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Uniqueness results for an inverse periodic transmission problem

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Abstract

The paper is devoted to the inverse problem of recovering a 2D periodic structure from scattered waves measured above and below the structure. We show that measurements corresponding to a finite number of refractive indices above or below the grating profile, uniquely determine the periodic interface in the inverse TE transmission problem. If a priori information on the height of the diffraction grating is available, then we also obtain upper bounds of the required number of wavenumbers by using the Courant-Weyl min-max principle for a fourth-order elliptic problem. This extends uniqueness results by Hettlich and Kirsch [11] to the inverse transmission problem.

1 Introduction

The problem of recovering the shape of periodic structures from measurements of scattered electromagnetic waves occurs in several applications of micro-optics [3], [13]. We assume the grating to be periodic in one direction and constant in the other, and consider the TE mode of polarization for the diffraction by a periodic interface between two materials. This corresponds to a two–dimensional quasi-periodic transmission problem for the Helmholtz equation. The goal of this paper is to study the uniqueness in the inverse problem of reconstructing the periodic interface. The uniqueness in the transmission problem is not solved in general, and is fundamental for reasonable numerical schemes. Here the grating is illuminated by an incident monochromatic plane wave, and data of the scattered field are taken on two lines lying above and below the grating profile, respectively.

Let the profile of the diffraction grating be given by

$$\Lambda_f := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = f(x_1)\}$$

with a 2π-periodic Lipschitz function, $f \in C^{0,1}_{per}$. Assume that the regions above and below Λ_f,

$$\Omega_f^\pm := \{x \in \mathbb{R}^2 : x_2 \gtrless f(x_1)\,,\ x_1 \in \mathbb{R}\},$$

are filled with materials of refractive indices (or wavenumbers) $k^\pm > 0$, $k^+ \neq k^-$. Suppose further that a plane wave given by

$$v^{in}(x) := \exp(i\alpha x_1 - i\beta x_2),\ (\alpha, \beta) = k^+(\sin\theta, \cos\theta)$$

is incident from the top, where $\theta \in (-\pi/2, \pi/2)$ is the incident angle. Then the diffracted field v^{sc} in the TE (transverse electric) mode satisfies the Helmholtz equations

$$\Delta v^{sc} + (k^\pm)^2 v^{sc} = 0 \quad \text{in} \quad \Omega_f^\pm \tag{1.1}$$

and the transmission conditions

$$[v]_{\Lambda_f} = [\partial_\nu v]_{\Lambda_f} = 0 \tag{1.2}$$

for the total field v given by

$$v = v^{sc} + v^{in} \quad \text{in} \quad \Omega_f^+ , \ v = v^{sc} \quad \text{in} \quad \Omega_f^- .$$

Here ν denotes the unit normal to Λ_f, and $[\cdot]_{\Lambda_f}$ stands for the jump across Λ_f. Moreover, v is assumed to be α-quasiperiodic

$$v(x_1 + 2\pi, x_2) = \exp(2i\alpha\pi)v(x_1, x_2) , \tag{1.3}$$

and v is required to satisfy radiation conditions as $x_2 \to \pm\infty$, i.e., the scattered field can be expanded as the infinite sums of plane waves

$$v^{sc}(x) = \sum_{n \in \mathbb{Z}} A_n^\pm \exp\{i(n+\alpha)x_1 \pm i\beta_n^\pm x_2\} ,$$
$$x_2 > \max(f) \quad \text{resp.} \quad x_2 < \min(f) , \tag{1.4}$$

with the Rayleigh coefficients $A_n^\pm \in \mathbb{C}$ and $\beta_n^\pm := \beta_n(\alpha, k^\pm)$ defined by

$$\beta_n(\alpha, k) := (k^2 - (n+\alpha)^2)^{1/2} , \ 0 \le \arg \beta_n(\alpha, k) < \pi .$$

The direct diffraction problem can be formulated as follows.

(DP): Given f, $k^\pm$ and v^{in}, determine $v = v_f \in H^1_{loc}(\mathbb{R}^2)$ satisfying (1.1)-(1.4).

It is known [4] that for $f \in C^{0,1}_{per}$ there is a unique solution of (DP) which satisfies $v \in H^2_{loc}(\mathbb{R}^2)$.

Our goal is to study the *inverse problem* or the *profile reconstruction problem*.

(IP): Given the refractive indices $k^\pm > 0$ and the incident angle $\theta \in (-\pi/2, \pi/2)$, determine the profile function $f \in C^{0,1}_{per}$ from the knowledge of the total fields

$$v_f(x_1, b^+), \quad v_f(x_1, b^-), \quad 0 \le x_1 \le 2\pi$$

for some $b^+ > \max(f)$, $b^- < \min(f)$.

So far the global uniqueness in problem (IP) is only known in the case of reflection gratings, i.e., for $\text{Im}\, k^- > 0$ (see [6]). On the other hand, for perfectly reflecting gratings (modeled by the Dirichlet problem), more complete uniqueness results were obtained. It was shown by Hettlich and Kirsch [11] that a finite number of incident waves are sufficient to recover the grating profile from the total field above the structure (on $x_2 = b^+$). In particular, one obtains the global uniqueness in the inverse Dirichlet problem if the (positive) wavenumber or the amplitude of the grating is sufficiently small. The proof is based on the Courant-Weyl min-max principle and the monotonicity of eigenvalues for the Laplacian. Global uniqueness results for any fixed wavenumber were established within the class of piecewise linear profiles [7], [8]. See also [1] and [2] for other uniqueness results in the Dirichlet problem.

The purpose of this paper is to extend the Hettlich-Kirsch method to our inverse transmission problem (IP), and we now state our main theorems.

Theorem 1.1 *Let $k^\pm > 0$ and $h > 0$ be such that*

$$h \max(k^+, k^-) < \pi. \tag{1.5}$$

Then, for sufficiently large $|b^\pm|$, the fields $v_f(\cdot, b^+)$ and $v_f(\cdot, b^-)$ corresponding to $k^\pm$ and θ determine the grating function f in problem (IP) uniquely if

$$f(t) \in [0, h] \quad for\ all \quad t \in \mathbb{R}.$$

Thus we obtain the global uniqueness in problem (IP) if both refractive indices or the amplitude of the profile are small. Furthermore, we prove that for any fixed maximal amplitude the profile function is uniquely determined by measurements for a finite number of wavenumbers k^+ or k^-.

Theorem 1.2 *Let $h > 0$ and $f, g \in C^{0,1}_{per}$ such that*

$$f(t), g(t) \in [0, h] \quad for\ all \quad t \in \mathbb{R}. \tag{1.6}$$

Furthermore, choose some

$$b^+ > \max\{f(t), g(t) : t \in \mathbb{R}\}, \quad b^- < \min\{f(t), g(t) : t \in \mathbb{R}\}. \tag{1.7}$$

Consider a fixed refractive index k^+ above the grating profile, and let θ be a fixed incident angle. The total fields are assumed to coincide, i.e.,

$$v_f(\cdot, b^+) = v_g(\cdot, b^+), \ v_f(\cdot, b^-) = v_g(\cdot, b^-) \quad in\ (0, 2\pi) \tag{1.8}$$

for N distinct wavenumbers $k_j^- \in (k^+, k_{max}]$, $j = 1, \ldots, N$, with $k_{max} > k^+$, where the integer N satisfies

$$N > \frac{h}{2}\, k^2_{max} + \frac{h}{\pi}\, \sqrt{k^2_{max} - (k^+)^2 \sin^2 \theta}\,. \tag{1.9}$$

Then f and g coincide. Furthermore, if the total fields coincide for N distinct wavenumbers $k_j^- \in (0, k^+)$ where

$$N > \frac{h}{\sqrt{2}}\, (k^+)^2 + \frac{h}{\pi}\, \sqrt{\sqrt{2}(k^+)^2 - (k^+)^2 \sin^2 \theta}\,, \tag{1.10}$$

then the assertion holds.

Theorem 1.3 *Let θ and $k^- > 0$ be fixed, and assume in addition to (1.6) and (1.7) that the end-points of f and g are fixed, i.e.,*

$$f(0) = g(0), \quad f(2\pi) = g(2\pi). \tag{1.11}$$

Then the relations (1.8) for N distinct wavenumbers $k_j^+ \in (k^-, k_{max}]$, $j = 1, \ldots, N$, with $k_{max} > k^-$ and

$$N > \frac{h}{2}\, k^2_{max} \tag{1.12}$$

imply $f = g$. Moreover, if the total fields coincide for N distinct wavenumbers $k_j^+ \in (0, k^-)$ where

$$N > \frac{h}{\sqrt{2}}\, (k^-)^2\,, \tag{1.13}$$

then the assertion holds.

Remark 1.1 *Assume additionally in Theorem 1.2 that the end-points of f and g are fixed. Then, as in (1.12) and (1.13), we can omit the second terms in estimates (1.9) and (1.10). In the special case $\theta = 0$, i.e. orthogonal incidence, one can prove a version of Theorem 1.3 without the restrictive assumption (1.11). In that case, estimate (1.12) has to be replaced by*

$$N > \frac{h}{2} \, k_{max}^2 + \frac{h}{\pi} \, k_{max} \, ,$$

and the bound (1.13) by

$$N > \frac{h}{\sqrt{2}} \, (k^-)^2 + \frac{h \sqrt[4]{2}}{\pi} \, k^- \, .$$

The proof of the above results is based on the Courant-Weyl min-max principle for a fourth-order elliptic problem; see Sections 3 and 4. However, in contrast to the Laplacian (cf. [11]), the corresponding eigenvalue problem may have negative eigenvalues, and it is more difficult to derive appropriate bounds for the positive and negative eigenvalues (see Section 3).

Unfortunately, if $\theta \neq 0$ and condition (1.11) is not fulfilled, then our approach to the reconstruction problem of Theorem 1.3 leads to a nonlinear eigenvalue problem in general. In fact, one obtains an operator polynomial of degree four in k^+, and it is not clear to us whether the monotonicity principle proved in [11] (for a certain second-order operator polynomial) can be extended to this case. However, it is possible to obtain at least a uniqueness result for an interval of wavenumbers; see Theorem 2.2 in the next section.

2   Uniqueness for an interval of wavenumbers. Reduction to an eigenvalue problem

First we shall prove that measurements on the two horizontal lines $\{x_2 = b^\pm\}$ for an interval of wavenumbers k^- uniquely determine the grating function f.

Theorem 2.1 *Let k^+ and θ be fixed, and let $f, g \in C_{per}^{0,1}$. If (1.8) holds for all wavenumbers $k^- \in [k_{min}, k_{max}]$ for some $0 < k_{min} < k_{max}$, then f and g coincide.*

Remark 2.1 *The proof of Theorem 2.1 shows that it is sufficient to assume (1.8) for an infinite set of wavenumbers k^- having a finite accumulation point. A corresponding remark applies to Theorem 2.2 below, with k^- and k^+ interchanged.*

Proof of Theorem 2.1. From (1.7) and (1.8) we obtain that v_f and v_g coincide in the region

$$\Omega^* := \{x \in \mathbb{R}^2 : x_2 \geq \max(f(x_1), g(x_1)), \ x_2 \leq \min(f(x_1), g(x_1))\}; \qquad (2.1)$$

see, e.g., [11]. Following that paper, we assume that $f \not\equiv g$ and distinguish between two cases.

Case 1. There exist $t_0, t_1 \in [0, 2\pi]$ with $f(t_0) = g(t_0)$ and $f(t_1) < g(t_1)$. Assume without loss of generality that $t_0 = 0$ and $t_1 \in (0, 2\pi)$, and define

$$a_1 \quad := \quad \inf\{\tau \in [0, t_1] : f(t) \neq g(t) \text{ for } t \in (\tau, t_1)\} \,,$$

$$a_2 \quad := \quad \sup\{\tau > t_1 : f(t) \neq g(t) \text{ for } t \in (t_1, \tau)\} \,.$$

Consider the domain

$$\Omega := \{x \in \mathbb{R}^2 : a_1 < x_1 < a_2, \ f(x_1) < x_2 < g(x_1)\} \tag{2.2}$$

which is bounded by two Lipschitz graphs. The function

$$u := v_f - v_g \tag{2.3}$$

belongs to $H^2_{loc}(\mathbb{R}^2)$ and satisfies the boundary conditions

$$u|_{\partial\Omega} = \partial_\nu u|_{\partial\Omega} = 0 \,. \tag{2.4}$$

Furthermore, u belongs to the space $\tilde{H}^2(\Omega)$, i.e., the function u extended by zero to the whole $\mathbb{R}^2$ is an element of the Sobolev space $H^2(\mathbb{R}^2)$. To see this, note that the restriction of u to the strip $\Pi = \{x \in \mathbb{R}^2 : a_1 < x_1 < a_2\}$ belongs to $H^2(\Pi)$ and vanishes in $\Pi \cap \Omega^*$ (cf. (2.1)). Hence $u = \partial_\nu u = 0$ on $\partial\Pi$, and we can extend u by zero to a function in $H^2(\mathbb{R}^2)$. Moreover, we have

Lemma 2.1 *The function u defined in (2.3) satisfies $u \in H^2_0(\Omega)$, where $H^2_0(\Omega)$ denotes the completion of $C^\infty_0(\Omega)$ in the norm of $H^2(\Omega)$.*

Proof of Lemma 2.1. If Ω has continuous boundary (cf. pp. 89-90 in [12] for the definition), then the assertion is a consequence of the relation $\tilde{H}^2(\Omega) = H^2_0(\Omega)$; see [10, Chap. 1.4.2] and, for a detailed proof, also [12, Thm. 3.2.9]. However, as a counter-example in [10, Chap. 1.2] shows, the boundary $\partial\Omega$ need not be continuous in general. In that case, for any small $\epsilon > 0$, we multiply u by a suitable cut-off function χ_ϵ supported in a substrip $\Pi_\epsilon = \{x \in \mathbb{R}^2 : a_1 + c\epsilon < x_1 < a_2 - c\epsilon\}$, with some $c > 0$ independent of ϵ. Then each $\chi_\epsilon u \in H^2_0(\Omega \cap \Pi_\epsilon)$ can be approximated (in H^2 norm) by functions from $C^\infty_0(\Omega)$. Note that $\Omega \cap \Pi_\epsilon$ even has Lipschitz boundary.

It remains to prove that $u \in H^2_0(\Pi)$ can be approximated by suitable functions $u_\epsilon = \chi_\epsilon u$. We choose $\chi \in C^\infty([0, \infty))$ with $0 \leq \chi \leq 1$ in $[0, \infty)$, $\chi|_{[0,1]} = 0$, $\chi|_{[2,\infty)} = 1$, and put $\chi_\epsilon(x) = \chi(\epsilon^{-1}\rho(x))$ with $\rho(x) = (x_1 - a_1)(a_2 - x_1) \sim \text{dist}(x, \partial\Pi)$. Then $u_\epsilon \in H^2_0(\Pi)$ and $u_\epsilon(x) = 0$ for $\rho(x) < \epsilon$, so that each u_ϵ is supported in a suitable substrip Π_ϵ. Moreover, using [10, Thm. 1.4.4.4] it is not difficult to verify that $u_\epsilon \to u$ in $H^2(\Pi)$ as $\epsilon \to 0$. ∎

From the transmission problems (DP) corresponding to v_f and v_g and from Lemma 2.1, we now obtain that the function u defined in (2.3) satisfies the variational equation

$$((\Delta + l)u, (\Delta + l)\phi) = \lambda((-\Delta - l)u, \phi) \quad \forall \phi \in H^2_0(\Omega) \,, \tag{2.5}$$

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with

$$l := (k^+)^2, \quad \lambda := (k^-)^2 - (k^+)^2$$

and the pairing

$$(u, \phi) := \int_\Omega u \overline{\phi}.$$

Note that

$$(\Delta + (k^+)^2)v_f = (\Delta + (k^-)^2)v_g = 0 \quad \text{in} \quad \Omega,$$

and u is a solution of the fourth-order equation

$$(\Delta + (k^-)^2)(\Delta + (k^+)^2)u = (\Delta + (k^+)^2)^2 u + \lambda(\Delta + (k^+)^2)u = 0$$

in Ω in the sense of distributions.

Consequently, in case 1 we have the nontrivial solution $u \in H_0^2(\Omega)$ of the eigenvalue problem (2.5) corresponding to the eigenvalue $\lambda \neq 0$. (For $u = 0$ in Ω, we would have $v_f = v_g$ implying $\lambda v_f = 0$, hence $v_f = 0$ in Ω and also in $\mathbb{R}^2$, which is a contradiction.)

Case 2. We can assume that $f(t) < g(t)$ for all $t \in \mathbb{R}$. Then the domain (2.2) has to be replaced by the periodic layer

$$\Omega := \{x \in \mathbb{R}^2 : 0 \leq x_1 \leq 2\pi, f(x_1) < x_2 < g(x_1)\}. \tag{2.6}$$

The function $u_f := \exp(-i\alpha x_1)v_f$, which is 2π-periodic in x_1, satisfies the Helmholtz equation

$$(\Delta_\alpha + k^2)u_f = 0 \quad \text{in} \quad \mathbb{R}^2, \quad \text{with} \quad k = k^\pm \quad \text{in} \quad \Omega_f^\pm, \tag{2.7}$$

where we use the notation

$$\nabla_\alpha := \nabla + i(\alpha, 0), \ \Delta_\alpha := \nabla_\alpha \cdot \nabla_\alpha = \Delta + 2i\alpha\partial_1 - \alpha^2,$$

and the corresponding transmission conditions

$$[u_f]_{\Lambda_f} = [\partial_\nu u_f]_{\Lambda_f} = 0$$

are included. The function $u_g := \exp(-i\alpha x_1)v_g$ satisfies an analogous transmission problem corresponding to the interface Λ_g.

Then the function

$$u := u_f - u_g \tag{2.8}$$

belongs to $H_{loc}^2(\mathbb{R}^2)$ and satisfies the boundary conditions (2.4) again. Moreover, u belongs to the space $H_{0,per}^2(\Omega)$, the completion with respect to the H^2 norm of all functions from $C_0^\infty(\Omega)$ that are 2π-periodic in x_1. Here we use a periodic version of

Lemma 2.1, the proof of which is simpler since Ω has Lipschitz boundary. Moreover, analogously to (2.5), the function u defined in (2.8) satisfies the eigenvalue problem

$$((\Delta_\alpha + l)u, (\Delta_\alpha + l)\phi) = \lambda((-\Delta_\alpha - l)u, \phi) \quad \forall \phi \in H^2_{0,per}(\Omega) \tag{2.9}$$

with $l := (k^+)^2$ and $\lambda := (k^-)^2 - (k^+)^2$.

Similarly to case 1, we have a nontrivial solution $u \in H^2_{0,per}(\Omega)$ of the eigenvalue problem (2.9) corresponding to the eigenvalue $\lambda \neq 0$.

To complete the proof of Theorem 2.1, we need some properties of the variational problems (2.5) and (2.9). Let $H := H^2_0(\Omega)$ in case 1 and $H := H^2_{0,per}(\Omega)$ in case 2, and introduce the following sesquilinear form on $H \times H$:

$$\langle u, \phi \rangle := ((\Delta_\alpha + l)u, (\Delta_\alpha + l)\phi), \tag{2.10}$$

with $\alpha = 0$ in case 1. In the following $\|\cdot\|$ will denote the norm in $L^2(\Omega)$.

1) $\sqrt{\langle u, u \rangle} = \|(\Delta_\alpha + l)u\|$ is an equivalent norm on H:
Since $\Delta_\alpha + l : H \to L^2(\Omega)$ is a compact perturbation of Δ which is injective with closed range, it remains to prove that $(\Delta_\alpha + l)u = 0$ and $u \in H$ imply $u = 0$. The latter follows from the boundary conditions (2.4) and the unique continuation for the operator $\Delta_\alpha + l$.

Consider the sesquilinear form

$$a(u, \phi) := ((-\Delta_\alpha - l)u, \phi) \quad \forall u, \phi \in H, \tag{2.11}$$

with $\alpha = 0$ in case 1, and define the operator $T : H \to H$ via

$$a(u, \phi) = \langle Tu, \phi \rangle \quad \forall u, \phi \in H. \tag{2.12}$$

2) $T : H \to H$ is self-adjoint and compact:
The first property follows from the relation

$$\langle Tu, \phi \rangle = \int_\Omega \left(\nabla_\alpha u \cdot \overline{\nabla_\alpha \phi} - l u \overline{\phi} \right) = \langle u, T\phi \rangle,$$

and the second using the compact imbeddings

$$H^1_0(\Omega) \subset L^2(\Omega), \quad H^1_{0,per}(\Omega) \subset L^2(\Omega)$$

in cases 1 and 2, respectively.

Using property 1) and (2.12), we can reformulate the eigenvalue problems (2.5) and (2.9) as

$$\langle u, \phi \rangle = \lambda \langle Tu, \phi \rangle \quad \forall \phi \in H$$

or, equivalently, $\langle u, \phi \rangle = \lambda((-\Delta_\alpha - l)u, \phi)$ for any $\phi \in H$, or

$$Tu = \mu u, \quad \text{with} \quad \mu := 1/\lambda. \tag{2.13}$$

Now we can conclude the proof of Theorem 2.1. Applying the standard theory of compact operators to (2.13), we obtain that in both cases 1 and 2 the set of eigenvalues λ of (2.5) and (2.9) is a discrete subset of $\mathbb{R}$. Therefore, there exists $k^- \in [k_{min}, k_{max}]$ such that the corresponding function u defined in (2.3) or (2.8) vanishes in Ω, which leads to a contradiction. $\blacksquare$

Now we prove the same result as Theorem 2.1 for an interval of wavenumbers k^+.

Theorem 2.2 *Let k^- and θ be fixed, and let $f, g \in C^{0,1}_{per}$. If (1.8) holds for all wavenumbers $k^+ \in [k_{min}, k_{max}]$ for some $0 < k_{min} < k_{max}$, then f and g coincide.*

Proof. We proceed as in the proof of the preceding theorem, and in case 2 we consider the variational problem (2.9) in the domain Ω (given by (2.6)) again. However, since $\alpha = k^+ \sin \theta$ depends on k^+, we have the following nonlinear eigenvalue problem in k^+:

$$\big((\Delta + 2ik^+ \sin\theta \, \partial_1 + (k^+)^2 \cos^2\theta)u, (\Delta + 2ik^+ \sin\theta \, \partial_1 + (k^+)^2 \cos^2\theta)\phi\big)$$

$$+\big((k^-)^2 - (k^+)^2\big)\big((\Delta + 2ik^+ \sin\theta \, \partial_1 + (k^+)^2 \cos^2\theta)u, \phi\big) = 0 \quad \forall \phi \in H \,.$$

Introducing the equivalent norm $\sqrt{\langle u, u \rangle} = \|\Delta u\|$ on H, we can write this problem in the form (cf. (2.10)-(2.12))

$$\mathcal{A}(\lambda)u := u + T_0 u + \sum_{j=1}^{4} \lambda^j T_j u = 0 \,, \quad \lambda := k^+ \,, \tag{2.14}$$

where T_j are compact (and self-adjoint) operators on H. The operator polynomial $\mathcal{A}(\lambda)$ is an analytic Fredholm operator function in λ, which is invertible for $\lambda = k^+ = k^-$ by property 1). Therefore, applying [9, Chap. 1, Thm. 5.1] to the operator function (2.14), we obtain that the set of eigenvalues k^+ is discrete. This finishes the proof in case 2.

In case 1 (the bounded domain Ω) and in case 2 (the periodic layer Ω) under orthogonal incidence (i.e., $\theta = 0$), it is sufficient to consider the linear eigenvalue problems (2.5) and (2.9) (with $\alpha = 0$), respectively. Then the proof of Theorem 2.2 follows from that of Theorem 2.1 by interchanging the roles of k^+ and k^-. ∎

In the following we restrict the discussion to the linear eigenvalue problem where the Courant-Weyl min-max principle can be applied.

3   The linear eigenvalue problem

In the following, we need further properties of the eigenvalue problems (2.5) and (2.9) where k^+ and $\alpha = k^+ \sin \theta$ are fixed. We also consider the equivalent eigenvalue problem (2.13).

In this section, we assume (1.6). We recall that $l = (k^+)^2$.

3) $\mu = 0$ is not an eigenvalue of (2.13):
Otherwise we would have

$$\langle Tu, \phi \rangle = ((-\Delta_\alpha - l)u, \phi) = 0 \quad \forall \phi \in H \,,$$

implying the last equality for all $\phi \in L^2(\Omega)$. Hence $(\Delta_\alpha + l)u = 0$ which gives $u = 0$ by property 1).

Applying the standard theory for compact operators to (2.13) again, we obtain from 2) and 3)

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Lemma 3.1 *For both problems (2.5) and (2.9), the infinite nondecreasing sequence of eigenvalues (counting multiplicities) takes the form*

$$\ldots \leq \lambda_{-n} \leq \ldots \leq \lambda_{-1} < 0 < \lambda_1 \leq \ldots \leq \lambda_n \leq \ldots \tag{3.1}$$

and can only have accumulation points at $\pm\infty$. Moreover, there is a corresponding orthonormal basis $\{u_j\}_{j\in\mathbb{Z}} \in H$ of eigenfunctions satisfying

$$\langle u_j, u_k \rangle = \lambda_j \langle Tu_j, u_k \rangle = \lambda_j((-\Delta_\alpha - l)u_j, u_k) = \delta_{jk}. \tag{3.2}$$

Here we set $\delta_{jj} = 1$ and $\delta_{jk} = 0$ if $j \neq k$.

We now present a version of the Courant-Weyl min-max principle for the operator T defined in (2.12).

Lemma 3.2 *For the positive eigenvalues $\mu_n = 1/\lambda_n$ of problem (2.13) we have*

$$\mu_n = \inf_{v_1,\ldots,v_{n-1}\in L^2(\Omega)} \sup_{\substack{0\neq\phi\in H \\ (\phi,v_1)=\ldots=(\phi,v_{n-1})=0}} \frac{\langle T\phi, \phi \rangle}{\langle \phi, \phi \rangle}. \tag{3.3}$$

Remark 3.1 *The assertion is well-known if in (3.3) elements $v_1, \ldots, v_{n-1} \in H$ with $\langle\phi, v_1\rangle = \ldots = \langle\phi, v_{n-1}\rangle = 0$ are taken; see, e.g., [5, p.133].*

Proof of Lemma 3.2. We proceed as in [11]. From Lemma 3.1 we have for $u = \sum_{j\in\mathbb{Z}} \alpha_j u_j \in H$

$$\langle Tu, u \rangle / \langle u, u \rangle = \sum_{j\in\mathbb{Z}} \mu_j |\alpha_j|^2 \bigg/ \sum_{j\in\mathbb{Z}} |\alpha_j|^2.$$

For arbitrary $v_1, \ldots, v_{n-1} \in L^2(\Omega)$ we can choose $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ such that

$$\hat{u} = \sum_{j=1}^{n} \alpha_j u_j, \quad \text{with} \quad (\hat{u}, v_k) = 0, \ k = 1, \ldots, n-1, \quad \langle\hat{u}, \hat{u}\rangle = 1,$$

and obtain

$$\sup_{\substack{0\neq u\in H \\ (u,v_1)=\ldots=(u,v_{n-1})=0}} \frac{\langle Tu, u \rangle}{\langle u, u \rangle} \geq \langle T\hat{u}, \hat{u} \rangle = \sum_{j=1}^{n} \mu_j |\alpha_j|^2 \geq \mu_n.$$

On the other hand, there exist $v_1, \ldots, v_{n-1} \in L^2(\Omega)$ such that

$$\langle Tu, u \rangle / \langle u, u \rangle \leq \mu_n \quad \text{for all} \quad u \in H \quad \text{with} \quad (u, v_1) = \ldots = (u, v_{n-1}) = 0.$$

Indeed, for $v_j := (-\Delta_\alpha - l)u_j$ we have

$$(u, v_j) = (u, (-\Delta_\alpha - l)u_j) = \mu_j \langle u, u_j \rangle,$$

so that $(u, v_j) = 0$ is equivalent to $\alpha_j = \langle u, u_j \rangle = 0$. Thus we obtain

$$\frac{\langle Tu, u \rangle}{\langle u, u \rangle} = \frac{\sum_{j\in\mathbb{Z}\setminus\{1,\ldots,n-1\}} \mu_j |\alpha_j|^2}{\sum_{j\in\mathbb{Z}\setminus\{1,\ldots,n-1\}} |\alpha_j|^2} \leq \mu_n,$$

and (3.3) is proved. ∎

Remark 3.2 *For the negative eigenvalues $\mu_{-n} = 1/\lambda_{-n}$ of problem (2.13) we have analogously*

$$\mu_{-n} = \sup_{\substack{v_1,\ldots,v_{n-1}\in L^2(\Omega)}} \inf_{\substack{0\neq\phi\in H \\ (\phi,v_1)=\ldots=(\phi,v_{n-1})=0}} \frac{\langle T\phi,\phi\rangle}{\langle\phi,\phi\rangle} . \tag{3.4}$$

The min-max principle from (3.3) and (3.4) can easily be reformulated for the positive and negative eigenvalues of (2.5) and (2.9):

Lemma 3.3 *The eigenvalues (3.1) satisfy*

$$\lambda_n = \sup_{\substack{v_1,\ldots,v_{n-1}\in L^2(\Omega)}} \inf_{\substack{\phi\in H,\ \langle T\phi,\phi\rangle>0 \\ (\phi,v_1)=\ldots=(\phi,v_{n-1})=0}} \frac{\langle\phi,\phi\rangle}{\langle T\phi,\phi\rangle} , \tag{3.5}$$

$$\lambda_{-n} = \inf_{\substack{v_1,\ldots,v_{n-1}\in L^2(\Omega)}} \sup_{\substack{\phi\in H,\ \langle T\phi,\phi\rangle<0 \\ (\phi,v_1)=\ldots=(\phi,v_{n-1})=0}} \frac{\langle\phi,\phi\rangle}{\langle T\phi,\phi\rangle} , \tag{3.6}$$

with $\langle T\phi,\phi\rangle = ((-\Delta_\alpha - l)\phi,\phi)$ and $\alpha = 0$ in case 1.

We now establish appropriate bounds for the positive and negative eigenvalues (3.1). In case 1 (the bounded domain Ω), let (Λ_n) denote the nondecreasing sequence of the Dirichlet eigenvalues (counting multiplicities) of the negative Laplacian in Ω, whereas in case 2 (the periodic layer Ω), (Λ_n) stands for the corresponding sequence of the Dirichlet eigenvalues of the periodic operator $-\Delta_\alpha$. We have

$$0 < \Lambda_1 \leq \ldots \leq \Lambda_n \leq \ldots, \quad \Lambda_n \to \infty,$$

and according to the Courant-Weyl min-max principle (see [5, Chap. 6])

$$\Lambda_n = \sup_{\substack{v_1,\ldots,v_{n-1}\in L^2(\Omega)}} \inf_{\substack{0\neq\phi\in\mathcal{H} \\ (\phi,v_1)=\ldots=(\phi,v_{n-1})=0}} \frac{\|\nabla_\alpha\phi\|^2}{\|\phi\|^2} , \tag{3.7}$$

where $\alpha = 0$ and $\mathcal{H} = H_0^1(\Omega)$ in case 1 and $\mathcal{H} = H_{0,per}^1(\Omega)$ in case 2.

Lemma 3.4 *We have $\lambda_n \geq \Lambda_n - l$ and $\lambda_{-n} \leq l - \frac{1}{l}\Lambda_n^2$ for all $n \in \mathbb{N}$.*

Proof. We first prove the inequality

$$\|\nabla_\alpha u\|^2 \leq \|\Delta_\alpha u\|\,\|u\|, \quad u \in H, \tag{3.8}$$

with $\alpha = 0$ in case 1. In that case, (3.8) is an immediate consequence of Parseval's equality and the obvious estimate

$$\int_{\mathbb{R}^2} |x|^2\,|\hat{u}(x)|^2 \leq \left(\int_{\mathbb{R}^2} |x|^4\,|\hat{u}(x)|^2\right)^{1/2} \left(\int_{\mathbb{R}^2} |\hat{u}(x)|^2\right)^{1/2}, \quad u \in C_0^\infty(\Omega),$$

where $\hat{u}$ denotes the Fourier transform of u. In case 2, the domain Ω defined by (2.6) is contained in the periodic cell $[0, 2\pi] \times (0, h)$, and it is sufficient to verify (3.8) for any infinitely smooth function u which is 2π-periodic in x_1 and h-periodic in x_2. Since

$$\nabla(\exp(i\alpha x_1)u) = \exp(i\alpha x_1) \nabla_\alpha u, \quad \Delta(\exp(i\alpha x_1)u) = \exp(i\alpha x_1) \Delta_\alpha u,$$

it is enough to prove the estimate $\|\nabla v\|^2 \leq \|\Delta v\| \, \|v\|$ for any infinitely smooth function v which is α-quasiperiodic in x_1 and h-periodic in x_2. This estimate follows easily by expanding v into the Fourier series

$$v = \sum_{m \in \mathbb{Z}^2} c_m \exp(i(m_1 + \alpha)x_1 + im_2(h/2\pi)x_2), \quad c_m \in \mathbb{C}, \quad m = (m_1, m_2),$$

and using the Cauchy-Schwarz inequality.

Looking for the positive eigenvalues λ_n, we now choose $\phi \in H$ such that

$$((-\Delta_\alpha - l)\phi, \phi) = \|\nabla_\alpha \phi\|^2 - l\|\phi\|^2 > 0$$

and estimate the quantity

$$A(\phi) := \frac{((\Delta_\alpha + l)\phi, (\Delta_\alpha + l)\phi)}{((-\Delta_\alpha - l)\phi, \phi)} \tag{3.9}$$

using (3.8):

$$A(\phi) = \frac{\|\Delta_\alpha \phi\|^2 - 2l\|\nabla_\alpha \phi\|^2 + l^2\|\phi\|^2}{\|\nabla_\alpha \phi\|^2 - l\|\phi\|^2} \geq \frac{1}{\|\phi\|^2} \frac{\|\nabla_\alpha \phi\|^4 - 2l\|\nabla_\alpha \phi\|^2\|\phi\|^2 + l^2\|\phi\|^4}{\|\nabla_\alpha \phi\|^2 - l\|\phi\|^2}$$

$$= \frac{\|\nabla_\alpha \phi\|^2 - l\|\phi\|^2}{\|\phi\|^2} = \frac{\|\nabla_\alpha \phi\|^2}{\|\phi\|^2} - l =: A_0(\phi) - l. \tag{3.10}$$

From (3.10), Lemma 3.3 and (3.7) we then obtain

$$\lambda_n = \sup_{v_1,\dots,v_{n-1} \in L^2(\Omega)} \inf_{\substack{\phi \in H, \ \langle T\phi, \phi \rangle > 0 \\ (\phi, v_1) = \dots = (\phi, v_{n-1}) = 0}} A(\phi)$$

$$\geq \sup_{v_1,\dots,v_{n-1} \in L^2(\Omega)} \inf_{\substack{0 \neq \phi \in \mathcal{H} \\ (\phi, v_1) = \dots = (\phi, v_{n-1}) = 0}} A_0(\phi) - l = \Lambda_n - l.$$

We now consider the negative eigenvalues λ_{-n} and choose $\phi \in H$ such that

$$((-\Delta_\alpha - l)\phi, \phi) = \|\nabla_\alpha \phi\|^2 - l\|\phi\|^2 < 0. \tag{3.11}$$

We have $\langle \phi, \phi \rangle = \|\Delta_\alpha \phi\|^2 - 2l\|\nabla_\alpha \phi\|^2 + l^2\|\phi\|^2 > 0$; see property 1) in Section 2. Therefore, using (3.11) and (3.8), we can estimate the quantity $A(\phi)$ (cf. (3.9),

(3.10)) as

$$
A(\phi) \;=\; \frac{\|\Delta_\alpha \phi\|^2 - 2l\|\nabla_\alpha \phi\|^2 + l^2\|\phi\|^2}{\|\nabla_\alpha \phi\|^2 - l\|\phi\|^2} \;\le\; \frac{\|\Delta_\alpha \phi\|^2 - 2l\|\nabla_\alpha \phi\|^2 + l^2\|\phi\|^2}{-l\|\phi\|^2}
$$

$$
=\; -\frac{\|\Delta_\alpha \phi\|^2}{l\|\phi\|^2} + 2\frac{\|\nabla_\alpha \phi\|^2}{\|\phi\|^2} - l \;\le\; -\frac{\|\Delta_\alpha \phi\|^2}{l\|\phi\|^2} + l \;\le\; l - \frac{\|\nabla_\alpha \phi\|^4}{l\|\phi\|^4}
$$

$$
=\; l - \frac{1}{l}(A_0(\phi))^2 \,.
$$

Together with Lemma 3.3 and (3.7), the last estimate implies

$$
\lambda_{-n} \;=\; \inf_{v_1,\dots,v_{n-1}\in L^2(\Omega)} \;\; \sup_{\substack{\phi\in H,\ \langle T\phi,\phi\rangle<0 \\ (\phi,v_1)=\dots=(\phi,v_{n-1})=0}} \;\; A(\phi)
$$

$$
\le\; l - \sup_{v_1,\dots,v_{n-1}\in L^2(\Omega)} \;\; \inf_{\substack{\phi\in H,\ \langle T\phi,\phi\rangle<0 \\ (\phi,v_1)=\dots=(\phi,v_{n-1})=0}} \;\; \frac{1}{l}(A_0(\phi))^2
$$

$$
\le\; l - \sup_{v_1,\dots,v_{n-1}\in L^2(\Omega)} \;\; \inf_{\substack{0\neq\phi\in\mathcal{H} \\ (\phi,v_1)=\dots=(\phi,v_{n-1})=0}} \;\; \frac{1}{l}(A_0(\phi))^2 \;=\; l - \frac{1}{l}\Lambda_n^2 \,. \qquad \blacksquare
$$

We conclude this section by presenting well known bounds for the Dirichlet eigenvalues (3.7); see [5], [11]. Let $N(c)$ denote the number of these eigenvalues which do not exceed $c > 0$. Recall that

$$
\Omega \subset \tilde{\Omega} := (0,2\pi)\times(0,h) \quad\text{and}\quad \Omega \subset \tilde{\Omega} := [0,2\pi]\times(0,h)\,,
$$

where Ω is given by (2.2) and (2.6) corresponding to the cases 1 and 2, respectively.

Lemma 3.5 *(i) The smallest Dirichlet eigenvalue always satisfies $\Lambda_1 \ge \pi^2/h^2$.*
(ii) We have the estimates

$$
N(c) \le \frac{h}{2}c \quad\text{and}\quad N(c) \le \frac{h}{2}c + \frac{h}{\pi}\sqrt{c-\alpha^2}
$$

in the cases 1 and 2, respectively.

The proof of Lemma 3.5 is based on the classical monotonicity principle (e.g., [5]) which gives that the eigenvalues Λ_n corresponding to Ω are not less than those of $\tilde{\Omega}$ denoted by $\tilde{\Lambda}_n$. In both cases, by separation of variables one can find the eigenvalues $\tilde{\Lambda}_n$ explicitly, which then implies the assertions (i) and (ii) (cf. the proof of Theorem 3.2 in [11]). In particular, in case 2 these eigenvalues are given by

$$
\tilde{\Lambda}_{pm} = (p+\alpha)^2 + m^2\pi^2/h^2 \quad\text{for all}\quad p\in\mathbb{Z},\, m\in\mathbb{N}.
$$

Then $N(c)$ is bounded by the number of gridpoints $(p,m)\in\mathbb{Z}^2$ in the upper half of the ellipse with axes $\sqrt{c}$, $h\sqrt{c}/\pi$ and centre $(-\alpha,0)$, and the number of the gridpoints is bounded by the area plus the number of the gridpoints $(0,m)$ on the vertical axis lying inside the ellipse. Thus we obtain the second estimate of (ii).

4   Proof of Theorems 1.1 to 1.3

Proof of Theorem 1.1. Arguing by contradiction, we assume that there are two profile functions $f \neq g$ such that the relations (1.6) - (1.8) hold, where $k^{\pm} > 0$ are fixed refractive indices satisfying condition (1.5). Let first $k^- > k^+$. Then $\lambda := (k^-)^2 - (k^+)^2$ is a positive eigenvalue of problem (2.5) or (2.9) with the corresponding eigenfunction u defined in (2.3) or (2.8); compare the proof of Theorem 2.1. On the other hand, from Lemmas 3.4 and 3.5 (i), we obtain the estimate

$$\lambda_1 + (k^+)^2 \geq \Lambda_1 \geq \pi^2/h^2$$

for the first positive eigenvalue λ_1 of (2.5) or (2.9). By $\lambda = (k^-)^2 - (k^+)^2 \geq \lambda_1$, this implies the inequality

$$(k^-)^2 \geq \lambda_1 + (k^+)^2 \geq \pi^2/h^2 \,,$$

which is a contradiction to condition (1.5).

Let $k^- < k^+$. Then the function u defined in (2.3) or (2.8) also satisfies the variational equation (2.5) or (2.9) with $l := (k^-)^2$ and $\lambda := (k^+)^2 - (k^-)^2 > 0$, i.e., with k^+ and k^- interchanged. (Note that k^+ and k^- are fixed.) Since λ is a positive eigenvalue of that problem, we obtain a contradiction as above. ∎

Proof of Theorem 1.2. We first consider case 2, i.e., the periodic layer Ω defined in (2.6). Assume that (1.8) holds for N distinct wavenumbers $k_j^- \in (k^+, k_{max}]$. Following the proof of Theorem 2.1, we have to estimate the number N^+ of eigenvalues of (2.9) satisfying

$$0 < \lambda := (k^-)^2 - (k^+)^2 \leq k_{max}^2 - (k^+)^2 \,.$$

By Lemma 3.4 the nth positive eigenvalue λ_n of (2.5) can be estimated as

$$\lambda_n + (k^+)^2 \geq \Lambda_n \,,$$

where Λ_n is the nth Dirichlet eigenvalue of the periodic operator $-\Delta_\alpha$ in Ω. Therefore, the number N^+ does not exceed the number of Λ_n bounded by k_{max}^2. Now it follows from Lemma 3.5 (ii) that N^+ is not greater than

$$N(k_{max}^2) \leq \frac{h}{2} k_{max}^2 + \frac{h}{\pi} \sqrt{k_{max}^2 - \alpha^2} \,, \quad \alpha = k^+ \sin\theta \,, \tag{4.1}$$

which proves the bound (1.9).

We now consider the case $k_j^- \in (0, k^+)$ and have to estimate the number N^- of eigenvalues of (2.9) with

$$-(k^+)^2 < \lambda := (k^-)^2 - (k^+)^2 < 0 \,. \tag{4.2}$$

Because of Lemma 3.4, the nth negative eigenvalue of (2.9) satisfies the inequality

$$|\lambda_{-n}| + (k^+)^2 \geq \frac{1}{(k^+)^2} \Lambda_n^2 \,. \tag{4.3}$$

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From (4.2) we see that N^- is equal to the number of $|\lambda_{-n}|$ which are smaller than $(k^+)^2$. Consequently, by (4.3), N^- does not exceed the number of Λ_n bounded by $\sqrt{2}\,(k^+)^2$. Applying Lemma 3.5 (ii) again, we obtain the estimate

$$N^- \le N\left(\sqrt{2}\,(k^+)^2\right) \le \frac{h}{\sqrt{2}}\,(k^+)^2 + \frac{h}{\pi}\,\sqrt{\sqrt{2}(k^+)^2 - \alpha^2}\,, \qquad (4.4)$$

which implies (1.10).

In case 1 where Ω is given by (2.2), we apply Lemma 3.4 to the eigenvalue problem (2.5). Moreover, we use the first estimate of Lemma 3.5 (ii) to obtain the bounds (4.1) and (4.4) without the square root terms, so that the second terms in estimates (1.9) and (1.10) can even be omitted. This finishes the proof of Theorem 1.2 and that of the first assertion of Remark 1.1. ∎

Proof of Theorem 1.3. Let θ and k^- be fixed, and consider a refractive index k^+ for which the relations (1.8) hold. By condition (1.11), the two profiles Λ_f and Λ_g intersect so that only case 1 can occur; see the proof of Theorem 2.1.

Moreover, the function u defined in (2.3) is an eigenfunction of the problem (2.5) with $l := (k^-)^2$ and $\lambda := (k^+)^2 - (k^-)^2$, where the domain Ω is given by (2.2). Interchanging the roles of k^+ and k^-, we can now proceed as in the proof of Theorem 1.2 in case 1. ∎

Finally, we note that the second assertion of Remark 1.1 can be proved as Theorem 1.2 in case 2.

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