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On some classes of limit cycles of planar dynamical systems

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Abstract

We consider two-dimensional smooth vector fields $dx/dt = P(x,y), dy/dt = Q(x,y)$ and estimate the maximal number of limit cycles with special properties which are defined by means of generalized Dulac and Cherkas functions. In case that P and Q are polynomials we present results about the weakend 16-th problem of Hilbert.

1 Introduction

We consider two-dimensional systems of autonomous differential equations

$$\frac{dx}{dt} = P(x,y),\ \frac{dy}{dt} = Q(x,y) \tag{1.1}$$

in some region $D \subset R^2$. Throughout the paper we assume that P and Q are continuously differentiable in D, that is, $P, Q \in C^1(D)$. If D is bounded and such that the boundary of D has no contact with the trajectories of system (1.1), then according to the skeleton method of A.A. Andronov and E.A. Leontovich [1] the topological structure of the trajectories of (1.1) in D is determined by the singular trajectories of system (1.1), that is, by the equilibria, the separatricies and the limit cycles of (1.1) in D.
There are sufficiently effective methods to study equilibria and separatricies, there are also methods to prove that (1.1) has no limit cycle or at least one limit cycle in D (see, e.g. [7, 29, 41, 42]), but there is no general method to localize all limit cycles and to prove the existence or absence of multiple limit cycles. Therefore, the problem to estimate the number of limit cycles of general systems (1.1) is an open problem, even in the case of polynomial systems

$$\frac{dx}{dt} = \sum_{i+j=0}^{n} a_{ij}x^i y^j,\ \frac{dy}{dt} = \sum_{i+j=0}^{n} b_{ij}x^i y^j \tag{1.2}$$

with real coefficients a_{ij}, b_{ij}.
It is well-known that the problem to estimate the maximal number $H(n)$ of limit cycles of the polynomial system (1.2) and to localize their relative position represents the famous 16-th problem of D. Hilbert [18]. Since linear systems (1.2) have no limit cycle, it holds $H(1) = 0$. But even in the case $n = 2$ we don't know any upper bound for $H(2)$.

H. Dulac [14] formulated in 1923 a theorem claiming that any individual system (1.2) can have only a finite number of limit cycles. Later, it has been noted that his proof is not correct. In the eighties of the last century, R. Bamon [3] and V. Romanovskij [31] gave a proof of the above mentioned theorem for the case $n = 2$. Yu. Ilyashenko [20] proved Dulac's theorem in 1991 for any n, another proof has been published by J. Ecalle [15] in the same year. But to this very day we do not know any upper bound for $H(n)$ even in the case $n = 2$.
L.S. Chen and M.S. Wang [6] and S.L. Shi [37] published examples of quadratic systems with four limit cycles. Thus, 4 is a lower bound for $H(2)$. Concerning $H(3)$ we have $H(3) \geq 11$, since H. Żołądek [43] gave a cubic system with eleven limit cycles (see also [29]).
Several authors considered the rate of growth of $H(n)$ as n increases. Yu. Ilyashenko [22] has established that $H(n)$ grows at least as n^2. C. Christopher and N. Lloyd [13] have improved this lower bound by $n^2 \log_2 n$. More information about the centennial history of Hilbert's 16-th problem can be found in [22].
Due to its theoretical and practical importance, Hilbert's 16-th problem has been included by S. Smale into the list of the 18 most important mathematical problems of the 21-th century [38].
The goal of this paper is to estimate the maximal number of limit cycles for system (1.1) belonging to some classes which are distinguished by the property that some expression does not change sign on a limit cycle, that is, we treat weakened versions of Hilbert's 16-th problem. Especially, we are able to estimate the number of regular limit cycles that have been introduced by P.N. Papusch [28]. The proofs are based on generalized Dulac and Cherkas functions.
Our paper is organized as follows: In section 2 we compose some known simplified or restricted versions of Hilbert's 16-th problem. Section 3 is devoted to generalized Dulac and Cherkas functions. In the final section 4 we present estimates of the maximal number of limit cycles with special properties.

2 Weakened versions of Hilbert's 16-th problem

2.1 The Hilbert-Arnold problem

We consider the autonomous differential system (1.2) in a compact region $G \subset D$ and assume that the set of admissible parameters $c_{ijkl} := (a_{ij}, b_{kl})$ represents a compact set $K \subset R^{n^2+3n+2}$. Additionally, we suppose that the boundary of G has no contact with the vector field $f := (P, Q)$ for all $c_{ijkl} \in K$. The Hilbert-Arnold problem can be formulated as follows: *Find an upper bound for the maximal number of limit cycles of system* (1.2) *when* c_{ijkl} *varies in* K.

Taking into account that under our assumptions the number of limit cycles of system (1.2) changes only when a limit cycle bifurcates from

(i) an equilibrium point (Andronov-bifurcation),

(ii) a multiple limit cycle (saddle-node bifurcation),

(iii) a polycycle (homoclinic, heteroclinic bifurcation),

then the main concern is to estimate the maximal number of limit cycles which can bifurcate from these singular trajectories (cyclicity of a singular trajectory).
We would like to mention a special program proposed by R. Roussarie [32] for solving the Hilbert-Arnold problem and that Yu. Ilyashenko and S. Yakovenko proved [21] that all elementary polycyles of a smooth generic family have finite cyclicity.

2.2 The infinitesimal Hilbert problem

Let $F(x,y)$ be a real polynomial of degree at most n, let $H(x,y)$ be a real polynomial of degree $m+1$. V. I. Arnold formulated in [2] the following problem:

Find an upper bound $V(n,m)$ for the number of real zeros of the function

$$I(c) = \int\int_{H(x,y)\le c} F(x,y)dx\ dy. \tag{2.1}$$

In the case $F(x,y) \equiv -(P_x(x,y) + Q_y(x,y))$ we get by means of Green's formula the following representation of the right hand side of (2.1)

$$I(c) = \int_{\Gamma_c} Q(x,y)dx - P(x,y)dy, \tag{2.2}$$

where Γ_c is the boundary of the region defined by $H(x,y) \le c$. Under the condition that $H(x,y) = c^*$ represents a limit cycle of system (1.1) we have $I(c^*) = 0$. Thus, an upper bound for the zeros of the Abelian integral (2.2) provides an upper bound for the maximal number of limit cycles of system (1.1).
We note that the infinitesimal Hilbert problem is closely related to the problem of determining an upper bound for the number of limit cycles $V(\max\{m,n\})$ of the perturbed Hamiltonian system

$$\frac{dx}{dt} = -\frac{\partial H}{\partial y} + \varepsilon P(x,y),\ \frac{dy}{dt} = \frac{\partial H}{\partial x} + \varepsilon Q(x,y), \tag{2.3}$$

where $0 < \varepsilon \ll 1$.

The relationship between both problems comes from the following two facts [24]:

1. If there exists a constant c^* such that $I(c^*) = 0$ and $I'(c^*) \neq 0$, then the oval Γ_{c^*} generates a family of hyperbolic limit cycles $\Gamma_{c^*,\varepsilon}$ of (2.3) for sufficiently small ε. On the other hand, if there exists a family of hyperbolic limit cycles Γ_ε of system (2.3), then as $\varepsilon \to 0$ we get the existence of a number c^* such that $I(c^*) = 0$.

2. The maximal number of isolated zeros c_k of $I(c) = 0$ (taking into account their multiplicity) is an upper bound for the number of limit cycles of system (2.3) with sufficiently small positive ε which tend to some closed orbits Γ_{c_k} of system (2.3) as $\varepsilon \to 0$.

The number $V(n,m)$ has been estimated only in case of some non-generic low-degree Hamiltonians (see, e.g., [19]). The first general result has been achieved by A. Varchenko [39] and A. Khovanskij [23]. They proved independently that $V(n,m)$ is finite, but they did not obtain an explicit expression for $V(n,m)$. The paper [24] contains the following result with respect to system (2.3):

Theorem 2.1 *Let* $H(x,y) := \frac{1}{2}y^2 + \frac{1}{m+1}x^{m+1}$, $P(x,y) \equiv 0$, *and let* $Q(x,y) := y\overline{Q}(x,y)$ *be a polynomial with degree at most* $n-1$. *Additionally, let* m *and* n *to be odd. Then the maximal number* $b(m,n)$ *of isolated zeroes (taking into account their multiplicity) of the Abelian integral* (2.2) *is*

$$b(m,n) = \begin{cases} \frac{(n+1)(n+3)}{8} - 1 & \text{if} \quad n \le m, \\ \frac{(m+1)(2n-m+3)}{8} - 1 & \text{if} \quad n \ge m. \end{cases}$$

Moreover, there are perturbations of system (2.3) such that $b(m,n)$ *continuous families of limit cycles exist. Consequently,*

$$b(m,n) \le V(m,n) \le H(\max\{m,n\}).$$

In general, the infinitesimal Hilbert problem remains open.

2.3 Abel's differential equation

If we consider system (1.2) in a neighborhood of a focus, then we can introduce polar coordinates and obtain a scalar differential equation whose right hand side is analytic in the radius with periodic coefficients. A truncation of the right hand side leads to an Abel's differential equation

$$\frac{dy}{dx} = y^n + \sum_{j=0}^{n-1} a_j(x)y^j, \ y \in R^1, \ x \in S^1, \tag{2.4}$$

where the coefficients a_j are continuous. The corresponding weakened Hilbert's problem is *to find an upper bound on the number of limit cycles of* (2.4). In [36] it has been proven that for $n \le 3$ the number of limit cycles of (2.4) is not greater than n. For $n \ge 4$, equation (2.4) can have arbitrarily many limit cycles [26]. If the coefficients a_j are trigonometric polynomials of degree not greater than m, than the bound should be expressed only by means of the numbers n and m. This problem is unsolved even for $m = 1$.

2.4 Liénard systems

We consider the Liénard system

$$\frac{dx}{dt} = y - F(x), \ \frac{dy}{dt} = -x, \tag{2.5}$$

where $F(x) = \sum_{i=1}^{2l+1} a_i x^i$. The weakened Hilbert problem consists in *finding an upper bound for the number of limit cycles of* (2.5) *which depends only on* l.

The following results are known:

Theorem 2.2 [25]. *System* (2.5) *with* $F(x) = a_1 x + a_2 x^2 + a_3 x^3$ *and* $a_1 a_3 < 0$ *has exactly one limit cycle. It is stable for* $a_1 < 0$ *and unstable for* $a_1 > 0$.

Theorem 2.3 [33]. *System* (2.5) *with* $F(x) = a_1 x + a_3 x^3 + a_5 x^5$ *has at most two limit cycles.*

Theorem 2.4 [4]. *The origin of system* (2.5) *has the maximal cyclicity* l*, that is, system* (2.5) *has at most* l *small limit cycles. There are coefficients* $a_1, a_3, a_5, \ldots, a_{2l+1}$ *with alternating sign such that* (2.5) *has* l *(small) limit cycles.*

Theorem 2.5 [29]. *For sufficiently small* ε *different from* 0*, system* (2.5) *with* $F(x) = \varepsilon \sum_{i=1}^{2l+1} a_i x^i$ *has at most* l *limit cycles. It has exactly* l *limit cycles if and only if the equation of degree* l

$$\frac{a_1}{2} + \frac{3a_3}{8}\varrho + \frac{5a_5}{16}\varrho^2 + \frac{35a_7}{128}\varrho^3 + \ldots + \begin{pmatrix} 2l+2 \\ l+1 \end{pmatrix} \frac{a_{2l+1}}{2^{2l+1}}\varrho^l = 0$$

has l *positive roots* $\varrho_j = r_j^2$, $j = 1, \ldots, l$*. In that case, the limit cycles tend to circles of radius* $r_j, j = 1, \ldots l$*, centered at the origin, as* $\varepsilon \to 0$.

S. Smale conjectures in the case of Theorem 2.5 that l is an upper bound on the number of limit cycles. This conjecture has been confirmed in the cases $l = 1, 2, 3, 4, 5$ by using Dulac functions (see [17, 12]).

We would like to mention that S. Lynch [26] computed the cyclicity of the origin of the Liénard system

$$\dot{x} = y, \ \dot{y} = -g(x) - f(x)y$$

for different degrees of the polynomials f and g using a Maple-package for some algebraic approach.

2.5 Algebraic limit cycles

Another approach to weaken Hilbert's 16th problem is to estimate the number of limit cycles with a special property. One possibility is to ask for limit cycles which are algebraic curves, that is, we want *to estimate the maximal number of limit cycles which are algebraic curves, and to estimate their maximal degree* . We recall that an irreducible algebraic curve defined by the equation $f(x,y)=0$ is called an invariant algebraic curve for system (1.2) if there exists a polynomial $g(x,y)$ such that the condition

$$\frac{\partial f}{\partial x}P+\frac{\partial f}{\partial y}Q=g(x,y)\,f(x,y)$$

holds. The following result has been established by N. Sadovskaia and R. Ramirez [34].

Theorem 2.6 *Suppose system* (1.2) *has* s $(s>1)$ *invariant nonsingular algebraic curves of degree* $m\geq 2$. *Then, the maximal number* $V(n)$ *of algebraic limit cycles of* (1.2) *satisfies* $V(n)=n-1$.

The problem of estimating the maximal degree $m(n)$ of an algebraic limit cycle for system (1.2) is open until now, even for quadratic systems ($n=2$). In that case only the inequality $m(2)\geq 4$ is known (see [5]).

3 Generalized Dulac function, Cherkas function

We consider system (1.1) in some region $D\subset R^2$ under the assumption that P and Q are continuously differentiable in D. A first step in finding an upper bound for the number of limit cycles of (1.1) in D is to characterize classes of systems (1.1) having no limit cycle in some subregion in D. The following theorem represents the classical criterion of Dulac.

Theorem 3.1 *Let* $G\subset D$ *be a simply-connected region, let* $f:=(P,Q)$. *If there exists a function* $B\in C^1(G)$ *such that* $div(Bf)$ *does not change sign in* G *and is not identically zero, then* (1.1) *has no closed orbit lying entirely in* G.

Remark 3.1 *The proof of Theorem* 3.1 *is based on Green's Theorem. For* $B\equiv 1$, *Theorem 3.1 represents the well-known criterion of Bendixson. In that case, under the additional conditions that* G *is bounded and that* $div f$ *is strictly positive or negative in the closure of* G *there is another proof Theorem* 3.1 *based on the Poincare-Bendixson theory (see [41]).*

Definition 3.1 *A function* $B\in C^1(D)$ *such that* $div(Bf)$ *has the same sign in some connected region* $G\subset D$ *and is not identically zero is called a Dulac function to system* (1.1) *in* G.

The following well-known result (see [40]) show that a Dulac function can be used to estimate the number of limit cycles of system (1.1) in some regions.

Theorem 3.2 *Let $G \subset D$ be a p-connected region. If there exists a Dulac function in G, then system* (1.1) *has at most $p-1$ limit cycles in G.*

The following definition provides some generalization of a Dulac function.

Definition 3.2 *Let $\Omega \subset D$ be a connected region containing finitely many equilibria $E_1, ..., E_s$ of system* (1.1). *Let $\tilde{\Omega} := \Omega \backslash E$, where $E := \{E_1, ..., E_s\}$. A function $B \in C^1(\tilde{\Omega})$ such that*

- *$div(Bf)$ is not identically zero and has the same sign in $\tilde{\Omega}$,*
- $$\lim_{\varepsilon \to 0} \int_{S_i^\varepsilon} (Bf, n_i) ds = 0,$$
  where S_i^ε is a circle with radius ε centered at E_i, n_i the unit-normal vector to S_i^ε, and $(,)$ denotes the scalar product in R^2,

is called a generalized Dulac function to system (1.1) *in Ω.*

As an example of a generalized Dulac function we may consider the function (see [35])

$$B(x,y) := \{P(x,y)^2 + Q(x,y)^2\}^{-1/2}.$$

In analogy to Theorem 3.1 and Theorem 3.2 we have:

Theorem 3.3 *Let Ω be a region as in Definition* 3.2, *moreover, let Ω be simply-connected. If there exists a generalized Dulac function in $\Omega \backslash E$, then* (1.1) *has no closed orbit lying entirely in Ω.*

Theorem 3.4 *Let Ω be a doubly-connected region. If there exists a generalized Dulac function in $\Omega \backslash E$, then system* (1.1) *has at most one limit cycle in Ω.*

Now we present a generalization of Dulac's criterion which is due to L.A. Cherkas [8]. For this purpose we introduce the following definition.

Definition 3.3 *A function $\Psi \in C^1(G)$, $G \subset D$, is called a Cherkas function of system* (1.1) *in G if there exists a real number $k \neq 0$ such that*

$$\Phi := k\Psi \; div \; f + \frac{\partial \Psi}{\partial x} P + \frac{\partial \Psi}{\partial y} Q > 0 \; (< 0) \quad in \quad G. \tag{3.1}$$

Remark 3.2 *The condition* (3.1) *can be relaxed by assuming that the function* Φ *vanishes on a set measure zero, where this set satisfies some additional condition.*

In what follows we recall some ways to construct a Cherkas function Ψ for system (1.2) with degree n.

1. As a polynomial of degree 2 in the case $n = 3$ in the full plane [9] and in case $n = 2$ in a half-plane [10];
2. As a polynomial of degree 3 in the full plane in the cases $n = 2$ [11] and $n = 4$ [16];
3. As the product $\Psi(x, y) = \Psi_1^{l_1}(x, y)...\Psi_m^{l_m}(x, y)$ in the case $n = 2$ [11] in the full plane such that we get for the corresponding Dulac function
$$B = \prod_{j=1}^{m} |\Psi_j(x, y)|^{\frac{1}{k_j}};$$
4. As a spline consisting of two polynomials of degree 2 for the case $n = 3$ [11];
5. As a spline consisting of four linear polynomials for system in the case $n = 3$ [11];
6. In the form $\Psi = \sum_{i=1}^{m} \Psi_i(x)y^{m-i}$, where $m - i$ is even in case $n = 2$ in a half-plane and for the Liénard system (2.5) in the full plane in the cases $l = 1, 2, 3, 4, 5$. [17], [12].

The following result gives a connection between a Cherkas function and a Dulac function (see [8]).

Lemma 3.1 *Let* $G \subset D$ *be connected, let* Ψ *be a Cherkas function of system* (1.1) *in* G. *Then* $B := |\Psi|^{1/k}$ *is a Dulac function in each subregion of* G, *where* Ψ *is positive or negative.*

We note that in case $k = 1$ we have

$$\Phi = \text{sign}\Psi \; div(|\Psi| f). \tag{3.2}$$

This relation suggests to introduce the notation of a generalized Cherkas function.

Definition 3.4 *A function* $\Psi \in C^1(\Omega)$, *where* Ω *is a subregion of* D *containing finitely many equilibria of* (1.1) *is called a generalized Cherkas function in* Ω *if there exists a real number* $k \neq 0$ *such that*

- *The function* Φ *defined in* (3.1) *satisfies*
$$\Phi(x, y) > 0 (< 0) \quad in \quad \Omega \backslash E,$$

•

$$\lim_{\varepsilon\to 0}\int_{S_i^\varepsilon}(|\Psi|^{1/k}f,n_i)ds=0.$$

Analogously to Lemma 3.1 we have

Lemma 3.2 *Let Ψ be a generalized Cherkas function of system (1.1) in Ω. Then $B:=|\Psi|^{1/k}$ is a generalized Dulac function in each subregion of Ω, where Ψ is positive or negative.*

For the sequel we introduce the set W by

$$W:=\{(x,y)\in G:\ \Psi(x,y)=0\}.$$

We denote by a branch of the curve W a subset of W which is either a simple closed curve (oval) or a simply connected subset which intersects ∂G in exactly two different points.

Lemma 3.3 *Let Ψ be a Cherkas function of system* (1.1) *in $G\subset D$. Then any trajectory of system* (1.1)*, which meets W, intersects W transversally.*

Proof. We denote by $\frac{d\Psi}{dt}$ the derivative of Ψ along system (1.1). From (3.1) we get

$$\frac{d\Psi}{dt}_{|\Psi=0}=\Phi_{|\Psi=0}\neq 0. \tag{3.3}$$

Thus, any trajectory of (1.1) which meets the curve W crosses W transversally. □

Lemma 3.4 *Let Ψ be a Cherkas function of system* (1.1) *in $G\subset D$. Then the curve W does not contain any equilibrium of system* (1.1)*.*

Proof. Let E be an equilibrium point of system (1.1) located on W. Then, by the definition of the function Φ in (3.1) we have $\Phi(E)=0$ which contradicts the inequalities in (3.1). □

Lemma 3.5 *Let Ψ be a Cherkas function of system* (1.1) *in $G\subset D$. Then the curve W consists in G of branches which do not meet.*

Proof. Assume W contains two branches W_1 and W_2 meeting in the point T. Then the trajectory γ_T of system (1.1) through T intersects W_1 and W_2 transversally. Then each trajectory close to the trajectory γ_T intersects W_1 and W_2. According to (3.3) and (3.1), $\frac{d\Psi}{dt}$ has the same sign on W_1 and W_2. Thus, γ_T cannot intersect W_1 and W_2. The obtained contradiction proves the Lemma. □

Remark 3.3 *By Lemma* 3.5 *the set* W *consists of two disjoint sets, the set* W_{cl} *consisting from ovals and the set* W_{nc} *consisting from non-closed branches.*

From the Lemmata 3.3 and 3.5 we get

Corollary 3.1 *Let* Ψ *be a Cherkas function of system* (1.1) *in* $G \subset D$. *Then* W *separates in* G *regions, where* Ψ *is positive from regions, where* Ψ *is negative.*

Furthermore, Lemma 3.3 implies the following result:

Lemma 3.6 *Let* Ψ *be a Cherkas function of system* (1.1) *in* $G \subset D$. *Then any limit cycle of system* (1.1) *which is entirely located in* G *does not intersect the curve* W.

Proof. Suppose system (1.1) has a limit cycle $\Gamma \subset G$ intersecting the curve W. Without loss of generality we can assume $\Phi > 0$ in G, otherwise we replace f by $-f$. According to Lemma 3.3, the limit cycle Γ intersects W transversally. From $\Phi > 0$ in G we get that the derivative of Ψ along system (1.1) is positive in any point G. Thus, the limit cycle Γ enters at any intersection point with W the region $\Psi > 0$ for increasing t. Consequently, Γ can meet W only once in G. But this contradicts the property that Γ is a closed curve in G. □

The following theorem represents a generalization of Dulac's criterion on the non-existence of a limit cycle.

Theorem 3.5 *Let* Ψ *be a Cherkas function for system* (1.1) *in* $G \subset D$. *Furthermore, we suppose that* W *decomposes* G *in simply connected subregions* $G_i, i = 1, ..., l$. *Then system* (1.1) *has no limit cycle in* G.

Proof. Let G_i be one of the simply connected subregions. By definition, Ψ is different from zero in G_i. Thus, by Lemma 3.1, $B := |\Psi|^{1/k}$ is a classical Dulac function in G_i, and, consequently, system (1.1) has no limit cycle located completely in G_i. By Lemma 3.6, any limit cycle of system (1.1) entirely located in G does not intersect W. Therefore, we can conclude that G contains no limit cycle of system (1.1). □

We note that in Theorem 3.5 the sign of the constant k in the expression Φ plays no role. This is due to the property that the subregions G_i are simply connected and Ψ does not vanish in these regions. The following generalization of Dulac's criterion admits that Ψ can change sign in a simply connected region, but then the sign of k is essential.

Theorem 3.6 *Let* $G \subset D$ *be a simply-connected region and let* Ψ *be a Cherkas function for system* (1.1) *in* G, *where* k *is positive. Then system* (1.1) *has no limit cycle in* G.

Proof. Without loss of generality we may assume $\Phi > 0$ in G. If W divides G only in simply connected subregions, then we can apply Theorem 3.5. Therefore, we suppose that the decomposition of G by W contains a multiply connected region $\tilde{G}$ where all interior boundaries are closed branches (ovals) of W.

Next we establish that there is no limit cycle of (1.1) in $\tilde{G}$ that does not surround an oval of W. If we suppose that (1.1) has a limit cycle Γ in $\tilde{G}$ that does not surround any oval of W, then Γ is located in a simply connected region G_1 of $\tilde{G}$, where Ψ is either positive or negative. This contradicts the property that $|\Psi|^{1/k}$ is a Dulac function in G_1.

Now, we assume that there is a limit cycle Γ of system (1.1) containing in its interior an oval W_0 of W. Without loss of generality we may assume that in the annulus $A_{W_0,\Gamma}$ formed by W_0 and Γ there is located neither an oval of W different from W_0 nor a limit cycle of (1.1) different from Γ. According to Lemma 3.6, Γ is entirely located either in a region where Ψ is strictly positive or negative.

First we consider the case that Γ is located in a region, where $\Psi > 0$. In that case, under our assumptions Ψ is positive in the interior of $A_{W_0,\Gamma}$, where W vanishes on W_0 but is positive on Γ. Since $d\Psi/dt$ is positive on W_0 we can conclude by Lemma 3.3 that the trajectories of (1.1) enter for increasing t transversally on W_0 the annulus $A_{W_0,\Gamma}$ which is positively invariant.

From (3.1) we get

$$\text{div } f = \frac{\Phi - \frac{d\Psi}{dt}}{k\Psi}. \tag{3.4}$$

Under our assumption we have

$$\int_\Gamma \text{div } f \ dt = \int_0^T \frac{1}{k\Psi}\Big(\Phi - \frac{d\Psi}{dt}\Big)dt = \int_0^T \frac{\Phi}{k\Psi}\ dt > 0, \tag{3.5}$$

hence, Γ is orbitally unstable. Therefore, the interior of $A_{W_0,\Gamma}$ must contain an attractor. Under our assumptions, we can conclude from to the Poincare-Bendixson theory that the attractor consists either from a stable equilibrium or from a stable polycycle (closed heteroclinic or homoclinic orbit).

From (3.4) we obtain that in any equilibrium point of system (1.1) in $A_{W_0,\Gamma}$ the inequality

$$div f = \frac{\Phi - \frac{d\Psi}{dt}}{k\Psi} = \frac{\Phi}{k\Psi} > 0$$

holds. Therefore, in $A_{W_0,\Gamma}$ there is no stable equilibrium and no stable polycycle. This contradiction proves that there is no limit cycle in the domain $\Psi > 0$ surrounding an oval of W. The case that Γ is lying in a region with $\Psi < 0$ can be treated similarly by replacing t by $-t$. □

From Theorem 3.6 it follows that we cannot replace a Dulac function by a Cherkas function in Theorem 3.1 without any restriction on k.

The following generalization of Theorem 3.2 requires assumption on the decomposition of the underlying region by the curve W.

Theorem 3.7 *Let $G \subset D$ be a p-connected region with the interior boundaries ∂G_i, $i = 1, ..., p-1$, and with the outer boundary ∂G_p. Let Ψ be a Cherkas function in G. If we additional assume that W has no oval in G, then system* (1.1) *has at most $p-1$ limit cycles in G.*

Proof. Under our assumptions and taking into account Remark 3.3, the set W consists only of non-closed branches intersecting the outer boundary ∂G_p of G in exactly two points. The decomposition of G by W yields the representation

$$G = \bigcup_{k_1=1}^{q_1} G_{k_1}^{(1)} \bigcup_{k_2=1}^{q_2} G_{k_2}^{(2)} ... \bigcup_{k_{p-1}=1}^{q_{p-1}} G_{k_{p-1}}^{(p-1)} \bigcup_{k_p=1}^{q_p} G_{k_p}^{(p)},$$

where the upper indices characterize the connectivity. It can easily be shown by induction that the relation

$$q_2 + 2q_3 + ... + (p-1)q_p = p - 1$$

is valid. Since Ψ is positive or negative in $G_{k_i}^{(i)}$, we can conclude by Lemma 3.1 that Ψ is a Dulac-function in each region $G_{k_i}^{(i)}$. Thus, $G_{k_i}^{(i)}$ contains at most $i-1$ limit cycles, and G not more than $p-1$ limit cycles. □

If we admit that W has ovals in G we have the following result:

Theorem 3.8 *Let $G \subset D$ be a p-connected region with the interior boundaries ∂G_i, $i = 1, ..., p-1$, and with the outer boundary ∂G_p. Let Ψ be a Cherkas function such that W has s ovals in G. Then system* (1.1) *has at most $p - 1 + s$ limit cycles in G.*

Proof. We denote the ovals by ∂O_1, ..., ∂O_s. In case $s = 0$ Theorem 3.8 coincides with Theorem 3.7.
Next we consider the case that no oval ∂O_j contains any interior boundary ∂G_k or another oval ∂O_l. Since Ψ is positive or negative in the region O_j bounded by any ∂O_j, we can conclude that no oval contains a limit cycle in its interior and that no limit cycle intersects any oval. Thus, the number of limit cycles in G and in the region $\tilde{G} := G \setminus \bigcup_{j=1}^{s} O_j$ is the same. Since $\tilde{G}$ is a $p+s$-connected region and since Ψ is either positive or negative on $\tilde{G}$, we get by Lemma 3.1 and Theorem 3.2 that G contains not more than $p - 1 + s$ limit cycles. If we admit that W contains also non-closed branches, then the assumptions of Theorem 3.7 are satisfied concerning the region $\tilde{G}$ such that it contains not more than $p - 1 + s$ limit cycles.
Next we consider the case that one oval, say ∂O_r surrounds m interior boundaries, while all other ovals satisfy the assumptions as above. Then the region bounded by ∂O_r and the m interior boundaries of G contains not more than m limit cycles. The region $\tilde{G} := G \setminus \bigcup_{j=1}^{s} O_j$ is now $p-1-m+s$-connected and contains not more than $p-1-m+s$ limit cycles.Hence, the region G contains at most $p-1+s$ limit cycles.

If we suppose that one oval surrounds another oval, then similar considerations lead to the same upper bound of limit cycles.

□

The proof of Theorem 3.6 does not work in case $k < 0$. The following result include the case $k < 0$.

Theorem 3.9 *Let $\Omega \subset D$ be a connected subset of R^2, and let Ψ be a Cherkas function for system (1.1) in Ω. Then in each p-connected subdomain $\tilde{\Omega}$ of Ω with $\partial\tilde{\Omega} \subset W \cup \partial\Omega$, that means any point of the boundary of $\tilde{\Omega}$ belongs to W or to $\partial\Omega$, the number of limit cycles is at most $p-1$ and any limit cycle in $\tilde{\Omega}$ is stable (unstable) if k sign $(\Psi\Phi) < 0$ (k sign $(\Psi\Phi) > 0$).*

Proof. The proof of this theorem follows the same line as in the proof of Theorem 3.6. □

Remark 3.4 *The substantial difference of our approach to estimate the number of limit cycles from the classical one is that we must not localize limit cycles because this localization follows from the topological analysis of the curve $\Psi(x, y) = 0$.*

Remark 3.5 *In the definition of the Cherkas function we can relax the condition $\Phi > 0 (< 0)$ by $\Phi \geq 0 (\leq 0)$, where we have to assume that Φ is not identically zero and that the curve $\Phi = 0$ will be intersected transversally by the trajectories of system (1.1), that is, $\Phi = 0$ is not a trajectory of* (1.1) [40].

4 Estimate of the number of limit cycles with special properties

In the preceding section we estimated the number of *all* limit cycles of (1.1) in some regions of the phase plane under the assumption that there exists an appropriate Dulac or Cherkas function. In section 2 we described some possibilities how we can relax the problem to estimate the number of *all* limit cycles in some given region. In what follows we present further possibilities to weaken Hilbert's 16-th problem.

At first we define some class of limit cycles by means of a function which can be considered as a generalized Dulac function.

Definition 4.1 *Let $\Omega \subset D$ be a simply connected region containing a finite number of equilibria $E_1, ..., E_s$. Let $\tilde{\Omega} := \Omega \backslash E$, where $E := \{E_1, ..., E_s\}$. Let there exists a function $\tilde{B} \in C^1(\tilde{\Omega})$ such that*

- *$div(\tilde{B}f)$ vanishes in $\tilde{\Omega}$ only on a set of measure 0.*
-
$$\lim_{\varepsilon\to 0}\int_{S_i^\varepsilon}(\tilde{B}f, n_i)ds = 0 \quad for \quad i = 1, ...s, \tag{4.1}$$
where S_i^ε is a circle with radius ε centered at E_i, n_i is the unit-normal vector to S_i^ε and $(.,.)$ denotes the scalar product in R^2.

We say a limit cycle Γ of (1.1) in Ω belongs to the class $\mathcal{B}$ if $div(\tilde{B}f)$ is positive or negative on Γ.

The following theorem provides an upper bound for the number of limit cycles belonging to the class $\mathcal{B}$.

Theorem 4.1 *The number of limit cycles of system (1.1) in Ω belonging to the class $\mathcal{B}$ is not greater than the number of closed curves of $div(\tilde{B}f) = 0$ in Ω, where $div(\tilde{B}f)$ changes sign.*

Proof. Suppose Γ is a limit cycle of (1.1) in Ω belonging to the class $\mathcal{B}$ and bounding the region Ω_Γ. If $div(\tilde{B}f)$ does not change sign in Ω_Γ, then we have

$$0 < |\int\int_{\Omega_\Gamma} div(\tilde{B}f)dxdy| = \int_\Gamma \tilde{B}(f,n)dt - \sum_{i=1}^{s}\int_{S_i}\tilde{B}(f,n_i)ds,$$

where n is the normal unit vector to Γ. By (4.1), the integral along S_i vanishes as ε tends to zero. Since Γ is a solution of (1.1), we have $(f,n) = 0$ on Γ. Hence, we get

$$0 < |\int\int_{\Omega_\Gamma} div(\tilde{B}f)dxdy| = 0.$$

The obtained contradiction proves that a limit cycle of (1.1) belonging to the class $\mathcal{B}$ contains in its interior at least one closed orbit of $div\tilde{B}f = 0$ on which $div\tilde{B}f$ changes sign.
Next we assume that Γ_1 and Γ_2 are two limit cycles of (1.1) belonging to the class $\mathcal{B}$, where Γ_1 is located in the interior of Γ_2. By the same way as before we can establish that there is a closed curve defined by $div\tilde{B}f = 0$, located between Γ_1 and Γ_2 and surrounding Γ_1, on which $div(\tilde{B}f)$ changes sign. This completes the proof. □

Remark 4.1 *If we consider the case*

$$\tilde{B}(x,y) \equiv \{P(x,y)^2 + Q(x,y)^2\}^{-1/2},$$

then it can be checked that $div\tilde{B}f$ *represents the curvature* $h(x,y)$ *of those trajectories which are orthogonal to the trajectories of system* (1.1), *that is which satisfy the differential system*

$$\frac{dx}{dt} = -Q(x,y), \ \frac{dy}{dt} = P(x,y) \tag{4.2}$$

(*see* [35]). *It is interesting to note that a limit cycle of system* (1.1) *on which* $h(x,y)$ *does not change sign represents a so-called regular limit cycle, a notation introduced by P.N. Papusch* [28].

The number N of regular limit cycles of a polynomial system (1.2) can be estimated as follows.

Theorem 4.2 *Let* P *and* Q *be polynomials of degree* n. *Then the number* N *of regular limit cycles can be estimated by*

$$N \leq \frac{9n^2 - 15n + 8}{2}. \tag{4.3}$$

Proof. According to Theorem 4.1 and taking into account the expression for $h(x,y)$ we get that N is not greater than the number of simple closed curves of

$$\bar{h}(x,y) := P^2 Q_y - PQ(P_y + Q_x) + Q^2 P_x.$$

Under our assumptions, this curve is an algebraic curve of degree $3n-1$. Due to a theorem of A. Harnack, a plane algebraic curve of genus p has not more than $p+1$ isolated connected curves. Since for a plane algebraic curve of degree m the relation

$$p + 1 \leq \frac{(m-1)(m-2)+2}{2} \tag{4.4}$$

holds, we get for $m = 3n-1$ the estimate (4.3). □

Remark 4.2 *If we know more about the location of the closed curves of* $\bar{h} = 0$, *then we can improve the estimate* (4.3). *For example, if we know that they form two groups of nested ovals, then there exists a straight line having at least two intersection points with each of these closed curves. According to the theorem of Bezout, this straight line has at most* $3n-1$ *intersection points with* $\bar{h} = 0$. *Therefore, there exist not more than* $[3n - 1/2]$ *regular limit cycles, where* $[z]$ *means the greatest integer not bigger than* z.

Next we define another class of limit cycles by means of a function Ψ which can be interpreted as a Cherkas function near a limit cycle.

Definition 4.2 *Let Ψ be continuously differentiable in D, let the function Φ be defined by* (3.1). *We say that a limit cycle Γ of system* (1.1) *in the region D belongs to the class L_Ψ if $\Phi(x,y)$ is positive or negative on Γ.*

It is obvious that a limit cycle of the class L_Ψ does not meet the curve $V := \{(x,y) \in D : \Phi(x,y) = 0\}$. The following lemma shows that Γ does also not intersect the curve W.

Lemma 4.1 *A limit cycle of the class L_Ψ does not intersect the curve W.*

Proof. Suppose Γ intersects W in the point T and assume $\Phi > 0$ on Γ. Then we get by (3.1) and Definition 4.2.

$$0 < \Phi(x,y)_{|T} = \frac{d\Psi}{dt}_{|T},$$

This inequality is valid for any intersection point of W and Γ. Thus, Γ cannot intersect W twice. The obtained contradiction proves the lemma. □

These considerations show that Ψ represents a Cherkas function near a limit cycle belonging to the class L_Ψ.

In what follows we will exploit these properties to estimate the number of limit cycles belonging to the class L_Φ.

Theorem 4.3 *Let $G \subset D$ be open and simply connected. Let $\Psi \in C^1(G)$. Then the number of limit cycles belonging to the class L_Ψ is not greater than the number of closed curves of $\Phi(x,y)\Psi(x,y) = 0$ in G.*

Proof. Let Γ be a limit cycle of (1.1) located in G on which Φ has constant sign. Without loss of generality we can assume $\Phi(x,y) > 0$ on Γ. By Lemma 4.1 Ψ is positive or negative on Γ. Without loss of generality we can assume $\Psi(x,y) > 0$ on Γ. Let G_Γ be the simply connected region bounded by Γ. Suppose $\Phi\Psi > 0$ in G_Γ, that is, there is no closed curve of $\Phi\Psi = 0$ in G_Γ. Then, by (3.4) and (3.5) any limit cycle and any equilibrium point in the interior of G_Γ have the same stability as Γ. This contradiction proves the theorem. □

Corollary 4.1 *Suppose the assumptions of Theorem* 4.3 *are satisfied. Furthermore we assume that Ψ is a Cherkas function in G. Then the number of limit cycles in Ω belonging to the class L_Ψ is not greater than the number of closed curves of $\Psi = 0$ in Ω.*

Corollary 4.2 *Suppose the assumptions of Theorem* 4.3 *are satisfied. Furthermore we assume $\Psi > 0$ or $\Psi < 0$ in G. Then the number of limit cycles in G belonging to the class L_Ψ is not greater than the number of closed curves of $\Phi = 0$ in G.*

Example 4.1 *Consider the differential system*

$$\begin{aligned} \frac{dx}{dt} &= -y + x(x^2 + y^2 - 1), \\ \frac{dy}{dt} &= x + y(x^2 + y^2 - 1). \end{aligned} \tag{4.5}$$

Using polar coordinates $x = r cos\varphi, y = r sin\varphi$, *system* (4.5) *is equivalent to*

$$\frac{dr}{dt} = r(r^2 - 1), \quad \frac{d\varphi}{dt} = 1, \tag{4.6}$$

and it is easy to see that (4.5) *has the unique limit cycle* $r = 1$. *From* (4.5) *we obtain*

$$P_x + Q_y \equiv 2(2r^2 - 1). \tag{4.7}$$

Since $P_x + Q_y < 0$ *for* $0 \leq r < 0.5$, *system* (4.5) *has no limit cycle in that region. Setting* $k = 1, \Psi(x, y) = r^2$, *we get from* (3.1) *and* (4.7)

$$\Phi(x) = 2r^2(3r^2 - 2).$$

From this relation it follows that $\Phi(x, y) = 0$ *has exactly one closed curve in the region* $r \geq 0.4$ *and which can be described by* $r = \sqrt{2/3}$. *Hence,*

$$\Phi(x, y) > 0 \quad \textit{for} \quad r > 0.83,$$

that is, all limit cycles of (4.5) *in that region belong to the class* L_Ψ. *Since* Ψ *is positve for* $r > 0.83$, *system* (4.5) *has by Corollary* 4.2 *system at most one limit cycle in that region. As* (4.6) *has exactly one limit cycle, this estimate cannot be improved.*

If we assume that P and Q are polynomials in x and y of degree n, and if we additionally suppose that Ψ is also a polynomial in x and y of degree m, then it follows from (3.1) that Φ is a polynomial of degree $s_0 := m + n - 1$ and $\Phi\Psi$ is a polynomial of degree $s := 2m + n - 1$. By (4.4) we have that the genus p of the algebraic curve $\Phi\Psi = 0$ satisfies

$$p + 1 \leq \frac{(s-1)(s-2) + 2}{2}.$$

Hence, we have the following result:

Theorem 4.4 *Let* P *and* Q *be polynomials of degree* n. *Then the number* N *of limit cycle of system* (1.2) *belonging to the class* L_Ψ *can be estimated by*

$$N \leq \frac{(s-1)(s-2) + 2}{2}. \tag{4.8}$$

From Theorem 4.4 we get:

Corollary 4.3 *Suppose the assumptions of Theorem 4.4 are satisfied. Furthermore we assume* $\Psi(x,y) > 0$ *or* $\Psi(x,y) < 0$ *in* G. *Then the number of limit cycles of system* (1.2) *belonging to the class* L_Ψ *satisfies*

$$N \leq \frac{(s_0-1)(s_0-2)+2}{2}. \tag{4.9}$$

Theorem 4.5 *Let* $G \subset D$ *be* p*-connected and let* $\Psi \in C^1(G)$. *Then the number of limit cycles in* G *belonging to the class* L_Ψ *is not greater than* $p-1+q$, *where* q *is the number of closed curves of* $\Psi\Phi = 0$ *in* G.

Proof. We consider the decomposition of Ω by the curve $\Psi\Phi = 0$. If we get $p-1$ doubly connected regions, then it is possible that there is a limit cycle of the class L_Ψ surrounding an interior boundary of G.

If we suppose that there are two limit cycles Γ_1 and Γ_2 of the class L_Ψ where one limit cycle is located in the interior of the other limit cycle, then it follows from the proof of Theorem 4.3 that an oval of $\Psi(x,y)\Phi(x,y) = 0$ must be located between Γ_1 and Γ_2. Taking into account Theorem 4.3, the proof of Theorem 4.5 is complete.

□

Corollary 4.4 *Suppose the assumptions of Theorem* 4.5 *are satisfied. Furthermore we assume that* Ψ *is a Cherkas function in* G. *Then the number of limit cycles in* G *belonging to the class* L_Ψ *is not greater than* $p-1+l$, *where* l *is the number of closed curves of* $\Psi(x,y) = 0$ *in* Ω.

Corollary 4.5 *Suppose the assumptions of Theorem* 4.5 *are satisfied. Furthermore we assume* $\Psi > 0$ *or* $\Psi < 0$ *in* G. *Then the number of limit cycles in* G *belonging to the class* L_Ψ *is not greater than* $p-1+r$, *where* r *is the number of closed curves of* $\Phi(x,y) = 0$ *in* G.

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