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Large deviation principle for the single point catalytic super-Brownian motion

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ABSTRACT. In the single point catalytic super-Brownian motion "particles" branch only if they meet the position of the single point catalyst. If the branching rate tends to zero, the model degenerates to the heat flow. We are concerned with large deviation probabilities related to this law of large numbers. To this aim the well-known explicit representation of the model by excursion densities is heavily used. The rate function is described by the Fenchel-Legendre transform of log-exponential moments described by a log-Laplace equation.

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1. INTRODUCTION

1.1. **Motivation.** Starting with Dawson and Fleischmann (1983) [1], a large area of spatial branching models in random media was developed. For recent surveys, see [3, 4, 13]. But there are only a few papers in this field which deal with large deviation probabilities, see Greven and den Hollander (1991,1992) [11, 12]. In particular, we do not know a single reference for a large deviation result in the important case of a branching model with a singular catalytic medium.

Here we pay attention to the simplest model of this kind: a super-Brownian motion $X = \{X_t : t \geq 0\}$ on the real line R with a single point catalyst $\varrho\delta_0$, where $\varrho > 0$ is a fixed constant, and δ_0 denotes the Dirac delta function centered at 0. This process was introduced in Dawson and Fleischmann (1994) [2], and further investigated in Dawson et al. (1995) [5], Dynkin (1995) [7], and Fleischmann and Le Gall (1995) [10]. Roughly speaking, here the tiny "particles" move as independent Brownian motions in R with migration constant $\varkappa > 0$, and split according to critical Feller's branching diffusion with rate ϱ, but only if they pass the position $0 \in \mathsf{R}$ of the single point catalyst. In order for the branching to happen, the particles have to accumulate enough local time at 0.

What law of large numbers on X we have in mind we are asking for related large deviations? Well, if the branching rate $\varrho > 0$ tends to zero, one expects that X approaches the heat flow. Is there a related large deviation principle?

For the classical case of a constant medium, that is if $\varrho\delta_0$ is replaced by ϱ, large deviation principles had been established in Fleischmann and Kaj (1994) [9], Fleischmann et al. (1996) [8], and Schied (1996) [14]. To deal with large deviations in a general catalytic situation however seems to be rather hopeless at the first sight. However in the present special case of the single point catalyst $\varrho\,\delta_0$ one can use a different explicit representation of the model found by Fleischmann and Le Gall (1995) [10], which we want briefly recall in the next subsection.

1.2. **The single point catalytic super-Brownian motion** X**.** More formally, the single point catalytic super-Brownian motion $X = \big(X,\, P_\mu,\, \mu \in \mathcal{M}(\mathsf{R})\big)$ is a (time-homogenous) continuous measure-valued Markov process determined by its log-Laplace transition functional

$$-\log P_\mu \mathrm{e}^{-\langle X_t, \varphi\rangle} \;=\; \big\langle \mu, w(0,\,\cdot\,)\big\rangle, \qquad \mu \in \mathcal{M}(\mathsf{R}), \quad t \geq 0, \quad \varphi \in \Phi_+(\mathsf{R}). \tag{1.1}$$

Here, $\Phi(\mathsf{R})$ denotes the separable Banach space of all continuous functions $\varphi : \mathsf{R} \to \mathsf{R}$ such that the limit $\lim_{|a|\uparrow\infty} \mathrm{e}^{|a|}\, \varphi(a)$ exists in R, furnished with the norm $\|\varphi\| := \sup_{a\in\mathsf{R}} \mathrm{e}^{|a|}\, |\varphi|(a)$, and $\Phi_+(\mathsf{R})$ is the cone of all non-negative members of $\Phi(\mathsf{R})$. Furthermore, $\langle \mu, \varphi\rangle$ denotes the integral $\int \mu(\mathrm{d}a)\, \varphi(a)$ (integrated over the whole space). $\mathcal{M}(\mathsf{R})$ is the set of all (non-negative) measures μ on R such that $\langle \mu, \varphi\rangle < \infty$ for all $\varphi \in \Phi_+(\mathsf{R})$. Equipped with the finest topology such that all these mappings $\mu \mapsto \langle \mu, \varphi\rangle$ are continuous, $\mathcal{M}(\mathsf{R})$ is a Polish space. P_μ indicates that X starts at time $t = 0$ with the measure $X_0 = \mu$. Finally, for t, φ fixed, $w = w_{t,\varphi} = \big\{w(s,a) : \, s \geq 0,\; a \in \mathsf{R}\big\}$ denotes the unique non-negative solution to the log-Laplace equation

$$w(s,a) \;=\; 1_{\{s<t\}} \int \mathrm{d}b\; p_{t-s}(b-a)\, \varphi(b) \;-\; \varrho \int_s^\infty \mathrm{d}r\; p_{r-s}(a)\, w^2(r,0), \tag{1.2}$$

$s \geq 0,\; a \in \mathsf{R},$ related to the formal partial differential equation

$$-\frac{\partial}{\partial s} w(s,a) \;=\; \frac{\varkappa^2}{2}\frac{\partial^2}{\partial a^2} w(s,a) \;-\; \varrho\,\delta_0(a)\, w^2(s,a), \qquad 0 < s < t, \quad a \in \mathsf{R}, \tag{1.3}$$

with terminal condition $w(t-,\,\cdot\,) = \varphi$. Here, p is the heat kernel in R with migration constant $\varkappa > 0$:

$$p_t(a) \;:=\; \frac{1}{\sqrt{2\pi\varkappa t}}\, \exp\Big[-\frac{a^2}{2\varkappa t}\Big], \qquad t > 0, \quad a \in \mathsf{R}. \tag{1.4}$$

From [2, Theorem 1.2.4] it is known that the measure-valued process X has a jointly continuous occupation density field denoted by $y = \big\{y(t,a) : t \geq 0,\; a \in \mathsf{R}\big\}$. That is,

$$\int_0^t \mathrm{d}s\; \langle X_s\,, \varphi\rangle \;=\; \int \mathrm{d}a\;\, y(t,a)\, \varphi(a), \qquad t \geq 0, \quad \varphi \in \Phi_+(\mathsf{R}). \tag{1.5}$$

In particular, $t \mapsto y(t,0)$ is a continuous non-decreasing function, determining a continuous random measure λ on R_+, which by Dawson et al. (1995) [5], or [10] is singular. On the other hand, by [2, Theorem 1.2.5 and Theorem 1.2.2], λ has carrying Hausdorff dimension one, and off the catalyst's position X has a jointly continuous density field denoted by $x = \big\{x(t,a) : t > 0,\; a \neq 0\big\}$.

For the moment, fix a (non-zero) initial measure $X_0 = \mu \in \mathcal{M}(\mathsf{R})$. The point is, that by [10, Theorem 1] there is an independent construction (in law) of the occupation density measure λ, namely as the *total occupation measure*

$$V := \int_0^\infty \mathrm{d}s\, U_s \tag{1.6}$$

of a *super-$\frac{1}{2}$-stable subordinator* $U = \{U_t : t \geq 0\}$ on R_+ starting from a particular measure $U_0 = \nu_\mu$ (see (1.13) below). More precisely, $U = \big(U,\, P_\nu,\, \nu \in \mathcal{M}(\mathsf{R}_+)\big)$

is the measure-valued continuous Markov process determined by its log-Laplace transition functional

$$-\log P_\nu \mathrm{e}^{-\langle U_t,\varphi\rangle} \;=\; \big\langle \nu, u(0,\cdot\,)\big\rangle, \qquad \nu\in\mathcal{M}(\mathsf{R}_+), \quad t\geq 0, \quad \varphi\in\Phi_+(\mathsf{R}_+), \quad (1.7)$$

where for t,φ fixed, $u=u_{t,\varphi}=\big\{u(s,a):\ s,a\in\mathsf{R}_+\big\}$ is the unique non-negative solution to the log-Laplace equation

$$u(s,a) \;=\; 1_{\{s<t\}}\int_0^\infty \mathrm{d}b\, q_{t-s}(b-a)\,\varphi(b) - \varrho\int_s^\infty \mathrm{d}r\int_0^\infty \mathrm{d}b\, q_{r-s}(b-a)\, u^2(r,b), \quad (1.8)$$

$s,a\in\mathsf{R}_+$. Here $\mathcal{M}(\mathsf{R}_+)$ and $\Phi(\mathsf{R}_+)$ are defined analogously to $\mathcal{M}(\mathsf{R})$ and $\Phi(\mathsf{R})$ using only the non-negative part R_+ of R, and q is the transition density of the $\frac{1}{2}$-stable subordinator on R:

$$q_s(a) \;:=\; 1_{\{a>0\}}\,\frac{s}{\sqrt{2\pi\varkappa a^3}}\,\exp\Big[-\frac{s^2}{2\varkappa a}\Big], \qquad s>0, \quad a\in\mathsf{R}, \qquad (1.9)$$

where additionally we formally set $q_0=\delta_0$. Note that V has log-Laplace functional

$$-\log P_\nu \mathrm{e}^{-\langle V,\varphi\rangle} \;=\; \langle \nu, v\rangle, \qquad \nu\in\mathcal{M}(\mathsf{R}_+), \quad \varphi\in\Phi_+(\mathsf{R}_+), \qquad (1.10)$$

where for φ fixed, $v=v_\varphi=\big\{v(a):\, a\in\mathsf{R}_+\big\}$ is the unique non-negative solution to the log-Laplace equation

$$v(a) \;=\; \int_a^\infty \mathrm{d}b\,\sqrt{\frac{\varkappa}{2\pi(b-a)}}\,\varphi(b) \;-\; \varrho\int_a^\infty \mathrm{d}b\,\sqrt{\frac{\varkappa}{2\pi(b-a)}}\,v^2(b), \qquad a\in\mathsf{R}_+, \quad (1.11)$$

(since

$$\int_0^\infty \mathrm{d}s\; q_s(a) \;=\; 1_{\{a>0\}}\sqrt{\frac{\varkappa}{2\pi a}}\,, \qquad a\in\mathsf{R}). \qquad (1.12)$$

From now on we assume that for $X_0=\mu\in\mathcal{M}(\mathsf{R})$ fixed, U starts with $U_0=\nu_\mu\in\mathcal{M}(\mathsf{R}_+)$ defined by

$$\langle \nu_\mu\,,\varphi\rangle \;:=\; \int\!\mu(\mathrm{d}a)\int_0^\infty \mathrm{d}r\; q_{|a|}(r)\,\varphi(r), \qquad \varphi\in\Phi_+(\mathsf{R}_+). \qquad (1.13)$$

Besides the alternative construction $\lambda\overset{\mathcal{L}}{=}V$, all the randomness of X is restored in V in the sense, that its density field x can explicitly be written (in law) using Brownian excursion densities of excursions starting from the catalyst's position 0. In particular, x satisfies the heat equation off the catalyst with the random singular boundary condition λ. To make this more precise, we introduce the transition density p^* of *Brownian motion killed at* 0:

$$p^*_t(a,b) \;:=\; 1_{\{ab>0\}}\big[p_t(b-a)-p_t(b+a)\big], \qquad (1.14)$$

$t>0$, $a,b\in\mathsf{R}$. Then according to the main result of [10], the *representation formula*

$$x_t(a) \;=\; \int\!\mu(\mathrm{d}b)\,p^*_t(b,a) \;+\; \int_{[0,t)}\lambda(\mathrm{d}s)\,q_{|a|}(t-s), \qquad (1.15)$$

$t>0$, $a\in\dot{\mathsf{R}}:=\mathsf{R}\backslash\{0\}$, holds P_μ-a.s. So the first term takes care of the initial particles which do not reach the catalyst, whereas the second term gives the contribution of particles born by branching at the catalyst's position and providing Brownian excursions away from 0.

1.3. **Main result.** Recall that $X =: X^\varrho$ depends on the branching rate $\varrho > 0$ and we want to let $\varrho \downarrow 0$. The corresponding large deviation principle formulated in terms of the related density fields $x =: x^\varrho$ is the content of the following theorem. Write

$$\Lambda_\varphi^\varrho(a) \;:=\; \Lambda_\varphi(a) \;:=\; \log P_{\delta_a} \mathrm{e}^{\langle V, \varphi \rangle}, \qquad \varphi \in \Phi(\mathsf{R}_+), \quad a \in \mathsf{R}_+, \tag{1.16}$$

for the log-exponential moments of the total occupation measure $V =: V^\varrho$ from (1.6). Recall that $X_0 = \mu \in \mathcal{M}(\mathsf{R}) \backslash \{0\}$ is fixed.

Theorem 1.1 (**Large deviation principle for** x^ϱ**).** *As* $\varrho \downarrow 0$, *the family* $\{x^\varrho : \varrho > 0\}$ *of density fields satisfies a large deviation principle on* $\mathbf{C} := \mathcal{C}\big((0,\infty) \times \dot{\mathsf{R}}\big)$ *with good convex rate function*

$$J(f) \;:=\; \sup_{\varphi \in \Phi(\mathsf{R}_+)} \Big(\langle \eta_f, \varphi \rangle \,-\, \big\langle \nu_\mu\,,\, \Lambda_\varphi^1 \big\rangle \Big), \qquad f \in \mathbf{C}, \tag{1.17}$$

with ν_μ *from* (1.13) *and* Λ_φ^1 *from* (1.16), *and where, for* $f \in \mathbf{C}$ *fixed,* $\eta_f = \eta \in \mathcal{M}(\mathsf{R}_+)$ *is the (unique) solution, if it exists, to the equation*

$$f_t(a) \;=\; \int \mu(\mathrm{d}b)\, p_t^*(b,a) \,+\, \int_{[0,t)} \eta(\mathrm{d}s)\, q_{|a|}(t-s), \qquad t > 0, \quad a \in \dot{\mathsf{R}}, \tag{1.18}$$

and otherwise we set $\eta_f := \infty$ *and* $J(f) := \infty$.

Here $\mathcal{C}_+\big((0,\infty) \times \dot{\mathsf{R}}\big)$ denotes the set of all non-negative continuous functions on the locally compact space $(0,\infty) \times \mathsf{R}\backslash\{0\}$ equipped with the topology of uniform convergence on compacta. Clearly, according to a general terminology (see, for instance, Dembo and Zeitouni (1993) [6, § 1.2]), the statement that $\{x^\varrho : \varrho > 0\}$ satisfies a *large deviation principle on* $\mathbf{C}$ *with rate function* J means that for each Borel subset Γ of $\mathbf{C}$,

$$-\inf_{f \in \Gamma^\circ} J(f) \leq \liminf_{\varrho \downarrow 0} \varrho \log P_\mu(x^\varrho \in \Gamma) \leq \limsup_{\varrho \downarrow 0} \varrho \log P_\mu(x^\varrho \in \Gamma) \leq -\inf_{f \in \bar{\Gamma}} J(f),$$

where Γ° and $\bar{\Gamma}$ denote the interior and the closure of Γ, respectively. Moreover, the rate function $J : \mathbf{C} \to [0,\infty]$ has to be lower semicontinuous, and it is called *good* if all of its level sets are compact.

The occurrence of the log-exponential moments Λ_φ^1 within the rate function seems to be a bit complicated, but they can actually be characterized as unique solutions of an equation, see Proposition 2.1 below.

1.4. **Method of proof.** Our approach to the proof of Theorem 1.1 is to establish first the following large deviation principle for $V = V^\varrho$ as $\varrho \downarrow 0$, and then to use the contraction principle based on the fact that the correspondence $\eta \mapsto f =: F(\eta)$ defined by (1.18) maps $\mathcal{M}(\mathsf{R}_+)$ continuously into $\mathbf{C}$ and is one-to-one (see Lemma 3.2 below).

Proposition 1.1 (**Large deviation principle for** V^ϱ**).** *Fix* $\nu \in \mathcal{M}(\mathsf{R}_+)$. *As* $\varrho \downarrow 0$, *the family* $\{V^\varrho : \varrho > 0\}$ *of total occupation measures of the super-*$\frac{1}{2}$*-stable subordinators* $U = (U, P_\nu) =: (U^\varrho, P_\nu^\varrho) = U^\varrho$ *satisfies a large deviation principle on* $\mathcal{M}(\mathsf{R}_+)$ *with good convex rate function*

$$I(\eta) \;:=\; \sup_{\varphi \in \Phi(\mathsf{R}_+)} \Big(\langle \eta, \varphi \rangle \,-\, \big\langle \nu,\, \Lambda_\varphi^1 \big\rangle \Big), \qquad \eta \in \mathcal{M}(\mathsf{R}_+), \tag{1.19}$$

with Λ_φ^1 *from* (1.16).

For the proof of this proposition in Subsection 3.1 we will exploit ideas from [9].

The remaining paper is organized as follows. In the next section, we deal with the log-Laplace equation related to the random measure V in a functional analytic setting. Using this we describe the exponential moments of V (Proposition 2.1). Then the large deviation proofs follow in Section 3.

2. Exponential moments

The purpose of this section is to verify that the random measure $\lambda \stackrel{\mathcal{L}}{=} V$ has some finite exponential moments.

2.1. **To equation (1.11).** Recall that via its log-Laplace functional, the random measure V is related to equation (1.11), we now want to deal with in a Banach space setting. Recall the separable Banach space $\Phi(\mathsf{R}_+)$ introduced in Subsection 1.2. Set

$$\mathcal{T}\varphi\,(a) \;:=\; \int_a^\infty \mathrm{d}b\, \sqrt{\frac{\varkappa}{2\pi(b-a)}}\;\varphi(b), \qquad \varphi\in\Phi(\mathsf{R}_+), \quad a\in\mathsf{R}_+. \tag{2.1}$$

Lemma 2.1 (**Continuity of $\mathcal{T}$**). *The functional $\mathcal{T}$ maps $\Phi(\mathsf{R}_+)$ continuously into itself.*

Proof. Introduce the reference function $\phi(a) := \mathrm{e}^{-a}$, $a\geq 0$. Clearly, $a\mapsto \mathcal{T}\varphi\,(a)$ is continuous. Moreover,

$$\frac{\mathcal{T}\varphi\,(a)}{\phi(a)} \;=\; \int_0^\infty \mathrm{d}b\,\sqrt{\frac{\varkappa}{2\pi b}}\;\frac{\varphi(a+b)}{\phi(a)}\,, \qquad a\geq 0. \tag{2.2}$$

Then $|\varphi|\leq\|\varphi\|\phi$ and dominated convergence imply that $\mathcal{T}\varphi$ belongs to $\Phi(\mathsf{R}_+)$. For $n\geq 1$,

$$\begin{aligned}\mathcal{T}\phi^n(a) \;&\leq\; \phi^n(a)\int_a^{a+1}\mathrm{d}b\,\sqrt{\frac{\varkappa}{2\pi(b-a)}} \;+\; \sqrt{\frac{\varkappa}{2\pi}}\int_{a+1}^\infty \mathrm{d}b\;\phi^n(b)\\ &\leq\; 2\sqrt{\frac{\varkappa}{2\pi}}\,\phi^n(a) + \frac{1}{n}\sqrt{\frac{\varkappa}{2\pi}}\,\phi^n(a+1) \;<\; 2\sqrt{\varkappa}\,\phi^n(a) \;\leq\; 2\sqrt{\varkappa}, \qquad a\geq 0.\end{aligned} \tag{2.3}$$

This gives

$$|\mathcal{T}\varphi| \;\leq\; 2\sqrt{\varkappa}\,\|\varphi\|\,\phi \quad\text{and}\quad \|\mathcal{T}\varphi\| \;\leq\; 2\sqrt{\varkappa}\,\|\varphi\|, \qquad \varphi\in\Phi(\mathsf{R}_+), \tag{2.4}$$

finishing the proof. □

Obviously, $\mathcal{F}$ defined by

$$\mathcal{F}(v,\varphi) \;:=\; v \;-\; \mathcal{T}\varphi \;+\; \varrho\,\mathcal{T}(v^2), \qquad (v,\varphi)\in\Phi(\mathsf{R}_+)\times\Phi(\mathsf{R}_+), \tag{2.5}$$

maps continuously into $\Phi(\mathsf{R}_+)$.

Lemma 2.2 (**Uniqueness**). *For each $\varphi\in\Phi(\mathsf{R}_+)$ there is at most one $v\in\Phi(\mathsf{R}_+)$ such that $\mathcal{F}(v,\varphi)=0$.*

Proof. Assume we have two such solutions v_1 and v_2, and set $\tilde v := v_1 - v_2$. Then

$$\tilde v \;+\; \varrho\,\mathcal{T}\big((v_1+v_2)\tilde v\big) \;=\; 0. \tag{2.6}$$

It suffices to show that for fixed $g\in\Phi(\mathsf{R}_+)$,

$$\tilde v \;+\; \mathcal{T}(g\tilde v) \;=\; 0 \quad\text{implies}\quad \tilde v = 0. \tag{2.7}$$

By assumption, $|g| \leq ||g||\phi \leq c\phi$, where with c we always denote a finite positive constant, which value might change from place to place (except it has an index referring to the equation number). Then from (2.7) this gives

$$|\tilde{v}|(a) \;\leq\; c\int_a^\infty \frac{\mathrm{d}b}{\sqrt{b-a}}\,\phi(b)\,|\tilde{v}|(b), \qquad a \geq 0. \tag{2.8}$$

Iterating this equation ones, interchanging the order of integration, and noting that

$$\int_a^r \mathrm{d}b\;\frac{1}{\sqrt{b-a}}\frac{1}{\sqrt{r-b}} \;\equiv\; \int_0^1 \frac{\mathrm{d}b}{\sqrt{b(1-b}} \;<\; 4, \qquad 0 \leq a \leq r, \tag{2.9}$$

we obtain

$$|\tilde{v}|(a) \;\leq\; c\int_a^\infty \mathrm{d}b\;\phi(b)\,|\tilde{v}|(b), \qquad a \geq 0. \tag{2.10}$$

Then necessarily $|\tilde{v}| = 0$. (To see this, use new variables $x := \mathrm{e}^{-a}$ and $y = \mathrm{e}^{-b}$, and apply Gronwall's inequality, for instance. Or pass to $\sup_{b\geq a}|\tilde{v}|(b)$ which then has to disappear for all sufficiently large a, etc.) This finishes the proof. □

For the moment fix $(v,\varphi) \in \Phi(\mathsf{R}_+) \times \Phi(\mathsf{R}_+)$. We now consider the Fréchet derivative of $\mathcal{F}(v,\varphi)$ at v :

$$D_v^1\mathcal{F}(v,\varphi)\tilde{v} \;=\; \tilde{v} + 2\varrho\mathcal{T}(v\tilde{v}), \qquad \tilde{v} \in \Phi(\mathsf{R}_+). \tag{2.11}$$

Lemma 2.3 (Fréchet derivative). *For $(v,\varphi) \in \Phi(\mathsf{R}_+)\times\Phi(\mathsf{R}_+)$ fixed, $D_v^1\mathcal{F}(v,\varphi)$ is a one-to-one mapping of $\Phi(\mathsf{R}_+)$ onto itself.*

Proof. By linearity, the one-to-one statement follows from (2.7). Now consider $g \in \Phi(\mathsf{R}_+)$. To finish the proof, we want to show that there is a $\tilde{v} \in \Phi(\mathsf{R}_+)$ such that $D_v^1\mathcal{F}(v,\varphi)\tilde{v} = g$, that is

$$\tilde{v}(a) + 2\varrho\mathcal{T}(v\tilde{v})\,(a) \;=\; g(a), \qquad a \geq 0. \tag{2.12}$$

We do this by decomposing R_+ into finitely many cells as follows. Fix $a_0 \geq 0$, introduce $\Phi\big([a_0,\infty)\big)$ similarly as $\Phi(\mathsf{R}_+)$, denoting the norm by $\|\cdot\|_0$, and consider $\mathcal{T}$ as a continuous operator on $\Phi\big([a_0,\infty)\big)$. Then $\mathcal{S}_0(\tilde{v}) := 2\varrho\mathcal{T}(v\tilde{v})$, $\tilde{v} \in \Phi\big([a_0,\infty)\big)$, is also a continuous linear operator on $\Phi\big([a_0,\infty)\big)$. Since $|v\tilde{v}| \leq \|v\|\,\|\tilde{v}\|_0\,\phi^2$, we have

$$|\mathcal{S}_0(\tilde{v})|\,(a) \;\leq\; 2\varrho\,\|v\|\,\|\tilde{v}\|_0\,\mathcal{T}\phi^2(a) \;\leq\; 4\sqrt{\varkappa}\,\varrho\,\|v\|\,\|\tilde{v}\|_0\,\phi^2(a), \qquad a \geq a_0\,, \tag{2.13}$$

where we used (2.3). Hence, $\|\mathcal{S}_0(\tilde{v})\|_0 \leq 4\sqrt{\varkappa}\,\varrho\,\|v\|\,\|\tilde{v}\|_0\,\phi(a_0)$. Choose $a_0 \geq 0$ so large that $4\sqrt{\varkappa}\,\varrho\,\|v\|\,\phi(a_0) < 1$. Then $\mathcal{S}_0$ is a contraction, and

$$\tilde{v}_0 \;:=\; (\mathcal{I}+\mathcal{S}_0)^{-1}g \;=\; \sum_{n=0}^{\infty}(-1)^n g^n \tag{2.14}$$

solves (2.12) on $[a_0,\infty)$ (cf. Zeidler (1986) [15, Theorem 1.B, p.32]).

If $a_0 > 0$, take $a_1 \in [0,a_0)$, and introduce the continuous linear operator $\mathcal{T}_1$ on $\mathcal{C}\big([a_1,a_0]\big)$ (with the supremum norm denoted by $\|\cdot\|_1$) by modifying (2.1):

$$\mathcal{T}_1\varphi\,(a) \;:=\; \int_a^{a_0}\mathrm{d}b\;\sqrt{\frac{\varkappa}{2\pi(b-a)}}\;\varphi(b), \qquad \varphi \in \mathcal{C}\big([a_1,a_0]\big), \quad a \in [a_1,a_0]. \tag{2.15}$$

Consider the equation

$$\tilde{v}(a) + 2\varrho\,\mathcal{T}_1(v\tilde{v})\,(a) \;=\; g(a) - 2\varrho \int_{a_0}^{\infty} \mathrm{d}b\,\sqrt{\frac{\varkappa}{2\pi(b-a)}}\,(v\tilde{v})(b), \qquad a \in [a_1, a_0]. \tag{2.16}$$

Set $\mathcal{S}_1(\tilde{v}) := 2\varrho\,\mathcal{T}_1(v\tilde{v})$, $\tilde{v} \in \mathcal{C}\big([a_1, a_0]\big)$ to get also a continuous linear operator on $\mathcal{C}\big([a_1, a_0]\big)$. Now $\|\mathcal{S}_1(\tilde{v})\|_1 \le 2\varrho\,\|v\|_\infty\,\|\tilde{v}\|_1\,\|\mathcal{T}_1(1)\|_1 = c\,\sqrt{a_0 - a_1}\,\|\tilde{v}\|_1$, where c is a constant independent of a_0, a_1. Choosing a_1 sufficiently close to a_0, the operator $\mathcal{S}_1$ is a contraction, and we can solve (2.16). Continuing in this way finitely many times, R_+ is exhausted. Putting together the constructed solutions, the proof is finished. □

Denote by $\hat{\Phi}(\mathsf{R}_+)$ the set of all $\varphi \in \Phi(\mathsf{R}_+)$ such that $\|\varphi\| < \frac{1}{8\kappa\varrho}$.

Lemma 2.4 (**All solutions**). *Denote by O the set of all $\varphi \in \Phi(\mathsf{R}_+)$ for which there is a $v = v_\varphi \in \Phi(\mathsf{R}_+)$ such that $\mathcal{F}(v,\varphi) = 0$. Then*

- **(a):** *O is open, and $\hat{\Phi}_+(\mathsf{R}_+) \subseteq O$.*
- **(b):** *The mapping $\varphi \to v_\varphi$ defined on O is analytic.*

Proof. **(a)** Fix $(v_0, \varphi_0) \in \Phi^2(\mathsf{R}_+)$ satisfying $\mathcal{F}(v_0, \varphi_0) = 0$. By Lemmas 2.2 and 2.3, the implicit function theorem (cf. [15, Theorem 4.B, p.150]) yields that there is an open set O_0 containing φ_0 such that there is a unique map $\varphi \mapsto v_\varphi$ defined on O_0 with $\mathcal{F}(v_\varphi, \varphi) = 0$. Hence, O is open.

Suppose $\varphi \in \hat{\Phi}_+(\mathsf{R}_+)$. We want to construct $v = v_\varphi \in \Phi(\mathsf{R}_+)$ such that $\mathcal{F}(v_\varphi, \varphi) = 0$, that is $v = \mathcal{T}\varphi - \varrho\,\mathcal{T}(v^2)$. For this purpose, define recursively

$$v_0 \;:=\; \mathcal{T}\varphi \quad \text{and} \quad v_{m+1} \;:=\; \mathcal{T}\varphi - \varrho\,\mathcal{T}(v_m^2), \quad m \ge 0. \tag{2.17}$$

Then

$$|v_m| \;\le\; 2\sqrt{\varkappa}\,\|\varphi\|\,\phi, \qquad m \ge 0. \tag{2.18}$$

In fact, we will show this by induction. For $m = 0$ this is true by (2.4). Assume it is valid for $0, \ldots, m$ for some $m \ge 0$. To get (2.18) for $m+1$, note that both terms in the definition (2.17) of v_{m+1} are non-negative, so we have

$$|v_{m+1}| \;\le\; \max\big(\mathcal{T}\varphi,\, \varrho\,\mathcal{T}(v_m^2)\big). \tag{2.19}$$

Hence, it suffices to show that the second term is bounded from above by $2\sqrt{\varkappa}\,\|\varphi\|\phi$. But by induction hypothesis (2.18) and (2.3),

$$\varrho\,\mathcal{T}(v_m^2) \;\le\; \varrho\,\big(2\sqrt{\varkappa}\,\|\varphi\|\big)^2\, 2\sqrt{\varkappa}\,\phi^2 \;\le\; 2\sqrt{\varkappa}\,\|\varphi\|\,\phi, \tag{2.20}$$

since

$$c_{(2.21)} \;:=\; 8\,\varkappa\,\varrho\,\|\varphi\| \;<\; 1 \tag{2.21}$$

by assumption on φ. Hence, (2.18) is proven.

Now (2.17), (2.18), and again (2.4) imply

$$\|v_{m+1} - v_m\|_0 \;\le\; \varrho\,\big\|\mathcal{T}\big((v_m + v_{m-1})(v_m - v_{m-1})\big)\big\|_0 \tag{2.22}$$

$$\le\; 4\sqrt{\varkappa}\,\varrho\,\|\varphi\|\,2\sqrt{\varkappa}\,\|v_m - v_{m-1}\|_0 \;=\; c_{(2.21)}\,\|v_m - v_{m-1}\|_0\,, \quad m \ge 1.$$

Letting $m \uparrow \infty$, by completeness of $\Phi(\mathsf{R}_+)$, we see that there is a v in $\Phi(\mathsf{R}_+)$ satisfying $v = \mathcal{T}\varphi - \varrho\,\mathcal{T}(v^2)$, that is, $\hat{\Phi}_+(\mathsf{R}_+) \subseteq O$.

(b) Note that the derivatives

$$D^1_\varphi \mathcal{F}(v,\varphi)\tilde{\varphi} \;=\; -\mathcal{T}(\tilde{\varphi}) \quad \text{and} \quad D^2_v \mathcal{F}(v,\varphi)(\tilde{v},\tilde{w}) \;=\; 2\varrho\,\mathcal{T}(\tilde{v}\tilde{w}), \tag{2.23}$$

$\tilde{\varphi}, \tilde{v}, \tilde{w} \in \Phi(\mathsf{R}_+)$, are independent of v, φ. Therefore, $\mathcal{F}(v,\varphi)$ is analytic in (v,φ), and hence (cf. [15, Corollary 4.23, p.151]) the mapping $\varphi \to v_\varphi$ defined on O is analytic, finishing the proof. □

2.2. **Exponential moments.** Recall the log-exponential moments Λ_φ defined in (1.16). Put

$$\tilde{O} := \{\varphi \in \Phi(\mathsf{R}_+):\ \Lambda_\varphi \in \Phi(\mathsf{R}_+)\}. \tag{2.24}$$

Recalling from Lemma 2.4 the set O of all solutions, let O_0 denote the largest connected subset of O containing $\hat{\Phi}_+(\mathsf{R}_+)$ (see the notation in front of Lemma 2.4).

Proposition 2.1 (Exponential moments). *We have $-O_0 \subseteq \tilde{O}$. As a consequence,*

$$\log P_\nu \mathrm{e}^{\langle V,\varphi\rangle} = \langle \nu, \Lambda_\varphi\rangle = \langle \nu, -v_{-\varphi}\rangle < \infty, \qquad \nu \in \mathcal{M}(\mathsf{R}_+), \quad \varphi \in -O_0\,. \tag{2.25}$$

Proof. First fix $\varphi \in \hat{\Phi}_+(\mathsf{R}_+)$. Then by Lemmas 2.4(a) and 2.2, for each $\theta \in [0,1]$ there is a unique $v_{\theta\varphi} \in \Phi(\mathsf{R}_+)$ satisfying $\mathcal{F}(v_{\theta\varphi}, \theta\varphi) = 0$, and $\theta\varphi \in O_0$. On the other hand, $-\Lambda_{-\theta\varphi} \geq 0$ uniquely solves (1.11) with φ replaced by $\theta\varphi$, and it obviously belongs to $\Phi(\mathsf{R}_+)$. Hence, $-\Lambda_{-\theta\varphi} = v_{\theta\varphi} \in \Phi_+(\mathsf{R}_+)$, in particular, $-\hat{\Phi}_+(\mathsf{R}_+) \subseteq \tilde{O}$.

For general $\varphi \in \hat{\Phi}(\mathsf{R}_+)$ and $\theta_1, \theta_2 \geq 0$, we have that $-\Lambda_{-\theta_1\varphi_+ - \theta_2\varphi_-} \in \Phi_+(\mathsf{R}_+)$ solves (1.11) with φ replaced by $\theta_1\varphi_+ + \theta_2\varphi_-$, where $\varphi_+ := \varphi \vee 0$ and $\varphi_- := -(\varphi \wedge 0)$. Writing $\boldsymbol{\theta} := (\theta_1, \theta_2)$, for the fixed $\varphi \in \hat{\Phi}(\mathsf{R}_+)$ we introduce

$$\begin{aligned} \tilde{\Theta} \,=\, \tilde{\Theta}(\varphi) &:= \Big\{\boldsymbol{\theta} \in \mathsf{R}^2:\ -\theta_1\varphi_+ - \theta_2\varphi_- \in \tilde{O}\Big\}, \\ \Theta_0 \,=\, \Theta_0(\varphi) &:= \Big\{\boldsymbol{\theta} \in \mathsf{R}^2:\ \theta_1\varphi_+ + \theta_2\varphi_- \in O_0\Big\}. \end{aligned} \tag{2.26}$$

Note that $[0, \frac{1}{2}]^2$ belongs to $\tilde{\Theta} \cap \Theta_0$, that Θ_0 is a connected open set, and that by Hölder's inequality, $\tilde{\Theta}$ is convex.

Fix $a \in \mathsf{R}_+$ and put

$$f_a(\boldsymbol{\theta}) := -\Lambda_{-\theta_1\varphi_+ - \theta_2\varphi_-}(a), \qquad \boldsymbol{\theta} \in \tilde{\Theta}, \tag{2.27a}$$

$$g_a(\boldsymbol{\theta}) := v_{\theta_1\varphi_+ + \theta_2\varphi_-}(a), \qquad \boldsymbol{\theta} \in \Theta_0\,. \tag{2.27b}$$

Then, f_a is analytic on the interior $\tilde{\Theta}^\circ$ of $\tilde{\Theta}$, and g_a is analytic on all of Θ_0. Moreover, $f_a = g_a$ on $[0, \frac{1}{2}]^2 \subseteq \tilde{\Theta} \cap \Theta_0$. Therefore, both are branches of a unique analytic function defined on $\tilde{\Theta} \cup \Theta_0$. Since a is arbitrary and $\tilde{\Theta}$ is maximal, we obtain $\Theta_0 \subseteq \tilde{\Theta}^\circ = \tilde{\Theta}$ and $-\Lambda_{-\theta_1\varphi_+ - \theta_2\varphi_-} = v_{\theta_1\varphi_+ + \theta_2\varphi_-}$ for $\boldsymbol{\theta} \in \Theta_0$.

Consider $\varphi \in O_0$. Then we have $(1,-1) \in \Theta_0(\varphi) \subseteq \tilde{\Theta}(\varphi)$, implying $-\varphi \in \tilde{O}$ and $-\Lambda_{-\varphi} = v_\varphi$. Finally, (2.25) follows from the branching property. □

3. Proof of the large deviation principles

3.1. **Proof of Proposition 1.1.** We start with the following scaling property. Fix $\nu \in \mathcal{M}(\mathsf{R}_+)$. Recall that the super-$\frac{1}{2}$-stable subordinator $U = (U, P_\nu) = (U^\varrho, P^\varrho_\nu) = U^\varrho$ and its total occupation measure $V = V^\varrho$ depend on the branching rate $\varrho > 0$.

Lemma 3.1 (Scaling of U). *Fix a constant $c > 0$. If U is distributed according to P^ϱ_ν, then cU has the law $P^{c\varrho}_{c\nu}$.*

Proof. For $t \geq 0$ fixed, denote by $u^{\varrho}_{t,\varphi}$ the unique solution of (1.8). Then by the Markov property, via (1.7) and (1.8), the claim in the lemma immediately follows from the identity $u^{\varrho}_{t,c\varphi} = c u^{c\varrho}_{t,\varphi}$, $\varphi \in \Phi_+(\mathsf{R}_+)$. □

By Lemma 3.1, we have $V^{\varrho} \stackrel{\mathcal{L}}{=} \varrho V^1$, provided that $U^1_0 = \varrho^{-1} U^{\varrho}_0$. Thus, the large deviation claims of Proposition 1.1 can be seen as statements on ϱV^1 under the law $P^1_{\varrho^{-1}\nu}$ of U^1. Consequently, setting $\varrho^{-1} =: R$, we look at the probabilities

$$g(R) := P^1_{R\nu}(R^{-1}V^1 \in A), \qquad \nu \in \mathcal{M}(\mathsf{R}_+), \quad \text{Borel } A \subseteq \mathcal{M}(\mathsf{R}_+), \tag{3.28}$$

as $R \uparrow \infty$. For the moment, fix such A. As in [9, Lemma 4.2.1], the branching property implies that $R \mapsto g(R)$ is supermultiplicative, provided that in addition A is convex. Moreover, if A is additionally open, then $g(R) > 0$ for some $R > 0$ yields that g is bounded away from 0 on some non-empty open interval. This can be seen as in [9, Lemma 4.2.3] by using the branching property, supermultiplicativity, and finiteness of some exponential moments according to Proposition 2.1.

We further proceed as in [9, Subsection 4.3]. Denote by $\mathfrak{A}$ the system of all non-empty, convex, and open subsets of $\mathcal{M}(\mathsf{R}_+)$. Then the mentioned supermultiplicativity implies that the function

$$R \mapsto \sigma(R) := -\log P^1_{R\nu}(R^{-1}V^1 \in A) \in [0, +\infty], \qquad A \in \mathfrak{A}, \tag{3.29}$$

is subadditive, and σ is either bounded on some non-empty open interval, or it is identically $+\infty$. Thus, the limit as $R \uparrow \infty$ exists in $[0, +\infty]$, we denote it by $\mathbf{I}(A)$. Using shrinking open balls and monotone limits, we can define $\mathbf{I}(\eta)$, $\eta \in \mathcal{M}(\mathsf{R}_+)$. Note that $\eta \mapsto \mathbf{I}(\eta) \in [0, +\infty]$ is lower semi-continuous and convex. Actually, the family $\{P^1_{R\nu}(R^{-1}V^1 \in \cdot\,) : R > 0\}$ satisfies a weak large deviation principle as $R \uparrow \infty$ with convex rate function $\mathbf{I}$. Moreover, based on Proposition 2.1, as in [9, Subsection 4.4] we get exponential tightness, hence the full large deviation principle for this family with good convex rate function $\mathbf{I}$.

Finally, specializing to $R = 1, 2, \ldots$, using the branching property and Cramér's theorem [9, Corollary 5.1.3], the uniqueness of rate functions gives $\mathbf{I} = I$ with I the Fenchel-Legendre transform as defined in (1.19). This finishes the proof of Proposition 1.1. □

3.2. **Proof of Theorem 1.1.** Recall that the occupation density measure λ^{ϱ} of X^{ϱ} at the catalyst's position 0, where $X^{\varrho}_0 = \mu \in \mathcal{M}(\mathsf{R}) \backslash \{0\}$, coincides in law with the total occupation measure V^{ϱ} of U^{ϱ}, if $U^{\varrho}_0 = \nu_{\mu}$. By the representation (1.15), x^{ϱ} is (in law) a functional F of V^{ϱ}. Having now available Proposition 1.1, by the contraction principle Theorem 1.1 will follow from the following lemma.

Lemma 3.2 (**Continuity of** F)**.** *The mapping* $\eta \mapsto f = F(\eta)$ *of* $\mathcal{M}(\mathsf{R}_+)$ *into* $\mathbf{C} = \mathcal{C}\big((0,\infty) \times \dot{\mathsf{R}}\big)$ *defined by* (1.18) *is continuous and one-to-one.*

Proof. Clearly, f is finite and continuous, i.e. belongs to $\mathbf{C}$. Also, if $(t_n, a_n, s_n) \to (t, a, s)$ as $n \uparrow \infty$ in $(0, \infty) \times \dot{\mathsf{R}} \times (0, \infty)$, then

$$1_{[0,t_n)}(s_n)\, q_{|a_n|}(t_n - s_n) \to 1_{[0,t)}(s)\, q_{|a|}(t - s). \tag{3.30}$$

Hence, F is continuous. Also, $\eta_1 \neq \eta_2$ certainly implies $F(\eta_1) \neq F(\eta_1)$, finishing the proof. □

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