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## Monte Carlo evaluation of American options using consumption processes

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**ABSTRACT.** Here we develop a new approach for pricing both continuous-time and discrete-time American options which is based on the fact that an American option is equivalent to a European option with a consumption process involved. This approach admits construction of an upper bound (a lower bound) on the true price using a lower bound (an upper bound) by Monte Carlo simulation. A number of effective estimators of the upper and lower bounds with reduced variance are proposed. The results obtained are supported by numerical experiments which look promising.

## 1. Introduction

It is well known that the value of a trading strategy for pricing European options satisfies the classical Cauchy problem for an equation of parabolic type. If an option depends on many stocks, the solution of the problem via any deterministic method is, as a rule, impossible in practice due to huge volume of computations. However, for constructing the hedging strategy we have to find only individual values of solutions to certain Cauchy problems at any instant. In such a situation a probabilistic approach becomes viable. It is based on probabilistic representations of solutions. Further, the ideas of weak sense numerical integration of stochastic differential equations (SDEs) are exploited and finally the Monte Carlo technique is applied. Much more complicated numerical problems arise in connection with multi-dimensional American options. On the one hand there is no alternative to a Monte Carlo approach but on the other hand the arising boundary value problems for partial differential problems are nonlinear and they do not have sufficiently constructive probabilistic representations from the numerical point of view.

Valuation and optimal exercise of American and Bermudan options are one of the most important problems both in theory and practice. Several approaches have been proposed in recent years to develop simulation technique for their pricing (see, e.g. [2, 3, 6, 9, 10, 11, 13, 14, 15, 16, 19, 20, 27, 29] and references therein). The recent papers [13, 14, 15, 16, 19, 27] are devoted to the dual minimax method for American and Bermudan options. The authors of [13] and [27] establish a dual representation of American option prices which allows them to compute upper bounds on several types of the options using Monte Carlo simulation. Another dual representation is established and used in [14]. These approaches involve expensive calculations connected with the maximization of expressions depending on the trajectories of the price process. The papers [15, 16] are devoted to the further development of the dual minimax approach and gives its efficient numeric performance.

Here we develop a new approach to pricing American options, which is based on the fact that an American option is equivalent to a European option with a consumption process involved. In the case of a continuous-time American option (see Sections 2 and 3) the consumption process is equal to zero in the continuation region of the American option and is equal to a known function in the exercise (stopping) region (we recall that these regions themselves are unknown). If an approximation of the exercise region is found and this approximation is wider than the true exercise region,

then the European option with a consumption process, which is nonnegative in the wider region, is an upper bound on the true price. In turn, the approximation of the exercise region is determined by a lower bound. Let  $f(t, x)$  be a payoff function of an American option and let its lower bound  $v(t, x)$  be equal, for example, to

$$(1.1) \quad v(t, x) = \max\{f(t, x), u_{eu}(t, x)\},$$

where  $u_{eu}(t, x)$  is the price of the underlying European option which, in principle, can easily be computed by the Monte Carlo method. The value of the upper bound  $V(t, x)$  at a position  $(t, x)$  is constructed by the Monte Carlo method in the following way. Let  $u(t, x)$  be the true option price and  $\mathcal{E}$  be the exercise region of the considered American option. Let

$$\mathcal{E}_v = \{(t, x) \in [0, T) \times \mathbf{R}_+^d \mid v(t, x) = f(t, x)\}.$$

Since  $v \leq u$ , we get that  $\mathcal{E} \subset \mathcal{E}_v$ . Now the upper bound  $V(t, x)$  is constructed as the European option with the consumption process determined by the set  $\mathcal{E}_v$ . The inequality  $V(t, x) \geq u(t, x)$  is ensured because we take the consumption processes for  $u$  and  $V$  to be equal on  $\mathcal{E}$  and the consumption process on  $\mathcal{E}_v \setminus \mathcal{E}$  to be nonnegative. We show that if two lower bounds  $v_1$  and  $v_2$  are such that  $v_1 \leq v_2 \leq u$ , then the corresponding upper bounds  $V_1, V_2$  satisfy the inequality  $V_1 \geq V_2 \geq u$ . The upper bound  $V$  is approximately constructed by using weak methods of numerical integration of SDEs and Monte Carlo simulation. The obtained estimate  $\hat{V}$  of  $V$  involves an error of numerical integration (bias of  $\hat{V}$ ) and a statistical error due to the Monte Carlo method. The first error can be reduced considerably by a proper choice of numerical integration scheme. We emphasize that our approach admits applying a number of known variance reduction methods for reducing the second error (see Section 4) which are of crucial importance for the effectiveness of any Monte Carlo procedure. In this connection we use the results of [21, 22, 24, 25, 26, 30] (see also [11] and references therein).

Thus, the new method of constructing upper bound for the price using some lower bound is developed. We note that we do not need any explicit analytical form of the lower bound, we only require the ability to compute its price at every position, for instance, by a Monte Carlo procedure.

In Section 5 we consider the discrete-time case. Here we also use the equivalence of a discrete-time American option to a European option with consumption process. The consumption process is again easily expressed through the characteristics of the price process and the continuation and exercise regions. Let  $(B_n, X_n) = (B_n, X_n^1, \dots, X_n^d)$ ,  $n = 0, 1, \dots, N$ , be the vector of prices at time  $n$  of a discrete-time financial model under consideration. Here  $B_n$  is the price of a scalar riskless asset and  $X_n$  is the price vector process of risky assets. Let  $f_n(x)$  be the profit made by exercising an American option at time  $n$  if  $X_n = x$ . It turns out that the discrete-time American option is equivalent to the European option with the payoff function  $f_N(x)$  and with

the consumption process  $\gamma_n(x)$  defined by

$$\gamma_n(x) = \left[ f_n(x) - B_n E \left( \frac{u_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) \right]^+, \quad n = 0, \dots, N-1,$$

where  $u_n(x)$  is the price of the American option. It is clear that if we take a lower bound (an upper bound) instead of  $u_{n+1}$  in this formula, we obtain a consumption process which leads to an upper bound (to a lower bound). Following this way and starting with a lower bound  $v^1$ , we obtain an upper bound  $V^1$ , then again a lower bound  $v^2$  and so on. This procedure gives us the sequences  $v_n^1(x) \leq v_n^2(x) \leq v_n^3(x) \leq \dots \leq u_n(x)$ , and  $V_n^1(x) \geq V_n^2(x) \geq \dots \geq u_n(x)$ . However each further step of the procedure requires labor-consuming calculations and in practice it is possible to realize only a few steps of this procedure. All the bounds  $v^i$  and  $V^i$  can, in principle easily, be evaluated by the Monte Carlo simulation. In this connection, much attention is given to variance reduction techniques and some effective and constructive methods reducing statistical errors are proposed (see Section 6). All the results obtained for discrete-time American options are carried over to the Bermudan options in Sections 5 and 6.

In this paper, our attention focuses on constructing the new numerical procedures and on their practical implementation. The results of numerical experiments (see Section 7) confirm efficiency of the proposed algorithms.

## 2. Preliminaries

We consider a multi-dimensional American style option in a generalized Black-Scholes framework

$$(2.1) \quad dX_s^i = X_s^i(a_i(s, X_s)ds + \sum_{j=1}^d \sigma_{ij}(s, X_s)dW_s^j), \quad X_0^i = x^i, \quad i = 1, \dots, d,$$

$$(2.2) \quad dB_s = r(s, X_s)B_s ds, \quad B_0 = 1, \quad 0 \leq s \leq T.$$

In (2.1)-(2.2), the process  $X = (X^1, \dots, X^d)$  is the price vector process of risky assets in  $\mathbf{R}_+^d$ ,  $B$  is the price of a scalar locally riskless asset,  $W = (W^1, \dots, W^d)$  is a  $d$ -dimensional standard Wiener process on a probability space  $(\Omega, \mathcal{F}, P)$ . As usual, the filtration generated by  $W$  is denoted by  $\{\mathcal{F}_s\}$ . It is assumed that  $a_i, \sigma_{ij}, r$  are sufficiently regular functions from  $[0, T] \times \mathbf{R}_+^d \rightarrow \mathbf{R}$ . Moreover, we assume that the matrix  $\sigma(t, x) = \{\sigma_{ij}(t, x)\}$  has full rank for every  $(t, x) \in [0, T] \times \mathbf{R}_+^d$ . Under these assumptions the model  $(B, X)$  constitutes a complete market, see e.g. [17, 28]. Due to the American style option contract, the holder has the right to exercise the option at any  $0 \leq t \leq T$ , yielding a payoff  $f(t, X_t)$ , where  $f$  is a nonnegative continuous function.

If we set  $a_i = r$ ,  $i = 1, \dots, d$ , in (2.1), we obtain the price process  $X$  in the risk neutral measure. We recall that with respect to the risk neutral measure the discounted process  $\tilde{X}(t) := e^{-\int_0^t r(s, X_s)ds} X(t)$  is a martingale and the price  $u(t, X_t)$  of

the American option is given by

$$(2.3) \quad u(t, X_t) = \sup_{\tau \in \mathcal{T}_{t,T}} E[e^{-\int_t^\tau r(s, X_s) ds} f(\tau, X_\tau) | \mathcal{F}_t],$$

where  $\mathcal{T}_{t,T}$  represents the set of stopping times  $\tau$  taking values in  $[t, T]$ .

For any time  $t$ , let us introduce the following differential operator  $A_t$  (we recall that  $a_i = r$ ,  $i = 1, \dots, d$ ):

$$A_t u(t, x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t, x) \frac{\partial^2 u}{\partial x^i \partial x^j}(t, x) + \sum_{i=1}^d b_i(t, x) \frac{\partial u}{\partial x^i}(t, x),$$

where

$$a_{ij}(t, x) = \sum_{k=1}^d x^i x^j \sigma_{ik}(t, x) \sigma_{jk}(t, x), \quad b_i(t, x) = x^i r(t, x).$$

We denote  $X_s^{t,x}$  (or  $X^{t,x}(s)$ ),  $s \geq T$ , the solution of (2.1) starting at the moment  $t$  from  $x$ :  $X_t^{t,x} = x$ .

It is known (see e.g. [5, 17, 31] and references therein) that if  $u(t, x)$  is a regular solution of the following system of partial differential inequalities:

$$(2.4) \quad \begin{aligned} \frac{\partial u}{\partial t} + A_t u - ru &\leq 0, \quad u \geq f, \quad (t, x) \in [0, T) \times \mathbf{R}_+^d, \\ \left( \frac{\partial u}{\partial t} + A_t u - ru \right) (f - u) &= 0, \quad (t, x) \in [0, T) \times \mathbf{R}_+^d, \\ u(T, x) &= f(T, x), \quad x \in \mathbf{R}_+^d, \end{aligned}$$

then

$$(2.5) \quad u(t, x) = \sup_{\tau \in \mathcal{T}_{t,T}} E[e^{-\int_t^\tau r(s, X_s^{t,x}) ds} f(\tau, X_\tau^{t,x})], \quad (t, x) \in [0, T) \times \mathbf{R}_+^d,$$

i.e., the solution of (2.4) is the price of the American option.

Consider the continuation region  $\mathcal{C}$ , the exercise (stopping) region  $\mathcal{E}$ , and the exercise boundary (critical price surface)  $\gamma$ :

$$\begin{aligned} \mathcal{C} &= \{(t, x) \in [0, T) \times \mathbf{R}_+^d \mid u(t, x) > f(t, x)\}, \\ \mathcal{E} &= \{(t, x) \in [0, T) \times \mathbf{R}_+^d \mid u(t, x) = f(t, x)\}, \\ \gamma &= \partial \mathcal{C} \cap \partial \mathcal{E}. \end{aligned}$$

Introduce the function

$$(2.6) \quad c(t, x) = \begin{cases} 0 & \text{if } (t, x) \in \mathcal{C}, \\ -\left( \frac{\partial f}{\partial t} + A_t f - rf \right) & \text{if } (t, x) \in \mathcal{E}. \end{cases}$$

It is clear from (2.4) that  $\frac{\partial u}{\partial t} + A_t u - ru = 0$  in  $\mathcal{C}$ ,  $u = f$  in  $\mathcal{E}$ , and  $c \geq 0$ . Consequently, the function  $u(t, x)$  is the solution of the following Cauchy problem

for the equation of parabolic type

$$(2.7) \quad \begin{aligned} \frac{\partial u}{\partial t} + A_t u - ru + c &= 0, \quad (t, x) \in [0, T) \times \mathbf{R}_+^d, \\ u(T, x) &= f(T, x), \quad x \in \mathbf{R}_+^d. \end{aligned}$$

This means that the considered American option is equivalent to the European option with the consumption process  $C$  which is defined by the consumption rate  $c(t, X(t))$ ,  $0 \leq t \leq T$ . Thus, the price  $u(t, x)$  can be represented as

$$(2.8) \quad u(t, x) = E[f(T, X^{t,x}(T))Y^{t,x,1}(T) + Z^{t,x,1,0}(T)],$$

where  $X$  satisfies (2.1) and the scalars  $Y$  and  $Z$  satisfy the equations

$$(2.9) \quad \begin{aligned} dY &= -r(s, X)Yds, \quad Y(t) = 1, \\ dZ &= c(s, X)Yds, \quad Z(t) = 0. \end{aligned}$$

### 3. Construction of upper bounds using consumption processes

We start with a lower bound  $v(t, x)$  for the true option price  $u(t, x)$ . Due to (2.5), for any  $\tau \in \mathcal{T}_{t,T}$  the function

$$(3.1) \quad v(t, x) = E[e^{-\int_t^\tau r(s, X_s^{t,x})ds} f(\tau, X_\tau^{t,x})], \quad (t, x) \in [0, T) \times \mathbf{R}_+^d,$$

is a lower bound. In particular, if we take  $\tau$  to be equal zero, then  $v(t, x) = f(t, x)$ , and if  $\tau = T$ ,  $v(t, x)$  is the price of the corresponding European option (without consumption). We can also define  $\tau$  with the help of any surface in  $[0, T) \times \mathbf{R}_+^d$  as the first stopping time when the process  $X_s^{t,x}$  reaches this surface. In all these cases the lower bound can be effectively evaluated using Monte Carlo simulation. Further, if  $v_1(t, x), \dots, v_k(t, x)$  are some lower bounds, then  $v(t, x) = \max_{1 \leq i \leq k} v_i(t, x)$  is also a lower bound. Henceforth we consider lower bounds satisfying the inequality  $v(t, x) \geq f(t, x)$ . Consider the following important example of lower bound. Introduce the grid

$$(3.2) \quad t = \theta_0 < \theta_1 < \dots < \theta_l \leq T$$

and the time

$$\begin{aligned} \theta(t, x) &= \inf \left\{ \theta_m \geq 0 : E[e^{-\int_t^{\theta_m} r(s, X_s^{t,x})ds} f(\theta_m, X_{\theta_m}^{t,x})] \right. \\ &\quad \left. = \max_{0 \leq i \leq l} E[e^{-\int_t^{\theta_i} r(s, X_s^{t,x})ds} f(\theta_i, X_{\theta_i}^{t,x})] \right\}. \end{aligned}$$

Clearly, the following formula gives us a lower bound

$$v(t, x) = E[e^{-\int_t^{\theta(t,x)} r(s, X_s^{t,x})ds} f(\theta(t, x), X_{\theta(t,x)}^{t,x})] \leq u(t, x).$$

For any lower bound  $v(t, x)$  we introduce the sets

$$\begin{aligned} \mathcal{C}_v &= \{(t, x) \in [0, T) \times \mathbf{R}_+^d \mid v(t, x) > f(t, x)\}, \\ \mathcal{E}_v &= \{(t, x) \in [0, T) \times \mathbf{R}_+^d \mid v(t, x) = f(t, x)\}. \end{aligned}$$

Clearly  $\mathcal{C}_v \subset \mathcal{C}$ ,  $\mathcal{E}_v \supset \mathcal{E}$ . Also consider also the set

$$\mathcal{D} = \{(t, x) \in [0, T) \times \mathbf{R}_+^d \mid -\left(\frac{\partial f}{\partial t} + A_t f - r f\right) \geq 0\}.$$

Due to (2.4), we have  $\mathcal{D} \supset \mathcal{E}$  and therefore  $\mathcal{E} \subset \mathcal{D} \cap \mathcal{E}_v$ . Introduce the function  $c_v(t, x)$ :

$$(3.3) \quad c_v(t, x) := -\left(\frac{\partial f}{\partial t} + A_t f - r f\right) \chi_{\mathcal{D} \cap \mathcal{E}_v}(t, x),$$

where  $\chi_{\mathcal{D} \cap \mathcal{E}_v}$  is the characteristic function of the set  $\mathcal{D} \cap \mathcal{E}_v$ . Since  $\mathcal{E} \subset \mathcal{D} \cap \mathcal{E}_v$ , we have  $c_v(t, x) \geq c(t, x)$ . Hence the price  $V(t, x)$  of the European option with the consumption, defined by  $c_v(t, x)$  instead of  $c(t, x)$ , exceeds  $u(t, x)$ , i.e.,  $V(t, x)$  is an upper bound for the true option price  $u(t, x)$ . Thus, having begun with a lower bound, we construct an upper bound. The upper bound  $V(t, x)$  is equal to

$$(3.4) \quad V(t, x) = E[f(T, X^{t,x}(T))Y^{t,x,1}(T) + Z^{t,x,1,0}(T)],$$

where  $X$  satisfies (see (2.1))

$$(3.5) \quad dX^i = X^i(r(s, X)ds + \sum_{j=1}^d \sigma_{ij}(s, X)dW^j(s)), \quad X^i(t) = x^i, \quad i = 1, \dots, d,$$

and the scalars  $Y$  and  $Z$  satisfy the equations

$$(3.6) \quad \begin{aligned} dY &= -r(s, X)Yds, \quad Y(t) = 1, \\ dZ &= c_v(s, X)Yds, \quad Z(t) = 0. \end{aligned}$$

As  $u - v \leq V - v$  and  $V - u \leq V - v$ , the difference  $V - v$  estimates the exactness of both the lower and upper bounds. We note that the more a lower bound  $v(t, x)$  is close to the true option price  $u(t, x)$  the more  $V(t, x)$  is close to  $u(t, x)$ , i.e., if  $v_1 \leq v_2 \leq u$  then  $V_1 \geq V_2 \geq u$ , where the upper bounds  $V_1, V_2$  correspond to the lower bounds  $v_1, v_2$  according to the approach under consideration. In particular, if  $v = u$  then  $V = u$ .

To evaluate  $V(t, x)$ , we simulate some approximate random variables  $\bar{X}^{t,x}(T), \bar{Y}^{t,x,1}(T), \bar{Z}^{t,x,1,0}(T)$  which can be obtained by using weak methods for numerical integration of SDEs [25]. The error of such an approximation is of order  $O(h^p)$ , where  $p$  is the order of weak convergence, depending on the specific method, and  $h$  is a time discretization step. For example, let us use an equidistant time discretization of the time interval  $[t, T]: t = t_0 < t_1 < \dots < t_L = T$  with step size  $h = (T - t)/L$  then the Euler method (its order  $p$  is equal to 1) with simplified simulation of Wiener processes applied to system (3.5), (3.6) gives

$$(3.7) \quad \begin{aligned} \bar{X}(t) &= x, \quad \bar{X}^i(t_{l+1}) = \bar{X}^i(t_l)(1 + r_l h + (\sigma_l \zeta_l)^i \sqrt{h}), \quad i = 1, \dots, d, \\ \bar{Y}(t) &= 1, \quad \bar{Y}(t_{l+1}) = \bar{Y}(t_l) - r_l \bar{Y}(t_l) h, \\ \bar{Z}(t) &= 0, \quad \bar{Z}(t_{l+1}) = \bar{Z}(t_l) + (c_v)_l \bar{Y}(t_l) h, \quad l = 0, \dots, L - 1. \end{aligned}$$



In (3.7)  $r_l$ ,  $\sigma_l$ , and  $(c_v)_l$  are values of the functions  $r$ ,  $\sigma$ ,  $c_v$  at  $(t_l, \bar{X}(t_l))$  and  $\zeta_l = (\zeta_l^1, \dots, \zeta_l^d)^\top$  is a vector of the random variables  $\zeta_l^j$  distributed by the law  $P(\zeta_l^j = \pm 1) = 1/2$  and independent for  $j = 1, \dots, d$ ,  $l = 0, \dots, L - 1$ .

Then, using the Monte Carlo approach, we get

$$(3.8) \quad \begin{aligned} V(t, x) &= E\xi \simeq E\bar{\xi} = E[f(T, \bar{X}^{t,x}(T))\bar{Y}^{t,x,1}(T) + \bar{Z}^{t,x,1,0}(T)] \\ &\simeq \frac{1}{M} \sum_{m=1}^M [f(T, {}_m\bar{X}(T)) {}_m\bar{Y}(T) + {}_m\bar{Z}(T)] := \hat{V}(t, x), \end{aligned}$$

where  $\xi = f(T, X^{t,x}(T))Y^{t,x,1}(T) + Z^{t,x,1,0}(T)$ ,  $\bar{\xi} = f(T, \bar{X}^{t,x}(T))\bar{Y}^{t,x,1}(T) + \bar{Z}^{t,x,1,0}(T)$ , and  ${}_m\bar{X}(t_l) = ({}_m\bar{X}^1(t_l), \dots, {}_m\bar{X}^d(t_l))^\top$ ,  ${}_m\bar{Y}(t_l)$ ,  ${}_m\bar{Z}(t_l)$ ,  $m = 1, \dots, M$ , are independent weak approximate trajectories of system (3.5), (3.6). So, the approximation  $\hat{V}(t, x)$  of  $V(t, x)$  involves two errors: the first one is due to the method of numerical integration (this error is the bias of  $\hat{V}(t, x)$ ) and the second one is a statistical error due to the Monte Carlo method (it is determined by the variance of  $\hat{V}(t, x)$ ). The first error can be reduced by a proper choice of numerical integration scheme and step size  $h$ . It is well known that decreasing the second error, i.e. variance reduction, is of crucial importance for effectiveness of any Monte Carlo procedure.

#### 4. Variance reduction methods in constructing upper bounds

Variance reduction methods can be derived from the generalized probabilistic representation for  $V(t, x)$  (see, e.g., the method of important sampling in [25, 26, 30], the method of control variates in [25, 26], and the combining method in [21, 22, 25]). Let us use the method of control variates. Along with the previous probabilistic representation for  $V(t, x)$  the generalized representation given by the formula (3.4) with  $X$ ,  $Y$ ,  $Z$ , satisfying the system

$$(4.1) \quad \begin{aligned} dX^i &= X^i(r(s, X)ds + \sum_{j=1}^d \sigma_{ij}(s, X)dW^j(s)), \quad X^i(t) = x^i, \quad i = 1, \dots, d, \\ dY &= -r(s, X)Yds, \quad Y(t) = 1, \\ dZ &= c_v(s, X)Yds + F^\top(s, X)YdW(s), \quad Z(t) = 0, \end{aligned}$$

is evidently true as well. In (4.1),  $F(s, x)$  is a column-vector of dimension  $d$  with good analytical properties but arbitrary otherwise. We see that the expectation  $E\xi_F$  in (3.4) does not depend on a choice of  $F$  (we note that  $\xi_F$  is equal to the previous expression for  $\xi$  but now it is calculated due to (4.1)). At the same time,  $Var\xi_F$  does depend on  $F$ . A suitable choice of  $F$  allows us to reduce the variance. It is known (see [21, 22, 25]) that

$$(4.2) \quad Var\xi_F = E \int_t^T (Y^{t,x,1}(s))^2 \sum_{j=1}^d \left( \sum_{i=1}^d X^i \sigma_{ij} \frac{\partial V}{\partial x^i} + F_j \right)^2 ds.$$

Therefore, if

$$(4.3) \quad F_j(s, x) = - \sum_{i=1}^d x^i \sigma_{ij}(s, x) \frac{\partial V}{\partial x^i}(s, x),$$

then  $Var \xi_F = 0$ . Of course, such a vector  $F$  cannot be constructed without knowing the function  $V$ . However, the function  $v$  is known. Suppose that the lower bound  $v$  is not too far from the true price  $u$ , then  $v$  is close to  $V$ . Choosing

$$(4.4) \quad F_j(s, x) = - \sum_{i=1}^d x^i \sigma_{ij}(s, x) \frac{\partial v}{\partial x^i}(s, x),$$

we obtain that  $Var \xi_F$  although is not zero but small. As  $\xi$  is close to  $\bar{\xi}$ , the variance  $Var \bar{\xi}_F$  is small as well. Therefore, if the estimate  $\hat{V}(t, x)$  is computed according to (3.8) with  $\bar{X}$  and  $\bar{Y}$  from (3.7) and  $\bar{Z}$  due to the formula

$$(4.5) \quad \bar{Z}(t) = 0, \quad \bar{Z}(t_{l+1}) = \bar{Z}(t_l) + (c_v)_l \bar{Y}(t_l) h + F^\top(t_l, \bar{X}(t_l)) \bar{Y}(t_l) \zeta_l \sqrt{h}, \quad l = 0, \dots, L-1,$$

with  $F$  from (4.4), then the variance  $Var \bar{\xi}_F$  is small.

The complexity of computing  $\hat{V}(t, x)$  depends on the complexity of computing the lower bound  $v(s, X)$  and (for variance reduction) its derivatives  $\frac{\partial v}{\partial x^i}(s, X)$ . If  $v(s, x)$  is known analytically, then the use of (4.5) is straightforward. Let us describe a way of reducing variance in the most typical situation when  $v(s, x)$  is unknown analytically however it can be evaluated by a Monte Carlo procedure. Let  $\hat{v}(t, x)$  be an estimator for  $v(t, x)$ . For instance, if  $v$  is the price of the underlying European option, then

$$(4.6) \quad \hat{v}(t, x) = \frac{1}{K} \sum_{k=1}^K f(T, {}_k\bar{X}(T)) {}_k\bar{Y}(T),$$

where  ${}_k\bar{X}(T)$ ,  ${}_k\bar{Y}(T)$  are simulated due to (3.7). We stress that for any position  $(t, x)$  the estimator  $\hat{v}(t, x)$  is computed by a procedure which is independent of the procedure for computing  $\hat{V}(t, x)$ .

There are many methods of evaluating the derivatives  $\frac{\partial v}{\partial x^i}(s, X)$  (see, e.g. [11, 21, 24] and references therein). In [24] a very simple method is justified. It makes use of evaluating only the values of  $v$  to evaluate deltas. This method rests on the finite difference formula

$$(4.7) \quad \frac{\partial v}{\partial x^i} = \frac{v(t, x^1, \dots, x^i + \Delta x^i, \dots, x^d) - v(t, x^1, \dots, x^i - \Delta x^i, \dots, x^d)}{2\Delta x^i} + O\left((\Delta x^i)^2\right).$$

Of course, in (4.7) we are forced to use the approximate values  $\hat{v}(t, x^1, \dots, x^i \pm \Delta x^i, \dots, x^d)$  instead of  $v(t, x^1, \dots, x^i \pm \Delta x^i, \dots, x^d)$ . Usually two errors appear in evaluating  $v$ : the error of numerical integration, say  $O(h^p)$ , and the statistical error

of the Monte Carlo method, say  $O(1/\sqrt{K})$ , i.e.,

$$v \simeq \hat{v} + O(h^p) + O(1/\sqrt{K}).$$

Therefore, the error  $R$  of the approximation

$$\frac{\partial v}{\partial x^i} \simeq \frac{\hat{v}(t, x^1, \dots, x^i + \Delta x^i, \dots, x^d) - \hat{v}(t, x^1, \dots, x^i - \Delta x^i, \dots, x^d)}{2\Delta x^i}$$

is evaluated, in general, by

$$R \sim O\left((\Delta x^i)^2\right) + O\left(\frac{h^p}{\Delta x^i}\right) + O\left(\frac{1}{\Delta x^i \sqrt{K}}\right).$$

Due to the presence of small  $\Delta x^i$  in the denominators, the difference approach seems to be not admissible. Fortunately, the more accurate arguments and the employment of the dependent realizations in simulation of  $\hat{v}(t, x^1, \dots, x^i + \Delta x^i, \dots, x^d)$  and  $\hat{v}(t, x^1, \dots, x^i - \Delta x^i, \dots, x^d)$  rehabilitate the difference approach. In [24] it is proved that the error of numerical integration by the weak Euler method ( $p = 1$ ) contributes to the total error of evaluation of the derivatives not  $O(h/\Delta x^i)$  but only  $O(h) + O(h^2/\Delta x^i)$ . Further, it is proved that the method of dependent realizations, which is close to using common random numbers for Monte Carlo estimators (see [6, 11, 24]), contributes just  $O(1/\sqrt{K})$  to the total error. Thus,

$$R \sim O\left((\Delta x^i)^2\right) + O(h) + O\left(\frac{h^2}{\Delta x^i}\right) + O\left(\frac{1}{\sqrt{K}}\right).$$

If we put  $\Delta x^i = \alpha_i h^{1/2}$ ,  $\alpha_i > 0$ , then

$$R \sim O(h) + O(1/\sqrt{K}).$$

Hence we get the same convergence rate in evaluating derivatives of a function as in evaluating the function itself.

Briefly, the method of dependent realizations consists in the following. For getting the estimator  $\hat{\partial}^i v(t, x)$  for  $\partial v(t, x)/\partial x^i$ ,  $K$  pairs of approximate trajectories are simulated, each pair consists of a trajectory starting from  $x + \alpha_i h^{1/2} e_i := (x^1, \dots, x^i + \alpha_i h^{1/2}, \dots, x^d)$  and a trajectory starting from  $x - \alpha_i h^{1/2} e_i := (x^1, \dots, x^i - \alpha_i h^{1/2}, \dots, x^d)$  at the moment  $t$ . The pairs are independent but the two trajectories of the same pair are dependent: they correspond to the same realization (as a rule, in the weak sense) of the Wiener process. For the above example of the European option (see (4.6)), the estimator  $\hat{\partial}^i v(t, x)$  looks as

$$\begin{aligned} \hat{\partial}^i v(t, x) &= \frac{1}{2\alpha_i h^{1/2}} \frac{1}{K} \sum_{k=1}^K [f(T, {}_k\bar{X}^{t, x + \alpha_i h^{1/2} e_i}(T)) {}_k\bar{Y}^{t, x + \alpha_i h^{1/2} e_i, 1}(T) \\ &\quad - f(T, {}_k\bar{X}^{t, x - \alpha_i h^{1/2} e_i}(T)) {}_k\bar{Y}^{t, x - \alpha_i h^{1/2} e_i, 1}(T)]. \end{aligned}$$

As the final result, the estimator  $\hat{V}(t, x)$  with reduced variance is computed by the formula

$$(4.8) \quad \hat{V}(t, x) = \frac{1}{M} \sum_{m=1}^M [f(T, {}_m\bar{X}(T)) {}_m\bar{Y}(T) + {}_m\bar{Z}(T)],$$

where  ${}_m\bar{X}(T)$ ,  ${}_m\bar{Y}(T)$  are simulated due to (3.7) and  ${}_m\bar{Z}(T)$  due to

$$(4.9) \quad \bar{Z}(t) = 0, \quad \bar{Z}(t_{l+1}) = \bar{Z}(t_l) + (c_v)_l \bar{Y}(t_l) h - \sum_{i=1}^d \bar{X}^i(t_l) \sigma_{ij}(t_l, \bar{X}(t_l)) \hat{\partial}^i v(t, x).$$

## 5. Discrete-time case

**5.1. American options.** In this section we consider a discrete-time financial model. Let

$$(B_n, X_n) = (B_n, X_n^1, \dots, X_n^d), \quad n = 0, 1, \dots, N,$$

be the vector of prices at time  $n$ , where  $B_n$  is the price of a scalar riskless asset (we assume that  $B_n$  is deterministic and  $B_0 = 1$ ) and  $X_n = (X_n^1, \dots, X_n^d)$  is the price vector process of risky assets. Let  $f_n(x)$  be the profit made by exercising an American option at time  $n$  if  $X_n = x$ . We assume that the modelling is based on the filtered space  $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{0 \leq n \leq N}, P)$ , where the probability  $P$  is the risk-neutral pricing probability for the problem, and that  $X_n$  is a Markov chain with respect to the filtration  $(\mathcal{F}_n)_{0 \leq n \leq N}$ . The discounted process  $\tilde{X}_n := X_n/B_n$  is a martingale with respect to the probability  $P$  and the price  $u_n(X_n)$  of the American option is given by

$$(5.1) \quad u_n(X_n) = \sup_{\nu \in \mathcal{T}_{n,N}} B_n E \left( \frac{f_\nu(X_\nu)}{B_\nu} \mathcal{F}_n \right),$$

where  $\mathcal{T}_{n,N}$  is the set of stopping times  $\nu$  taking values in  $\{n, n+1, \dots, N\}$ .

The value process  $u_n$  can be determined by induction as follows (Snell envelope):

$$(5.2) \quad u_N(x) = f_N(x),$$

$$u_n(x) = \max \left[ f_n(x), B_n E \left( \frac{u_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) \right], \quad n = N-1, \dots, 0.$$

We see that in theory the problem of evaluating  $u_0(x)$ , the price of the discrete-time American option, is easily solved using iteration procedure (5.2). However, if  $X$  is high dimensional, the iteration procedure is not practical.

For  $0 \leq n \leq N-1$  the equation (5.2) can be rewritten in the form

$$(5.3) \quad u_n(x) = B_n E \left( \frac{u_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) + \left[ f_n(x) - B_n E \left( \frac{u_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) \right]^+.$$

Introduce the functions

$$(5.4) \quad \gamma_n(x) = \left[ f_n(x) - B_n E \left( \frac{u_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) \right]^+, \quad n = N-1, \dots, 0.$$

Due to (5.3), we have

$$\begin{aligned} u_{N-1}(X_{N-1}) &= B_{N-1} E \left( \frac{f_N(X_N)}{B_N} \middle| \mathcal{F}_{N-1} \right) + \gamma_{N-1}(X_{N-1}), \\ u_{N-2}(X_{N-2}) &= B_{N-2} E \left( \frac{u_{N-1}(X_{N-1})}{B_{N-1}} \middle| \mathcal{F}_{N-2} \right) + \gamma_{N-2}(X_{N-2}) \\ &= B_{N-2} E \left( \frac{f_N(X_N)}{B_N} \middle| \mathcal{F}_{N-2} \right) + B_{N-2} E \left( \frac{\gamma_{N-1}(X_{N-1})}{B_{N-1}} \middle| \mathcal{F}_{N-2} \right) + \gamma_{N-2}(X_{N-2}). \end{aligned}$$

Continuing in the same way, we get

$$(5.5) \quad u_n(X_n) = B_n E \left( \frac{f_N(X_N)}{B_N} \middle| \mathcal{F}_n \right) + B_n \sum_{k=1}^{N-(n+1)} E \left( \frac{\gamma_{N-k}(X_{N-k})}{B_{N-k}} \middle| \mathcal{F}_n \right) + \gamma_n(X_n), \quad n = 0, \dots, N-1.$$

Putting  $X_0 = x$  and recalling that  $B_0 = 1$ , we obtain

$$(5.6) \quad u_0(x) = E \left( \frac{f_N(X_N)}{B_N} \right) + \gamma_0(x) + \sum_{n=1}^{N-1} E \left( \frac{\gamma_n(X_n)}{B_n} \right).$$

The formula (5.6) gives the value of the European option with the payoff function  $f_N(x)$  and with the consumption process  $\gamma_n$  defined by (5.4). For the considered American option, the exercise (stopping) set  $\mathcal{E}$  can be determined as (see (5.2)):

$$(5.7) \quad \mathcal{E} = \left\{ (n, x) : f_n(x) > B_n E \left( \frac{u_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) \right\},$$

i.e., similar to the continuous case, the consumption process does not vanish in stopping set only (see (5.3)).

The obtained result on the equivalence of the discrete-time American option to the European option with the consumption process cannot be used directly because  $u_n(x)$  and, consequently,  $\gamma_n(x)$  are unknown. We take advantage of the discovered connection in the following way.

Let  $v_n(x)$  be a lower bound for the true option price  $u_n(x)$ . We introduce the functions

$$(5.8) \quad \gamma_{n,v}(x) = \left[ f_n(x) - B_n E \left( \frac{v_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) \right]^+, \quad n = 0, \dots, N-1.$$

Clearly,

$$\gamma_{n,v}(x) \geq \gamma_n(x).$$

Hence the price  $V_n(x)$  of the European option with the payoff function  $f_N(x)$  and with the consumption process  $\gamma_{n,v}(x)$  is an upper bound:  $V_n(x) \geq u_n(x)$ .

Conversely, if  $V_n(x)$  is an upper bound for the true option price  $u_n(x)$  and

$$(5.9) \quad \gamma_{n,V}(x) = \left[ f_n(x) - B_n E \left( \frac{V_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) \right]^+, \quad n = 0, \dots, N-1,$$

then the price  $v_n(x)$  of the European option with the consumption process  $\gamma_{n,V}(x)$  is a lower bound.

Thus, starting from a lower bound  $v_n^1(x)$ , one can construct the upper bound  $V_n^1(x)$  as the European option with the corresponding consumption process  $\gamma_{n,v^1}(x)$  (we do not require that  $v_n^1(x)$  itself is equipped with any consumption process). Then it is possible to construct the lower bound  $v_n^2(x)$  with the consumption process  $\gamma_{n,V^1}(x)$ . Consider the lower bound  $\tilde{v}_n^2(x) = \max\{v_n^1(x), v_n^2(x)\} \geq v_n^1(x)$  (we do not equip  $\tilde{v}_n^2(x)$  with any consumption process) and construct the upper bound  $V_n^2(x)$  as the European option with the consumption process  $\gamma_{n,\tilde{v}^2}(x)$ . Clearly,  $V_n^2(x) \leq V_n^1(x)$ . Then we construct the lower bound  $v_n^3(x)$  with the consumption process  $\gamma_{n,V^2}(x)$  (we need not in  $\tilde{v}_n^3(x)$  since  $v_n^3(x) \geq v_n^2(x)$ ), and so on. This procedure gives us the sequences

$$(5.10) \quad \begin{aligned} v_n^1(x) &\leq \tilde{v}_n^2(x) \leq v_n^3(x) \leq \dots \leq u_n(x), \\ V_n^1(x) &\geq V_n^2(x) \geq \dots \geq u_n(x). \end{aligned}$$

However, each further step of the procedure requires labor-consuming calculations and in practice it is possible to realize only a few steps of this procedure. In the capacity of  $v_n^1(x)$  one can propose the European option with payoff function  $f_N(x)$  (this option is equipped with zero consumption process). In this case  $\tilde{v}_n^2(x) = v_n^2(x)$ . Another proposition for  $v_n^1(x)$ :

$$(5.11) \quad v_n^1(x) = \max \left\{ f_n(x), B_n E \left( \frac{f_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) \right\}.$$

This  $v_n^1(x)$  is one of the simplest lower bounds. In addition we note that the quantity

$$B_n E \left( \frac{f_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right)$$

can be very often evaluated exactly.

If a lower bound  $v_n(x)$  is known, the upper bound  $V_n(x)$  can be, in general easily, evaluated by the Monte Carlo simulation:

$$(5.12) \quad V_n(x) \simeq \frac{1}{M} \sum_{m=1}^M \frac{f_N(mX_N^{n,x})}{B_N} + \gamma_{n,v}(x) + \frac{1}{M} \sum_{m=1}^M \sum_{k=n+1}^{N-1} \frac{\gamma_{k,v}(mX_k^{n,x})}{B_k} := \hat{V}_n(x)$$

In (5.12)  $mX_k$ ,  $m = 1, \dots, M$ , are independent trajectories of the Markov chain  $X$ . A lower bound is evaluated analogously if an upper bound is known.

As an important example of the discrete-time financial model, let us consider the Markov chain  $(B_n, X_n) = (B_n, X_n^1, \dots, X_n^d)$ ,  $n = 0, 1, \dots, N$ , generated by the system

$$(5.13) \quad \begin{aligned} B_0 &= 1, \quad B_{n+1} = (1 + r_n h) B_n, \\ X_0 &= x, \quad X_{n+1}^i = X_n^i (1 + r_n h + (\sigma_n(X_n) \zeta_n^i)^2 h^{1/2}), \quad i = 1, \dots, d, \end{aligned}$$

where  $h > 0$  is a sufficiently small constant,  $r_n \geq 0$  are scalars,  $\sigma_n(x)$  are matrices of dimension  $d \times d$ ,  $\zeta_n = (\zeta_n^1, \dots, \zeta_n^d)^\top$  is a vector of two point random variables  $\zeta_n^j$ , distributed by the law  $P(\zeta_n^j = \pm 1) = 1/2$  and independent in  $j = 1, \dots, d$ ,  $n = 0, \dots, N - 1$ . We see, that the system for the Markov chain  $X$  coincides with the discretization scheme (3.7) for  $\bar{X}$ . Let us denote by  ${}_l \zeta = ({}_l \zeta^1, \dots, {}_l \zeta^d)$ ,  $l = 1, \dots, 2^d$ , all the different values of this vector which components take the values  $\pm 1$  and by  ${}_l X(n, x, h)$  the values of  $X_{n+1}$  from (5.13) for  $X_n = x$ ,  $\zeta_n = {}_l \zeta$ . For any function  $g(x)$ , the conditional expectation  $B_n E \left( \frac{g(X_{n+1})}{B_{n+1}} \middle| X_n = x \right)$  can be found exactly:

$$(5.14) \quad B_n E \left( \frac{g(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) = \frac{1}{1 + r_n h} \frac{1}{2^d} \sum_{l=1}^{2^d} g({}_l X(n, x, h)).$$

Therefore for this example, the functions  $\gamma_{n,v}(x)$  from (5.8),  $\gamma_{n,V}(x)$  from (5.9), and  $v_n^1(x)$  from (5.11) can be expressed exactly through  $v_n(x)$ ,  $V_n(x)$ , and  $f_n(x)$ .

Let us compare the consumption rate  $c(t, x)$  given by (2.6) in the continuous-time case and the consumption process  $\gamma_0(x)$  defined by (5.4) in the corresponding discrete-time case in a heuristic manner. For simplicity we consider  $d = 1$ . We set  $t_0 = t$ ,  $X_0 = x$ ,  $n = 0$ . Let  $(t, x) \in \text{Int}\mathcal{E}$ . If  $h$  is sufficiently small, then  $u_0(x) \simeq u(t, x) = f(t, x)$ ,  $u_1(X_1) \simeq u(t + h, X_1) = f(t + h, X_1)$ . Further (see (5.4)),

$$\begin{aligned} \gamma_0(x) &= \left[ f_0(x) - B_0 E \left( \frac{u_1(X_1)}{B_1} \middle| X_0 = x \right) \right]^+ \\ &= \left[ f_0(x) - \frac{1}{1 + r_0 h} E(f(t + h, X_1) \middle| X_0 = x) \right]^+. \end{aligned}$$

Since  $f_0(x) = f(t, x)$ ,  $X_1 = x(1 + rh + \sigma\zeta h^{1/2})$ , we get

$$\begin{aligned} \gamma_0(x) &= \left[ f(t, x) - \frac{1}{1 + rh} \frac{1}{2} (f(t + h, x(1 + rh + \sigma\zeta h^{1/2})) \right. \\ &\quad \left. + f(t + h, x(1 + rh - \sigma\zeta h^{1/2}))) \right]^+ \\ &= - \left( \frac{\partial f}{\partial t} + A_t f - r f \right) h + O(h^2) \simeq c(t, x) h, \end{aligned}$$

i.e., we obtain the expected correspondence (see (2.6)).

**5.2. Bermudan options.** As before we consider the discrete-time model

$$(B_n, X_n) = (B_n, X_n^1, \dots, X_n^d), \quad n = 0, 1, \dots, N.$$

However, now an investor can exercise his right only at time belonging to the set of stopping times  $S = \{i_1, \dots, i_l\}$  within  $\{0, 1, \dots, N\}$  where  $i_l = N$ . The price  $u_n(X_n)$  of the Bermudan option is given by

$$(5.15) \quad u_n(X_n) = \sup_{\nu \in \mathcal{T}_{S \cap [n, N]}} B_n E \left( \frac{f_\nu(X_\nu)}{B_\nu} \middle| \mathcal{F}_n \right),$$

where  $\mathcal{T}_{S \cap [n, N]}$  is the set of stopping times  $\nu$  taking values in  $\{i_1, \dots, i_l\} \cap \{n, n+1, \dots, N\}$ .

The value process  $u_n$  is determined as follows:

$$(5.16) \quad u_N(x) = f_N(x),$$

$$u_n(x) = \begin{cases} \max \left[ f_n(x), B_n E \left( \frac{u_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) \right], & n \in S, \\ B_n \left( \frac{u_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right), & n \notin S. \end{cases}$$

Thus, we obtain that the Bermudan option is equivalent to the European option with the payoff function  $f_N(x)$  and with the consumption process  $\gamma_n$  defined by

$$(5.17) \quad \gamma_n(x) = \begin{cases} \left[ f_n(x) - B_n E \left( \frac{u_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) \right]^+, & n \in S, \\ 0, & n \notin S. \end{cases}$$

From here all the results obtained in this section for discrete-time American options can be carried over to the Bermudan options. For example, if  $v_n(x)$  is a lower bound of the true option price  $u_n(x)$ , the price  $V_n(x)$  of the European option with the payoff function  $f_N(x)$  and with the consumption process

$$(5.18) \quad \gamma_{n,v}(x) = \begin{cases} \left[ f_n(x) - B_n E \left( \frac{v_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) \right]^+, & n \in S, \\ 0, & n \notin S. \end{cases}$$

is an upper bound:  $V_n(x) \geq u_n(x)$ .

## 6. Variance reduction in the discrete-time case

The statistical error of the approximation  $\hat{V}_n(x)$  for  $V_n(x)$  is determined by variance of  $\hat{V}_n(x)$ . To reduce the statistical error, one can use both the method of important sampling and the method of control variates. Let us consider the method of control variates. To this aim we need in a generalized probabilistic representation for  $V_n(x)$ .

Let  $P_n(x, dy)$ ,  $n \geq 1$ , be one-step transition functions of the Markov chain  $X_n$ , i.e.,

$$P(X_n \in dy | X_{n-1}) = P_n(X_{n-1}, dy), \quad n = 1, 2, \dots$$

In the case of a homogeneous Markov chain all the one-step transition functions coincide and equal to  $P(x, dy) = P_1(x, dy) = \dots = P_n(x, dy)$ .



Clearly,  $V_n(x)$  is the solution to the Cauchy problem for the following difference integral equation

$$(6.1) \quad V_N(x) = f_N(x),$$

$$(6.2) \quad V_n(x) = \frac{B_n}{B_{n+1}} \int V_{n+1}(y) P_{n+1}(x, dy) + \gamma_{n,v}(x), \quad n \leq N-1.$$

A probabilistic representation for  $V_n(x)$  is given by the formula (see [23])

$$(6.3) \quad V_n(x) = E\left[\frac{B_n}{B_N} f_N(X_N^{n,x}) + Z_N^{n,x}\right],$$

where the scalar  $Z_{n+k}^{n,x}$ ,  $k = 0, 1, \dots$ , is governed by the equation

$$(6.4) \quad Z_{n+k+1}^{n,x} = Z_{n+k}^{n,x} + \frac{B_n}{B_{n+k}} \gamma_{n+k,v}(X_{n+k}^{n,x}), \quad Z_n^{n,x} = 0.$$

Let  $\tilde{V}_n(x)$  be the solution to the Cauchy problem

$$(6.5) \quad \tilde{V}_N(x) = \tilde{f}(x),$$

$$(6.6) \quad \tilde{V}_n(x) = \frac{B_n}{B_{n+1}} \int \tilde{V}_{n+1}(y) P_{n+1}(x, dy) + \tilde{\gamma}_n(x), \quad n \leq N-1.$$

We have

$$(6.7) \quad \tilde{V}_n(x) = E\left[\frac{B_n}{B_N} \tilde{f}(X_N^{n,x}) + \tilde{Z}_N^{n,x}\right],$$

where the scalar  $\tilde{Z}_{n+k}^{n,x}$ ,  $k = 0, 1, \dots$ , satisfies the equation

$$(6.8) \quad \tilde{Z}_{n+k+1}^{n,x} = \tilde{Z}_{n+k}^{n,x} + \frac{B_n}{B_{n+k}} \tilde{\gamma}_{n+k}(X_{n+k}^{n,x}), \quad \tilde{Z}_n^{n,x} = 0.$$

Let us denote

$$\xi = \frac{B_n}{B_N} f_N(X_N^{n,x}) + Z_N^{n,x}, \quad \tilde{\xi} = \frac{B_n}{B_N} \tilde{f}(X_N^{n,x}) + \tilde{Z}_N^{n,x}.$$

We have

$$(6.9) \quad V_n(x) = \alpha \tilde{V}_n(x) + E(\xi - \alpha \tilde{\xi}),$$

where  $\alpha$  is a constant.

The formula (6.9) gives a generalized probabilistic representation for  $V_n(x)$ . If using (6.9) (and considering  $V_n(x)$  to be known) the Monte Carlo error is determined by  $Var(\xi - \alpha \tilde{\xi})$  instead of  $Var\xi$  if using (6.3). We have

$$Var(\xi - \alpha \tilde{\xi}) = Var\xi + \alpha^2 Var\tilde{\xi} - 2\alpha Cov(\xi, \tilde{\xi}).$$

The optimal  $\alpha$  is

$$\alpha^{opt} = \frac{Cov(\xi, \tilde{\xi})}{Var\tilde{\xi}}$$

and

$$Var(\xi - \alpha^{opt} \tilde{\xi}) = Var\xi - \frac{Cov^2(\xi, \tilde{\xi})}{Var\tilde{\xi}} \leq Var\xi.$$

In practice, the value  $\alpha^{opt}$  can be evaluated during a numerical experiment.

If  $\alpha = 1$ , then

$$Var(\xi - \tilde{\xi}) = Var\xi + Var\tilde{\xi} - 2Cov(\xi, \tilde{\xi})$$

and method (6.9) is effective if the covariance between  $\xi$  and  $\tilde{\xi}$  is large.

Let a lower bound  $v_n(x)$ , which is the price of a European option with the consumption process  $\tilde{\gamma}_n(x)$ , be taken in the capacity of  $\tilde{V}_n$ , i.e.,  $v_n(x)$  satisfies the Cauchy problem

$$(6.10) \quad v_n(x) = \frac{B_n}{B_{n+1}} \int v_{n+1}(y) P_{n+1}(x, dy) + \tilde{\gamma}_n(x), \quad n \leq N-1,$$

$$(6.11) \quad v_N(x) = f_N(x).$$

Hence

$$v_n(x) = E\left[\frac{B_n}{B_N} f_N(X_N^{n,x}) + \tilde{Z}_N^{n,x}\right]$$

with  $\tilde{Z}_{n+k}^{n,x}$ ,  $k = 0, 1, \dots$ , satisfying the equation (6.8), and

$$V_n(x) = v_n(x) + E\zeta,$$

where

$$\zeta = Z_N^{n,x} - \tilde{Z}_N^{n,x}.$$

Now the Monte Carlo estimator  $\hat{V}_n(x)$  for  $V_n(x)$  has the form

$$(6.12) \quad \hat{V}_n(x) = v_n(x) + \frac{1}{M} \sum_{m=1}^M ({}_m Z_N^{n,x} - {}_m \tilde{Z}_N^{n,x}).$$

The bias of this estimator is equal to zero and the variance is equal to

$$(6.13) \quad Var\hat{V}_n(x) = Var\zeta \simeq \frac{1}{M} \sum_{m=1}^M ({}_m Z_N^{n,x} - {}_m \tilde{Z}_N^{n,x})^2 - \left[ \frac{1}{M} \sum_{m=1}^M ({}_m Z_N^{n,x} - {}_m \tilde{Z}_N^{n,x}) \right]^2.$$

Clearly, if  $v_n(x)$  is close to  $u_n(x)$ , the estimator (6.12) is much better than the direct estimator

$$(6.14) \quad \hat{V}_n(x) = \frac{1}{M} \sum_{m=1}^M \left[ \frac{B_n}{B_N} f_N({}_m X_N^{n,x}) + {}_m Z_N^{n,x} \right].$$

Analogously, we can reduce the variance when we construct a lower bound using an upper bound. We emphasize that the proposed variance reduction method requires both lower and upper bounds equipped with the corresponding consumption processes. If  $v_n^1$  in the procedure (5.10) is equipped with the corresponding consumption process, then  $\tilde{v}_n^2 = v_n^2$  and all the bounds  $V_n^1$ ,  $v_n^2$ ,  $V_n^2$  and so on can be constructed using the proposed variance reduction method. If  $v_n^1$  is not equipped with a consumption process, then to reduce variance in constructing  $V_n^1$  we need another approach. But for further reducing one can again use the proposed method. For example, for the variance reduction in constructing  $V_n^2$ , we can use  $v_n^2$  or  $V_n^1$  which are equipped with the corresponding consumption processes (but, in general, not  $\tilde{v}_n^2$ ). Let us return to reducing variance in constructing  $V_n^1$  if  $v_n^1$  is not equipped

with a consumption process. Evidently,  $v_n(x) = v_n^1(x)$  satisfies the Cauchy problem (6.10)-(6.11) with

$$(6.15) \quad \tilde{\gamma}_n(x) = v_n^1(x) - B_n E \left( \frac{v_{n+1}^1(X_{n+1})}{B_{n+1}} \middle| X_n = x \right).$$

We note that  $\tilde{\gamma}_n(x)$  from (6.15) can be negative. Nevertheless, the estimator of the form (6.12) can again be better than the direct estimator (6.14).

For evaluating  $\gamma$  due to formula (5.8) ((5.9)), we need at any step  $n$  to compute the conditional expectation of the form  $E \left( \frac{v_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right)$  ( $E \left( \frac{V_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right)$ ). It can be done either exactly (see, for example, formula (5.14)) or by simulation:

$$(6.16) \quad E \left( \frac{v_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) \simeq \frac{1}{K} \sum_{k=1}^K \frac{v_{n+1}({}_k X_{n+1}^{n,x})}{B_{n+1}},$$

where  ${}_k X_{n+1}^{n,x}$ ,  $k = 1, \dots, K$ , are independent realizations of the state of the Markov chain  $X$  at the moment  $n + 1$  starting from  $x$  at the step  $n$ . Thus, unlike the continuous case, where the consumption defined by  $c_v$ , is computed explicitly, we have to compute  $\gamma$  by simulation in the discrete-time case. Fortunately, the computation (6.16) is rather inexpensive because of simulating the Markov chain at a single step only, i.e., computing  $\gamma$  is ‘almost explicit’.

## 7. Numerical examples

**7.1. An American put on a single asset.** Let us consider an American put on a single log-Brownian asset, which price is given by

$$X_t = X_0 \exp(\sigma W_t + (r - \sigma^2/2)t),$$

with  $r$  denoting as usual the riskless rate of interest, assumed constant, and  $\sigma$  denoting the constant volatility, and which payoff function  $f(t, x) = (K - x)^+$ . No closed-form solution for the price is known, but there are various numerical methods which give accurate approximations to the price.

The aim of this subsection is to investigate the performance of continuous consumption process method (abbreviated by CCP) in this setup. The results of simulation for the case of the initial lower bound approximation (1.1) are reported in Table 7.1. The parameters values are  $K = 100$ ,  $\sigma = 0.4$ ,  $r = 0.06$  and  $T = 0.5$ , with  $X_0$  varying as shown in the first column of the table. The true values of the American option are quoted from the article [1]. The fifth column gives values of the upper bound estimated as (see (3.8))

$$(7.1) \quad \hat{V}(0, X_0) = \frac{1}{M} \sum_{m=1}^M V^{(m)}(0, X_0),$$

TABLE 7.1. Upper bound for the standard one-dimensional American put with parameters  $K = 100$ ,  $r = 0.06$ ,  $T = 0.5$ , and  $\sigma = 0.4$  obtained by CCP and CCP&VR methods. Values for different numbers of Monte Carlo simulations and different initial stocks are presented.

$X_0$	Lower Bound (1.1)	American Option (True Value)	M	Upper Bound (CCP)	Upper Bound (CCP&VR)
80	20.6893	21.6059	$10^2$	$24.9425 \pm 3.6261$	$22.1791 \pm 0.2430$
			$10^4$	$22.3643 \pm 0.3500$	$22.2012 \pm 0.0234$
			$10^6$	$22.1862 \pm 0.0347$	$22.2120 \pm 0.0024$
90	14.4085	14.9187	$10^2$	$18.1221 \pm 3.2926$	$15.1391 \pm 0.2630$
			$10^4$	$15.4796 \pm 0.3176$	$15.2712 \pm 0.0277$
			$10^6$	$15.2724 \pm 0.0314$	$15.2932 \pm 0.0028$
100	9.6642	9.9458	$10^2$	$12.6247 \pm 2.8720$	$9.9633 \pm 0.2818$
			$10^4$	$10.3200 \pm 0.2742$	$10.1067 \pm 0.0298$
			$10^6$	$10.1123 \pm 0.0271$	$10.1240 \pm 0.0030$
110	6.2797	6.4352	$10^2$	$8.4518 \pm 2.4013$	$6.3575 \pm 0.3162$
			$10^4$	$6.6991 \pm 0.2266$	$6.4909 \pm 0.0311$
			$10^6$	$6.5051 \pm 0.0223$	$6.5119 \pm 0.0031$
120	3.9759	4.0611	$10^2$	$5.6270 \pm 1.8924$	$3.9788 \pm 0.2959$
			$10^4$	$4.2647 \pm 0.1813$	$4.0889 \pm 0.0298$
			$10^6$	$4.1094 \pm 0.0178$	$4.1133 \pm 0.0029$

where

$$(7.2) \quad V^{(m)}(0, X_0) = f(T, {}_m\bar{X}(T)) {}_m\bar{Y}(T) + {}_m\bar{Z}(T)$$

with  $\bar{X}$ ,  $\bar{Y}$ ,  $\bar{Z}$  being defined in (3.7) and  $h$  being equal to 0.01. All the values are given together with their 5% confidence intervals:

$$(7.3) \quad R_{MC} = \frac{2}{\sqrt{M}} \left[ \frac{1}{M} \sum_{m=1}^M (V^{(m)}(0, X_0))^2 - \left( \frac{1}{M} \sum_{m=1}^M V^{(m)}(0, X_0) \right)^2 \right]^{1/2}.$$

The sixth column in turn shows results for the same set of parameters and the same initial lower bound when in addition the variance reduction (VR) technique is employed. Since the European option and its derivative can be computed analytically, the same holds for the function  $F$  in (4.4). More precisely,  $F$  is given in this case by

$$(7.4) \quad F(s, x) = \begin{cases} -x\sigma \frac{\partial u_{eu}}{\partial x}(s, x), & u_{eu}(s, x) \geq (K - x)^+, \\ x\sigma \cdot \chi_{\{x < K\}}, & u_{eu}(s, x) < (K - x)^+. \end{cases}$$

Hence, variance reduction can be done in this case efficiently by using the weak integration scheme (4.5) for  $\bar{Z}(T)$  in (7.2).

**7.2. Upper bounds depending on choice of lower ones.** It is known that the price  $u^*(x)$  of the American put in the case of infinite time horizon (see [17]) is equal to

$$u^*(x) = \begin{cases} K - x, & x \leq x^*, \\ (K - x^*) \left(\frac{x}{x^*}\right)^{-\gamma}, & x > x^*, \end{cases}$$

where  $\gamma = 2r/\sigma^2$ ,  $x^* = K\gamma/(1 + \gamma)$ .

Let us return to the American put with a finite time horizon. Clearly,  $u^*(x)$  is an upper bound and

$$\begin{aligned} \mathcal{E}_{u^*} &= \{(t, x) \in [0, T) \times \mathbf{R}_+^d \mid u^*(x) = f(x)\} \\ &= \{(t, x) \in [0, T) \times \mathbf{R}_+^d \mid x \leq x^*\} \end{aligned}$$

belongs to  $\mathcal{E}$ :  $\mathcal{E}_{u^*} \subset \mathcal{E}$  (recall that  $f(x) = (K - x)^+$ ). Besides  $\mathcal{E}_{u^*} \subset \mathcal{D}$ . Therefore

$$c_{u^*}(t, x) := - \left( \frac{\partial f}{\partial t} + A_t f - r f \right) \chi_{\mathcal{E}_{u^*}}(t, x)$$

is less than  $c(t, x)$ :  $c_{u^*}(t, x) \leq c(t, x)$  and the price  $u_*(t, x)$  of the European option with the consumption  $c_{u^*}(t, x)$  is a lower bound. The lower bound

$$v_*(t, x) = \max\{f(x), u_*(t, x)\}$$

is larger than that given by (1.1) and, consequently, it is more preferable for the considered example.

In Table 7.2 we compute values of  $v_*(t, x)$  at the point  $(0, X_0)$  for the standard one-dimensional American put with parameters  $K = 100$ ,  $r = 0.06$ ,  $T = 5$ , and  $\sigma = 0.2$  using CCP&VR approach. For the purpose of variance reduction, one possible way would be to define in (4.4)

$$F(s, x) = -x\sigma \frac{du^*}{dx}(x).$$

But numerical experiments suggest using rather the lower bound (1.1) as an approximation for  $v_*$ . So we simulate  $M$  trajectories using the system (4.1) with  $F$  given by (7.4) and the size of the integration step  $h$  being equal to 0.1. Note that for  $X_0 = 80$  we have  $f(X_0) > u_*(X_0)$  and hence  $v_*(0, X_0) = f(X_0) = 20$ .

Now we are able to obtain the upper bound using the lower bound  $v_*(t, x)$  constructed in the way described above. In Table 7.3 we present values of this bound (third column) for the standard one-dimensional American put (the parameters are the same as before).  $M_1 = 1000$  paths were used to estimate the lower bound  $v_*$  and  $M_2 = 1000$  paths for estimating the upper bound itself. Improvements are clearly observable.

**7.3. Bermudan put on a single asset.** In this section we turn to Bermudan put on a single asset in the Black-Scholes framework and, consequently, we will use discrete consumption process approach (DCP) described in Section 5.

TABLE 7.2. Lower bounds for the standard one-dimensional American put with parameters  $K = 100$ ,  $r = 0.06$ ,  $T = 5$ , and  $\sigma = 0.2$ . The first lower bound is given by (1.1) while the second one is obtained by CCP&VR method using the upper approximation  $u^*$ . Values for different initial stocks and different numbers  $M$  of simulated paths are presented.

$X_0$	Lower Bound $v(0, X_0)$	M	Lower Bound $v_*(0, X_0)$
80	20	$10^2$	20
		$10^4$	20
		$10^6$	20
90	10	$10^2$	$13.1654 \pm 1.3341$
		$10^4$	$12.0265 \pm 0.1406$
		$10^6$	$11.9013 \pm 0.0140$
100	5.6968	$10^2$	$7.6077 \pm 1.0398$
		$10^4$	$8.0171 \pm 0.1045$
		$10^6$	$7.9671 \pm 0.0106$
110	4.11214	$10^2$	$5.1080 \pm 0.8355$
		$10^4$	$5.3793 \pm 0.0867$
		$10^6$	$5.4280 \pm 0.0085$
120	2.96985	$10^2$	$3.9085 \pm 0.6127$
		$10^4$	$3.6431 \pm 0.0709$
		$10^6$	$3.6578 \pm 0.0070$

TABLE 7.3. Upper bounds for the standard one-dimensional American put with parameters  $K = 100$ ,  $r = 0.06$ ,  $T = 5$ , and  $\sigma = 0.2$  obtained by CCP method with different initial lower approximations.

$X_0$	Upper Bound ((1.1) based)	Upper Bound ( $v_*$ based)
80	$26.8186 \pm 0.1292$	$24.1685 \pm 0.3304$
90	$19.5623 \pm 0.1202$	$15.9458 \pm 0.3532$
100	$12.7508 \pm 0.1053$	$10.7251 \pm 0.2581$
110	$8.7377 \pm 0.0884$	$7.1022 \pm 0.1974$
120	$6.0040 \pm 0.0727$	$5.0116 \pm 0.1528$

We assume throughout this section that the points  $\theta_0 < \dots < \theta_l$  corresponding to exercise periods of the option considered are equidistant,  $\delta = \theta_i - \theta_{i-1}$  and  $\theta_0 = 0$ ,  $\theta_l = T$ . Further, in this one can approximate case the underlying asset process by a Markov chain  $(B_n, X_n)$  generated by the system (5.13), where  $h = \delta/L$  for some natural number  $L$  and  $N = Ll$ . Now the consumption process corresponding

to the Bermudan option is given by (see Section 5.1)

$$(7.5) \quad \gamma_n(x) = \begin{cases} \left[ f_n(x) - B_n E \left( \frac{u_{n+1}(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) \right]^+, & n \in \{0, L, 2L, \dots, (l-1)L\}, \\ 0, & n \notin \{0, L, 2L, \dots, (l-1)L\}, \end{cases}$$

where  $u_{n+1}(X_{n+1})$  is the option price of the option at the time  $\theta_k + h$  if  $n = kL$ . Let us consider the Black-Scholes model with parameters  $K = 100$ ,  $r = 0.06$ ,  $T = 5$ ,  $\sigma = 0.4$  and the Bermudan option with  $L = 5$  and  $l = 10$ , i.e.  $\delta = 0.5$ ,  $h = 0.1$ . In the third column of Table 7.4 we present values of the corresponding upper bound at the point  $(0, X_0)$  estimated as (see (6.12))

$$(7.6) \quad \hat{V}_n^1(X_n) = v_n^1(X_n) + \frac{1}{M_1} \sum_{m=1}^{M_1} m \Delta_N^{n, X_n},$$

where

$${}_m \Delta_N^{n, X_n} = ({}_m Z_N^{n, X_n} - \tilde{Z}_N^{n, X_n})$$

with  $Z$  and  $\tilde{Z}$  defined in (6.4) and (6.8) correspondingly, and  $\gamma_{n,v}, \tilde{\gamma}_n$  given by

$$\gamma_{n,v^1}(x) = \begin{cases} \left[ f_n(x) - B_n E \left( \frac{v_{n+1}^1(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) \right]^+, & n \in \{0, L, 2L, \dots, (l-1)L\}, \\ 0, & n \notin \{0, L, 2L, \dots, (l-1)L\}, \end{cases}$$

$$\tilde{\gamma}_n(x) = \begin{cases} v_n^1(x) - B_n E \left( \frac{v_{n+1}^1(X_{n+1})}{B_{n+1}} \middle| X_n = x \right), & n \in \{0, L, 2L, \dots, (l-1)L\}, \\ 0, & n \notin \{0, L, 2L, \dots, (l-1)L\}. \end{cases}$$

Here (1.1) is used as a lower bound,  $v_n^1(X_n)$  is equal to the value of this lower bound at the point  $(nh, X_n)$  and  $M_1 = 10^4$ . All values are given together with their 5% confidence intervals

$$\hat{V}_0^1(X_0) \pm \frac{2}{\sqrt{M_1}} \left[ \frac{1}{M_1} \sum_{m=1}^{M_1} ({}_m \Delta_N^{0, X_0})^2 - \left( \frac{1}{M_1} \sum_{m=1}^{M_1} m \Delta_N^{0, X_0} \right)^2 \right]^{1/2}.$$

Let us now make one step further and compute the new lower bound  $v_0^2(x)$  (and then  $\tilde{v}_0^2(x) = \max\{v_0^1(X_0), v_0^2(X_0)\}$ ) using the constructed upper bound  $V_n^1(x)$ . First, we define the process  $Z$  using  $\gamma_{n,v^1}$  given by

$$\gamma_{n,V^1}(x) = \begin{cases} \left[ f_n(x) - B_n E \left( \frac{V_{n+1}^1(X_{n+1})}{B_{n+1}} \middle| X_n = x \right) \right]^+, & n \in \{0, L, 2L, \dots, (l-1)L\}, \\ 0, & n \notin \{0, L, 2L, \dots, (l-1)L\}, \end{cases}$$

and similarly process  $\tilde{Z}$  by means of  $\tilde{\gamma}_n = \gamma_{n,v^1}$  as described in Section 6. Upon constructing  $M_2$  independent realizations of these processes, the estimated value of the new lower bound at the point  $(0, X_0)$  is given by

$$(7.7) \quad \hat{v}_0^2(X_0) = \hat{V}_0^1(X_0) + \frac{1}{M_2} \sum_{m=1}^{M_2} m \delta_N^{0, X_0},$$

TABLE 7.4. Bounds for the standard one-dimensional Bermudan put with parameters  $K = 100$ ,  $r = 0.06$ ,  $T = 5$ ,  $\sigma = 0.4$ ,  $L = 5$  and  $l = 10$ . Results from the two consecutive steps of DCP&VR method are reported.

$X_0$	Lower Bound (1.1)	Upper Bound (DCP&VR)	New Lower Bound (DCP&VR)
60	40	45.5692±0.1010	41.5142±0.1129
70	30	40.4180±0.1028	37.7906±0.1137
80	23.7333	35.5510±0.0944	33.3436±0.1048
90	20.9902	30.6261±0.0988	28.8452±0.1064
100	18.6459	25.5120±0.0941	24.0281±0.1025
110	16.6311	23.2165±0.0969	22.0260±0.1025
120	14.8903	20.5668±0.0875	19.5517±0.0932

where

$${}_m\delta_N^{0,X_0} = ({}_mZ_N^{0,X_0} - {}_m\tilde{Z}_N^{0,X_0}).$$

We see that the Monte Carlo error for  $v_0^2(X_0)$  is a sum of two errors one coming from the first summand in (7.7) and another one from the second term. While the first error depends crucially on the choice of  $M_1$ , the second one is mainly determined by  $M_2$  and is stable with respect to the number of simulations  $M_3$  used for calculation of the process  $\gamma_{n,V^1}$  at each time step. So, we take  $M_1 = 10^4$  for estimating  $V_0^1(X_0)$ , set  $M_2 = 10^4$ ,  $M_3 = 100$  and construct 5% confidence intervals for  $v_0^2(X_0)$  as

$$\hat{v}_0^2(X_0) \pm 2 \left[ \frac{1}{M_2^2} \sum_{m=1}^{M_2} ({}_m\delta_N^{0,X_0})^2 - \frac{1}{M_2} \left( \frac{1}{M_2} \sum_{m=1}^{M_2} {}_m\delta_N^{0,X_0} \right)^2 + \frac{1}{M_1^2} \sum_{m=1}^{M_1} ({}_m\Delta_N^{0,X_0})^2 - \frac{1}{M_1} \left( \frac{1}{M_1} \sum_{m=1}^{M_1} {}_m\Delta_N^{0,X_0} \right)^2 \right]^{1/2}.$$

The corresponding values  $\tilde{v}_0^2(X_0) = \max\{v_0^1(X_0), \hat{v}_0^2(X_0)\}$  are given in the forth column of the Table 7.4. The new lower bound together with the upper bound gives already comparatively tight bounds for the true value of the Bermudan option.

## 8. Conclusions

In this paper we present the new approach for evaluation of American and Bermudan options based, in a sense, on the decomposition of these options into the European option and some consumption process. This decomposition together with its probabilistic representation allows us to construct an efficient sequential method for improving the initial approximation by interchanging between lower and upper bounds at each step of the algorithm. The approach seems to be constructive and, in principle, to be easily implementable. Different types of variance reduction techniques



can be incorporated into algorithms within the framework of this approach. Since the methods obtained are Monte Carlo based, high-dimensional problems become tractable.

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