widehat{\epsilon}_1(\sigma), \widehat{\epsilon}_2(\sigma), i))$

$$|E_{\widehat{\epsilon}_1(\sigma),\sigma_1,i}| < \widehat{\epsilon}_2(\sigma). \tag{A.43}$$

Let $\sigma_1(\sigma) \in (0, \min_{i=1,2,3}\{\sigma_1^0(\widehat{\epsilon}_1(\sigma), \widehat{\epsilon}_2(\sigma), i)\})$ be such that $\sigma_1(\sigma) \to 0$ for $\sigma \to 0$. Further, there exists $\sigma_2^0(\widehat{\epsilon}_1(\sigma), \widehat{\epsilon}_3(\sigma), \sigma_1(\sigma), i) > 0 : \forall \sigma_2 \in (0, \sigma_2^0(\widehat{\epsilon}_1(\sigma), \widehat{\epsilon}_3(\sigma), \sigma_1(\sigma), i))$

$$|E_{\widehat{\epsilon}_1(\sigma),\sigma_1(\sigma),\sigma_2,i}| < \widehat{\epsilon}_3(\sigma) \tag{A.44}$$

for $i = 1,2,3$. Let, now, $\sigma_2(\sigma) \in (0, \min_{i=1,2,3}\{\sigma_2^0(\widehat{\epsilon}_1(\sigma), \widehat{\epsilon}_3(\sigma), \sigma_1(\sigma), i)\})$ be such that $\sigma_2(\sigma) \to 0$ for $\sigma \to 0$. We define the functions

$$t_{i,\sigma} = \frac{\widehat{t}_{i,\sigma_1(\sigma),\sigma_2(\sigma)}}{|\widehat{\mathbf{t}}_{\sigma_1(\sigma),\sigma_2(\sigma)}|},\ \ i = 1,2,3. \tag{A.45}$$

Since

$$|\widehat{t}_{i,\sigma_1(\sigma),\sigma_2(\sigma)}| \geq \sigma_1(\sigma) > 0 \text{ on } (0,l) \tag{A.46}$$

as a result of (A.40), the estimate

$$|\widehat{\mathbf{t}}_{\sigma_1(\sigma),\sigma_2(\sigma)}|^2 \overset{(A.1)}{=} |\mathbf{t}|^2 + |\widehat{\mathbf{t}}_{\sigma_1(\sigma),\sigma_2(\sigma)}|^2 - |\mathbf{t}|^2$$

$$= 1 + \sum_{i=1}^{3} (\widehat{t}_{i,\sigma_1(\sigma),\sigma_2(\sigma)} - t_i)(\widehat{t}_{i,\sigma_1(\sigma),\sigma_2(\sigma)} + t_i) \geq 1 - 6\widehat{\epsilon}_1(\sigma) \tag{A.47}$$

on $(0,l) \setminus (E_{\widehat{\epsilon}_1(\sigma),\sigma_1(\sigma)} \cup E_{\widehat{\epsilon}_1(\sigma),\sigma_1(\sigma),\sigma_2(\sigma)})$ enables us to deduce from (A.45) that

$$\mathbf{t}_\sigma \to \mathbf{t} \text{ in measure on } (0,l). \tag{A.48}$$

We can easily deduce from (A.37), (A.45) and (A.46) that the functions $\mathbf{t}_\sigma$ are piecewise continuous with finitely many points of discontinuity, because the number of intervals $J_{I^+_{t_i,\sigma_1},k}$ is for all $\sigma_1 \in (0,1)$ finite . Analogously we can define the function $\mathbf{n}_\sigma$. □

In the next lemma, we discuss the orthogonality of the approximating vectors.

Lemma A.6 *Let the functions* $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b} \in L^\infty(0,l)^3$ *be constructed in the proof of Lemma A.3. Then there exist piecewise continuous functions* $\check{\mathbf{t}}_\sigma$, $\check{\mathbf{n}}_\sigma$, $\check{\mathbf{b}}_\sigma$, $\sigma \in (0,1)$, *with a finite set* D_σ *of points of discontinuity such that*

$$|\check{\mathbf{t}}_\sigma| = |\check{\mathbf{n}}_\sigma| = |\check{\mathbf{b}}_\sigma| = 1,\ \check{\mathbf{t}}_\sigma \perp \check{\mathbf{n}}_\sigma \perp \check{\mathbf{b}}_\sigma \text{ in } [0,l] \setminus D_\sigma \tag{A.49}$$

for all $\sigma \in (0,1)$ *and*

$$\check{\mathbf{t}}_\sigma \to \mathbf{t},\ \check{\mathbf{n}}_\sigma \to \mathbf{n},\ \check{\mathbf{b}}_\sigma \to \mathbf{b} \text{ in measure on } [0,l] \tag{A.50}$$

for $\sigma \to 0$.

P r o o f: Let us define the set $E_{\frac{1}{12},\sigma} = \bigcup_{i=1}^3 E_{\frac{1}{12},\sigma,t_i} \cup E_{\frac{1}{12},\sigma,n_i}$, where $|t_{i,\sigma} - t_i| \geq \frac{1}{12}$ on $E_{\frac{1}{12},\sigma,t_i}$ and $|n_{i,\sigma} - n_i| \geq \frac{1}{12}$ on $E_{\frac{1}{12},\sigma,n_i}$, where the functions $t_{i,\sigma}$ and $n_{i,\sigma}$ are constructed in the previous lemma. From (A.24), it follows that

$$|E_{\frac{1}{12},\sigma}| \to 0 \text{ for } \sigma \to 0. \tag{A.51}$$

We can easily see that then

$$\sup_{\sigma\in(0,1)} \|(\mathbf{t}_\sigma, \mathbf{n}_\sigma)\|_{L^\infty([0,l]\setminus E_{\frac{1}{12},\sigma})} \leq \sup_{\sigma\in(0,1)} \|(\mathbf{t}_\sigma - \mathbf{t}, \mathbf{n}_\sigma)\|_{L^\infty([0,l]\setminus E_{\frac{1}{12},\sigma})}$$

$$+ \sup_{\sigma\in(0,1)} \|(\mathbf{t}, \mathbf{n}_\sigma - \mathbf{n})\|_{L^\infty([0,l]\setminus E_{\frac{1}{12},\sigma})} \leq \frac{1}{2}. \tag{A.52}$$

(A.51) enables us to use Lemma A.4 with $I = \emptyset$ and to construct the open intervals $J_{E_{\frac{1}{12},\sigma},k} \subset (0,l)$ such that

$$E_{\frac{1}{12},\sigma} \subset \bigcup_{k=1}^{\infty} J_{E_{\frac{1}{12},\sigma},k}, \tag{A.53}$$

where $J_{E_{\frac{1}{12},\sigma},k} = (a_{E_{\frac{1}{12},\sigma},k}, b_{E_{\frac{1}{12},\sigma},k})$ and

$$|\bigcup_{k=1}^{\infty} J_{E_{\frac{1}{12},\sigma},k}| \to 0 \text{ for } \sigma \to 0. \tag{A.54}$$

We can suppose without loss of generality that the intervals $J_{E_{\frac{1}{12},\sigma},k}$ are mutually disjoint and that

$$dist(J_{E_{\frac{1}{12},\sigma},k_1}, J_{E_{\frac{1}{12},\sigma},k_2}) = \min\{|a_{E_{\frac{1}{12},\sigma},k_1} - b_{E_{\frac{1}{12},\sigma},k_2}|, |a_{E_{\frac{1}{12},\sigma},k_2} - b_{E_{\frac{1}{12},\sigma},k_1}| > 0$$

for $k_1 \neq k_2$ arbitrary. Further, we take a family of positive indices $\{\widehat{\sigma}_k(\sigma)\}_{k=1}^{\infty}$, $\sigma \in (0,1)$, such that

$$\sum_{k=1}^{\infty} \widehat{\sigma}_k(\sigma) \to 0 \tag{A.55}$$

for $\sigma \to 0$. We define the intervals $J^*_{E_{\frac{1}{12},\sigma},k} = (a_{E_{\frac{1}{12},\sigma},k} - \widehat{\sigma}_k(\sigma), b_{E_{\frac{1}{12},\sigma},k} + \widehat{\sigma}_k(\sigma)) \cap (0,l)$. From (A.53)–(A.55), it follows that

$$|E_{\frac{1}{12},\sigma}| \le |\bigcup_{k=1}^{\infty} J_{E_{\frac{1}{12},\sigma},k}| \le |\bigcup_{k=1}^{\infty} J^*_{E_{\frac{1}{12},\sigma},k}|$$

$$= |\bigcup_{k=1}^{\infty} J_{E_{\frac{1}{12},\sigma},k} \cup \bigcup_{k=1}^{\infty} (a_{E_{\frac{1}{12},\sigma},k} - \widehat{\sigma}_k(\sigma), a_{E_{\frac{1}{12},\sigma},k}) \cup \bigcup_{k=1}^{\infty} (b_{E_{\frac{1}{12},\sigma},k}, b_{E_{\frac{1}{12},\sigma},k} + \widehat{\sigma}_k(\sigma))|$$

$$\le |\bigcup_{k=1}^{\infty} J_{E_{\frac{1}{12},\sigma},k}| + 2\sum_{k=1}^{\infty} \widehat{\sigma}_k(\sigma) \to 0 \tag{A.56}$$

for $\sigma \to 0$.

Now, using Proposition 3.1 we can construct the piecewise continuous normal and binormal vector functions $\ddot{\mathbf{n}}_\sigma$ and $\ddot{\mathbf{b}}_\sigma$ to the curve $\mathbf{\Phi}_\sigma$, where we put

$$\mathbf{\Phi}_\sigma(x_1) = \int_0^{x_1} \mathbf{t}_\sigma(z_1)\, dz_1 + \mathbf{\Phi}(0),\ x_1 \in [0,l].$$

It is easy to see that the functions $\ddot{\mathbf{n}}_\sigma$ and $\ddot{\mathbf{b}}_\sigma$ have the same points of discontinuity as $\mathbf{t}_\sigma$.

Let us denote $\widehat{\mathbf{n}}^0_\sigma = n_{i,\sigma} - (\mathbf{n}_\sigma, \mathbf{t}_\sigma) t_{i,\sigma}$ on $(0,l)$. Now, we define the function $\widehat{\mathbf{n}}^{k+1}_\sigma$ in this way

$$\widehat{n}^{k+1}_{i,\sigma} =$$

$$\begin{cases} \widehat{n}^k_{i,\sigma} \text{ on } [0,l] \setminus [a_{E_{\frac{1}{12},\sigma},k} - \widehat{\sigma}_k(\sigma), b_{E_{\frac{1}{12},\sigma},k} + \widehat{\sigma}_k(\sigma)] \\ (1 - \widehat{l}_{i,1,\sigma,k})\widehat{n}^k_{i,\sigma} + \widehat{l}_{i,1,\sigma,k}\ddot{n}_{i,\sigma} \text{ on } [a_{E_{\frac{1}{12},\sigma},k} - \widehat{\sigma}_k(\sigma), a_{E_{\frac{1}{12},\sigma},k}] \\ \ddot{n}_{i,\sigma} \text{ on } [a_{E_{\frac{1}{12},\sigma},k}, b_{E_{\frac{1}{12},\sigma},k}] \\ (1 - \widehat{l}_{i,2,\sigma,k})\ddot{n}_{i,\sigma} + \widehat{l}_{i,2,\sigma,k}\widehat{n}^k_{i,\sigma} \text{ on } [b_{E_{\frac{1}{12},\sigma},k}, b_{E_{\frac{1}{12},\sigma},k} + \widehat{\sigma}_k(\sigma)] \end{cases} \tag{A.57}$$

where $\widehat{l}_{i,m,\sigma}$, $i = 1,2,3$, $m = 1,2$ and $k = 1,2,\ldots$, are linear functions such that

$$\widehat{l}_{i,1,\sigma,k}(a_{E_{\frac{1}{12},\sigma},k} - \widehat{\sigma}_k(\sigma)) = 0,\ \widehat{l}_{i,1,\sigma}(a_{E_{\frac{1}{12},\sigma},k}) = 1,$$

$$\widehat{l}_{i,2,\sigma,k}(b_{E_{\frac{1}{12},\sigma},k}) = 0,\ \widehat{l}_{i,2,\sigma,k}(b_{E_{\frac{1}{12},\sigma},k} + \widehat{\sigma}_k(\sigma)) = 1.$$

The definition (A.57) together with (A.55) enable us to deduce that $\forall\widehat{\epsilon} > 0\ \exists k_1(\widehat{\epsilon},\sigma) \in \mathbb{N}_0\colon \forall k_2 \geq k_1(\widehat{\epsilon},\sigma)$ and for arbitrary but fixed σ

$$\Big|x_1 \in (0,l):\ |\widehat{\mathbf{n}}_\sigma^{k_1(\widehat{\epsilon},\sigma)}(x_1) - \widehat{\mathbf{n}}_\sigma^{k_2}(x_1)| > 0\Big| = |\bigcup_{k=k_1(\widehat{\epsilon},\sigma)}^{k_2-1} J^*_{E_{\frac{1}{12},\sigma},k}|$$

$$\leq |\bigcup_{k=k_1(\widehat{\epsilon},\sigma)}^{k_2-1} J_{E_{\frac{1}{12},\sigma},k}| + 2\sum_{k=k_1(\widehat{\epsilon},\sigma)}^{k_2-1} \widehat{\sigma}_k(\sigma) < \widehat{\epsilon}. \tag{A.58}$$

Then, from boundedness $\|\widehat{\mathbf{n}}_\sigma^k\|_\infty \leq 2$, we conclude that the sequence $\{\widehat{\mathbf{n}}_\sigma^k\}_{k=1}^\infty$ is for arbitrary but fixed $\sigma \in (0,1)$ a Cauchy and thus convergent sequence in $L^p(0,l)^3$, $p \in [1,\infty)$, which implies the convergence in measure, i.e. there exits a function $\widehat{\mathbf{n}}_\sigma$ such that

$$\widehat{\mathbf{n}}_\sigma^k \to \widehat{\mathbf{n}}_\sigma \text{ in measure on } (0,l) \tag{A.59}$$

for $k \to \infty$.

It is easy to see from (A.52) and (A.57) that the vector $\widehat{\mathbf{n}}_\sigma$ is orthogonal to $\mathbf{t}_\sigma$ and all functions $\widehat{n}_{i,\sigma}$, $i = 1,2,3$, cannot be equal to zero at the same point, if

$$\begin{aligned}&\widehat{n}_{i,\sigma}^k(a_{E_{\frac{1}{12},\sigma},k} - \frac{\widehat{\sigma}_k(\sigma)}{2}) \neq -\ddot{n}_{i,\sigma}(a_{E_{\frac{1}{12},\sigma},k} - \frac{\widehat{\sigma}_k(\sigma)}{2})\\ &\widehat{n}_{i,\sigma}^k(b_{E_{\frac{1}{12},\sigma},k} + \frac{\widehat{\sigma}_k(\sigma)}{2}) \neq -\ddot{n}_{i,\sigma}(b_{E_{\frac{1}{12},\sigma},k} + \frac{\widehat{\sigma}_k(\sigma)}{2})\end{aligned} \tag{A.60}$$

for some i, $i = 1,2,3$, and $k = 0,1,\ldots$. We refer the reader to (3.17)–(3.19) for the idea of the modification of the definition (A.57) in the case that one of the conditions in (A.60) does not hold. It is obvious from (A.57) that the function $\widehat{\mathbf{n}}_\sigma$ has again finitely many points of discontinuity, because the functions $\mathbf{n}_\sigma - (\mathbf{n}_\sigma, \mathbf{t}_\sigma)\mathbf{t}_\sigma$ and $\ddot{\mathbf{n}}_\sigma$ have finitely many points of discontinuity.

From (A.24), (A.56), (A.57) and (A.59), it follows that

$$\mathbf{t}_\sigma \perp \widehat{\mathbf{n}}_\sigma,\ |\widehat{\mathbf{n}}_\sigma| > 0 \text{ and } \widehat{\mathbf{n}}_\sigma \to \mathbf{n} \tag{A.61}$$

in measure on $(0,l)$ for $\sigma \to 0$. Taking

$$\check{\mathbf{n}}_\sigma = \frac{\widehat{\mathbf{n}}_\sigma}{|\widehat{\mathbf{n}}_\sigma|} \tag{A.62}$$

and denoting $\check{\mathbf{t}}_\sigma = \mathbf{t}_\sigma$, we get from (A.24), (A.52), (A.56), (A.57), (A.59), (A.61) and (A.62) that

$$\check{\mathbf{t}}_\sigma \to \mathbf{t} \text{ and } \check{\mathbf{n}}_\sigma \to \mathbf{n} \text{ in measure on } (0,l) \tag{A.63}$$

and the function $\check{\mathbf{n}}_\sigma$ is piecewise continuous with finitely many points of discontinuity.

Defining the functions

$$\check{\mathbf{b}}_\sigma = \check{\mathbf{t}}_\sigma \times \check{\mathbf{n}}_\sigma,\ \ \check{\mathbf{\Phi}}_\sigma(x_1) = \int_0^{x_1} \mathbf{t}_\sigma(z_1)\ dz_1 + \mathbf{\Phi}(0),\ \ x_1 \in (0,l),$$

the proof is finished. □

Remark A.7 To obtain the smooth approximation of the functions $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b} \in L^\infty(0,l)^3$, we can use Proposition 3.2 and we get again (3.2)–(3.5) with the convergence in measure instead of the pointwise convergence.

Remark A.8 Theorem 8.8 remains valid for $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b} \in L^\infty(0,l)^3$. The only thing we must change in the previous proofs, is the application of the Lebesgue theorem, for instance in (7.79). But it is easy to check that

$$\left(\int_0^l |\mathbf{t}_\epsilon - \mathbf{t}|^2 \|(\partial_1 \mathbf{U}_*, \mathbf{t})(x_1)\|^2_{H^{-1}(S)}\ dx_1\right)^{\frac{1}{2}} \to 0$$

for $\mathbf{t}_\epsilon \to \mathbf{t}$ in measure and for $\epsilon \to 0$, because we have the estimate

$$\left(\int_0^l |\mathbf{t}_\epsilon - \mathbf{t}|^2 \|(\partial_1 \mathbf{U}_*, \mathbf{t})(x_1)\|^2_{H^{-1}(S)}\ dx_1\right)^{\frac{1}{2}}$$

$$\leq \widehat{\epsilon}\|(\partial_1 \mathbf{U}_*, \mathbf{t})(x_1)\|_{L^2(0,l;H^{-1}(S))} + 2\left(\int_{I_{\widehat{\epsilon},\epsilon}} \|(\partial_1 \mathbf{U}_*, \mathbf{t})(x_1)\|^2_{H^{-1}(S)}\ dx_1\right)^{\frac{1}{2}},$$

where $|\mathbf{t}_\epsilon - \mathbf{t}| > \widehat{\epsilon}$ on $I_{\widehat{\epsilon},\epsilon}$ and $|I_{\widehat{\epsilon},\epsilon}| \to 0$ for $\epsilon \to 0$. We can replace analogously the application of the Lebesgue theorem in (7.72), etc. Therefore, one can obtain the same results for $W^{1,\infty}(0,l)^3$ curved rods.

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