Conditional Excursion Representation
for a Class of Interacting Superprocesses

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Abstract

A class of interacting superprocesses, called superprocesses with dependent spatial motion (SDSMs), has been introduced and characterized in Wang [22] and Dawson et al. [7]. In this paper, we give a construction or an excursion representation of the non-degenerate SDSM with immigration by making use of a Poisson system associated with the conditional excursion laws of the SDSM. As pointed out in Wang [22], the multiplicative property or summable property is lost for SDSMs and immigration SDSMs. However, summable property is the foundation of excursion representation. This raises a sequence of technical difficulties. The main tool we used is the conditional log-Laplace functional technique that gives the conditional summability, the conditional excursion law, and the Poisson point process for the construction of the immigration SDSMs.

1 Introduction

Wang ([21] [22]) and Dawson et al [7] have considered a model of interacting branching particle systems in which each individual is governed by a stochastic differential equation (SDE):

$$dx_i(t) = c(x_i(t))dB_i(t) + \int_{\mathbb{R}} h(y - x_i(t))W(dt, dy), \quad t \geq 0, i = 1, 2, \cdots \qquad (1.1)$$

where B_1, B_2, $\cdots$ are independent Brownian motions and W is a Brownian sheet or a space-time white noise which is independent of B_i's, $c \in C_b^2(\mathbb{R})$, $h \in C_b^2(\mathbb{R})$ is square-integrable with square-integrable derivative. Above $C_b^k(\mathbb{R})$ denotes the set of functions that, together with their derivatives up to order k inclusive, are bounded continuous on $\mathbb{R}$. The subset of non-negative elements of $C_b^k(\mathbb{R})$ is denoted by $C_b^k(\mathbb{R})^+$. Suppose that $\sigma \in C_b^2(\mathbb{R})^+$ is the branching rate for all particles in the system. Let $M(\mathbb{R})$ be the space of all finite measures on $\mathbb{R}$ with weak topology and let $W^E := C([0, \infty), M(\mathbb{R}))$. Then the corresponding limit superprocesses, which are called superprocesses with dependent spatial motion (SDSMs), of such interacting branching systems can be constructed and characterized as the unique solution on W^E of the martingale problem associated with the operator $\mathcal{L} = \mathcal{A} + \mathcal{B}$ defined as follows:

$$
\begin{aligned}
\mathcal{A}F(\mu) &= \frac{1}{2} \int_{\mathbb{R}} a(x) \frac{d^2}{dx^2} \frac{\delta F(\mu)}{\delta \mu(x)} \mu(dx) \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^2} \rho(x - y) \frac{d^2}{dxdy} \frac{\delta^2 F(\mu)}{\delta \mu(x)\delta \mu(y)} \mu(dx)\mu(dy) \qquad (1.2)
\end{aligned}
$$

$$\mathcal{B}F(\mu) = \frac{1}{2} \int_{\mathbb{R}} \sigma(x) \frac{\delta^2 F(\mu)}{\delta \mu(x)^2} \mu(dx), \qquad (1.3)$$

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where $a(x) = c^2(x) + \rho(0)$ and

$$\rho(x) := \int_{\mathbb{R}} h(y - x)h(y)dy. \qquad (1.4)$$

For more details, see Wang [22] and Dawson et al. [7]. The SDSM can also be uniquely characterized by the following simplified martingale problem: $\forall \phi \in \mathcal{S}(\mathbb{R})$, the space of the infinitely differentiable functions that, together with all their derivatives, are rapidly decreasing at infinity,

$$M_t(\phi) \equiv \langle \phi, X_t \rangle - \langle \phi, \mu \rangle - \frac{1}{2} \int_0^t \langle a\phi'', X_u \rangle \, du$$

is a martingale with quadratic variation process

$$\langle M(\phi) \rangle_t = \int_0^t \langle \sigma\phi^2, X_u \rangle \, du + \int_0^t du \int_{\mathbb{R}^2} \rho(y - z)\phi'(y)\phi'(z)X_u(dy)X_u(dz),$$

where $\phi'(x) = \frac{d}{dx}\phi(x)$ and $\phi''(x) = \frac{d^2}{dx^2}\phi(x)$. Regarding the investigation of this model in different assumptions, it was proved in Wang [21] that X_t is absolutely continuous and Dawson et al [9] derived a stochastic partial differential equation (SPDE) for the density process for the case that $c(\cdot) \equiv \epsilon > 0$. The absolute continuity was proved in Dawson et al [7] under the weaker assumption $|c(\cdot)| \geq \epsilon > 0$. Wang [24] proved the convergence of SDSM to a super-Brownian motion in the case that $c(\cdot) \equiv \epsilon > 0$ and h converges to a singular function. When $c(\cdot) \equiv 0$, Wang ([21], [23]) proved that X_t is purely atomic and Dawson et al [8] derived a degenerated SPDE and proved the existence of a unique strong solution. Li et al. [16] has constructed a singular, degenerate SPDE that connected to coalescing Brownian motions.

For a general term of the excursion representation of a stochastic process, we mean a construction or representation of the stochastic process in terms of excursions and Poisson point processes. The first breakthrough in the excursion representation of Brownian motion was the seminal paper of Itô [12], although some ideas were already, at an intuitive level, in the work of Lévy, it was Itô who put the subject on a firm mathematical basis, thus supplying another cornerstone to Probability Theory. Since then, there are many important progresses in this field (see Dawson and Li [6] and references therein). In the present paper the following problems are tackled: Recently Dawson and Li [6] gives an excursion representation of the immigration SDSM with $c \equiv 0$ (degenerate SDSM) and Fu and Li [10] gives an excursion representation of immigration Dawson-Watanabe process. This naturally raises a question: *Given $c \in C_b^2(\mathbb{R})$ satisfing $c(\cdot) \geq \epsilon > 0$, can we give an excursion representation of the immigration SDSM with this non-degenerate coefficient c?*

In the case that $c \equiv 0$, the SDSM is a purely-atomic measure-valued process. The corresponding location processes of the purely-atomic SDSM are (smooth) coalescing stochastic flows which never meet each other if their initial locations are separated. Over each stochastic flow or location process, it is just one dimensional Feller branching diffusion. [6] has well used this purely-atomic property of the degenerate SDSM and transformed the question of the excursion representation into the one dimensional case. Then, the problem of the excursion representation of the degenerate SDSM is solved based on the results of the one dimensional continuous state branching processes (for more details, See Dawson and Li [6] and Pitman and Yor [18]). In the case of immigration Dawson-Watanabe process, the motions of particles are independent and infinite divisible property holds. Then,

the cluster representation of the Dawson-Watanabe process gives the entrance law (See Dawson [5], Fu and Li [10]). For our case, $c(\cdot)$ is a strictly elliptic function (i.e. there exists an $\epsilon > 0$ such that $c(\cdot) \geq \epsilon$). This corresponds to the non-degenerate case. In this case, the SDSM has density at each $t > 0$. On the other hand, the motions of particles of SDSM are always dependent based on the assumption of coefficient $h \not\equiv 0$ and ρ is twice continuously differentiable with ρ' and ρ'' bounded. Thus, to give an excursion representation for the non-degenerate SDSM with immigration, our first difficulty is how to construct the entrance law in this case. We already mentioned that in this case we don't have cluster representation any more due to the dependency of particles' motions. An even more challenging difficulty for us is the loss of summable property of the SDSM due to interaction. The summable property is just the multiplicative property which is the fact that a measure-valued Markov process has summable property if two such processes start at μ_1 and μ_2, respectively, then their sum is equal in law to the same process starting at $\mu_1 + \mu_2$. This is an obvious consequence of the independent behavior of the particles in the population. The summable property is the key to the successes in the study of the independent measure-valued Markov processes whatever the approaches to different problems be. In particular for the immigration and the excursion representation of the measure-valued Markov processes, even the definition of immigration process is directly given based on the summable property (See Li-Shiga [14] Definition 1.1). Nevertheless, it is obvious that an interacting particle system has lost such a property and is only allowed instantaneous perturbation of motion, branching, and immigration due to loss of independent behavior of individuals. For more information, see the counter example given in Wang [22] for the summable property of SDSM. From above analysis, this seems to us that to find the excursion representation of the non-degenerate SDSM would be hopeless. However, the following intuitive idea brought sunshine into our dull struggling with this problem. The idea is looking the common space-time white noise or the Brownian sheet as a shared random environment or a common living ground. If we freeze the shared random environment (conditioned on the space-time white noise), the individual particles are still independent and summable. This is called conditional summability. We will see that the successful realization of this plan is based on the construction of the conditional or stochastic log-Laplace functional which is a unique strong solution of a nonlinear, backward stochastic partial differential equation. This stochastic log-Laplace functional plays the role for SDSM same as the log-Laplace functional does for the super-Brownian motion. The basic idea, thus, can be intuitively explained as follows: When freezing the random environment, the SDSM becomes a generalized, inhomogeneous super-Brownian motion if condition $c(\cdot) \geq \epsilon > 0$ holds. Thus, all the results are intuitively natural generalization of the classical super-Brownian motion under conditional argument. However, this is by no means that everything is straight forward copy from the case of super-Brownian motion after freezing the random environment. Actually, although the stochastic log-Laplace functional technique can provide conditional summable property, it raises lot of new challenging problems related to the nonlinear SPDE and other issues. We will see that the stochastic log-Laplace functional will serve as a basic tool for a sequence of works. (See Xiong [26], [25] and Li et al [15] for motivations and more details of conditional log-Laplace functionals).

For fixed integers $m \geq 1$ and $0 \leq k \leq \infty$, let $C^k(\mathbb{R}^m)$ be the set of functions on $\mathbb{R}^m$ having continuous derivatives up to order k and $C_\partial^k(\mathbb{R}^m)$ be the set of functions in $C^k(\mathbb{R}^m)$, which, together with their derivatives up to order k, can be extended continuously to

$\bar{\mathbb{R}}^m := \mathbb{R}^m \cup \{\partial\}$, the one point compactification of $\mathbb{R}^m$. Let $C_0^k(\mathbb{R}^m)$ denote the set of functions in $C^k(\mathbb{R}^m)$, which, together with their derivatives up to order k, vanish at infinity. $C_c^k(\mathbb{R}^m)$ stands for the set of functions in $C_0^k(\mathbb{R}^m)$, which, together with their derivatives up to order k, have compact support. Let $B(\mathbb{R})$ denote all the Borel functions on $\mathbb{R}$. $L^2(\mathbb{R})$ denotes the Hilbert space of square integrable function classes with inner product $\langle \cdot, \cdot \rangle_0$ and norm $\| \cdot \|_0$. If h is a function of x, we use h' to denote either $\frac{d}{dx} h$ or $\partial_x h$. Let $\mathbb{H}^m(\mathbb{R})$ denote the Sobolev space of classes of functions that, together with their derivatives in the sense of distribution up to m^{th} order, are square integrable on $\mathbb{R}$ with norm defined by

$$\|\phi\|_m := \sqrt{\sum_{i=0}^{m} \|\partial_x^i \phi\|_0^2}, \qquad \phi \in \mathbb{H}^m(\mathbb{R}),$$

where $\| \cdot \|_0$ is the norm of $L^2(\mathbb{R})$. $\{\mathbb{H}^m(\mathbb{R}) : m \geq 0\}$ are Hilbert spaces. In particular, we have $\mathbb{H}^0(\mathbb{R}) = L^2(\mathbb{R})$. $C_b^k(\mathbb{R}) \bigcap \mathbb{H}^k(\mathbb{R})$ denotes the set of functions that, together with their bounded continuous derivatives up to order k, are square integrable. For $f \in B(\mathbb{R})$ and $\mu \in M(\mathbb{R})$, set $\langle f, \mu \rangle = \int_{\mathbb{R}} f d\mu$. Above notations will keep same meaning throughout the paper.

To simply the statement of each theorem, here we give a statement that put several required conditions together.

Basic Condition (A): *In our model, we assume that the coefficients $\sigma \in C_b^2(\mathbb{R})$, $h \in C_b^2(\mathbb{R}) \bigcap \mathbb{H}^2(\mathbb{R})$, $c \in C_b^2(\mathbb{R})$, and there exist constants $\epsilon > 0$ and $\sigma_b > \sigma_a > 0$ such that $c^2(x) \geq \epsilon$ and $\sigma_a \leq \sigma(x) \leq \sigma_b$.*

This article is organized as follows: In section 2, we discuss the stochastic log-Laplace equation, the existence of its unique strong solution, as well as the regularity of the solution. We generalize the results discussed in Li et al. [15] based on the results of Kurtz-Xiong [13] and Rozovskii [19]. We derive the ψ-semigroup property. In section 3, the conditional or stochastic log-Laplace functionals for SDSM will be discussed. In section 4, we will investigate the conditional generalized super-Brownian motion and give some basic tools for the conditional excursion representation of the SDSM. Section 5 will discuss the construction of the conditional entrance law for SDSM. The conditional excursion law of the SDSM, and the conditional excursion representation of the immigration SDSM (SDSMI) is discussed in section 6.

2   Stochastic log-Laplace equations and ψ-semigroups

Recall that we have assumed the basic condition (A) for the model coefficients. Let $W(ds, dx)$ be a space-time white noise. For any given initial data $\phi \in L^2(\mathbb{R})$, we consider the following forward non-linear SPDE:

$$\begin{aligned} \psi_t(x) &= \phi(x) + \int_0^t \left[\frac{1}{2} a(x) \partial_{xx}^2 \psi_s(x) - \frac{1}{2} \sigma(x) \psi_s(x)^2 \right] ds \\ &\quad + \int_0^t \int_{\mathbb{R}} h(y - x) \partial_x \psi_s(x) W(ds, dy), \qquad t \geq 0, \end{aligned} \qquad (2.1)$$

and the following backward nonlinear SPDE:

$$\psi_{r,t}(x) = \phi(x) + \int_r^t \left[\frac{1}{2} a(x) \partial_{xx}^2 \psi_{s,t}(x) - \frac{1}{2} \sigma(x) \psi_{s,t}(x)^2 \right] ds$$

$$+ \int_r^t \int_{\mathbb{R}} h(y-x) \partial_x \psi_{s,t}(x) \cdot W(ds, dy), \qquad t \geq r \geq 0, \qquad (2.2)$$

where "·" denotes the backward Itô stochastic integral.

In the following, first we discuss SPDEs (2.1) and (2.2) with initial data $\phi \in L^2(\mathbb{R})$. Although $\phi \equiv 1 \notin L^2(\mathbb{R})$, we will need the solutions of SPDEs (2.1) and (2.2) with initial data $\phi \equiv 1$. We will handle this case by a monotone convergence method. Let $L^2([0,T], \mathcal{P}, L^2(\mathbb{R}))$ stand for the space of all classes of predictable random mappings from $[0,T]$ to $L^2(\mathbb{R})$, which are square integrable with respect to measure $\ell \times \mathbb{P}$, where ℓ is Lebesgue measure on $\mathbb{R}$.

Definition 2.1 *For any given initial data $\phi \in L^2(\mathbb{R})$, a stochastic process $u \in L^2([0,T], \mathcal{P}, L^2(\mathbb{R}))$ is called a generalized solution of equation (2.1) if, for every $f \in C_c^\infty(\mathbb{R})$, the space of infinitely differentiable with compact support, it satisfies the following equation :*

$$\langle u_t, f \rangle_0 = \langle \phi, f \rangle_0 + \int_0^t \frac{1}{2} \Big(-\langle au_s', f' \rangle_0 - \langle a'u_s' + \sigma u_s^2, f \rangle_0 \Big) ds$$

$$+ \int_0^t \int_{\mathbb{R}} \langle h(y-\cdot)u_s', f \rangle_0 dW(ds, dy), \qquad \text{for any } t \geq 0, \ \mathbb{P}\text{-a.s..} \quad (2.3)$$

For the case that $\phi \equiv 1$, a continuous stochastic process u is called a generalized solution of equation (2.1) if, for every $f \in C_c^\infty(\mathbb{R})$, it satisfies the following equation :

$$\int_{\mathbb{R}} u_t(x)f(x)dx = \int_{\mathbb{R}} \phi(x)f(x)dx + \int_0^t \frac{1}{2} \Big(\int_{\mathbb{R}} u_s(x)(a(x)f'(x))'dx$$

$$+ \int_{\mathbb{R}} u_s(x)(a'(x)f(x))'dx - \int_{\mathbb{R}} \sigma(x)u_s^2(x)f(x)dx \Big) ds$$

$$- \int_0^t \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} u_s(x)\frac{\partial}{\partial x}(h(y-x)f(x))dx \Big] dW(ds, dy), \quad t \geq 0, \ \mathbb{P}\text{-a.s.} \quad (2.4)$$

For any $r \in [0,T]$, a stochastic process $v_{r,\cdot} \in L^2([r,T], \mathcal{P}, L^2(\mathbb{R}))$ is called a generalized solution of equation (2.2) if, for every $f \in C_c^\infty(\mathbb{R})$, it satisfies the following equation :

$$\langle v_{r,t}, f \rangle_0 = \langle \phi, f \rangle_0 + \int_r^t \frac{1}{2} \Big(-\langle av_{r,s}', f' \rangle_0 - \langle a'v_{r,s}' + \sigma v_{r,s}^2, f \rangle_0 \Big) ds$$

$$+ \int_r^t \int_{\mathbb{R}} \langle h(y-\cdot)v_{r,s}', f \rangle_0 \cdot dW(ds, dy), \qquad \text{for any } t \geq r, \ \mathbb{P}\text{-a.s..} \quad (2.5)$$

For the case that $\phi \equiv 1$, a continuous stochastic process u is called a generalized solution of equation (2.2) if, for every $f \in C_c^\infty(\mathbb{R})$, it satisfies the following equation :

$$\int_{\mathbb{R}} v_{r,t}(x)f(x)dx = \int_{\mathbb{R}} \phi(x)f(x)dx + \int_r^t \frac{1}{2} \Big(\int_{\mathbb{R}} v_{r,s}(x)(a(x)f'(x))'dx$$

$$+ \int_{\mathbb{R}} v_{r,s}(x)(a'(x)f(x))'dx - \int_{\mathbb{R}} \sigma(x)v_{r,s}^2(x)f(x)dx \Big) ds$$

$$- \int_r^t \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} v_{r,s}(x)\frac{\partial}{\partial x}(h(y-x)f(x))dx \Big] \cdot dW(ds, dy), t \geq r, \mathbb{P}\text{-a.s.} \quad (2.6)$$

Remark: In order to use Rozovskii's results (see [19]), above definition, we have used the divergent form of the principle term since $a \in C_b^2(\mathbb{R})$.

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For the existence, the uniqueness, and the regularity of solution of the SPDEs (2.1) and (2.2), we generalize two theorems from Li et al. [15] in the following.

Theorem 2.1 *Suppose that the basic condition* (A) *holds. Then, for any $\phi \in \{C_b(\mathbb{R})^+ \cap \mathbb{H}^1(\mathbb{R})\}$, equation (2.1) has a unique $C_b(\mathbb{R}) \cap \mathbb{H}^1(\mathbb{R})$-valued, non-negative, strong solution $\{\psi_t : t \geq 0\}$. Furthermore, if $\phi \equiv 1$, equation (2.1) has a unique $C_b(\mathbb{R})$-valued, non-negative, strong solution $\{\psi_t : t \geq 0\}$. For any $\phi \in \{C_b(\mathbb{R})^+ \cap \mathbb{H}^1(\mathbb{R})\} \cup \{1\}$, $\|\psi_t\|_a \leq \|\phi\|_a$ holds $\mathbb{P} - a.s.$ for all $t \geq 0$, where $\|\phi\|_a$ is the supremum of ϕ.*

Proof: For the case that $\phi \in \{C_b(\mathbb{R})^+ \cap \mathbb{H}^1(\mathbb{R})\}$, see the proof of Theorem 4.1 of Li et al. [15]. For $\phi \equiv 1$, let $\{\phi_n : n \geq 1\} \subset \{C_b(\mathbb{R})^+ \cap \mathbb{H}^1(\mathbb{R})\}$ be a monotone increasing sequence such that the following conditions are satisfied: $0 \leq \phi_n \leq 1$ and $\lim_{n\to\infty} \phi_n(x) = 1$ for all $x \in \mathbb{R}$. Let ψ_t^n be the unique solution of (2.1) with initial function ϕ_n. Let $u_t(x) = \psi_s^{n+1}(x) - \psi_s^n(x)$. Define

$$d_s(x) := \sigma(x)(\psi_s^{n+1}(x) + \psi_s^n(x)), \tag{2.7}$$

which is nonnegative. Then u_t is a solution to the following SPDE:

$$\begin{aligned}
u_t(x) &= \phi_{n+1}(x) - \phi_n(x) + \int_0^t \left[\frac{1}{2}a(x)\partial_{xx}^2 u_s(x) - \frac{1}{2}d_s(x)(u_s(x))\right] ds \\
&\quad + \int_0^t \int_{\mathbb{R}} h(y-x)\partial_x u_s(x)W(ds, dy), \qquad t \geq 0.
\end{aligned} \tag{2.8}$$

Note that $d_s(x)$ is not a Lipschitz function, the results of Kurtz-Xiong [13] is not directly applicable. However, the existence of a nonnegative solution given by the particle representation is still true. To prove $u_t(x) \geq 0$, we only need to prove the uniqueness for the solution of (2.8). Let $v_t(x)$ be the difference of any two solutions of (2.8). Then

$$\begin{aligned}
v_t(x) &= \int_0^t \left(\frac{1}{2}a(x)\partial_x^2 v_s(x) - \frac{1}{2}d_s(x)v_s(x)\right) ds \\
&\quad + \int_0^t \int_{\mathbb{R}} h(y-x)\partial_x v_s(x)W(ds\,dy).
\end{aligned}$$

By an argument similar to the proof of Lemma 4.2 in [15], we can show that there exists a finite nonnegative K such that

$$\mathbb{E}\|v_t\|_0^2 \leq K \int_0^t \mathbb{E}\|v_s\|_0^2 ds.$$

Then, Gronwall's inequality implies $v = 0$.

According to Kurtz-Xiong [13], equation (2.8) has a unique, non-negative solution. By direct calculation, we will see that $(\psi_t^{n+1} - \psi_t^n)$ is a solution of (2.8). Thus $\{\psi_t^n(x)\}$ is a bounded increasing sequence for each $x \in \mathbb{R}$. By definition 2.1, the limit is a unique solution of (2.1). ∎

Theorem 2.2 *Suppose that the basic condition* (A) *holds. Then, for any $\phi \in \{C_b(\mathbb{R})^+ \cap \mathbb{H}^1(\mathbb{R})\}$, equation (2.2) has a unique $C_b(\mathbb{R}) \cap \mathbb{H}^1(\mathbb{R})$-valued, non-negative, strong solution $\{\psi_{r,t} : t \geq r \geq 0\}$. Furthermore, if $\phi \equiv 1$, equation (2.2) has a unique $C_b(\mathbb{R})$-valued, non-negative, strong solution $\{\psi_{r,t} : t \geq r \geq 0\}$. For any $\phi \in \{C_b(\mathbb{R})^+ \cap \mathbb{H}^1(\mathbb{R})\} \cup \{1\}$, $\|\psi_{r,t}\|_a \leq \|\phi\|_a$ holds $\mathbb{P} - a.s.$ for all $t \geq r \geq 0$, where $\|\phi\|_a$ is the supremum of ϕ.*

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Proof: For any $\phi \in \{C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})\}$, see the proof of Theorem 4.2 of Li et al. [15]. For $\phi \equiv 1$, the existence and the uniqueness can be proved by an argument similar to the proof of Theorem 2.1. ∎

Since the solution of (2.1) depends on the initial function $\phi(\cdot)$, we can rewrite the solution of (2.1) as $\psi_t(x) = \psi_t(x, \phi)$. Based on this new notation, we say that $\psi_t(x, \phi)$, the solution of (2.1), defines a ψ-semigroup if there exists a set $N \subset \Omega$ such that $\mathbb{P}(N) = 0$ and for any $\phi \in C(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})$ and $0 \leq s \leq t$,

$$\psi_{s+t}(x, \phi) = \psi_t(x, \psi_s(\cdot, \phi)) \tag{2.9}$$

holds for all $\omega \notin N$. Based on this definition, we have following theorem.

Theorem 2.3 *Suppose that the basic condition* (A) *holds. Then, for any* $\phi \in \{C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})\}$, *equation* (2.1) *has a unique* $\{C_b(\mathbb{R}) \bigcap \mathbb{H}^1(\mathbb{R})\}$-*valued, non-negative, strong solution* $\{\psi_t : t \geq 0\}$. *Moreover, the solution defines a* ψ-*semigroup.*

Proof: Based on the Theorem 2.1, we only need to prove the semigroup property. Let ψ_t be the unique strong solution of (2.1). Then, for any nonnegative s, t, we have

$$\begin{aligned}
\psi_s(x) &= \phi(x) + \int_0^s \left[\frac{1}{2}a(x)\partial_{xx}^2\psi_u(x) - \frac{1}{2}\sigma(x)\psi_u(x)^2\right]du \\
&\quad + \int_0^s \int_{\mathbb{R}} h(y-x)\partial_x\psi_u(x)W(du, dy), \qquad s \geq 0.
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\psi_{s+t}(x) &= \phi(x) + \int_0^{s+t} \left[\frac{1}{2}a(x)\partial_{xx}^2\psi_u(x) - \frac{1}{2}\sigma(x)\psi_u(x)^2\right]du \\
&\quad + \int_0^{s+t} \int_{\mathbb{R}} h(y-x)\partial_x\psi_u(x)W(du, dy), \qquad s \geq 0 \quad t \geq 0.
\end{aligned} \tag{2.11}$$

(2.11) minus (2.10). Then, we get

$$\begin{aligned}
\psi_{s+t}(x) - \psi_s(x) &= \int_s^{s+t} \left[\frac{1}{2}a(x)\partial_{xx}^2\psi_u(x) - \frac{1}{2}\sigma(x)\psi_u(x)^2\right]du \\
&\quad + \int_s^{s+t} \int_{\mathbb{R}} h(y-x)\partial_x\psi_u(x)W(du, dy), \qquad s \geq 0 \quad t \geq 0
\end{aligned} \tag{2.12}$$

For any one-dimensional Borel set A, any $0 \leq u \leq v$, and any fixed $s \geq 0$, define $W^s([u, v], A) = W([u+s, v+s], A)$ as the s-shifted space-time white noise of W. Since $a(x)$, $\sigma(x)$, h are time homogeneous, we can reform (2.12) to get

$$\begin{aligned}
\psi_{s+t}(x) &= \psi_s(x) + \int_0^t \left[\frac{1}{2}a(x)\partial_{xx}^2\psi_{s+u}(x) - \frac{1}{2}\sigma(x)\psi_{s+u}(x)^2\right]du \\
&\quad + \int_0^t \int_{\mathbb{R}} h(y-x)\partial_x\psi_{s+u}(x)W^s(du, dy), \qquad s \geq 0 \quad t \geq 0.
\end{aligned} \tag{2.13}$$

By the uniqueness of the strong solution of (2.1), for any fixed $s \geq 0$, $\psi_{s+t}(x, \phi)$ is the unique strong solution of (2.12). On the other hand, for the same fixed $s \geq 0$, just following the same idea to prove Theorem 2.1 we can prove that (2.13) has a unique strong solution

which is just $\psi_t(x, \psi_s(\cdot, \phi))$ since the initial value is $\psi_s(\cdot, \phi)$. This obviously gives that for any fixed $s \geq 0$,

$$\psi_{s+t}(x, \phi) = \psi_t(x, \psi_s(\cdot, \phi)), \qquad t \geq 0, \tag{2.14}$$

holds for all $\omega \notin N$ with $\mathbb{P}(N) = 0$. The existence of the set N comes from the continuity of the unique strong solution of (2.1) and (2.13). ∎

Same as the forward case, since the solution of (2.2) depends on the initial value $\phi(\cdot)$, we can rewrite the solution of (2.2) as $\psi_{s,t}(x) = \psi_{s,t}(x, \phi)$. Based on this new notation, we say that $\psi_{s,t}(x, \phi)$, the solution of the backward equation (2.2), defines a backward ψ-semigroup if there exists a N such that $\mathbb{P}(N) = 0$ and for any $\phi \in C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})$ and $0 \leq r \leq s \leq t$,

$$\psi_{r,t}(x, \phi) = \psi_{r,s}(x, \psi_{s,t}(\cdot, \phi)) \tag{2.15}$$

holds for all $\omega \notin N$. Based on this definition, we have following theorem.

Theorem 2.4 *Suppose that the basic condition* (A) *holds. Then, for any* $\phi \in \{C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})\}$, *equation* (2.2) *has a unique* $C_b(\mathbb{R}) \bigcap \mathbb{H}^1(\mathbb{R})$-*valued, non-negative, strong solution* $\{\psi_{r,t} : t \geq r \geq 0\}$. *Moreover, the solution of* (2.2) *defines a backward* ψ-*semigroup.*

Proof: Based on the Theorem 2.2, we only need to prove that the $C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})$-valued strong solution $\{\psi_{r,t} : t \geq r \geq 0\}$ defines a backward ψ-semigroup. Let $\psi_{r,t}(x) = \psi_{r,t}(x, \phi)$ be the unique strong solution of (2.2). Then, for any nonnegative t, s, v, we have

$$\psi_{t-s-v,t}(x) = \phi(x) + \int_{t-s-v}^{t} \left[\frac{1}{2} a(x) \partial_{xx}^2 \psi_{u,t}(x) - \frac{1}{2} \sigma(x) \psi_{u,t}(x)^2 \right] du$$
$$+ \int_{t-s-v}^{t} \int_{\mathbb{R}} h(y-x) \partial_x \psi_{u,t}(x) \cdot W(du, dy), \qquad t \geq s+v, \tag{2.16}$$

and

$$\psi_{t-s,t}(x) = \phi(x) + \int_{t-s}^{t} \left[\frac{1}{2} a(x) \partial_{xx}^2 \psi_{u,t}(x) - \frac{1}{2} \sigma(x) \psi_{u,t}(x)^2 \right] du$$
$$+ \int_{t-s}^{t} \int_{\mathbb{R}} h(y-x) \partial_x \psi_{u,t}(x) \cdot W(du, dy), \qquad t \geq s \geq 0. \tag{2.17}$$

That (2.16) minus (2.17) gives

$$\psi_{t-s-v,t}(x) - \psi_{t-s,t}(x) = \int_{t-s-v}^{t-s} \left[\frac{1}{2} a(x) \partial_{xx}^2 \psi_{u,t}(x) - \frac{1}{2} \sigma(x) \psi_{u,t}(x)^2 \right] du$$
$$+ \int_{t-s-v}^{t-s} \int_{\mathbb{R}} h(y-x) \partial_x \psi_{u,t}(x) \cdot W(du, dy), \quad t \geq v + s. \tag{2.18}$$

In (2.18), for fixed t and s, let v change in $t - s \geq v \geq 0$. Then, $\psi_{t-s,t}(x)$ is treated as the initial value. Since the upper limit of the integrals in the right hand side is $t - s$ and the stochastic integral is backward, $t - s$ must be the backward initial time for the solution. Then, without loss any generality, we can reform (2.18) to get

$$\psi_{t-s-v,t-s}(x) - \psi_{t-s,t}(x) = \int_{t-s-v}^{t-s} \left[\frac{1}{2} a(x) \partial_{xx}^2 \psi_{u,t-s}(x) - \frac{1}{2} \sigma(x) \psi_{u,t-s}(x)^2 \right] du$$
$$+ \int_{t-s-v}^{t-s} \int_{\mathbb{R}} h(y-x) \partial_x \psi_{u,t-s}(x) \cdot W(du, dy), \quad t \geq v + s. \tag{2.19}$$

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By the uniqueness of the strong solution of (2.2), for any fixed $s \geq 0$ and $t - s \geq 0$, $\psi_{t-s-v,t}(x, \phi)$ is the unique strong solution of (2.18). On the other hand, for the same fixed $s \geq 0$ and $t - s \geq 0$, just following the same idea to prove Theorem 2.2 we can prove that (2.19) has a unique strong solution which is just $\psi_{t-s-v,t-s}(x, \psi_{t-s,t}(\cdot, \phi))$ since the initial value is $\psi_{t-s,t}(\cdot, \phi)$. This obviously gives that for any fixed $s \geq 0$ and $t - s \geq 0$

$$\psi_{t-s-v,t}(x, \phi) = \psi_{t-s-v,t-s}(x, \psi_{t-s,t}(\cdot, \phi)), \qquad t - s \geq v \geq 0, \qquad (2.20)$$

holds for all $\omega \notin N$ with $\mathbb{P}(N) = 0$. The existence of the set N comes from the continuity of the unique strong solution of (2.2) and (2.19). $\blacksquare$

3 Conditional log-Laplace functionals

In order to construct conditional entrance laws and conditional excursion laws, we need some conclusions and notations from conditional log-Laplace functional. Same as log-Laplace functional for Dawson-Watanabe processes, it will be demonstrated that conditional log-Laplace functional is a very powerful tool to handle the models in which Brownian branching particles move in a random medium.

First, let us characterize the SDSM as a unique weak solution of a SPDE.

Theorem 3.1 *Assume that $c \in C_b^2(\mathbb{R})$, $c^2(x) \geq \epsilon > 0$, $\sigma \in C_b(\mathbb{R})$, $h \in C_b^2(\mathbb{R}) \bigcap \mathbb{H}^2(\mathbb{R})$. Then, for any given $\mu \in M(\mathbb{R})$ and any $\phi \in C^2(\mathbb{R})$, the following SPDE has a unique, continuous weak solution $\{X_t : t \geq 0\}$:*

$$\langle \phi, X_t \rangle = \langle \phi, \mu \rangle + \frac{1}{2} \int_0^t \langle a\phi'', X_s \rangle ds$$
$$+ \int_0^t \int_{\mathbb{R}} \phi(y) Z(ds, dy) + \int_0^t \int_{\mathbb{R}} \langle h(y - \cdot)\phi', X_s \rangle W(ds, dy), \qquad (3.1)$$

where $W(ds, dx)$ is a space-time white noise and $Z(ds, dy)$ is an orthogonal martingale measure which is orthogonal to $W(ds, dy)$ and has covariation measure $\sigma(y) X_s(dy) ds$.

Proof: See the proofs of Theorem 1.2 of [9] or Theorem 3.1 of [15] for the construction of the weak solution of the equation (3.1). The uniqueness follows from the duality argument and the continuity of the solution follows from the Proposition 2 of Bakry-Emery [2] $\blacksquare$

It is obvious that above $\{X_t : t \geq 0\}$ also solves the $(\mathcal{L}, \delta_\mu)$-martingale problem.

Let $(\mathcal{G}_t)_{t \geq 0}$ denote the filtration generated by $\{W(ds, dy)\}$, $\{Z(ds, dy)\}$ and $\{X_s(dy)\}$. By Theorem 2.4, for any $\phi \in C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})$ the backward equation (2.2) has a unique strong solution $\psi_{r,t}$. The following are two main results of the conditional log-Laplace functionals.

Theorem 3.2 *Suppose that the basic condition* (A) *holds. Then, for any $t \geq r \geq 0$ and $\phi \in C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})$, we have a.s.*

$$\langle \phi, X_t \rangle = \langle \psi_{r,t}, X_r \rangle + \int_r^t \int_{\mathbb{R}} \psi_{s,t}(x) Z(ds, dx) + \frac{1}{2} \int_r^t \langle \sigma \psi_{s,t}^2, X_s \rangle ds. \qquad (3.2)$$

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Proof: See the proof of the Lemma 5.3 of [15]. ∎

Theorem 3.3 *Suppose that the basic condition* (A) *holds. Let* $\mathbb{E}^W$ *denote the conditional expectation of* $\{X_t : t \geq 0\}$ *given the space-time white noise* $W(ds, dy)$*. Then, for* $t \geq r \geq 0$ *and* $\phi \in C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})$*, we have a.s.*

$$\mathbb{E}^W \{ e^{-\langle \phi, X_t \rangle} | \mathcal{G}_r \} = \exp \left\{ - \langle \psi_{r,t}, X_r \rangle \right\}. \tag{3.3}$$

In particular, if $\phi \equiv 1$*, we have a.s.*

$$\mathbb{E}^W \{ e^{-\langle 1, X_t \rangle} | \mathcal{G}_r \} = \exp \left\{ - \langle \Upsilon_{r,t}(x), X_r \rangle \right\}, \tag{3.4}$$

where $\Upsilon_{r,t}(x)$ *is the unique, nonnegative,* $C_b(\mathbb{R})$*-valued solution of* (2.2) *with initial data* $\phi \equiv 1$*. Consequently,* $\{X_t : t \geq 0\}$ *is a diffusion process with Feller transition semigroup* $(Q_t)_{t \geq 0}$ *given by*

$$\int_{M(\mathbb{R})} e^{-\langle \phi, \nu \rangle} Q_t(\mu, d\nu) = \mathbb{E} \exp \left\{ - \langle \psi_{0,t}, \mu \rangle \right\}. \tag{3.5}$$

Proof: For $\phi \in C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})$, see the proof of Theorem 5.1 of [15]. For $\phi \equiv 1$, the conclusion follows from a monotone convergence sequence method. ∎

4   Conditional Generalized Super-Brownian Motion

In this section, we derive some new properties of the conditional SDSM which are similar to that of super-Brownian motion. These properties are necessary for the conditional excursion representation of the SDSMI. First, let us consider the following backward SPDE:

$$
\begin{aligned}
T_{r,t}(x) \;=\; & \phi(x) + \int_r^t \left[\frac{1}{2} a(x) \partial_{xx}^2 T_{s,t}(x) \right] ds \\
& + \int_r^t \int_{\mathbb{R}} h(y-x) \partial_x T_{s,t}(x) \cdot W(ds, dy), \qquad t \geq r \geq 0, \tag{4.1}
\end{aligned}
$$

where "·" denotes the backward stochastic integral. (4.1) is just (2.2) with $\sigma(\cdot) \equiv 0$. Thus, according to Theorem 2.4, for any $\phi \in C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})$, the backward equation (4.1) has a unique $C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})$-valued strong solution $\{T_{r,t} : t \geq r \geq 0\}$ which is continuous in t. Moreover, the solution of (4.1) defines a backward ψ-semigroup. In order to give a better estimate of the solution of (4.1) by using the results of Rozovskii [19], here we introduce some new notations. Let $\{h_j : j = 1, 2, \cdots\}$ be a complete orthonormal system of $L^2(\mathbb{R})$. Then

$$W_j(t) = \int_0^t \int_{\mathbb{R}} h_j(y) W(ds, dy), \qquad t \geq 0$$

defines a sequence of independent standard Brownian motions $\{W_j : j = 1, 2, \cdots\}$. For $\epsilon > 0$ let

$$W^\epsilon(dt, dx) = \sum_{j=1}^{[1/\epsilon]} h_j(x) W_j(dt) dx, \qquad s \geq 0, x \in \mathbb{R}.$$

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Let $L^2([0,T], \mathcal{P}, \mathbb{H}^k)$ denote the space of $\mathbb{H}^k$-valued, predictable, square-integrable stochastic processes and $C([0,T], \mathcal{P}, \mathbb{H}^k)$ denote the space of $\mathbb{H}^k$-valued, strongly continuous stochastic processes. Assume that $c \in C_b^2(\mathbb{R})$, $c^2(x) \geq \epsilon > 0$, $h \in C_b^2(\mathbb{R}) \bigcap \mathbb{H}^2(\mathbb{R})$. For any $\phi \in \{C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})\}$, by Rozovskii ([19] p133, Theorem 2), the equation

$$
\begin{aligned}
T_{r,t}^\epsilon(x) \quad = \quad & \phi(x) + \int_r^t \left[\frac{1}{2}a(x)\partial_x^2 T_{r,s}^\epsilon(x)\right] ds \\
& + \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x T_{r,s}^\epsilon(x) \cdot W^\epsilon(ds, dy), \qquad t \geq r \geq 0, \qquad (4.2)
\end{aligned}
$$

has a unique solution $T_{r,t}^\epsilon(x) \in L^2([0,T], \mathcal{P}, \mathbb{H}^2) \bigcap C([0,T], \mathcal{P}, \mathbb{H}^1)$ and the following inequality

$$
\mathbb{E} \sup_{s \in [r,T]} \|T_{r,s}^\epsilon\|_2^2 \leq K\mathbb{E}\|\phi\|_2^2
$$

holds. By a limit argument similar to the proof of Rozovskii ([19] p111, Theorem 2), we can get that $T_{r,s}(x)$, the solution of (4.1), satisfies

$$
\mathbb{E} \sup_{s \in [r,T]} \|T_{r,s}\|_2^2 \leq K\mathbb{E}\|\phi\|_2^2. \qquad (4.3)
$$

In the following, in order to emphasize some special points, we use $T_{r,t}(x) = T_{r,t}(x, \phi)$ and $\psi_{r,t}(x) = \psi_{r,t}(x, \phi)$ to denote the unique solution of (4.1) and (2.2), respectively. Similar to the super-Brownian motion case, for any $\phi \in C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})$ we consider the following stochastic equation:

$$
\Psi_{r,t}(x) = T_{r,t}(x, \phi) - \frac{1}{2}\int_r^t T_{r,s}[\sigma(x)(\Psi_{s,t}(x))^2] ds, \qquad (4.4)
$$

where $T_{r,t}(x, \phi)$ is the unique strong solution of (4.1). From the inequality (4.3), we can prove that the equation (4.4) has a unique solution by the Picard iterative scheme. Then, we have following theorem.

Theorem 4.1 *Suppose that the basic condition* (A) *holds. Then, for any* $\phi \in C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})$, (4.4) *has a unique strong solution that defines a* ψ-*semigroup for all* $\omega \notin N$ *with* $\mathbb{P}(N) = 0$. *Let* $\{\psi_{r,t} : 0 \leq r \leq t\}$ *denote the unique strong solution of the equation* (4.4). *Then, for each* $\mu \in M(\mathbb{R})$ *and given space-time white noise* W *there is a unique conditional probability measure* Q_μ^W *on* $W^E \equiv C([0, \infty), M(\mathbb{R}))$ *such that*

$$
\int_{W^E} e^{-\langle\phi, w_t\rangle} Q_\mu^W(dw) = \exp\left\{-\langle\psi_{0,t}, \mu\rangle\right\} \qquad \omega \notin N, \qquad (4.5)
$$

holds and the coordinate process $\{w_t : t \geq 0\}$ *on* W^E *under the system* $\{Q_\mu^W : \mu \in M(\mathbb{R})\}$ *defines a conditional diffusion process, called conditional generalized super-Brownian motion, with transition semigroup* $\{(Q_{r,t}^W) : t \geq r \geq 0\}$ *given by*

$$
\int_{M(\mathbb{R})} e^{-\langle\phi, \nu\rangle} Q_{r,t}^W(\mu, d\nu) = \exp\left\{-\langle\psi_{r,t}, \mu\rangle\right\}, \qquad \omega \notin N. \qquad (4.6)
$$

Furthermore, (4.4) *is equivalent to* (2.2). *Thus,* $\{\psi_{r,t} : 0 \leq r \leq t\}$ *in* (4.6) *is also the unique strong solution of* (2.2).

Proof: The existence of the unique measure Q_μ^W on W^E and the conditional transition semigroup $\{(Q_{r,t}^W) : t \geq r \geq 0\}$ such that (4.5) and (4.6) hold follows from the construction of the conditional log-Laplace functional for SDSMI with $b = 0 = m$ given in the section 5 in [15]. Let $T_{r,t}(x) = T_{r,t}(x, \phi)$ be the unique strong solution of (4.1), which is just the strong solution of (2.2) with $\sigma(\cdot) \equiv 0$. By Theorem 2.4, we know that $\{T_{r,t} : t \geq r \geq 0\}$ is a backward ψ- semigroup.

To complete the proof of the theorem, it suffices to prove that (4.4) is equivalent to (2.2). To this end, in the following we prove that given a solution of (4.4), we can change the form of (4.4) into that of (2.2) by a stochastic Fubini theorem (See Theorem 2.6 of Walsh [20]) as follows: For any $\phi \in C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})$, let $\psi_{r,t}(x)$ be a solution of (4.4). Thus, we have

$$
\begin{aligned}
\psi_{r,t}(x) \ &= \ T_{r,t}(x) - \frac{1}{2}\int_r^t T_{r,s}[\sigma(x)(\psi_{s,t}(x))^2]ds & (4.7)\\
&= \int_r^t [\frac{1}{2}a(x)\partial_{xx}^2 T_{u,t}(x)]du + \int_r^t \int_\mathbb{R} h(y-x)\partial_x T_{u,t}(x) \cdot W(du,dy)\\
&\quad - \frac{1}{2}\int_r^t [\sigma(x)(\psi_{s,t}(x))^2]ds - \frac{1}{2}\int_r^t \left\{ \int_r^s \frac{1}{2}a(x)\partial_{xx}^2 T_{u,s}[\sigma(x)(\psi_{s,t}(x))^2]du \right.\\
&\quad + \left. \int_r^s \int_\mathbb{R} h(y-x)\partial_x T_{u,s}[\sigma(x)(\psi_{s,t}(x))^2] \cdot W(du,dy) \right\} ds + \phi(x)\\
&= \int_r^t [\frac{1}{2}a(x)\partial_{xx}^2 T_{u,t}(x)]du - \frac{1}{2}\int_r^t \left\{ \int_u^t \frac{1}{2}a(x)\partial_{xx}^2 T_{u,s}[\sigma(x)(\psi_{s,t}(x))^2]ds \right\} du\\
&\quad + \phi(x) - \frac{1}{2}\int_r^t [\sigma(x)(\psi_{s,t}(x))^2]ds + \int_r^t \int_\mathbb{R} h(y-x)\partial_x T_{u,t}(x) \cdot W(du,dy)\\
&\quad - \frac{1}{2}\int_r^t \int_\mathbb{R} \left\{ \int_u^t h(y-x)\partial_x T_{u,s}[\sigma(x)(\psi_{s,t}(x))^2]ds \right\} \cdot W(du,dy)\\
&= \int_r^t [\frac{1}{2}a(x)\partial_{xx}^2 \psi_{u,t}(x)]du - \frac{1}{2}\int_r^t [\sigma(x)(\psi_{s,t}(x))^2]ds\\
&\quad + \int_r^t \int_\mathbb{R} h(y-x)\partial_x \psi_{u,t}(x) \cdot W(du,dy) + \phi(x),
\end{aligned}
$$

which gives (2.2) and the proof is complete. ∎

Theorem 4.2 *Suppose that the basic condition (A) holds. For any $\phi \in C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})$, let $Q_{r,t}^W(\mu, d\nu)$ be the transition semigroup constructed from theorem 4.1 and $T_{r,t}(x) = T_{r,t}(x, \phi)$ be the unique strong solution of (4.1). Then, we have*

$$
\int_{M(\mathbb{R})} \langle \phi, \nu \rangle Q_{r,t}^W(\mu, d\nu) = \langle T_{r,t}, \mu \rangle, \qquad \omega \notin N \tag{4.8}
$$

and

$$
\int_{M(\mathbb{R})} \langle \phi, \nu \rangle^2 Q_{r,t}^W(\mu, d\nu) = \langle T_{r,t}, \mu \rangle^2 + \int_r^t \langle T_{r,s}[\sigma(x)(T_{s,t}(x))^2], \mu \rangle ds, \qquad \omega \notin N \tag{4.9}
$$

Proof: For any non-negative real number λ, let $\psi_{r,t}^\lambda(x)$ be the unique strong solution of the following equation:

$$
\psi_{r,t}^\lambda(x) = T_{r,t}(\lambda\phi(x)) - \frac{1}{2}\int_r^t T_{r,s}[\sigma(x)(\psi_{s,t}^\lambda(x))^2]ds \tag{4.10}
$$

which is just the equation (4.4) with initial value $\lambda\phi(x)$ and where $T_{r,t}(\lambda\phi(x))$ or $T_{r,t}^{\lambda}(x)$ is the unique strong solution of the equation

$$T_{r,t}^{\lambda}(x) = \lambda\phi(x) + \int_r^t \left[\frac{1}{2}a(x)\partial_{xx}^2 T_{s,t}^{\lambda}(x)\right]ds$$

$$+ \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x T_{s,t}^{\lambda}(x) \cdot W(ds,dy), \qquad t \geq r \geq 0, \qquad (4.11)$$

where "$\cdot$" denotes the backward stochastic integral. From (4.11) and (4.10), we have $\psi_{r,t}^{\lambda}(x)|_{\lambda=0} \equiv 0$. Now we prove that $T_{r,t}^{\lambda}(x)$ and $\psi_{r,t}^{\lambda}(x)$ are differentiable with respect to λ in the norm $\|\cdot\|_0$. First, we can directly check that $T_{r,t}^{\lambda}(x) = \lambda T_{r,t}(x)$ is a solution of (4.11). Since (4.11) has uniqueness, it is obvious that $T_{r,t}^{\lambda}(x)$ is differentiable with respect to λ and $\frac{\partial}{\partial\lambda}T_{r,t}^{\lambda}(x) = T_{r,t}(x)$ and $\frac{\partial^2}{\partial\lambda^2}T_{r,t}^{\lambda}(x) \equiv 0$. Let $Z_{s,t}^{\lambda}(x) = \lambda^{-1}\psi_{s,t}^{\lambda}(x) - T_{s,t}(x)$. According to (4.10) and (4.11), we have

$$Z_{r,t}^{\lambda}(x) = -(2\lambda)^{-1}\int_r^t T_{r,s}[\sigma(x)\psi_{s,t}^{\lambda}(x)^2]ds \qquad (4.12)$$

$$= -\frac{1}{2}\int_r^t T_{r,s}[\sigma(x)\lambda^{-1}\psi_{s,t}^{\lambda}(x)^2 - \sigma(x)\psi_{s,t}^{\lambda}(x)T_{s,t}(x) + \sigma(x)\psi_{s,t}^{\lambda}(x)T_{s,t}(x)]ds$$

$$= -\frac{1}{2}\int_r^t T_{r,s}[\sigma(x)\psi_{s,t}^{\lambda}(x)Z_{s,t}^{\lambda}]ds - \frac{1}{2}\int_r^t T_{r,s}[\sigma(x)\psi_{s,t}^{\lambda}(x)T_{s,t}(x)]ds$$

Since $\|\psi_{s,t}^{\lambda}(x)\|_a \leq \lambda\|\phi\|_a$, $\|T_{r,s}(x)\|_a \leq \|\phi\|_a$, Gronwall's inequality yields

$$\mathbb{E}\|Z_{r,t}^{\lambda}\|_0^2 \to 0 \qquad \text{as } \lambda \to 0.$$

Let

$$u_{r,t}(x) = -\int_r^t T_{r,s}[\sigma(x)T_{s,t}(x)^2]ds$$

and

$$u_{s,t}^{\lambda}(x) = \lambda^{-2}[\psi_{s,t}^{2\lambda}(x) - 2\psi_{s,t}^{\lambda}(x)] - u_{s,t}(x).$$

According to (4.10) and (4.11), we have

$$u_{r,t}^{\lambda}(x) = \lambda^{-2}\left\{-\frac{1}{2}\int_r^t T_{r,s}[\sigma(x)\psi_{s,t}^{2\lambda}(x)^2]ds\right. \qquad (4.13)$$

$$\left. + \int_r^t T_{r,s}[\sigma(x)\psi_{s,t}^{\lambda}(x)^2]ds\right\} + \int_r^t T_{r,s}[\sigma(x)T_{s,t}(x)^2]ds$$

$$= \int_r^t T_{r,s}\left[\sigma(x)\left\{T_{s,t}(x)^2 + \left[\frac{\psi_{s,t}^{\lambda}(x)}{\lambda}\right]^2 - 2\left[\frac{\psi_{s,t}^{2\lambda}(x)}{2\lambda}\right]^2\right\}\right]ds$$

Since $\|\psi_{s,t}^{\lambda}(x)\|_a \leq \lambda\|\phi\|_a$, $\|T_{r,s}(x)\|_a \leq \|\phi\|_a$, and

$$\mathbb{E}\|Z_{r,t}^{\lambda}\|_0^2 \to 0 \qquad \text{as } \lambda \to 0,$$

we get

$$\mathbb{E}\|u_{r,t}^{\lambda}\|_0^2 \to 0 \qquad \text{as } \lambda \to 0.$$

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Therefore, we have $\frac{\partial \psi_{r,t}^{\lambda}(x)}{\partial \lambda}|_{\lambda=0} = T_{r,t}(x)$ and

$$\frac{\partial^2 \psi_{r,t}^{\lambda}(x)}{\partial \lambda^2}|_{\lambda=0} = -\int_r^t T_{r,s}[\sigma(x)(T_{s,t}(x))^2]ds.$$

Then, the conclusion follows from taking derivative with respect to λ in the following equation

$$\int_{M(\mathbb{R})} e^{-\langle \lambda\phi,\nu\rangle} Q_{r,t}^W(\mu,d\nu) = \exp\left\{ - \langle \psi_{r,t}^{\lambda},\mu\rangle \right\}, \qquad \omega \notin N, \qquad (4.14)$$

and then set $\lambda = 0$. $\blacksquare$

5   Conditional Entrance Laws for SDSM

Before we start to construct the conditional entrance law for SDSM, we first give a required lemma which gives the monotonicity of the solution of the SPDE (2.1) in the coefficient $\sigma(\cdot)$.

Lemma 5.1 *Suppose that $h \in C_b^2(\mathbb{R}) \bigcap \mathbb{H}^2(\mathbb{R})$, $c \in C_b^2(\mathbb{R})$, $\sigma_i(x) \in C_b^2(\mathbb{R}), i = 1, 2$, and $0 \le \sigma_1(x) \le \sigma_2(x)$. For $i = 1, 2$, let $\psi_t^i(x)$ be the unique strong solution of the following equation:*

$$\psi_t^i(x) = \phi(x) + \int_0^t \left(\frac{1}{2}a(x)\partial_x^2 \psi_s^i(x) - \frac{1}{2}\sigma_i(x)\psi_s^i(x)^2 \right) ds$$

$$+ \int_0^t \int_{\mathbb{R}} h(y - x)\partial_x \psi_s^i(x) W(ds\,dy).$$

Then $\psi_t^1(x) \ge \psi_t^2(x)$.

Proof: Let $u_t(x) = \psi_t^1(x) - \psi_t^2(x)$. Then

$$u_t(x) = \int_0^t \left(\frac{1}{2}a(x)\partial_x^2 u_s(x) - \frac{1}{2}d_s(x)u_s(x) \right) ds$$

$$+ \int_0^t \int_{\mathbb{R}} h(y - x)\partial_x u_s(x) W(ds\,dy)$$

$$+ \int_0^t c_s(x)ds, \qquad (5.1)$$

where

$$d_s(x) = \sigma_1(x)(\psi_s^1(x) + \psi_s^2(x)) \ge 0,$$

and

$$c_s(x) = \frac{1}{2}(\sigma_2(x) - \sigma_1(x))\psi_s^2(x)^2 \ge 0.$$

Similar to (2.8) we can show that (5.1) has at most one solution.

For $\phi \ge 0$, we consider equation

$$y_t(x) = \phi(x) + \int_r^t \left(\frac{1}{2}a(x)\partial_x^2 y_s(x) - \frac{1}{2}d_s(x)y_s(x) \right) ds$$

$$+ \int_r^t \int_{\mathbb{R}} h(y - x)\partial_x y_s(x) W(ds\,dy). \qquad (5.2)$$

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Denote the solution to (5.2) by $T^c_{r,t}(x,\phi)$. By Kurtz-Xiong [13], we know that $T^c_{r,t}(x,\phi) \geq 0$. Now we claim that

$$\theta_t(x) = \int_0^t T^c_{r,t}(x,c_r)dr \geq 0 \tag{5.3}$$

is a solution to (5.1). From (5.2), we have

$$T^c_{r,t}(x,c_r) = c_r(x) + \int_r^t \left(\frac{1}{2}a(x)\partial_x^2 T^c_{r,s}(x,c_r) - \frac{1}{2}d_s(x)T^c_{r,s}(x,c_r) \right) ds$$
$$+ \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x T^c_{r,s}(x,c_r)W(dsdy).$$

Hence

$$\theta_t(x) = \int_0^t c_r(x)dr + \int_0^t \int_r^t \left(\frac{1}{2}a(x)\partial_x^2 T^c_{r,s}(x,c_r) - \frac{1}{2}d_s(x)T^c_{r,s}(x,c_r) \right) ds dr$$
$$+ \int_0^t \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x T^c_{r,s}(x,c_r)W(dsdy)dr$$
$$= \int_0^t c_r(x)dr + \int_0^t \int_0^s \left(\frac{1}{2}a(x)\partial_x^2 T^c_{r,s}(x,c_r) - \frac{1}{2}d_s(x)T^c_{r,s}(x,c_r) \right) dr ds$$
$$+ \int_0^t \int_{\mathbb{R}} \int_0^s h(y-x)\partial_x T^c_{r,s}(x,c_r)dr W(dsdy)$$
$$= \int_0^t c_r(x)dr + \int_0^t \left(\frac{1}{2}a(x)\partial_x^2 \theta_s(x) - \frac{1}{2}d_s(x)\theta_s(x) \right) ds$$
$$+ \int_0^t \int_{\mathbb{R}} h(y-x)\partial_x \theta_s(x)W(dsdy).$$

This finishes the proof of the claim, and hence, the proof of the lemma. $\blacksquare$

Let $\{Q^{\circ,W}_{r,t} : t \geq r \geq 0\}$ be the restriction of $\{Q^W_{r,t} : t \geq r \geq 0\}$ on $M(\mathbb{R})^\circ := M(\mathbb{R}) \setminus \{0\}$.

Theorem 5.2 *Suppose that the basic condition* (A) *holds. For any* $\phi \in C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})$, *let* $\{\psi_{r,t} : 0 \leq r \leq t\}$ *be the unique strong solution of the backward equation* (2.2) *which defines a* ψ-*semigroup for all* $\omega \notin N$ *with* $\mathbb{P}(N) = 0$. *Then, for each* $x \in \mathbb{R}$, *any* $t > r \geq 0$, *and for all* $\omega \notin N$, *there is a unique finite random measure* $L^W_{r,t}(x,d\nu)$ *on* $M(\mathbb{R})^\circ$ *such that*

$$\int_{M(\mathbb{R})^\circ} (1 - e^{-\langle\phi,\nu\rangle})L^W_{r,t}(x,d\nu) = \psi_{r,t}(x) = \psi_{r,t}(x,\phi) \tag{5.4}$$

holds. Furthermore, $L^W_{r,t}(x,d\nu)$ *is an entrance law for* $\{Q^{\circ,W}_{r,t} : t \geq r \geq 0\}$, *i.e. for each* $x \in \mathbb{R}$, *any* $t > s > r \geq 0$, *and for all* $\omega \notin N$, *we have*

$$L^W_{r,t}(x,d\nu) = L^W_{r,s}(x,\cdot) \circ Q^{\circ,W}_{s,t}(\cdot,d\nu), \tag{5.5}$$

where the right hand side is the convolution measure defined by

$$L^W_{r,s}(x,\cdot) \circ Q^{\circ,W}_{s,t}(\cdot,d\nu) = \int_{M(\mathbb{R})^\circ} Q^{\circ,W}_{s,t}(\mu,d\nu)L^W_{r,s}(x,d\mu).$$

Proof: Recall that $\{Q^{\circ,W}_{r,t}(\mu,d\nu) : t \geq r \geq 0\}$ is the transition semigroup of the conditional generalized super-Brownian motion given space-time white noise W and restricted

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on $M(\mathbb{R})^\circ$. First, we prove the tightness of $\{\frac{1}{\varepsilon}Q_{r,t}^{\circ,W}(\varepsilon\delta_x,\cdot)\}$. Note that, for any real posi-
tive number $\eta > 0$, $\{\mu \in M(\mathbb{R}) : \langle 1 + |x|, \mu\rangle \le \eta\}$ is a compact subset in $M(\mathbb{R})$. We only
need to show that

$$\mathbb{E}\lim_{\eta\to\infty}\sup_{\varepsilon>0}\frac{1}{\varepsilon}Q_{r,t}^{\circ,W}(\varepsilon\delta_x,\{\mu \in M(\mathbb{R}) : \langle 1 + |x|, \mu\rangle > \eta\}) = 0 \tag{5.6}$$

and

$$\sup_{\varepsilon>0}\frac{1}{\varepsilon}Q_{r,t}^{\circ,W}(\varepsilon\delta_x, M(\mathbb{R})^\circ) < \infty, \qquad \text{for} \quad t > r \ge 0. \tag{5.7}$$

Let $\{\gamma_n\}$ be a sequence of functions in $C_b(\mathbb{R})^+ \cap \mathbb{H}^1(\mathbb{R})$, which increasingly converge to
$1 + |x|$. From Theorem 4.2, we have

$$
\begin{aligned}
\text{LHS of } (5.6) &= \lim_{\eta\to\infty}\mathbb{E}\sup_{\varepsilon>0}\frac{1}{\varepsilon}Q_{r,t}^{\circ,W}(\varepsilon\delta_x,\{\mu \in M(\mathbb{R}) : \langle 1 + |x|, \mu\rangle > \eta\}) \\
&\le \lim_{\eta\to\infty}\mathbb{E}\sup_{\varepsilon>0}\frac{1}{\eta}\int_{M(\mathbb{R})}\langle 1 + |x|, \mu\rangle\frac{1}{\varepsilon}Q_{r,t}^{\circ,W}(\varepsilon\delta_x, d\mu) \\
&= \lim_{\eta\to\infty}\lim_{n\to\infty}\mathbb{E}\sup_{\varepsilon>0}\frac{1}{\eta}\int_{M(\mathbb{R})}\langle \gamma_n, \mu\rangle\frac{1}{\varepsilon}Q_{r,t}^{\circ,W}(\varepsilon\delta_x, d\mu) \\
&= \lim_{\eta\to\infty}\lim_{n\to\infty}\mathbb{E}\frac{1}{\eta}T_{r,t}(x,\gamma_n) \\
&= \lim_{\eta\to\infty}\frac{1}{\eta}\mathbb{E}_x(1 + |\xi_t|) = 0,
\end{aligned}
$$

where the fourth equality follows from (4.1) and ξ_t is the driftless diffusion process with
diffusion coefficient $\sqrt{a(x)}$.

Recall that we have assumed that $0 < \sigma_a \le \sigma(x) \le \sigma_b < \infty$, where σ_a and σ_b are constants.
Let $\psi_{r,t}^a(x)$ and $\psi_{r,t}^b(x)$ denote the unique, non-negative solution of (2.2) with $\sigma(x)$ replaced
by σ_a and σ_b, respectively. By Lemma 5.1, we know that $0 \le \psi_{r,t}^b(x) \le \psi_{r,t}(x) \le \psi_{r,t}^a(x)$.
Let $Q_{r,t}^{W,\sigma_a}(\mu, d\nu)$ denote the conditional transition probability of the SDSM with branching
rate σ_a given Brownian sheet W. Since $0 < \sigma_a \le \sigma(x)$, we have

$$\int_{M(\mathbb{R})}e^{-\lambda\langle 1,\nu\rangle}Q_{r,t}^W(\mu, d\nu) \le \int_{M(\mathbb{R})}e^{-\lambda\langle 1,\nu\rangle}Q_{r,t}^{W,\sigma_a}(\mu, d\nu), \qquad 0 \le \lambda. \tag{5.8}$$

Let $\psi_{r,t}(\lambda)$ be the unique solution of (2.2) with $\phi(\cdot) \equiv \lambda$ and $\psi_{r,t}^{\sigma_a}(\lambda)$ be the unique solution
of (2.2) with $\phi(\cdot) \equiv \lambda$ and $\sigma(\cdot) \equiv \sigma_a$. Then,

$$
\begin{aligned}
Q_{r,t}^W(\mu, \{0\}) &= \lim_{\lambda\to\infty}\int_{M(\mathbb{R})}e^{-\lambda\langle 1,\nu\rangle}Q_{r,t}^W(\mu, d\nu) \\
&= \lim_{\lambda\to\infty}e^{-\langle\psi_{r,t}(\lambda),\mu\rangle} \\
&\ge \lim_{\lambda\to\infty}e^{-\langle\psi_{r,t}^{\sigma_a}(\lambda),\mu\rangle} \\
&= Q_{r,t}^{W,\sigma_a}(\mu, \{0\}) \tag{5.9}
\end{aligned}
$$

since $\sigma_a \le \sigma(x)$ and $\psi_{r,t}(\lambda) \le \psi_{r,t}^{\sigma_a}(\lambda)$. From (5.8), we get

$$\int_{M(\mathbb{R})^0}e^{-\lambda\langle 1,\nu\rangle}Q_{r,t}^W(\mu, d\nu) \le \int_{M(\mathbb{R})^0}e^{-\lambda\langle 1,\nu\rangle}Q_{r,t}^{W,\sigma_a}(\mu, d\nu), \qquad 0 \le \lambda. \tag{5.10}$$

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Since the right hand side of (5.8) is the Laplace transformation of a continuous state branching process with generator

$$\mathcal{G}f = \frac{\sigma_a}{2}\frac{\partial^2}{\partial x^2}f.$$

From the theory of continuous state branching process (see [1]), we can get that

$$\int_{M(\mathbb{R})^\circ} e^{-\lambda\langle 1,\nu\rangle}Q_{r,t}^{W,\sigma_a}(\mu,d\nu) = \int_{M(\mathbb{R})} e^{-\lambda\langle 1,\nu\rangle}Q_{r,t}^{W,\sigma_a}(\mu,d\nu) - \int_{\{0\}} e^{-\lambda\langle 1,\nu\rangle}Q_{r,t}^{W,\sigma_a}(\mu,d\nu)$$

$$= \exp\{-\frac{\langle 1,\mu\rangle\lambda}{1+\sigma_a\lambda(t-r)/2}\} - \exp\{-\frac{2\langle 1,\mu\rangle}{\sigma_a(t-r)}\}. \quad (5.11)$$

Thus, we have

$$\lim_{\epsilon\downarrow 0}\int_{M(\mathbb{R})^\circ} e^{-\lambda\langle 1,\nu\rangle}\frac{1}{\epsilon}Q_{r,t}^{W,\sigma_a}(\epsilon\delta_x,d\nu) = \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}[\exp\{-\frac{\epsilon\lambda}{1+\sigma_a\lambda(t-r)/2}\} - \exp\{-\frac{2\epsilon}{\sigma_a(t-r)}\}]$$

$$= \lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}[\frac{-\epsilon\lambda}{1+\sigma_a\lambda(t-r)/2} + \frac{2\epsilon}{\sigma_a(t-r)}]$$

$$= -\frac{\lambda}{1+\sigma_a\lambda(t-r)/2} + \frac{2}{\sigma_a(t-r)}. \quad (5.12)$$

This proves (5.7) and hence, the desired tightness.

For any convergence sequence $\frac{1}{\epsilon_k}Q_{r,t}^{\circ,W}(\epsilon_k\delta_x,\cdot)$, we have

$$\int_{M(\mathbb{R})^\circ}(1-e^{-\langle\phi,\nu\rangle})\frac{1}{\epsilon_k}Q_{r,t}^{\circ,W}(\epsilon_k\delta_x,d\nu) \quad (5.13)$$

$$= \int_{M(\mathbb{R})}(1-e^{-\langle\phi,\nu\rangle})\frac{1}{\epsilon_k}Q_{r,t}^{W}(\epsilon_k\delta_x,d\nu)$$

$$= \frac{1}{\epsilon_k} - \frac{1}{\epsilon_k}\int_{M(\mathbb{R})}e^{-\langle\phi,\nu\rangle}Q_{r,t}^{W}(\epsilon_k\delta_x,d\nu)$$

$$= \frac{1}{\epsilon_k} - \frac{1}{\epsilon_k}\exp\{-\epsilon_k\psi_{r,t}(x)\}$$

$$\longrightarrow \psi_{r,t}(x) \qquad \text{as } k\to\infty,$$

and especially (5.13) implies the limit is independent of choice of the convergent subsequences.

Thus, by Theorem 25.10 and the Corollary of [4], the unique limit of $(1/\epsilon)Q_{r,t}^{\circ,W}(\epsilon\delta_x,d\nu)$ exists, which is denoted by $L_{r,t}^{W}(x,d\nu)$. Especially, for any $t > r \geq 0$, $L_{r,t}^{W}(x,d\nu)$ is a finite random measure by ([3] p.94 Theorem 2). Then, for any $x \in \mathbb{R}$, $t \geq r \geq 0$, and $\omega \notin N$, $L_{r,t}^{W}(x,d\nu)$ is a σ-finite measure on $M(\mathbb{R})^\circ$ such that

$$\int_{M(\mathbb{R})^\circ}(1-e^{-\langle\phi,\nu\rangle})L_{r,t}^{W}(x,d\nu) = \psi_{r,t}(x) = \psi_{r,t}(x,\phi). \quad (5.14)$$

The uniqueness of $L_{r,t}^{W}(x,d\nu)$ comes from (5.14).

Now we are going to check that $L_{r,t}^{W}(x,d\nu)$ is an entrance law of $Q_{r,t}^{\circ,W}$. Since for any

$t \geq s > r \geq 0$, and for all $\omega \notin N$,

$$\int_{M(\mathbb{R})^\circ} \int_{M(\mathbb{R})^\circ} (1 - e^{-\langle \phi, \nu \rangle}) Q_{s,t}^{\circ,W}(\mu, d\nu) L_{r,s}^W(x, d\mu) \qquad (5.15)$$

$$= \int_{M(\mathbb{R})^\circ} \int_{M(\mathbb{R})} (1 - e^{-\langle \phi, \nu \rangle}) Q_{s,t}^W(\mu, d\nu) L_{r,s}^W(x, d\mu)$$

$$= \int_{M(\mathbb{R})^\circ} (1 - e^{-\langle \psi_{s,t}, \mu \rangle}) L_{r,s}^W(x, d\mu)$$

$$= \psi_{r,s}(x, \psi_{s,t}(\cdot, \phi)) = \psi_{r,t}(x, \phi)$$

$$= \int_{M(\mathbb{R})^\circ} (1 - e^{-\langle \phi, \mu \rangle}) L_{r,t}^W(x, d\mu).$$

Thus, (5.5) is proved. $\blacksquare$

Remark: Another way to prove this theorem is using the property that $\{Q_\mu^W : \mu \in M(\mathbb{R})\}$ is conditionally infinitely divisible. Then, by the canonical representation (See Dawson((1993) Theorem 3.3.1 and Section 11.5), for each $x \in \mathbb{R}$ there exists a set of σ-finite random measures $L_{r,t}^W(x, d\nu)$ for $t \geq r \geq 0$ on $M(\mathbb{R})^\circ$ and there exists a set N such that $\mathbb{P}(N) = 0$ and

$$\int_{M(\mathbb{R})^\circ} (1 - e^{-\nu(f)}) L_{r,t}^W(x, d\nu) = \psi_{r,t}(x), \qquad t \geq r \geq 0, x \in \mathbb{R}, f \in C_b(\mathbb{R})^+ \cap \mathbb{H}^1(\mathbb{R}) \quad (5.16)$$

holds for each $\omega \notin N$.

6 Excursion representation for the SDSM with immigration

Immigration processes associated with the SDSM were studied in [17, 15]. Based on the results on conditional entrance laws developed in the last section, we here give a representation for the sample paths of the immigration SDSM.

Let $\mathbb{W}$ denote the totality of continuous path $w \in C((0, \infty), M(\mathbb{R}))$ that takes values from $M(\mathbb{R})^\circ := M(\mathbb{R}) \setminus \{0\}$ in some interval $(\alpha(w), \beta(w)) \subset (0, \infty)$ and takes the value zero elsewhere. Let $(\mathcal{B}(\mathbb{W}), \mathcal{B}_t(\mathbb{W}))$ stand for the natural filtration of $\mathbb{W}$. For $a \geq 0$ let $\mathbb{W}_a$ denote the subset of $\mathbb{W}$ consisting of path w with $\alpha(w) = a$. Let us fix a typical sample of W out of a null set so that the corresponding semigroup $(Q_{s,t}^W)_{t \geq r}$ is defined by (4.6). By Theorem 5.2, $(L_{a,t}^W(x, \cdot))_{t > a}$ is an entrance law for the Markov semigroup $(Q_{s,t}^{\circ,W})_{t \geq r}$ at a. Then there is a unique σ-finite Borel measure Q_a^x on $C((a, \infty), M(\mathbb{R}))$ such that

$$Q_a^x(w_{t_1} \in d\nu_1, \cdots, w_{t_n} \in d\nu_n) = L_{a,t_1}^W(x, d\nu_1) Q_{t_1,t_2}^{\circ,W}(\nu_1, d\nu_2) \cdots Q_{t_{n-1},t_n}^{\circ,W}(\nu_{n-1}, d\nu_n) \quad (6.1)$$

for $a < t_1 < t_2 < \cdots < t_n$ and $\nu_1, \nu_2, \cdots, \nu_n \in M(\mathbb{R})^\circ$. The existence of this measure follows from a result proved in [11] in the setting of right processes. In particular, for $t \geq a$ we have

$$\int_{\mathbb{W}_a} (1 - e^{-\langle \phi, w_t \rangle}) Q_a^x(dw) = \psi_{a,t}(x, f). \qquad (6.2)$$

Roughly speaking, under Q_a^x the coordinate process $\{w_t : t > a\}$ is a conditional diffusion process with transition semigroup $(Q_{r,t}^W)_{t \geq r}$ and one-dimensional distributions $(L_{a,t}^W(x, \cdot))_{t > a}$.

We may and do regard Q_a^x as a σ-finite measure on $\mathbb{W}$ supported by $\mathbb{W}_a$. Now we fix $\mu, m \in M(\mathbb{R})$. By considering an extension of the original probability space, we may define the random objects $N_\mu^W(dx, dw)$ and $N_m^W(ds, dx, dw)$ that, conditioned upon the white noise W, are independent Poisson random measures with intensity measures $\mu(dx)Q_0^x(dw)$ and $m(dx)Q_s^x(dw)ds$, respectively.

For $t \geq 0$, let $\mathcal{F}_t^0$ be the σ-algebra generated by the $\mathbb{P}$-null sets and the random variables

$$\{N_\mu^W(J \times A) : J \in \mathcal{B}(\mathbb{R}), A \in \mathcal{B}_t(\mathbb{W})\}, \tag{6.3}$$

and let $\mathcal{F}_t^1$ be the σ-algebra generated by the $\mathbb{P}$-null sets and the random variables

$$\{N_m^W(J \times A) : J \in \mathcal{B}([0,r] \times \mathbb{R}), A \in \mathcal{B}_t(\mathbb{W}_r), 0 \leq r \leq t\}. \tag{6.4}$$

Define $\bar{\mathcal{G}}_t := \mathcal{F}_t^0 \vee \mathcal{F}_t^1$ which is the σ-algebra generated by $\mathcal{F}_t^0 \cup \mathcal{F}_t^1$. We define the measure-valued processes

$$X_t := \int_{\mathbb{R}} \int_{\mathbb{W}} w_t N_\mu^W(dx, dw), \quad t \geq 0, \tag{6.5}$$

and

$$I_t := \int_{(0,t]} \int_{\mathbb{R}} \int_{\mathbb{W}} w_t N_m^W(ds, dx, dw), \quad t \geq 0. \tag{6.6}$$

Theorem 6.1 *Let $Y_t = X_t + I_t$. Then, for any $h \in C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})$ we have*

$$\mathbb{E}^W \left(\exp\left(-\langle h, Y_t \rangle\right) \Big| \bar{\mathcal{G}}_r \right) = \exp\left\{ -\langle \psi_{r,t}(\cdot, h), Y_r \rangle - \int_r^t \langle \psi_{s,t}(\cdot, h), m \rangle ds \right\}. \tag{6.7}$$

Therefore, $\{Y_t : t \geq 0\}$ is a diffusion process with transition semigroup $(U_t)_{t \geq 0}$ given by

$$\int_{M(\mathbb{R})} e^{-\langle h, \nu \rangle} U_t(\mu, d\nu) = \mathbb{E} \exp\left\{ -\langle \psi_{0,t}(\cdot, h), \mu \rangle - \int_0^t \langle \psi_{s,t}(\cdot, h), m \rangle ds \right\} \tag{6.8}$$

and $\{Y_t : t \geq 0\}$ is just the immigration SDSM constructed as $(\mathcal{J}, \mathcal{D}(\mathcal{J})$-martingale problem in [17].

Proof: Let $t \geq r \geq 0$ and $\mathbb{E}^W$ denote the conditional expectation given W. For nonnegative $f \in \mathcal{B}(\mathbb{R}) \times \mathcal{B}_r(\mathbb{W})$, and nonnegative $g \in \mathcal{K}_r$, where $\mathcal{K}_r$ is the σ-algebra $\sigma\{J \times A : J \in \mathcal{B}([0,s] \times \mathbb{R}), A \in \mathcal{B}_r(\mathbb{W}_s), 0 \leq s \leq r\}$, and $h \in C_b(\mathbb{R})^+ \bigcap \mathbb{H}^1(\mathbb{R})$ we can use an expression for the Laplace transforms of Poisson random measures and Theorem 4.1 to get

$$\mathbb{E}^W \exp\left\{ -\int_{\mathbb{R}} \int_{\mathbb{W}} f(x,w) N_\mu^W(dx, dw) - \int_{(0,r]} \int_{\mathbb{R}} \int_{\mathbb{W}} g(s,x,w) N_m^W(ds, dx, dw) - \langle h, Y_t \rangle \right\}$$

$$= \mathbb{E}^W \exp\left\{ -\int_{\mathbb{R}} \int_{\mathbb{W}} [f(x,w) + \langle h, w_t \rangle] N_\mu^W(dx, dw) \right.$$

$$\left. -\int_{(0,r]} \int_{\mathbb{R}} \int_{\mathbb{W}} [g(s,x,w) + \langle h, w_t \rangle] N_m^W(ds, dx, dw) \right.$$

$$- \int_{(r,t]} \int_{\mathbb{R}} \int_{\mathbb{W}} \langle h, w_t \rangle N_m^W(ds, dx, dw) \Big\}$$

$$= \exp \Big\{ - \int_{\mathbb{R}} \mu(dx) \int_{\mathbb{W}} (1 - \exp[-f(x,w) - \langle h, w_t \rangle]) Q_0^x(dw)$$

$$- \int_0^r ds \int_{\mathbb{R}} m(dx) \int_{\mathbb{W}} (1 - \exp[-g(s,x,w) - \langle h, w_t \rangle]) Q_s^x(dw)$$

$$- \int_r^t ds \int_{\mathbb{R}} m(dx) \int_{\mathbb{W}} (1 - \exp[-\langle h, w_t \rangle]) Q_s^x(dw) \Big\}$$

$$= \exp \Big\{ - \int_{\mathbb{R}} \mu(dx) \int_{\mathbb{W}} (1 - \exp[-f(x,w) - \langle \psi_{r,t}(\cdot, h), w_r \rangle]) Q_0^x(dw)$$

$$- \int_0^r ds \int_{\mathbb{R}} m(dx) \int_{\mathbb{W}} (1 - \exp[-g(s,x,w) - \langle \psi_{r,t}(\cdot, h), w_r \rangle]) Q_s^x(dw)$$

$$- \int_r^t ds \int_{\mathbb{R}} \psi_{s,t}(x, h) m(dx) \Big\}$$

$$= \mathbb{E}^W \exp \Big\{ - \int_{\mathbb{R}} \int_{\mathbb{W}} [f(x,w) + \langle \psi_{r,t}(\cdot, h), w_r \rangle] N_\mu^W(dx, dw)$$

$$- \int_{(0,r]} \int_{\mathbb{R}} \int_{\mathbb{W}} [g(s,x,w) + \langle \psi_{r,t}(\cdot, h), w_r \rangle] N_m^W(ds, dx, dw)$$

$$- \int_r^t \langle \psi_{s,t}(\cdot, h), m \rangle ds \Big\}$$

$$= \mathbb{E}^W \exp \Big\{ - \int_{\mathbb{R}} \int_{\mathbb{W}} f(x,w) N_\mu^W(dx, dw) - \int_{(0,r]} \int_{\mathbb{R}} \int_{\mathbb{W}} g(s,x,w) N_m^W(ds, dx, dw)$$

$$- \langle \psi_{r,t}(\cdot, h), Y_r \rangle - \int_r^t \langle \psi_{s,t}(\cdot, h), m \rangle ds \Big\},$$

which yields (6.7) and (6.8).

By (6.8), Theorem 5.1 of [15], and the uniqueness of immigration SDSM, $\{Y_t : t \geq 0\}$ is just the unique solution of (5.1) of [15] with $b \equiv 0$, which is just the immigration SDSM constructed as $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$-martingale problem in [17] and the proof is complete. ∎

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