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A functional law of large numbers for Boltzmann type stochastic particle systems

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Abstract. A large system of particles is studied. Its time evolution is determined as the superposition of two components. The first component is the independent motion of each particle. The second component is the random interaction mechanism between pairs of particles. The intensity of the interaction depends on the state of the system and is assumed to be bounded.

Convergence of the empirical measures is proved as the number of particles tends to infinity. The limiting deterministic measure–valued function is characterized as the unique solution of a nonlinear equation of the Boltzmann type.

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1. Introduction

The basic equation of the kinetic theory of dilute (monatomic) gases is the Boltzmann equation (originally published in [5])

$$\frac{\partial}{\partial t} p(t,x,v) + (v, \nabla_x)\, p(t,x,v) + (\beta(x,v), \nabla_v)\, p(t,x,v) = \tag{1.1}$$
$$\int_{\mathcal{R}^3} dw \int_{\mathcal{S}^2} de\, B(v,w,e) \left[p(t,x,v^*)\, p(t,x,w^*) - p(t,x,v)\, p(t,x,w)\right].$$

This equation describes the time evolution of a density function $p(t,x,v)$ that depends on a time variable $t \geq 0$, on coordinates $x \in G \subset \mathcal{R}^3$ representing the possible positions of the gas particles, and on coordinates $v \in \mathcal{R}^3$ representing the possible velocities of the gas particles. The function B is called the collision kernel, and the function β describes an external force acting on the particles. The symbol ∇ denotes the gradient (the vector of the partial derivatives), and $(.,.)$ is the scalar product in three–dimensional Euclidean space $\mathcal{R}^3$. The symbols de and dw denote the uniform surface measure on the unit sphere $\mathcal{S}^2$ and the Lebesgue measure on $\mathcal{R}^3$, respectively. The objects v^* and w^* are defined as

$$v^* = v + e(e, w-v), \quad w^* = w + e(e, v-w), \tag{1.2}$$

where $v, w \in \mathcal{R}^3$, $e \in \mathcal{S}^2$. They are interpreted as the post–collision velocities of two particles with the pre–collision velocities v and w. The transformation (1.2) preserves momentum ($v^*+w^* = v+w$) and energy ($\|v^*\|^2+\|w^*\|^2 = \|v\|^2+\|w\|^2$). We refer to [7] concerning more information about the Boltzmann equation.

To turn to the discussion of stochastic models related to the Boltzmann equation, we introduce a Markov process

$$\left(X_i(t), V_i(t)\right)_{i=1}^n, \quad t \geq 0, \tag{1.3}$$

with the infinitesimal generator of the form

$$\mathcal{A}(\Phi)(\bar{z}) = \sum_{i=1}^n \left[(v_i, \nabla_{x_i}) + (\beta(x_i, v_i), \nabla_{v_i})\right](\Phi)(\bar{z}) + \tag{1.4}$$
$$\frac{1}{2n} \sum_{1 \leq i \neq j \leq n} \int_{\mathcal{S}^2} \left[\Phi(J(\bar{z}, i, j, e)) - \Phi(\bar{z})\right] \alpha(x_i, v_i, x_j, v_j, e)\, de\,,$$

where Φ is an appropriate test function, $\bar{z} = (x_i, v_i)_{i=1}^n$, $x_i \in G$, $v_i \in \mathcal{R}^3$, and

$$[J(\bar{z}, i, j, e)]_k = \begin{cases} (x_k, v_k) & \text{, if } k \neq i, j\,, \\ (x_i, v_i + e(e, v_j - v_i)) & \text{, if } k = i\,, \\ (x_j, v_j + e(e, v_i - v_j)) & \text{, if } k = j\,, \end{cases} \tag{1.5}$$

is a collision transformation based on (1.2). The function β is the external force appearing in Eq. (1.1), and the function α is related to the collision kernel B.

Stochastic particle systems of the form (1.3)–(1.5) have been investigated for a long time. The first reference seems to be the paper by Leontovich [17], which was brought to the author's attention by Ivanov and Rogazinskij [12]. Among other things, Leontovich pointed out the problem that has become known later as the problem of propagation of chaos. Let $p_n(t, x_1, v_1, \ldots, x_n, v_n)$ be the n–particle distribution function of the process (1.3), and let $p_{n|k}$ denote the corresponding marginal distributions. What Leontovich found was the following: If (in the limit $n \to \infty$)

$$p_{n|2}(t, x_1, v_1, x_2, v_2) = p_{n|1}(t, x_1, v_1)\, p_{n|1}(t, x_2, v_2)$$

and

$$\alpha(x, v, y, w, e) = \kappa(x-y)\, B(v, w, e)\,, \tag{1.6}$$

where κ denotes Dirac's delta–function, then (in the limit $n \to \infty$) the function $p_{n|1}(t, x, v)$ solves the Boltzmann equation (1.1).

In his famous paper on the mathematical foundation of kinetic theory of gases [13], Kac gave a precise notion of the problem: if a certain factorization property (the "chaos" property) holds at time zero, namely

$$\lim_{n\to\infty} p_{n|k}(0, x_1, v_1, \ldots, x_k, v_k) = \prod_{i=1}^{k} \lim_{n\to\infty} p_{n|1}(0, x_i, v_i)\,, \tag{1.7}$$

then this property remains true at any time (it "propagates"), i.e.

$$\lim_{n\to\infty} p_{n|k}(t, x_1, v_1, \ldots, x_k, v_k) = \prod_{i=1}^{k} \lim_{n\to\infty} p_{n|1}(t, x_i, v_i)\,, \quad \forall t > 0\,,$$

and the function $\lim_{n\to\infty} p_{n|1}(t,x,v)$ solves the Boltzmann equation (1.1). Kac also proved propagation of chaos for a specific model (Kac's caricature of a Maxwellian gas, cf. [19]).

However, Kac considered only the so–called spatially homogeneous case (more precisely, the case when both gradient terms in Eq. (1.1) disappear). This fact influenced the development of the theory of stochastic particle systems related to the Boltzmann equation quite strongly. Research in this field was restricted to the spatially homogeneous case during a long period after Kac's paper [13]. We refer to [14], [15], [20], [28], [25] ,[26], [24], [29], [9], [3] (cf. [11], [27] concerning up to (that) date reviews and additional reference lists). The spatially inhomogeneous case has been treated during the last decade in [23], [6], [10], [1], [21], [18], [16], [2].

Skorokhod [23, Ch. 2] considered a Markov process $Z(t) = (Z_i(t))_{i=1}^n$ (describing it via stochastic differential equations with respect to Poisson measures) with the generator

$$\mathcal{A}(\Phi)(\bar{z}) = \sum_{i=1}^{n} (b(z_i), \nabla_{z_i})(\Phi)(\bar{z}) + \frac{1}{2n} \sum_{1\le i\ne j\le n} \int_\Theta \Big[\Phi(J(\bar{z},i,j,\vartheta)) - \Phi(\bar{z})\Big]\, \pi(d\vartheta)\,, \tag{1.8}$$

where Φ is an appropriate test function, $\bar{z} = (z_i)_{i=1}^n$, $z_i \in \mathcal{Z}$, and

$$[J(\bar{z},i,j,e)]_k = \begin{cases} z_k & \text{, if } \ k \ne i,j\,, \\ z_i + f(z_i,z_j,\vartheta) & \text{, if } \ k = i\,, \\ z_j + f(z_j,z_i,\vartheta) & \text{, if } \ k = j\,. \end{cases} \tag{1.9}$$

The symbol $\mathcal{Z}$ denotes the state space of a single particle, Θ is a parameter set, f is a function on $\mathcal{Z}\times\mathcal{Z}\times\Theta$, and π is a measure on Θ.

This model is more general than the Leontovich model (1.4)–(1.5) as far as the gradient terms and the jump transformation J are concerned. However, the distribution π of the jump parameter ϑ does not depend on the state $\bar{z}$.

Let

$$\mu^{(n)}(t) = \frac{1}{n}\sum_{i=1}^{n} \delta_{Z_i(t)} \tag{1.10}$$

be the empirical measures associated with the Markov process $Z(t)$, where the symbol δ_z denotes the Dirac measure concentrated in z.

Skorokhod proved that the empirical measures (1.10) converge (for any t) to a deterministic limit $\lambda(t)$ which satisfies the equation

$$\frac{d}{dt}\int_{\mathcal{Z}}\varphi(z)\,\lambda(t,dz)=\int_{\mathcal{Z}}(b(z),\nabla_z)(\varphi)(z)\,\lambda(t,dz)+ \tag{1.11}$$
$$\int_{\mathcal{Z}}\int_{\mathcal{Z}}\left\{\int_{\Theta}\Big[\varphi(z_1+f(z_1,z_2,\vartheta))-\varphi(z_1)\Big]\,\pi(d\vartheta)\right\}\lambda(t,dz_1)\,\lambda(t,dz_2)\,,$$

for appropriate test functions φ.

It turns out (cf., e.g., [26]) that the chaos property (1.7) (i.e., the asymptotic factorization) is equivalent to the convergence in distribution of the empirical measures (considered as random variables with values in the space of measures on $\mathcal{Z}$) to a deterministic limit. In this setup, it is natural to study the convergence not only for fixed t, but also in the space of measure–valued functions of t (functional law of large numbers) (cf. [22], [25], [29], [11]).

In a recent paper [2], the authors considered a Markov process (1.3) with the infinitesimal generator

$$\mathcal{A}(\Phi)(\bar{z})=\sum_{i=1}^{n}(v_i,\nabla_{x_i})\,(\Phi)(\bar{z})+ \tag{1.12}$$
$$\frac{1}{2n}\sum_{1\le i\neq j\le n}\int_{\mathcal{R}^3}\int_{\mathcal{R}^3}\Big[\Phi(J(\bar{z},i,j,\tilde{v}_1,\tilde{v}_2))-\Phi(\bar{z})\Big]\,Q(v_i,v_j,d\tilde{v}_1,d\tilde{v}_2)\,,$$

where Φ is an appropriate test function, $\bar{z}=(x_i,v_i)_{i=1}^{n}$, $x_i\in\mathcal{R}^3$, $v_i\in\mathcal{R}^3$, and

$$\Big[J(\bar{z},i,j,\tilde{v}_1,\tilde{v}_2)\Big]_k=\begin{cases}(x_k,v_k)\,, & \text{if } \ k\neq i,j\,,\\ (x_i,\tilde{v}_1)\,, & \text{if } \ k=i\,,\\ (x_j,\tilde{v}_2)\,, & \text{if } \ k=j\,.\end{cases} \tag{1.13}$$

The symbol Q denotes a generalized collision kernel.

Under the assumption that the velocities belong to a bounded set, a functional law of large numbers was proved.

As compared with the Leontovich model (1.4)–(1.5), the case

$$\alpha(x_1,v_1,x_2,v_2,e)=B(v_1,v_2,e)$$

is covered by the above model. But still the distribution of the jump parameters $\tilde{v}_1,\tilde{v}_2$ does not depend on the coordinates that perform a drift (i.e., x_i,x_j, in this case).

In the present paper, we prove a functional law of large numbers for a model, which includes both the Leontovich model (1.4)–(1.5) and the Skorokhod model (1.8)–(1.9). The paper is organized as follows.

The main results are formulated in two theorems in Section 2. The first theorem is concerned with the description of the limiting equation showing existence and uniqueness of the solution. The second theorem studies the behaviour of the empirical measures showing convergence to a measure–valued function determined by the limiting equation.

Section 3 contains some technical preparations concerning random variables with values in metric spaces.

Section 4 is devoted to the study of the limiting equation. In particular, existence and uniqueness of the solution are proved. In the spatially homogeneous case, this solution reduces to the so–called Wild's sum (cf. [30], [19], [25]).

Section 5 concerns some properties of the basic Markov particle system. Relative compactness of the empirical measures is proved.

In Section 6 we give the proof of the convergence theorem. The main idea is (as in [23]) to approximate the Markov process by a pure jump process, to study the convergence of the approximate system, and to control the error resulting from the approximation.

Finally, Section 7 contains some remarks concerning the results and their possible or rather desirable generalizations.

2. Main Results

In the first part of this section we introduce what we call a Boltzmann type stochastic particle system. This is a Markov process, which is determined by two basic components. The first component is a transition function U_0 describing the independent motion of the particles (called the "free flow" in the Boltzmann context). The second component is a generalized collision kernel Q describing the pairwise interaction among the particles (interpreted as "collisions" in the Boltzmann case). In the second part of this section we consider an equation that determines the limit of the empirical measures and give a theorem concerning existence and uniqueness of the solution. This result is valid for rather general U_0 and (bounded) Q. In the third part we introduce several restrictions concerning U_0 and Q, and give a convergence theorem.

Let $(\mathcal{Z}, r)$ be a locally compact separable metric space (r denoting the metric) and $\mathcal{B}_{\mathcal{Z}}$ denote the Borel–σ–algebra. Let $B(\mathcal{Z})$ be the Banach space of bounded Borel measurable functions on $\mathcal{Z}$ with $\|\varphi\| = \sup_{z\in\mathcal{Z}} |\varphi(z)|$, and let $\hat{C}(\mathcal{Z})$ denote the subspace of continuous functions vanishing at infinity. Furthermore, let $\mathcal{M}(\mathcal{Z})$ be the space of finite, positive measures on $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$ and $\mathcal{P}(\mathcal{Z})$ denote the space of probability measures on $(\mathcal{Z}, \mathcal{B}_{\mathcal{Z}})$.

Let $U_0(t, z, \Gamma)$ be a transition function on $[0,\infty) \times \mathcal{Z} \times \mathcal{B}_{\mathcal{Z}}$ with the properties (cf. [8, Ch. 4, Sect. 1])

$$U_0(t,z) \in \mathcal{P}(\mathcal{Z}), \quad \forall (t,z) \in [0,\infty) \times \mathcal{Z}, \tag{2.1}$$

$$U_0(0,z) = \delta_z, \quad \forall z \in \mathcal{Z}, \tag{2.2}$$

$$U_0(.,.,\Gamma) \text{ is measurable}, \quad \forall \Gamma \in \mathcal{B}_{\mathcal{Z}}, \tag{2.3}$$

$$U_0(t+s,z,\Gamma) = \int_{\mathcal{Z}} U_0(s,\tilde{z},\Gamma)\, U_0(t,z,d\tilde{z}), \tag{2.4}$$
$$\forall s,t \in [0,\infty), \quad \forall z \in \mathcal{Z}, \quad \forall \Gamma \in \mathcal{B}_{\mathcal{Z}}.$$

Let $Q(z_1, z_2, \Gamma_1, \Gamma_2)$ be a function on $\mathcal{Z} \times \mathcal{Z} \times \mathcal{B}_{\mathcal{Z}} \times \mathcal{B}_{\mathcal{Z}}$ with the properties

$$Q(z_1,z_2,.,\Gamma),\, Q(z_1,z_2,\Gamma,.) \in \mathcal{M}(\mathcal{Z}), \quad \forall z_1,z_2 \in \mathcal{Z},\, \Gamma \in \mathcal{B}_{\mathcal{Z}}, \tag{2.5}$$

$$Q(.,.,\Gamma_1,\Gamma_2) \text{ is measurable}, \quad \forall \Gamma_1,\Gamma_2 \in \mathcal{B}_{\mathcal{Z}}. \tag{2.6}$$

Using U_0 and Q, we introduce a Markov process $Z(t) = (Z_i(t))_{i=1}^n$ with the state space $\mathcal{Z}^n$ and the generator

$$\mathcal{A}(\Phi)(\bar{z}) = \sum_{i=1}^n A_{0,z_i}(\Phi)(\bar{z}) + \tag{2.7}$$
$$\frac{1}{n} \sum_{1\le i\ne j\le n} \int_{\mathcal{Z}} \int_{\mathcal{Z}} \Big[\Phi(J(\bar{z},i,j,\tilde{z}_1,\tilde{z}_2)) - \Phi(\bar{z})\Big]\, Q(z_i,z_j,d\tilde{z}_1,d\tilde{z}_2),$$

where $\bar{z} = (z_i)_{i=1}^n$, $z_i \in \mathcal{Z}$, A_0 denotes the generator corresponding to the transition function U_0, Φ is an appropriate test function, and

$$\Big[J(\bar{z},i,j,\tilde{z}_1,\tilde{z}_2)\Big]_k = \begin{cases} z_k, & \text{if } k \ne i,j, \\ \tilde{z}_1, & \text{if } k = i, \\ \tilde{z}_2, & \text{if } k = j. \end{cases} \tag{2.8}$$

We call the process $Z(t)$ a Boltzmann type stochastic particle system. Some basic properties of this process will be studied in Section 5.

We assume

$$Q(z_1, z_2, \mathcal{Z}, \mathcal{Z}) \leq C_{Q,max}\,, \quad \forall z_1, z_2 \in \mathcal{Z}\,, \tag{2.9}$$

and introduce a kernel

$$\begin{aligned} Q_{max}(z_1, z_2, \Gamma_1, \Gamma_2) &= Q(z_1, z_2, \Gamma_1, \Gamma_2) + \\ &[C_{Q,max} - Q(z_1, z_2, \mathcal{Z}, \mathcal{Z})]\, \delta_{z_1}(\Gamma_1)\, \delta_{z_2}(\Gamma_2)\,, \quad z_1, z_2 \in \mathcal{Z}\,, \quad \Gamma_1, \Gamma_2 \in \mathcal{B}_{\mathcal{Z}}\,. \end{aligned} \tag{2.10}$$

Furthermore, we define a function

$$T_0(t)^*(\mu)(\Gamma) = \int_{\mathcal{Z}} \mu(dz)\, U_0(t, z, \Gamma)\,, \quad \mu \in \mathcal{M}(\mathcal{Z})\,, \quad \Gamma \in \mathcal{B}_{\mathcal{Z}}\,, \tag{2.11}$$

and a function

$$\begin{aligned} K_{max}(\mu_1, \mu_2)(\Gamma) &= \int_{\mathcal{Z}} \int_{\mathcal{Z}} \Big[Q_{max}(z_1, z_2, \Gamma, \mathcal{Z}) + \\ &Q_{max}(z_1, z_2, \mathcal{Z}, \Gamma)\Big]\, \mu_1(dz_1)\, \mu_2(dz_2)\,, \quad \mu_1, \mu_2 \in \mathcal{M}(\mathcal{Z})\,, \quad \Gamma \in \mathcal{B}_{\mathcal{Z}}\,. \end{aligned} \tag{2.12}$$

It is easy to realize that (2.11) defines an operator

$$T_0(t)^* \,:\, \mathcal{M}(\mathcal{Z}) \to \mathcal{M}(\mathcal{Z})\,,$$

and that (2.12) defines an operator

$$K_{max} \,:\, \mathcal{M}(\mathcal{Z}) \times \mathcal{M}(\mathcal{Z}) \to \mathcal{M}(\mathcal{Z})\,,$$

justifying the notations. Some additional properties of these operators will be established in Section 4.

Using the operators $T_0(t)^*$ and K_{max}, we introduce the equation

$$\lambda(t) = e^{-c_0\, t}\, T_0(t)^*(\lambda_0) + \int_0^t e^{-c_0\,(t-s)}\, T_0(t-s)^*\, K_{max}(\lambda(s), \lambda(s))\, ds\,, \tag{2.13}$$

where $t \in [0, \infty)\,,\ \lambda_0 \in \mathcal{M}(\mathcal{Z})$, and

$$c_0 = 2\, C_{Q,max}\, \lambda_0(\mathcal{Z})\,. \tag{2.14}$$

Theorem 2.1 *Assume the generalized collision kernel Q satisfies (2.9). Suppose $\lambda_0 \in \mathcal{M}(\mathcal{Z})$, and let c_0 be given in (2.14).*

Define, for $t \in [0,\infty)$, $k \geq 1$,

$$\nu_1(t)(\lambda_0) = e^{-c_0 t}\, T_0(t)^*(\lambda_0)\,, \tag{2.15}$$

and

$$\nu_{k+1}(t)(\lambda_0) = \sum_{i=1}^{k} \int_0^t e^{-c_0\,(t-s)}\, T_0(t-s)^*\, K_{max}(\nu_i(s)(\lambda_0), \nu_{k+1-i}(s)(\lambda_0))\, ds\,. \tag{2.16}$$

Then the series

$$W(t)(\lambda_0) = \sum_{k=1}^{\infty} \nu_k(t)(\lambda_0)\,, \quad t \in [0,\infty)\,, \tag{2.17}$$

converges in $\mathcal{M}(\mathcal{Z})$ in the total variation norm.

The function $\lambda(t) = W(t)(\lambda_0)$ is the unique solution of Eq. (2.13).

Example 2.2 (Wild's sum) *Consider the case, when the free flow degenerates, i.e. $U_0(t,z) = \delta_z\,,\ \forall t > 0\,,\ \forall z \in \mathcal{Z}\,.$ Notice that*

$$\begin{aligned}
&\int_{\mathcal{Z}} \varphi(z)\, K_{max}(\mu,\mu)(dz) = 2\, C_{Q,max}\, \mu(\mathcal{Z}) \int_{\mathcal{Z}} \varphi(z)\, \mu(dz) + \\
&\quad \int_{\mathcal{Z}} \int_{\mathcal{Z}} \Big\{ \int_{\mathcal{Z}} \int_{\mathcal{Z}} \big[\varphi(\tilde z_1) + \varphi(\tilde z_2) - \varphi(z_1) - \varphi(z_2)\big] \times \\
&\quad Q(z_1, z_2, d\tilde z_1, d\tilde z_2) \Big\}\, \mu(dz_1)\, \mu(dz_2)\,, \quad \forall \varphi \in B(\mathcal{Z})\,, \quad \forall \mu \in \mathcal{M}(\mathcal{Z})\,.
\end{aligned} \tag{2.18}$$

Thus, Eq. (2.13) takes the form

$$\begin{aligned}
\frac{d}{dt} \int_{\mathcal{Z}} \varphi(z)\, \lambda(t,dz) = &\int_{\mathcal{Z}} \int_{\mathcal{Z}} \Big\{ \int_{\mathcal{Z}} \int_{\mathcal{Z}} \big[\varphi(\tilde z_1) + \varphi(\tilde z_2) - \varphi(z_1) - \varphi(z_2)\big] \times \\
&Q(z_1, z_2, d\tilde z_1, d\tilde z_2) \Big\}\, \lambda(t,dz_1)\, \lambda(t,dz_2)\,, \quad \lambda(0) = \lambda_0\,,
\end{aligned} \tag{2.19}$$

where φ is an arbitrary bounded measurable function. Eq. (2.19) is a weak version of the spatially homogeneous Boltzmann equation (cf. [28], [25], [11]).

The solution $\lambda(t)$ given in (2.17) has now the representation

$$\lambda(t) = \sum_{k=1}^{\infty} e^{-c_0 t}(1 - e^{-c_0 t})^{k-1} \tilde{\nu}_k , \quad t \in [0,\infty) , \tag{2.20}$$

where $\tilde{\nu}_1 = \lambda_0$, $\tilde{\nu}_{k+1} = \frac{1}{c_0 k} \sum_{i=1}^{k} K_{max}(\tilde{\nu}_i, \tilde{\nu}_{k+1-i})$, $k \geq 1$, *and* c_0 *is defined in (2.14). The series on the right-hand side of (2.20) is called Wild's sum (cf. [25], with* $\lambda_0(\mathcal{Z}) = 1$, $C_{Q,max} = \frac{1}{2}$*).*

We introduce now certain restrictions on the basic components U_0 and Q of the generator (2.7)–(2.8).

Let $T_0(t)$ denote the semigroup of operators on $B(\mathcal{Z})$ associated with the transition function U_0 . We suppose that $T_0(t)$ is a Feller semigroup (cf. [8, Ch. 4, Sect. 2]), i.e.

$$T_0(t)(\varphi) \in \hat{C}(\mathcal{Z}) , \quad \forall \varphi \in \hat{C}(\mathcal{Z}) , \tag{2.21}$$

and

$$\lim_{t\to 0} \|T_0(t)(\varphi) - \varphi\| = 0 , \quad \forall \varphi \in \hat{C}(\mathcal{Z}) . \tag{2.22}$$

Furthermore, we assume that the transition function U_0 is determined as

$$U_0(t,z,\Gamma) = \delta_{F(t,z)}(\Gamma) , \quad \Gamma \in \mathcal{B}_{\mathcal{Z}} , \tag{2.23}$$

where $F(t,z)$ is a mapping from $[0,\infty) \times \mathcal{Z}$ into $\mathcal{Z}$ such that

$$F(0,z) = z , \quad \forall z \in \mathcal{Z} , \tag{2.24}$$

and

$$F(t+s,z) = F(s,F(t,z)) , \quad \forall t,s \in [0,\infty) , \quad \forall z \in \mathcal{Z} . \tag{2.25}$$

Concerning the free flow F , we assume

$$r(F(t,z),F(s,\tilde{z})) \leq C_F \left[r(z,\tilde{z}) + |t-s| \, (1 + r_0(z) + r_0(\tilde{z})) \right] , \quad \forall t,s \in [0,\infty) , \ \forall z,\tilde{z} \in \mathcal{Z} , \tag{2.26}$$

where the function r_0 is defined as

$$r_0(z) = r(z,z_0) , \quad z \in \mathcal{Z} , \quad \text{for some } z_0 \in \mathcal{Z} , \tag{2.27}$$

and r is the metric in the space $\mathcal{Z}$.

We suppose that the collision kernel Q has the form

$$Q(z_1, z_2, d\tilde{z}_1, d\tilde{z}_2) = \int_\Theta \delta_{f_1(z_1,z_2,\vartheta)}(d\tilde{z}_1)\, \delta_{f_2(z_1,z_2,\vartheta)}(d\tilde{z}_2)\, q(z_1, z_2, \vartheta)\, \pi(d\vartheta)\,, \quad (2.28)$$

where δ is the Dirac measure, Θ is a parameter set, q, f_1, f_2 are appropriate functions on $\mathcal{Z} \times \mathcal{Z} \times \Theta$, and π is a σ–finite measure on Θ.

Concerning the function q, we assume

$$q(z_1, z_2, \vartheta) \le q_{max}(\vartheta)\,, \quad \forall z_1, z_2 \in \mathcal{Z}\,, \quad \forall \vartheta \in \Theta\,, \quad (2.29)$$

where

$$\int_\Theta q_{max}(\vartheta)\, \pi(d\vartheta) = C_{Q,max} < \infty\,, \quad (2.30)$$

and

$$\int_\Theta |q(z, z_1, \vartheta) - q(\tilde{z}, \tilde{z}_1, \vartheta)|\, \pi(d\vartheta) \le C_{q,L}\,[r(z, \tilde{z}) + r(z_1, \tilde{z}_1)]\,, \quad \forall z, z_1, \tilde{z}, \tilde{z}_1 \in \mathcal{Z}\,. \quad (2.31)$$

Concerning the functions f_i, $i = 1, 2$, we assume

$$\int_\Theta r(f_i(z, z_1, \vartheta), f_i(\tilde{z}, \tilde{z}_1, \vartheta))\, q_{max}(\vartheta)\, \pi(d\vartheta) \le C_{f,L}\,[r(z, \tilde{z}) + r(z_1, \tilde{z}_1)]\,, \quad \forall z, z_1, \tilde{z}, \tilde{z}_1 \in \mathcal{Z}\,, \quad (2.32)$$

$$\int_\Theta r(f_i(\tilde{z}_0, \tilde{z}_0, \vartheta), \tilde{z}_0)\, q_{max}(\vartheta)\, \pi(d\vartheta) < \infty\,, \quad \text{for some } \tilde{z}_0 \in \mathcal{Z}\,, \quad (2.33)$$

and

$$\int_\Theta \varphi(f_i(.,.,\vartheta))\, q_{max}(\vartheta)\, \pi(d\vartheta) \in \hat{C}(\mathcal{Z} \times \mathcal{Z})\,, \quad \forall \varphi \in \hat{C}(\mathcal{Z})\,. \quad (2.34)$$

Theorem 2.3 *Let $Z(t) = (Z_i(t))_{i=1}^n$ be a Markov process with the generator (2.7)–(2.8), where U_0 is defined in (2.23) and Q is defined in (2.28). Let $\mu^{(n)}(t)$ be the empirical measures defined in (1.10) and $\lambda(t)$ be the solution of Eq. (2.13).*

Suppose that the assumptions (2.21), (2.22), (2.26) concerning the free flow F and the assumptions (2.29)–(2.34) concerning the functions q, f_1, f_2 are satisfied.

If

$$\lim_{n\to\infty} \mathcal{E}^{(n)} \varrho(\mu^{(n)}(0), \lambda(0)) = 0\,, \tag{2.35}$$

then

$$\lim_{n\to\infty} \mathcal{E}^{(n)} \sup_{0\le s\le t} \varrho(\mu^{(n)}(s), \lambda(s)) = 0\,, \quad \forall t \ge 0\,, \tag{2.36}$$

where ϱ is any bounded metric equivalent to weak convergence in $\mathcal{P}(\mathcal{Z})$, and $\mathcal{E}^{(n)}$ denotes mathematical expectation.

Finishing this section, we consider two examples.

With Q defined in (2.28), the generator (2.7)–(2.8) takes the form

$$\mathcal{A}(\Phi)(\bar z) = \sum_{i=1}^{n} A_{0,z_i}(\Phi)(\bar z) + \frac{1}{n} \sum_{1\le i\ne j\le n} \int_\Theta \Big[\Phi(J(\bar z, i, j, \vartheta)) - \Phi(\bar z)\Big]\, q(z_i, z_j, \vartheta)\, \pi(d\vartheta)\,, \tag{2.37}$$

where $\bar z = (z_i)_{i=1}^n$, $z_i \in \mathcal{Z}$, and

$$\Big[J(\bar z, i, j, \vartheta)\Big]_k = \begin{cases} z_k & , \text{ if } \ k \ne i, j\,, \\ f_1(z_i, z_j, \vartheta)\,, \text{ if } & k = i\,, \\ f_2(z_i, z_j, \vartheta)\,, \text{ if } & k = j\,. \end{cases} \tag{2.38}$$

Let the free flow F be defined as the solution to a system of ordinary differential equations

$$\frac{d}{dt} F(t, z) = b(F(t, z))\,, \quad t \in (0, \infty)\,, \quad F(0, z) = z\,,$$

where b is a mapping from $\mathcal{Z}$ into $\mathcal{Z}$, and $\mathcal{Z} = \mathcal{R}^d$ (r is the Euclidean metric). Then the free flow generator A_0 takes the form

$$A_0(\varphi)(z) = (b(z), \nabla_z)\,. \tag{2.39}$$

In this case the properties (2.21), (2.22) of the semigroup $T_0(t)$ are consequences of assumption (2.26) (assuming $z_0 = 0$, without loss of generality). To show this, we consider $\varphi \in \hat{C}(\mathcal{Z})$ and note that $T_0(t)(\varphi)(z) = \varphi(F(t,z))$ by (2.23).

The function $T_0(t)(\varphi)$ is bounded and continuous, because of (2.26) with $s=t$. Thus, (2.21) is fulfilled if

$$\lim_{\|z\|\to\infty} \|F(t,z)\| = \infty\,, \quad \forall t > 0\,. \tag{2.40}$$

It follows from (2.26) and (2.24) that

$$\|F(t,z) - z\| \le C_F\, t\,(1 + 2\,\|z\|)\,. \tag{2.41}$$

Consequently,

$$\|F(t,z)\| \ge \|z\| - C_F\, t\,(1 + 2\,\|z\|) = \|z\|\,(1 - 2\,C_F\, t) - C_F\, t\,,$$

and

$$\|F(t,z)\| \ge \frac{1}{2}\,\|z\| - \frac{1}{4}\,, \quad \forall z \in \mathcal{Z}\,, \quad \forall t \le \frac{1}{4\,C_F}\,. \tag{2.42}$$

Thus, (2.40) follows, for small t, from (2.42) and, for arbitrary t, from the semigroup property.

To establish (2.22), we estimate

$$\begin{aligned} \|T_0(t)(\varphi) - \varphi\| \;&\le\; \sup_{\|z\|\le R} |T_0(t)(\varphi)(z) - \varphi(z)| \\ &\quad + \sup_{\|z\|>R} |T_0(t)(\varphi)(z)| + \sup_{\|z\|>R} |\varphi(z)|\,. \end{aligned}$$

The third term on the right–hand side is small for large R, the second term is small for large R uniformly in t because of (2.42), and the first term tends to zero as $t \to 0$ for fixed R, because of (2.41). Thus, (2.22) follows.

Example 2.4 (Skorokhod model) *The generator (1.8)–(1.9) has the form (2.37)–(2.38) with A_0 given in (2.39), $q(z_1,z_2,\vartheta)=\frac{1}{2}$, and*

$$f_1(z_1,z_2,\vartheta) = z_1 + f(z_1,z_2,\vartheta)\,, \quad f_2(z_1,z_2,\vartheta) = z_2 + f(z_2,z_1,\vartheta)\,. \tag{2.43}$$

Condition (2.26) is fulfilled if b satisfies the global Lipschitz condition

$$\|b(z) - b(\tilde{z})\| \le C_{b,L}\,\|z - \tilde{z}\|\,, \quad \forall z, \tilde{z} \in \mathcal{Z}\,. \tag{2.44}$$

Example 2.5 (Leontovich model) *The generator (1.4)-(1.5) has the form (2.37)-(2.38) with A_0 given in (2.39),*

$$\mathcal{Z} = \mathcal{R}^3 \times \mathcal{R}^3 \,, \quad z = (x, v) \,, \quad z_1 = (x_1, v_1) \,, \quad z_2 = (x_2, v_2) \,,$$

$$b(z) = (v, \beta(x, v)) \,, \qquad \Theta = \mathcal{S}^2 \,, \qquad \pi(d\vartheta) = de \,,$$

$$q(z_1, z_2, e) = \frac{1}{2}\alpha(x_1, v_1, x_2, v_2, e) \,,$$

and f_1, f_2 given in (2.43) with

$$f(z_1, z_2, e) = (0, e(e, v_2 - v_1)) \,. \tag{2.45}$$

Condition (2.44) is fulfilled if β satisfies a global Lipschitz condition, in particular, if $\beta = 0$.

Conditions (2.32) and (2.33) are fulfilled for the functions f_1, f_2 defined in (2.43) and (2.45). Condition (2.34) is a consequence of the energy conservation property of the Boltzmann collision transformation defined in (1.2).

3. Technical preparations

In this section we introduce some notations and prove several technical assertions concerning random variables with values in metric spaces.

Let (S, d) be a metric space (d denoting the metric) and $\mathcal{B}_S$ denote the σ-algebra of Borel subsets of S. Let $B(S)$ be the Banach space of bounded Borel measurable functions on S with $\|f\| = \sup_{x \in S} |f(x)|$, and $\bar{C}(S)$ be the subspace of bounded continuous functions. For $f \in \bar{C}(S)$, we denote

$$\|f\|_L = \max\left(\sup_{x \in S} |f(x)| \,, \quad \sup_{x, y \in S \,, x \neq y} \frac{|f(x) - f(y)|}{d(x, y)}\right) \,. \tag{3.1}$$

Furthermore, $\mathcal{M}(S)$ is the space of finite, positive measures on S, and $\mathcal{P}(S)$ is the space of probability measures on S. We denote

$$\langle f, \nu \rangle = \int_S f(y) \, \nu(dy) \,, \quad \text{where } f \in B(S) \,, \ \nu \in \mathcal{M}(S) \,.$$

On $\mathcal{M}(S)$, we consider the distance induced by the total variation norm on the space of finite signed Borel measures (cf. [8, p. 495]),

$$\|\nu_1 - \nu_2\| = \sup_{f \in B(S):\|f\| \leq 1} |\langle f, \nu_1 \rangle - \langle f, \nu_2 \rangle| \,. \tag{3.2}$$

On $\mathcal{P}(S)$, we consider the bounded Lipschitz metric

$$\varrho_L(P_1, P_2) = \sup_{f \in \bar{C}(S):\|f\|_L \leq 1} |\langle f, P_1 \rangle - \langle f, P_2 \rangle| \,, \tag{3.3}$$

which is equivalent to weak convergence (cf. [8, p. 150]).

Let $(\xi^{(n)})$ be a sequence of random variables with values in S, i.e. of measurable mappings from a probability space $(\Omega^{(n)}, \mathcal{F}^{(n)}, Prob^{(n)})$ into the space S. Let $(P^{(n)})$ denote the associated probability distributions on $(S, \mathcal{B}_S)$. The sequence $(\xi^{(n)})$ is said to converge in distribution to the S–valued random variable ξ if the sequence $(P^{(n)})$ converges weakly to the probability distribution P associated with ξ. Weak convergence (as $n \to \infty$) is denoted by $P^{(n)} \Rightarrow P$ and convergence in distribution (as $n \to \infty$) by $\xi^{(n)} \Rightarrow \xi$. Mathematical expectation with respect to $Prob^{(n)}$ is denoted by $\mathcal{E}^{(n)}$.

Lemma 3.1 (extension of the space) *Let S, S_1 be metric spaces such that $S \subset S_1$ and S has the relative topology. Let $\xi^{(n)}$, ξ be random variables with values in S.*

Then $\xi^{(n)}$, ξ are random variables with values in S_1 and

$$\xi^{(n)} \Rightarrow \xi \text{ in } S \quad \text{if and only if} \quad \xi^{(n)} \Rightarrow \xi \text{ in } S_1 \,. \tag{3.4}$$

Proof. It can be checked easily (cf. [4, Add. II]) that

$$\mathcal{B}_S = \{\Gamma_1 \cap S \,:\, \Gamma_1 \in \mathcal{B}_{S_1}\}\,.$$

Consequently, we have

$$\{\omega \,:\, \xi^{(n)}(\omega) \in \Gamma_1\} = \{\omega \,:\, \xi^{(n)}(\omega) \in \Gamma_1 \cap S\} \in \mathcal{F}^{(n)}, \quad \forall \Gamma_1 \in \mathcal{B}_{S_1}\,.$$

Assertion (3.4) is proved in analogy with [8, Ch. 3, Cor. 3.2], where the case $S \in \mathcal{B}_{S_1}$ was considered. Let $P^{(n)}$, P and $P_1^{(n)}$, P_1 denote the measures associated with $\xi^{(n)}$, ξ on S and S_1, respectively. Obviously,

$$P_1^{(n)}(\Gamma_1) = P^{(n)}(\Gamma_1 \cap S)\,, \quad P_1(\Gamma_1) = P(\Gamma_1 \cap S)\,, \quad \forall \Gamma_1 \in \mathcal{B}_{S_1}\,. \tag{3.5}$$

We use the criterion (e) from [8, Ch. 3, Th. 3.1], which reads $P^{(n)} \Rightarrow P$ on S if and only if $\liminf_{n\to\infty} P^{(n)}(G) \geq P(G)$, for all open subsets $G \subset S$.

Let $P^{(n)} \Rightarrow P$ on S and G_1 be an open subset of S_1. Then $G_1 \cap S$ is open in S. Using (3.5), we obtain

$$\liminf_{n\to\infty} P_1^{(n)}(G_1) = \liminf_{n\to\infty} P^{(n)}(G_1 \cap S) \geq P(G_1 \cap S) = P_1(G_1).$$

Let $P_1^{(n)} \Rightarrow P_1$ on S_1 and G be an open subset of S. Then $G = G_1 \cap S$, for some open subset $G_1 \subset S_1$. Using (3.5), we obtain

$$\liminf_{n\to\infty} P^{(n)}(G) = \liminf_{n\to\infty} P^{(n)}(G_1 \cap S) = \\ \liminf_{n\to\infty} P_1^{(n)}(G_1) \geq P_1(G_1) = P(G_1 \cap S) = P(G).$$

This completes the proof. □

Lemma 3.2 (convergence in distribution to a constant) *Let $(\xi^{(n)})$ be a sequence of random variables with values in a metric space (S,d) and $(P^{(n)})$ be the sequence of the associated probability distributions on $(S,\mathcal{B}_S)$. Let $y \in S$ be a fixed element and $\bar{d} = \min(d,1)$.*

Then

$$P^{(n)} \Rightarrow \delta_y \quad \text{if and only if} \quad \lim_{n\to\infty} \mathcal{E}^{(n)}\bar{d}(\xi^{(n)},y) = 0. \tag{3.6}$$

Proof. Suppose $P^{(n)} \Rightarrow \delta_y$. Since the function $f(x) = \bar{d}(x,y)$ is bounded and continuous on S, one obtains

$$\lim_{n\to\infty} \mathcal{E}^{(n)}\bar{d}(\xi^{(n)},y) = \lim_{n\to\infty} \langle f, P^{(n)}\rangle = \langle f, \delta_y\rangle = 0.$$

The second part of the assertion follows from [8, Ch. 3, Cor. 3.3]. □

Corollary 3.3 *Let (S,d) be a metric space and d_1 be a metric inducing the same topology. Let $(\xi^{(n)})$ be a sequence of random variables with values in S and $y \in S$ be a fixed element.*

Then $\lim_{n\to\infty} \mathcal{E}^{(n)}\bar{d}(\xi^{(n)},y)=0$ *if and only if* $\lim_{n\to\infty} \mathcal{E}^{(n)}\bar{d}_1(\xi^{(n)},y)=0$.

Proof. The assertion follows immediately from Lemma 3.2, since the left-hand side of (3.6) depends only on the topology in S. □

Corollary 3.4 *Let (S, d) be a separable metric space. Let $(\xi^{(n)}(t))$ be a sequence of stochastic processes with index set Δ and state space S. Let y be a deterministic mapping from Δ into S.*

Then

$$\left(\xi^{(n)}(t_1), \ldots, \xi^{(n)}(t_k)\right) \Rightarrow (y(t_1), \ldots, y(t_k)), \quad \forall (t_1, \ldots, t_k) \subset \Delta, \tag{3.7}$$

if and only if

$$\xi^{(n)}(t) \Rightarrow y(t), \quad \forall t \in \Delta. \tag{3.8}$$

Proof. Notice that $\left(\xi^{(n)}(t_1), \ldots, \xi^{(n)}(t_k)\right)$ is a random variable with values in S^k (cf. [8, p. 50]). According to Lemma 3.2, (3.7) is equivalent to

$$\lim_{n\to\infty} \mathcal{E}^{(n)} \max_{1\le i\le k} \bar{d}(\xi^{(n)}(t_i), y(t_i)) = 0 .$$

This is fulfilled if $\lim_{n\to\infty} \mathcal{E}^{(n)} \bar{d}(\xi^{(n)}(t_i), y(t_i)) = 0$, $\forall i = 1, \ldots, k$, which is assured by (3.8) and Lemma 3.2. □

Let $D_E[0,\infty)$ be the space of right continuous functions from $[0,\infty)$ into a metric space (E, ϱ) having left limits (cf. [8, Ch. 3, Sect. 5]), and let $C_E[0,\infty)$ denote the space of continuous functions from $[0,\infty)$ into E. The space $D_E[0,\infty)$ is topologized with the Skorokhod metric d_E. Furthermore, let $\bar{\varrho} = \min(1, \varrho)$, and

$$\hat{d}_u(x, y) = \int_0^\infty e^{-t} \sup_{0\le s\le t} \bar{\varrho}(x(s), y(s))\, dt \tag{3.9}$$

be the uniform metric on $D_E[0,\infty)$.

Corollary 3.5 *Let $(E, \varrho), (E_1, \varrho_1)$ be metric spaces such that $E \subset E_1$ and ϱ is the restriction of ϱ_1. Let $\xi^{(n)}, \xi$ be random variables with values in $D_E[0,\infty)$.*

Then $\xi^{(n)}$, ξ are random variables with values in $D_{E_1}[0,\infty)$, and

$$\xi^{(n)} \Rightarrow \xi \text{ in } D_E[0,\infty) \quad \text{if and only if} \quad \xi^{(n)} \Rightarrow \xi \text{ in } D_{E_1}[0,\infty).$$

Proof. The assertion follows immediately from Lemma 3.1, since

$$S = D_E[0,\infty) \subset D_{E_1}[0,\infty) = S_1 ,$$

and the topology in S is equivalent to the relative topology induced from S_1. □

Lemma 3.6 (uniform convergence) *Let (E,ϱ) be a metric space, and $\bar{\varrho} = \min(1,\varrho)$. Let $(\xi^{(n)})$ be a sequence of random variables with values in $D_E[0,\infty)$ and $y \in C_E[0,\infty)$.*

Then the following are equivalent

$$\lim_{n\to\infty} \mathcal{E}^{(n)} d_E(\xi^{(n)}, y) = 0\,, \tag{3.10}$$

$$\lim_{n\to\infty} \mathcal{E}^{(n)} \hat{d}_u(\xi^{(n)}, y) = 0\,, \tag{3.11}$$

$$\lim_{n\to\infty} \mathcal{E}^{(n)} \sup_{0\le s\le t} \bar{\varrho}(\xi^{(n)}(s), y(s)) = 0\,, \quad \forall t > 0\,. \tag{3.12}$$

Proof. It follows from definition (3.9) that (3.12) implies (3.11). It is easy to see from the definition of the Skorokhod metric (cf. [8, p. 117]) that $d_E \le \hat{d}_u$. Consequently, (3.11) implies (3.10). It remains to show that (3.10) implies (3.12).

We introduce a real-valued function $f(x) = \sup_{0\le s\le t} \bar{\varrho}(x(s), y(s))$ on $D_E[0,\infty)$, and notice that the function f is continuous at the point y. This is a consequence of [8, Ch. 3, Lemma 10.1].

The function f is also measurable on $(D_E[0,\infty), d_E)$. Really, the function $\bar{\varrho}(x(s), y(s))$ is measurable for any fixed s, because it is the superposition of a measurable mapping $(\pi_s(x) = x(s))$ from $D_E[0,\infty)$ into E, and a continuous function $(f_1(z) = \bar{\varrho}(z, y(s)))$ from E into $\mathcal{R}$. The $\sup_{0\le s\le t}$ equals the supremum over a dense set because of the cadlag–property of x.

Since (3.10) is equivalent to $P^{(n)} \Rightarrow \delta_y$, according to Lemma 3.2, we obtain from [4, Ch. 1, Th. 5.2] that

$$\lim_{n\to\infty} \mathcal{E}^{(n)} f(\xi^{(n)}) = \lim_{n\to\infty} \langle f, P^{(n)} \rangle = \langle f, \delta_y \rangle = 0\,.$$

This completes the proof. □

Let $\mathcal{Z}$ be a locally compact separable metric space and $\hat{C}(\mathcal{Z}) \subset \bar{C}(\mathcal{Z})$ denote the subspace of continuous functions vanishing at infinity. We introduce

$$\mathcal{M}_c(\mathcal{Z}) = \{\nu \in \mathcal{M}(\mathcal{Z}) \,:\, \nu(\mathcal{Z}) \le c\}\,, \quad c \in (0,\infty)\,.$$

Let (φ_k) be a dense subset of $\hat{C}(\mathcal{Z})$. On $\mathcal{M}_c(\mathcal{Z})$, we consider the metric

$$\tilde{\varrho}(\nu_1,\nu_2) = \sum_{k=1}^{\infty} \frac{1}{2^k} \min\Big(1, |\langle \varphi_k, \nu_1\rangle - \langle \varphi_k, \nu_2\rangle|\Big)\,. \tag{3.13}$$

Remark 3.7 *Notice that* $\lim_{n\to\infty} \tilde{\varrho}(\nu^{(n)},\nu)=0$ *if and only if*

$$\lim_{n\to\infty} \langle \varphi, \nu^{(n)} \rangle = \langle \varphi, \nu \rangle \,, \quad \forall \varphi \in \hat{C}(\mathcal{Z}) \,,$$

where $\nu^{(n)}$, $\nu \in \mathcal{M}_c(\mathcal{Z})$. *The space* $(\mathcal{M}_c(\mathcal{Z}), \tilde{\varrho})$ *is separable and complete. On* $\mathcal{P}(\mathcal{Z})$, *the metric* $\tilde{\varrho}$ *is equivalent to weak convergence.*

Corollary 3.8 *Let* $\xi^{(n)}, \xi$ *be random variables with values in* $D_{\mathcal{P}(\mathcal{Z})}[0,\infty)$.
Then $\xi^{(n)}$, ξ *are random variables with values in* $D_{\mathcal{M}_1(\mathcal{Z})}[0,\infty)$, *and*

$$\xi^{(n)} \Rightarrow \xi \text{ in } D_{\mathcal{P}(\mathcal{Z})}[0,\infty) \quad \text{if and only if} \quad \xi^{(n)} \Rightarrow \xi \text{ in } D_{\mathcal{M}_1(\mathcal{Z})}[0,\infty) \,.$$

Proof. The assertion follows from Corollary 3.5 and Remark 3.7, since

$$\mathcal{P}(\mathcal{Z}) = E \subset E_1 = \mathcal{M}_1(\mathcal{Z}) \,. \quad \square$$

Lemma 3.9 (empirical measures as random variables) *Let* $\xi = (\xi_i)_{i=1}^n$ *be a random variable with values in* $D_{\mathcal{Z}^n}[0,\infty)$, *for some* $n = 1, 2, \ldots$.
Then the mapping ν *defined as*

$$\nu(t) = \frac{1}{n} \sum_{i=1}^{n} \delta_{\xi_i(t)} \,, \quad t \in [0,\infty) \,,$$

is a random variable with values in $D_{\mathcal{P}(\mathcal{Z})}[0,\infty)$.

Proof. Consider the mapping $\psi : \mathcal{Z}^n \to \mathcal{P}(\mathcal{Z})$ defined as

$$\psi(\bar{z}) = \frac{1}{n} \sum_{i=1}^{n} \delta_{z_i} \,, \quad \bar{z} = (z_i)_{i=1}^n \,,$$

and the associated mapping $\hat{\psi} : D_{\mathcal{Z}^n}[0,\infty) \to D_{\mathcal{P}(\mathcal{Z})}[0,\infty)$ defined as

$$\hat{\psi}(x)(t) = \psi(x(t)) \,, \quad x \in D_{\mathcal{Z}^n}[0,\infty) \,, \quad t \in [0,\infty) \,.$$

The mapping ψ is continuous. If $\lim_{N\to\infty} \bar{z}^{(N)} = \bar{z}$, then $\lim_{N\to\infty} z_i^{(N)} = z_i$, $\forall i = 1,\ldots,n$, and $\delta_{z_i^{(N)}} \Rightarrow \delta_{z_i}$. Thus, $\psi(\bar{z}^{(N)}) \Rightarrow \psi(\bar{z})$. Therefore, the mapping $\hat{\psi}$ is continuous too (cf. [8, p. 151]), and the mapping $\nu = \hat{\psi}(\xi)$ is measurable. $\square$

Lemma 3.10 (relative compactness criterion) *Let* $(\xi^{(n)})$ *be a sequence of random variables with values in* $D_{\mathcal{M}_1(\mathcal{Z})}[0,\infty)$.

The following condition is sufficient for the relative compactness of the sequence $(\xi^{(n)})$:

$$\lim_{\Delta t\to 0}\limsup_{n\to\infty}\mathcal{E}^{(n)}\max_{i\,:\,t_i<T}\sup_{s\in[t_i,t_{i+1})}\tilde{\varrho}(\xi^{(n)}(s),\xi^{(n)}(t_i))=0\,,\quad \forall T>0\,, \tag{3.14}$$

where $t_i=i\,\Delta t$, $i=0,1,\ldots$, *and* $\tilde{\varrho}$ *is defined in (3.13).*

Proof. We apply [8, Ch. 3, Cor. 7.4]. The first condition (compact containment) is fulfilled because the space $\mathcal{M}_1(\mathcal{Z})$ itself is compact. The second condition is

$$\forall\eta>0\,,\quad \forall T>0\quad \exists\varepsilon>0\,:\,\limsup_{n\to\infty}Prob^{(n)}(w(\xi^{(n)},\varepsilon,T)\geq\eta)\leq\eta\,,$$

where (cf. [8, p. 122])

$$w(y,\varepsilon,T)=\inf_{(t_i)\in\Xi_{\varepsilon,T}}\max_i\sup_{s,t\in[t_i,t_{i+1})}\tilde{\varrho}(y(s),y(t))\,,\quad y\in D_{\mathcal{M}_1(\mathcal{Z})}[0,\infty)\,,$$

and

$$\Xi_{\varepsilon,T}=\{(t_i)\,:\,0=t_0<t_1<\ldots<t_{n-1}<T\leq t_n\,,\ \min_{1\leq i\leq n}(t_i-t_{i-1})>\varepsilon\,,\ n\geq 1\}\,.$$

From Chebyshev's inequality, it follows that the condition

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\mathcal{E}^{(n)}w(\xi^{(n)},\varepsilon,T)=0\,,\quad \forall T>0\,, \tag{3.15}$$

is sufficient. Note that w becomes larger if one chooses a concrete partition (t_i), in particular, $t_i=i\,\Delta t$, $i=0,1,\ldots$, $\Delta t>\varepsilon$, instead of the infimum. From the obvious inequality

$$\sup_{s,t\in[t_i,t_{i+1})}\tilde{\varrho}(y(s),y(t))\leq 2\sup_{s\in[t_i,t_{i+1})}\tilde{\varrho}(y(s),y(t_i))\,,$$

we obtain

$$w(y,\varepsilon,T)\leq 2\max_{i\,:\,t_i<T}\sup_{s\in[t_i,t_{i+1})}\tilde{\varrho}(y(s),y(t_i))\,.$$

Consequently, (3.15) follows from (3.14). □

4. Properties of the limiting equation

In this section we study Eq. (2.13), which is to characterize the limit of the empirical measures. First we show some properties of the operators $T_0(t)^*$ and K_{max} and check that the definitions (2.15)–(2.16) are correct. Then we prove Theorem 2.1. Finally, we study an approximation to the solution of the limiting equation.

Lemma 4.1 (integration of measure–valued functions) *Let E_1, E_2, and S be metric spaces. Let $\mu(x_1,x_2,\Gamma)$ be a function on $E_1\times E_2\times\mathcal{B}_S$ such that $\mu(x_1,x_2)\in\mathcal{M}(S)$, $\forall x_1\in E_1$, $\forall x_2\in E_2$, and $\mu(x_1,x_2,\Gamma)$ is measurable in (x_1,x_2), $\forall\Gamma\in\mathcal{B}_S$. Let α be a measure on E_2 such that*

$$\int_{E_2}\mu(x_1,x_2,S)\,\alpha(dx_2)\le C_{\mu,\alpha}\,,\qquad\forall x_1\in E_1\,. \tag{4.1}$$

Then the function $\hat{\mu}$ on $E_1\times\mathcal{B}_S$ defined as

$$\hat{\mu}(x_1,\Gamma)=\int_{E_2}\mu(x_1,x_2,\Gamma)\,\alpha(dx_2) \tag{4.2}$$

has the properties

$$\hat{\mu}(x_1)\in\mathcal{M}(S)\,,\qquad\forall x_1\in E_1\,, \tag{4.3}$$

$$\hat{\mu}(x_1,S)\le C_{\mu,\alpha}\,,\qquad\forall x_1\in E_1\,, \tag{4.4}$$

$$\hat{\mu}(x_1,\Gamma)\text{ is measurable in }x_1\,,\qquad\forall\Gamma\in\mathcal{B}_S\,. \tag{4.5}$$

Proof. Using the definition (4.2), we prove (4.3) showing σ–additivity:

$$\begin{aligned}\hat{\mu}(x_1,\cup_{n=1}^{\infty}\Gamma_n) &= \int_{E_2}\mu(x_1,x_2,\cup_{n=1}^{\infty}\Gamma_n)\,\alpha(dx_2)=\int_{E_2}\sum_{n=1}^{\infty}\mu(x_1,x_2,\Gamma_n)\,\alpha(dx_2)\\ &= \sum_{n=1}^{\infty}\int_{E_2}\mu(x_1,x_2,\Gamma_n)\,\alpha(dx_2)=\sum_{n=1}^{\infty}\hat{\mu}(x_1,\Gamma_n)\,.\end{aligned}$$

Assertion (4.4) is an obvious consequence of (4.1). Assertion (4.5) follows from the definition and Fubini's theorem. □

Lemma 4.2 *The function $T_0(t)^*(\lambda_0)(\Gamma)$ defined in (2.11) has the properties*

$$T_0(t)^*(\lambda_0) \in \mathcal{M}(\mathcal{Z}), \quad \forall t \in [0,\infty), \ \forall \lambda_0 \in \mathcal{M}(\mathcal{Z}), \tag{4.6}$$

$$T_0(t)^*(\lambda_0)(\mathcal{Z}) = \lambda_0(\mathcal{Z}), \quad \forall t \in [0,\infty), \ \forall \lambda_0 \in \mathcal{M}(\mathcal{Z}), \tag{4.7}$$

$$T_0(t)^*(\lambda_0)(\Gamma) \text{ is measurable in } t, \quad \forall \Gamma \in \mathcal{B}_S, \ \forall \lambda_0 \in \mathcal{M}(\mathcal{Z}), \tag{4.8}$$

$$\|T_0(t)^*(\nu_1) - T_0(t)^*(\nu_2)\| \le \|\nu_1 - \nu_2\|, \ \forall \nu_1, \nu_2 \in \mathcal{M}(\mathcal{Z}). \tag{4.9}$$

Proof. Properties (4.6) and (4.8) follow from Lemma 4.1, with $\mu = U_0$ and $\alpha = \lambda_0$. Here, assumptions (2.1), (2.3) concerning U_0 are used. Properties (4.7) and (4.9) follow from the definitions (2.11) and (3.2). □

Lemma 4.3 *The function $K_{max}(\nu_1, \nu_2)(\Gamma)$ defined in (2.12) has the properties*

$$K_{max}(\nu_1, \nu_2) \in \mathcal{M}(\mathcal{Z}), \quad \forall \nu_1, \nu_2 \in \mathcal{M}(\mathcal{Z}), \tag{4.10}$$

$$K_{max}(\nu_1, \nu_2)(\mathcal{Z}) = 2\, C_{Q,max}\, \nu_1(\mathcal{Z})\, \nu_2(\mathcal{Z}), \quad \forall \nu_1, \nu_2 \in \mathcal{M}(\mathcal{Z}), \tag{4.11}$$

$$\begin{aligned} \|K_{max}(\nu_1, \nu_2) - K_{max}(\tilde{\nu}_1, \tilde{\nu}_2)\| \le \Big[\|\nu_1 - \tilde{\nu}_1\| + \|\nu_2 - \tilde{\nu}_2\|\Big] \times \\ 2\, C_{Q,max} \max(\nu_1(\mathcal{Z}), \nu_2(\mathcal{Z}), \tilde{\nu}_1(\mathcal{Z}), \tilde{\nu}_2(\mathcal{Z})), \quad \forall \nu_1, \nu_2, \tilde{\nu}_1, \tilde{\nu}_2 \in \mathcal{M}(\mathcal{Z}). \end{aligned} \tag{4.12}$$

Proof. Property (4.10) follows from Lemma 4.1, with

$$\mu((z_1, z_2), \Gamma) = Q_{max}(z_1, z_2, \Gamma, \mathcal{Z}) \quad \text{and} \quad \alpha(dz_1, dz_2) = \nu_1(dz_1)\, \nu_2(dz_2).$$

Here, assumptions (2.5), (2.6) concerning Q are used. Properties (4.11) and (4.12) follow from the definitions (2.12), (2.10), and (3.2). □

Next we consider the functions $\nu_k(t)$ defined in (2.15)–(2.16). Their properties are studied by means of the following lemma.

Lemma 4.4 (measurability) *Let E, S_1, S_2 be separable metric spaces, and $\varphi \in B(E \times S_1 \times S_2)$. Let $\mu_i(x,\Gamma)$, $i = 1,2$, be functions on $E \times \mathcal{B}_{S_i}$ such that $\mu_i(x) \in \mathcal{M}(S_i)$, $\mu_i(x,S_i) \le C_\mu$, $\forall x \in E$, and*

$$\mu_i(x,\Gamma) \text{ is measurable in } x\,, \quad \forall \Gamma \in \mathcal{B}_{S_i}\,. \tag{4.13}$$

Then the function

$$\int_{S_2}\int_{S_1} \varphi(x,z_1,z_2)\,\mu_1(x,dz_1)\,\mu_2(x,dz_2)$$

is measurable in x.

Proof. We apply [8, App., Th. 4.3]. Consider the set of functions

$$H = \{\varphi \in B(E \times S_1 \times S_2) \,:\, \int_{S_2}\int_{S_1} \varphi(x,z_1,z_2)\,\mu_1(x,dz_1)\,\mu_2(x,dz_2) \in B(E)\}\,.$$

The set H is linear and contains constants because of (4.13). Furthermore, we consider the system of sets

$$\mathcal{S} = \{\Gamma \times \Gamma_1 \times \Gamma_2 \,:\, \Gamma \in \mathcal{B}_E\,,\, \Gamma_1 \in \mathcal{B}_{S_1}\,,\, \Gamma_2 \in \mathcal{B}_{S_2}\}\,.$$

We have $A_1 \cap A_2 \in \mathcal{S}$ if A_1, $A_2 \in \mathcal{S}$. Furthermore, $\P_A \in H$ if $A \in \mathcal{S}$, and the set H is closed with respect to bp–convergence.

Therefore, H contains all bounded $\sigma(\mathcal{S})$–measurable functions. Since

$$\sigma(\mathcal{S}) = \mathcal{B}_E \times \mathcal{B}_{S_1} \times \mathcal{B}_{S_2} = \mathcal{B}_{E\times S_1 \times S_2}\,,$$

because of the separability, we obtain $H = B(E \times S_1 \times S_2)$. □

Lemma 4.5 *The integration in (2.16) is well-defined in the sense of (4.2).*

The functions $\nu_k(t)(\lambda_0)$, $k \ge 1$, $t \in [0,\infty)$, defined in (2.15)–(2.16) have the following properties:

$$\nu_k(t)(\lambda_0) \in \mathcal{M}(\mathcal{Z})\,, \tag{4.14}$$

$$\nu_k(t)(\lambda_0)(\mathcal{Z}) = e^{-c_0\,t}\,(1-e^{-c_0\,t})^{k-1}\,\lambda_0(\mathcal{Z})\,, \tag{4.15}$$

$$\nu_k(t)(\lambda_0)(\Gamma) \text{ is measurable in } t\,, \quad \forall \Gamma \in \mathcal{B}_{\mathcal{Z}}\,. \tag{4.16}$$

Proof. The proof is performed by induction on k. For $k=1$, the assertions follow from the properties (4.6)–(4.8) of the operator $T_0(t)^*$.

Suppose the properties (4.14)–(4.16) are fulfilled for some $k \geq 1$. Consider $\mu(s,t) = T_0(t-s)^*(\bar{\mu}(s))$, where $\bar{\mu}(s) = K_{max}(\nu_i(s)(\lambda_0), \nu_{k+1-i}(s)(\lambda_0))$ and i is fixed.

First we notice that $\bar{\mu}(s) \in \mathcal{M}(\mathcal{Z})$ and the function $\bar{\mu}(s,\Gamma)$ is measurable in s. This follows from the definition (2.12) of K_{max} and Lemma 4.4 with

$$\begin{aligned} \varphi(z_1, z_2) &= Q_{max}(z_1, z_2, \Gamma, \mathcal{Z}) + Q_{max}(z_1, z_2, \mathcal{Z}, \Gamma), \\ \mu_1(s) &= \nu_i(s)(\lambda_0), \quad \mu_2(s) = \nu_{k+1-i}(s)(\lambda_0). \end{aligned}$$

Then we see that $\mu(s,t) \in \mathcal{M}(\mathcal{Z})$ and the function $\mu(s,t,\Gamma)$ is measurable in (s,t). This follows from the definition (2.11) of $T_0(t)^*$ and Lemma 4.4 with

$$\varphi((t,s),z_1) = U_0(t-s, z_1, \Gamma), \quad \mu_1((t,s)) = \bar{\mu}(s), \quad \mu_2((t,s),\mathcal{Z}) = 1.$$

Therefore, Lemma 4.1 can be applied to the function $\mu(s,t)$, and the properties (4.14) and (4.16) follow for $k+1$.

Using (4.11) and the induction hypothesis, we obtain

$$\begin{aligned} \nu_{k+1}(t)(\lambda_0)(\mathcal{Z}) &= \sum_{i=1}^{k} \int_0^t e^{-c_0\,(t-s)}\, 2\, C_{Q,max}\, \nu_i(s)(\mathcal{Z})\, \nu_{k+1-i}(s)(\mathcal{Z})\, ds \\ &= \sum_{i=1}^{k} \int_0^t e^{-c_0\,(t-s)}\, 2\, C_{Q,max}\, \lambda_0(\mathcal{Z})^2\, e^{-2\,c_0\,s}\, (1-e^{-c_0\,s})^{k-1}\, ds \\ &= \lambda_0(\mathcal{Z})\, e^{-c_0\,t} \int_0^t k\, c_0\, e^{-c_0\,s}\, (1-e^{-c_0\,s})^{k-1}\, ds. \end{aligned}$$

Thus, property (4.15) follows for $k+1$. □

Lemma 4.6 *Define, for* $k=1,2,\ldots$, $t \in [0,\infty)$, *and* $\lambda_0 \in \mathcal{M}(\mathcal{Z})$,

$$\lambda_k(t) = \sum_{i=1}^{k} \nu_i(t)(\lambda_0) \tag{4.17}$$

and

$$\hat{\lambda}_k(t) = e^{-c_0\,t}\, T_0(t)^*(\lambda_0) + \int_0^t e^{-c_0\,(t-s)}\, T_0(t-s)^*\, K_{max}(\lambda_k(s), \lambda_k(s))\, ds. \tag{4.18}$$

Then

$$\lambda_{k+1}(t) \leq \hat{\lambda}_k(t) \leq \lambda_{2\,k}(t). \tag{4.19}$$

Proof. It follows from (4.18) and (4.17) that

$$\begin{aligned}\hat{\lambda}_k(t) &= e^{-c_0 t}\, T_0(t)^*(\lambda_0) + \\ &\sum_{i=1}^{k}\sum_{j=1}^{k}\int_0^t e^{-c_0\,(t-s)}\, T_0(t-s)^*\, K_{max}(\nu_i(s)(\lambda_0), \nu_j(s)(\lambda_0))\, ds\,.\end{aligned}$$

Omitting in the double sum the terms with $i+j > k+1$, and changing the summation variables ($i+j=l$), we obtain

$$\begin{aligned}\hat{\lambda}_k(t) &\geq e^{-c_0 t}\, T_0(t)^*(\lambda_0) + \\ &\sum_{l=2}^{k+1}\sum_{i=1}^{l-1}\int_0^t e^{-c_0\,(t-s)}\, T_0(t-s)^*\, K_{max}(\nu_i(s)(\lambda_0), \nu_{l-i}(s)(\lambda_0))\, ds \\ &= e^{-c_0 t}\, T_0(t)^*(\lambda_0) + \sum_{l=2}^{k+1}\nu_l(t)(\lambda_0) = \lambda_{k+1}(t)\,,\end{aligned}$$

according to (2.15), (2.16), and (4.17). The other inequality in (4.19) is proved in an analogous way. □

Proof of Theorem 2.1. First we show the convergence of the series (2.17). Since $\nu_i(t)(\lambda_0) \in \mathcal{M}(\mathcal{Z})$, $\forall i$, it is sufficient to prove convergence of the masses. This follows from Lemma 4.5 via the estimate

$$\sup_{0\leq s\leq t} \|\lambda(s) - \lambda_k(s)\| \leq \lambda_0(\mathcal{Z})\,(1-e^{-c_0 t})^k\,. \tag{4.20}$$

It remains to prove that $\lambda(t)$ satisfies Eq. (2.13). It follows from Lemma 4.6 and the obvious inequality $\lambda_k(t) \leq \lambda(t)$ that

$$\|\hat{\lambda}_k(t) - \lambda(t)\| \leq \|\lambda_{k+1}(t) - \lambda(t)\|\,.$$

Thus,

$$\lim_{k\to\infty}\sup_{0\leq s\leq t} \|\hat{\lambda}_k(s) - \lambda(s)\| = 0\,,\ \forall t > 0\,,$$

according to (4.20). Taking the limit $k\to\infty$ in (4.18) shows that $\lambda(t)$ satisfies Eq. (2.13). Uniqueness of the solution follows from the Lipschitz properties (4.9) and (4.12) of the operators $T_0(t)^*$ and K_{max}, respectively, and from Gronwall's inequality. □

In the remainder of this section, we will study an approximation to the solution of Eq. (2.13).

To this end, we introduce an approximation $\tilde{A}_0^{(N)}$ to the free flow generator A_0, defined as

$$\tilde{A}_0^{(N)} = N\,[T_0(\frac{1}{N}) - I]\,, \quad N = 1,2,\dots\,. \tag{4.21}$$

Here, $T_0(t)$ is the semigroup of operators on $B(\mathcal{Z})$ associated with the transition function U_0, and I denotes the identity operator.

Lemma 4.7 *Let $\varphi \in B(\mathcal{Z})$ be such that*

$$\lim_{t\to 0} \|T_0(t)(\varphi) - \varphi\| = 0\,. \tag{4.22}$$

Let $\tilde{T}_0^{(N)}(t)$ be the semigroup corresponding to the generator $\tilde{A}_0^{(N)}$ defined in (4.21).

Then

$$\lim_{N\to\infty} \|\tilde{T}_0^{(N)}(t)(\varphi) - T_0(t)(\varphi)\| = 0\,, \quad \forall t \geq 0\,. \tag{4.23}$$

Proof. Since $\tilde{A}_0^{(N)}$ is bounded, the corresponding semigroup has the form

$$\tilde{T}_0^{(N)}(t) = \exp(t\,\tilde{A}_0^{(N)}) = \exp(-N\,t)\sum_{k=0}^{\infty}\frac{(N\,t)^k}{k!}T_0(\frac{k}{N})\,. \tag{4.24}$$

Therefore, we obtain the estimate

$$\begin{aligned}
&\|\tilde{T}_0^{(N)}(t)(\varphi) - T_0(t)(\varphi)\| \leq \\
&\leq \exp(-N\,t)\sum_{k=0}^{\infty}\frac{(N\,t)^k}{k!}\|T_0(\frac{k}{N})(\varphi) - T_0(t)(\varphi)\| \\
&\leq \exp(-N\,t)\sum_{k\,:\,|k-Nt|>\alpha_N}\frac{(N\,t)^k}{k!}\|T_0(\frac{k}{N})(\varphi) - T_0(t)(\varphi)\| + \\
&\quad \exp(-N\,t)\sum_{k\,:\,|k-Nt|\leq\alpha_N}\frac{(N\,t)^k}{k!}\|T_0(\frac{k}{N})(\varphi) - T_0(t)(\varphi)\| \\
&\leq 2\,\|\varphi\|\,\frac{t\,N}{(\alpha_N)^2} + \sup_{s\,:\,|s-t|\leq\frac{\alpha_N}{N}}\|T_0(s)(\varphi) - T_0(t)(\varphi)\| \\
&\leq 2\,\|\varphi\|\,\frac{t\,N}{(\alpha_N)^2} + \sup_{0\leq h\leq\frac{\alpha_N}{N}}\|T_0(h)(\varphi) - \varphi\|\,.
\end{aligned}$$

Now assertion (4.23) follows from (4.22) for an appropriate choice of α_N. $\square$

Let $\tilde{\lambda}^{(N)}(t)$ denote the solution of Eq. (2.13) with T_0 replaced by $\tilde{T}_0^{(N)}$, i.e.

$$\tilde{\lambda}^{(N)}(t) = e^{-c_0\, t}\, \tilde{T}_0^{(N)}(t)^*(\lambda_0) + \int_0^t e^{-c_0\,(t-s)}\, \tilde{T}_0^{(N)}(t-s)^*\, K_{max}(\tilde{\lambda}^{(N)}(s), \tilde{\lambda}^{(N)}(s))\, ds\,. \tag{4.25}$$

Lemma 4.8 (approximation of the solution) *Suppose*

$$K_{max} \quad \textit{is continuous on } \mathcal{M}_c(\mathcal{Z}) \times \mathcal{M}_c(\mathcal{Z})\,, \qquad c = \lambda_0(\mathcal{Z})\,, \tag{4.26}$$

and $T_0(t)$ satisfies (2.21) and (2.22).

Then $\quad \lim_{N\to\infty} \varrho_L(\tilde{\lambda}^{(N)}(t), \lambda(t)) = 0\,, \quad \forall t \in [0,\infty)\,.$

Proof. According to Theorem 2.1, we have

$$\tilde{\lambda}^{(N)}(t) = \sum_{i=1}^{\infty} \tilde{\nu}_i^{(N)}(t)(\lambda_0)\,, \tag{4.27}$$

where $\quad \tilde{\nu}_1^{(N)}(t)(\lambda_0) = e^{-c_0\, t}\, \tilde{T}_0^{(N)}(t)^*(\lambda_0)$, and, for $k \ge 1$,

$$\tilde{\nu}_{k+1}^{(N)}(t)(\lambda_0) = \sum_{i=1}^{k} \int_0^t e^{-c_0\,(t-s)}\, \tilde{T}_0^{(N)}(t-s)^*\, K_{max}(\tilde{\nu}_i^{(N)}(s)(\lambda_0), \tilde{\nu}_{k+1-i}^{(N)}(s)(\lambda_0))\, ds\,. \tag{4.28}$$

Since the masses are identical, it is sufficient to show

$$\lim_{N\to\infty} \langle \varphi, \tilde{\lambda}^{(N)}(t) \rangle = \langle \varphi, \lambda(t) \rangle\,, \quad \forall \varphi \in \hat{C}(\mathcal{Z})\,.$$

It follows from (4.27) that $\quad \langle \varphi, \tilde{\lambda}^{(N)}(t) \rangle = \sum_{i=1}^{\infty} \langle \varphi, \tilde{\nu}_i^{(N)}(t)(\lambda_0) \rangle$. Since there is a majorant uniformly in N, it is sufficient to show

$$\lim_{N\to\infty} \langle \varphi, \tilde{\nu}_i^{(N)}(t)(\lambda_0) \rangle = \langle \varphi, \nu_i(t)(\lambda_0) \rangle\,, \quad \forall i, \quad \forall \varphi \in \hat{C}(\mathcal{Z})\,. \tag{4.29}$$

We proceed by induction on i. For $i = 1$, assertion (4.29) follows from Lemma 4.7. The function under the integral in (4.28) is uniformly bounded.

The masses of the measures $\tilde{\nu}_i^{(N)}(s)(\lambda_0)$ are bounded by the constant c because of (4.15). According to assumption (4.26), to perform the induction step, it is sufficient to show

$$\lim_{N\to\infty} \langle \varphi, \tilde{T}_0^{(N)}(t)^*(\bar{\nu}_N)\rangle = \langle \varphi, T_0(t)^*(\bar{\nu})\rangle\,,$$

for any sequence $\bar{\nu}_N$ converging to $\bar{\nu}$ in $\mathcal{M}_c(\mathcal{Z})$. We have

$$|\langle \varphi, \tilde{T}_0^{(N)}(t)^*(\bar{\nu}_N)\rangle - \langle \varphi, T_0(t)^*(\bar{\nu})\rangle| \le \tag{4.30}$$
$$|\langle \tilde{T}_0^{(N)}(t)(\varphi), \bar{\nu}_N\rangle - \langle T_0(t)(\varphi), \bar{\nu}_N\rangle| + |\langle T_0(t)(\varphi), \bar{\nu}_N\rangle - \langle T_0(t)(\varphi), \bar{\nu}\rangle|\,.$$

The second term on the right–hand side of (4.30) tends to zero because of assumption (2.21). The first term can be estimated by the term

$$c\,\|\tilde{T}_0^{(N)}(t)(\varphi) - T_0(t)(\varphi)\|\,,$$

which tends to zero as $N \to \infty$ according to Lemma 4.7. □

5. Properties of the Markov process

In this section we study some properties of the Markov process with the generator (2.7)–(2.8). In particular, we establish relative compactness of the empirical measures defined in (1.10).

Let $\mathcal{Y}$ be a locally compact separable metric space. Let $S_0(t,y,\Gamma)$ be a transition function on $[0,\infty)\times\mathcal{Y}\times\mathcal{B}_{\mathcal{Y}}$ (cf. the properties (2.1)–(2.4)), and $\mathcal{S}_0(t)$ denote the corresponding semigroup on $B(\mathcal{Y})$. Suppose $H(y,\Gamma)$ is a function on $\mathcal{Y}\times\mathcal{B}_{\mathcal{Y}}$ such that $H(y)\in\mathcal{M}(\mathcal{Y})$, $H(y,\mathcal{Y})\le C_{H,max}$, $\forall y\in\mathcal{Y}$, and $H(.,\Gamma)$ is measurable, $\forall\Gamma\in\mathcal{B}_{\mathcal{Y}}$. Let $\mathcal{H}$ denote the operator on $B(\mathcal{Y})$ defined as

$$\mathcal{H}(\varphi)(y) = \int_{\mathcal{Y}} [\varphi(\tilde{y}) - \varphi(y)]\, H(y, d\tilde{y})\,, \quad y\in\mathcal{Y}\,.$$

Lemma 5.1 *Suppose*

$$\mathcal{S}_0(t)(\varphi)\in\hat{C}(\mathcal{Y})\,, \quad \forall t\ge 0\,, \quad \forall\varphi\in\hat{C}(\mathcal{Y})\,, \tag{5.1}$$

and

$$\lim_{t\to 0}\mathcal{S}_0(t)(\varphi) = \varphi\,, \quad \forall\varphi\in\hat{C}(\mathcal{Y})\,. \tag{5.2}$$

Suppose

$$\mathcal{H}(\varphi) \in \hat{C}(\mathcal{Y}), \quad \forall \varphi \in \hat{C}(\mathcal{Y}). \tag{5.3}$$

Then

$$\mathcal{S}(t)(\varphi) = \mathcal{S}_0(t)(\varphi) + \sum_{l=1}^{\infty} \int_0^t dt_1 \ldots \int_0^{t_{l-1}} dt_l \\ \mathcal{S}_0(t-t_1)\,\mathcal{H} \ldots \mathcal{S}_0(t_{l-1}-t_l)\,\mathcal{H}\,\mathcal{S}_0(t_l)(\varphi), \tag{5.4}$$

defines a Feller semigroup on $\hat{C}(\mathcal{Y})$ (cf. [8, Ch. 4, Sect. 2]). The generator of $\mathcal{S}(t)$ has the form $\mathcal{G}_0 + \mathcal{H}$, where $\mathcal{G}_0$ is the generator corresponding to $\mathcal{S}_0(t)$.

Proof. Let $\varphi \in \hat{C}(\mathcal{Y})$, and denote $\psi_1(t) = \mathcal{S}_0(t)(\varphi)$,

$$\psi_{k+1}(t) = \mathcal{S}_0(t)(\varphi) + \int_0^t \mathcal{S}_0(t-s)\,\mathcal{H}(\psi_k(s))\,ds, \quad k \geq 1. \tag{5.5}$$

The function $\psi_1(t)$ is continuous in $\hat{C}(\mathcal{Y})$ because of (5.1) and (5.2). It is easy to show that the function $\mathcal{S}_0(t-s)\,\mathcal{H}(\psi_k(s))$ is continuous with respect to s, provided that $\psi_k(s)$ is continuous. Thus, the integration in (5.5) is defined as the integration of continuous Banach–space–valued functions (cf. [8, Ch. 1, Sect. 1]). Furthermore, the function $\int_0^t \mathcal{S}_0(t-s)\,\mathcal{H}(\psi_k(s))\,ds$ is continuous with respect to t so that the definition (5.5) is correct.

One obtains that $\mathcal{S}(t)(\varphi) = \lim_{k\to\infty} \psi_k(t)$, and $\mathcal{S}(t)$ is an operator in $\hat{C}(\mathcal{Y})$. The estimate

$$\|\mathcal{S}(t)(\varphi) - \varphi\| \leq \|\mathcal{S}_0(t)(\varphi) - \varphi\| + e^{C_{H,max}\,t} - 1 \tag{5.6}$$

yields continuity at $t = 0$.

To prove the remaining properties of the operators $\mathcal{S}(t)$ (positive contraction semigroup), we use the approximation result established in Lemma 4.7,

$$\lim_{N\to\infty} \|\tilde{\mathcal{S}}_0^{(N)}(t)(\varphi) - \mathcal{S}_0(t)(\varphi)\| = 0, \quad \forall \varphi \in \hat{C}(\mathcal{Y}).$$

Using (5.4), we derive an analogous result for $\mathcal{S}(t)$,

$$\lim_{N\to\infty} \|\tilde{\mathcal{S}}^{(N)}(t)(\varphi) - \mathcal{S}(t)(\varphi)\| = 0, \quad \forall \varphi \in \hat{C}(\mathcal{Y}).$$

Now, the remaining properties of $\mathcal{S}(t)$ follow from the corresponding properties of $\tilde{\mathcal{S}}^{(N)}(t)$.

Finally, we calculate the generator of $\mathcal{S}(t)$. Since $\mathcal{S}(t)(\varphi)$ satisfies the equation

$$\mathcal{S}(t)(\varphi) = \mathcal{S}_0(t)(\varphi) + \int_0^t \mathcal{S}_0(t-s)\,\mathcal{H}\,\mathcal{S}(s)(\varphi)\,ds\,, \tag{5.7}$$

one obtains

$$\begin{aligned} \frac{1}{t}\Big[\mathcal{S}(t)(\varphi) - \varphi\Big] &= \frac{1}{t}\Big[\mathcal{S}_0(t)(\varphi) - \varphi\Big] + \mathcal{S}_0(t)\,\mathcal{H}(\varphi) + \\ &\quad \frac{1}{t}\int_0^t \Big[\mathcal{S}_0(t-s)\,\mathcal{H}\,\mathcal{S}(s)(\varphi) - \mathcal{S}_0(t)\,\mathcal{H}(\varphi)\Big]\,ds\,. \end{aligned} \tag{5.8}$$

The first term on the right–hand side of (5.8) tends to $\mathcal{G}_0(\varphi)$, the second term tends to $\mathcal{H}(\varphi)$, because of (5.2) and (5.3). The third term is estimated as follows,

$$\begin{aligned} &\frac{1}{t}\Big\|\int_0^t \Big[\mathcal{S}_0(t-s)\,\mathcal{H}\,\mathcal{S}(s)(\varphi) - \mathcal{S}_0(t)\,\mathcal{H}(\varphi)\Big]\,ds\Big\| \le \\ &\le \frac{1}{t}\int_0^t \|\mathcal{S}_0(t-s)\,\mathcal{H}\,\mathcal{S}(s)(\varphi) - \mathcal{S}_0(t-s)\,\mathcal{H}(\varphi)\|\,ds + \\ &\quad \frac{1}{t}\int_0^t \|\mathcal{S}_0(t-s)\,\mathcal{H}(\varphi) - \mathcal{S}_0(t)\,\mathcal{H}(\varphi)\|\,ds \\ &\le 2\,C_{H,max} \sup_{0\le s\le t} \|\mathcal{S}(s)(\varphi) - \varphi\| + \sup_{0\le s\le t} \|\mathcal{S}_0(s)\,\mathcal{H}(\varphi) - \mathcal{H}(\varphi)\|\,. \end{aligned} \tag{5.9}$$

The right–hand side of (5.9) tends to zero as $t \to 0$, because of (5.6), (5.2), and (5.3). □

Remark 5.2 *According to [8, Ch. 4, Th. 2.7], there exists a process $Y(t)$ with sample paths in $D_{\mathcal{Y}}[0,\infty)$ corresponding to the semigroup $\mathcal{S}(t)$. This process can be considered as a random variable with values in $D_{\mathcal{Y}}[0,\infty)$ (cf. [8, p. 128]).*

Lemma 5.3 (martingale representation) *Let $Z^{(n)}(t)$ be a Markov process with the generator (2.7)–(2.8). Suppose Q satisfies (2.9). Let $\mu^{(n)}(t)$ denote the empirical measures as defined in (1.10).*

Let $\varphi \in \hat{C}(\mathcal{Z})$ be such that φ, $\varphi^2 \in \mathcal{D}(A_0)$ (the domain of the generator A_0).

Then the following representation holds,

$$\langle \varphi, \mu^{(n)}(t)\rangle = \int_0^t \langle A_0(\varphi), \mu^{(n)}(s)\rangle \, ds + \int_0^t \langle \varphi, K_{max}(\mu^{(n)}(s), \mu^{(n)}(s))\rangle \, ds$$
$$-2\, C_{Q,max} \int_0^t \langle \varphi, \mu^{(n)}(s)\rangle \, ds + R^{(n)}(\varphi, t) + M^{(n)}(\varphi, t)\,, \qquad (5.10)$$

where K_{max} is defined in (2.12),

$$|R^{(n)}(\varphi, t) - R^{(n)}(\varphi, s)| \le \frac{1}{n}\, 4\, \|\varphi\|\, C_{Q,max}\, (t-s) \quad a.s.\,, \qquad (5.11)$$

and $M^{(n)}$ is a martingale such that

$$\mathcal{E}^{(n)}[M^{(n)}(\varphi, t) - M^{(n)}(\varphi, s)]^2 \le \qquad (5.12)$$
$$\left\{\frac{2}{n}\, \|\varphi\, A_0(\varphi)\| + \frac{1}{n}\, \|A_0(\varphi^2)\| + \frac{1}{n}\, 16\, \|\varphi\|^2\, C_{Q,max}\right\} (t-s)\,.$$

Proof. If $\Phi \in \mathcal{D}(\mathcal{A})$, then the process

$$M(t) = \Phi(Z(t)) - \int_0^t \mathcal{A}(\Phi)(Z(s))\, ds \qquad (5.13)$$

is a martingale (cf., e.g., [8, Ch. 4, Prop. 1.7]). Moreover, if $\Phi^2 \in \mathcal{D}(\mathcal{A})$, then one can show that

$$\mathcal{E}[M(t) - M(s)]^2 = \mathcal{E} \int_s^t [\mathcal{A}(\Phi^2) - 2\, \Phi\, \mathcal{A}(\Phi)](Z(u))\, du\,. \qquad (5.14)$$

We introduce the notations

$$\mathcal{A}_0^{(n)}(\Phi)(\bar{z}) = \sum_{i=1}^n A_{0,z_i}(\Phi)(\bar{z})\,, \qquad (5.15)$$

and

$$\mathcal{Q}^{(n)}(\Phi)(\bar{z}) \;=\; \frac{1}{n} \sum_{1\le i\ne j\le n} \int_{\mathcal{Z}} \int_{\mathcal{Z}} \Big[\Phi(J(\bar{z}, i, j, \tilde{z}_1, \tilde{z}_2)) - \Phi(\bar{z})\Big] \times \qquad (5.16)$$
$$Q(z_i, z_j, d\tilde{z}_1, d\tilde{z}_2)\,, \quad \bar{z} = (z_i)_{i=1}^n\,,\ z_i \in \mathcal{Z}\,,$$

where J is defined in (2.8).

We apply (5.13)–(5.14) to the generator (2.7)–(2.8), which, with the above notations, takes the form

$$\mathcal{A}^{(n)} = \mathcal{A}_0^{(n)} + \mathcal{Q}^{(n)}\,, \tag{5.17}$$

and to the function

$$\Phi^{(n)}(\bar{z}) = \frac{1}{n}\sum_{i=1}^{n} \varphi(z_i)\,. \tag{5.18}$$

Notice that

$$\Phi^{(n)}(Z^{(n)}(t)) = \langle \varphi, \mu^{(n)}(t)\rangle\,. \tag{5.19}$$

Taking into account (2.8), we obtain

$$\Phi^{(n)}(J(\bar{z},i,j,\tilde{z}_1,\tilde{z}_2)) = \Phi^{(n)}(\bar{z}) + \frac{1}{n}\Big[\varphi(\tilde{z}_1)+\varphi(\tilde{z}_2)-\varphi(z_i)-\varphi(z_j)\Big]\,, \tag{5.20}$$

and

$$\begin{aligned}[]
&[\Phi^{(n)}]^2(J(\bar{z},i,j,\tilde{z}_1,\tilde{z}_2)) = [\Phi^{(n)}]^2(\bar{z}) + 2\,\Phi^{(n)}(\bar{z})\times \\
&\quad \frac{1}{n}\Big[\varphi(\tilde{z}_1)+\varphi(\tilde{z}_2)-\varphi(z_i)-\varphi(z_j)\Big] + \frac{1}{n^2}\Big[\varphi(\tilde{z}_1)+\varphi(\tilde{z}_2)-\varphi(z_i)-\varphi(z_j)\Big]^2.
\end{aligned} \tag{5.21}$$

By (5.16), (5.20), and (5.21), we have

$$\begin{aligned}
&\mathcal{Q}^{(n)}(\Phi^{(n)})(\bar{z}) = \\
&\quad \frac{1}{n^2}\sum_{1\le i\ne j\le n}\int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big[\varphi(\tilde{z}_1)+\varphi(\tilde{z}_2)-\varphi(z_i)-\varphi(z_j)\Big]\,Q(z_i,z_j,d\tilde{z}_1,d\tilde{z}_2)
\end{aligned} \tag{5.22}$$

and

$$\begin{aligned}
&\mathcal{Q}^{(n)}([\Phi^{(n)}]^2)(\bar{z}) = 2\,\Phi^{(n)}(\bar{z})\,\mathcal{Q}^{(n)}(\Phi^{(n)})(\bar{z}) + \\
&\quad \frac{1}{n^3}\sum_{1\le i\ne j\le n}\int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big[\varphi(\tilde{z}_1)+\varphi(\tilde{z}_2)-\varphi(z_i)-\varphi(z_j)\Big]^2\,Q(z_i,z_j,d\tilde{z}_1,d\tilde{z}_2)\,.
\end{aligned} \tag{5.23}$$

Furthermore, it follows from (5.18) and (5.15) that

$$A_{0,z_i}(\Phi^{(n)})(\bar{z}) = \frac{1}{n}A_0(\varphi)(z_i)$$

and

$$\mathcal{A}_0^{(n)}(\Phi^{(n)})(\bar{z}) = \frac{1}{n}\sum_{i=1}^{n} A_0(\varphi)(z_i)\,. \tag{5.24}$$

Since $[\Phi^{(n)}]^2(\bar{z}) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j\neq i}\varphi(z_i)\,\varphi(z_j) + \frac{1}{n^2}\sum_{i=1}^{n}\varphi^2(z_i)$, we have

$$\begin{aligned} A_{0,z_k}([\Phi^{(n)}]^2)(\bar{z}) \;=\; & \frac{1}{n^2}\sum_{j\neq k} A_0(\varphi)(z_k)\,\varphi(z_j) + \\ & \frac{1}{n^2}\sum_{i\neq k}\varphi(z_i)\,A_0(\varphi)(z_k) + \frac{1}{n^2}A_0(\varphi^2)(z_k)\,, \end{aligned} \tag{5.25}$$

and, by (5.15),

$$\begin{aligned} \mathcal{A}_0^{(n)}([\Phi^{(n)}]^2)(\bar{z}) &= \frac{2}{n^2}\sum_{k=1}^{n}\sum_{j\neq k} A_0(\varphi)(z_k)\,\varphi(z_j) + \frac{1}{n^2}\sum_{k=1}^{n} A_0(\varphi^2)(z_k) \\ &= 2\,\Phi^{(n)}(\bar{z})\,\mathcal{A}_0^{(n)}(\Phi^{(n)})(\bar{z}) - \frac{2}{n^2}\sum_{k=1}^{n} A_0(\varphi)(z_k)\,\varphi(z_k) + \frac{1}{n^2}\sum_{k=1}^{n} A_0(\varphi^2)(z_k)\,. \end{aligned} \tag{5.26}$$

Consequently, by (5.17), (5.22), and (5.24), we obtain

$$\begin{aligned} &\mathcal{A}^{(n)}(\Phi^{(n)})(Z^{(n)}(s)) = \int_{\mathcal{Z}} A_0(\varphi)(z)\,\mu^{(n)}(s,dz) \\ &+ \int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big\{\int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big[\varphi(\tilde{z}_1)+\varphi(\tilde{z}_2)-\varphi(z_1)-\varphi(z_2)\Big]\times \\ &\qquad Q(z_1,z_2,d\tilde{z}_1,d\tilde{z}_2)\Big\}\mu^{(n)}(s,dz_1)\,\mu^{(n)}(s,dz_2) \\ &- \frac{1}{n}\int_{\mathcal{Z}}\Big\{\int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big[\varphi(\tilde{z}_1)+\varphi(\tilde{z}_2)-2\,\varphi(z)\Big]\,Q(z,z,d\tilde{z}_1,d\tilde{z}_2)\Big\}\mu^{(n)}(s,dz)\,. \end{aligned} \tag{5.27}$$

Furthermore, by (5.17), (5.23), and (5.26), we have

$$\begin{aligned} &\Big[\mathcal{A}^{(n)}([\Phi^{(n)}]^2) - 2\,\Phi^{(n)}\,\mathcal{A}^{(n)}(\Phi^{(n)})\Big](Z^{(n)}(u)) = \\ &= \Big[\mathcal{A}_0^{(n)}([\Phi^{(n)}]^2) - 2\,\Phi^{(n)}\,\mathcal{A}_0^{(n)}(\Phi^{(n)}) + \\ &\qquad \mathcal{Q}^{(n)}([\Phi^{(n)}]^2) - 2\,\Phi^{(n)}\,\mathcal{Q}^{(n)}(\Phi^{(n)})\Big](Z^{(n)}(u)) \\ &= -\frac{2}{n}\int_{\mathcal{Z}}\varphi(z)\,A_0(\varphi)(z)\,\mu^{(n)}(u,dz) + \frac{1}{n}\int_{\mathcal{Z}} A_0(\varphi^2)(z)\,\mu^{(n)}(u,dz) + \end{aligned} \tag{5.28}$$

$$\frac{1}{n}\int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big\{\int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big[\varphi(\tilde z_1)+\varphi(\tilde z_2)-\varphi(z_1)-\varphi(z_2)\Big]^2 \times$$
$$Q(z_1,z_2,d\tilde z_1,d\tilde z_2)\Big\}\mu^{(n)}(u,dz_1)\,\mu^{(n)}(u,dz_2)\,-$$
$$\frac{1}{n^2}\int_{\mathcal{Z}}\Big\{\int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big[\varphi(\tilde z_1)+\varphi(\tilde z_2)-2\,\varphi(z)\Big]^2 Q(z,z,d\tilde z_1,d\tilde z_2)\Big\}\mu^{(n)}(u,dz)\,.$$

The representation (5.10) follows from (5.13), (5.19), (5.27), and (2.18), with

$$R^{(n)}(\varphi,t)=$$
$$\int_0^t \frac{1}{n}\int_{\mathcal{Z}}\Big\{\int_{\mathcal{Z}}\int_{\mathcal{Z}}\Big[\varphi(\tilde z_1)+\varphi(\tilde z_2)-2\,\varphi(z)\Big]\,Q(z,z,d\tilde z_1,d\tilde z_2)\Big\}\mu^{(n)}(s,dz)\,ds\,.$$

The estimate (5.12) follows from (5.14) and (5.28). □

Lemma 5.4 (relative compactness) *Let $Z^{(n)}(t)$ be a Markov process with the generator (2.7)–(2.8) and sample paths in the space $D_{\mathcal{Z}^n}[0,\infty)$. Suppose Q satisfies (2.9). Let $\mu^{(n)}(t)$ denote the empirical measures as defined in (1.10).*

Suppose (φ_k) is a dense subset of $\hat C(\mathcal{Z})$ such that

$$\varphi_k\,,\ \varphi_k^2\in\mathcal{D}(A_0)\,,\quad \forall k\ge 1\,. \tag{5.29}$$

Then the sequence $\mu^{(n)}$ is relatively compact as a sequence of random variables with values in the space $D_{(\mathcal{M}_1(\mathcal{Z}),\tilde\varrho)}[0,\infty)$, where $\tilde\varrho$ is defined in (3.13).

Proof. According to Lemma 3.10, it is sufficient to check condition

$$\lim_{\Delta t\to 0}\limsup_{n\to\infty}\mathcal{E}^{(n)}\max_{i\,:\,t_i<T}\ \sup_{t\in[t_i,t_{i+1})}\ \tilde\varrho(\mu^{(n)}(t),\mu^{(n)}(t_i))=0\,,\quad \forall T>0\,, \tag{5.30}$$

where $t_i=i\,\Delta t$, $i=0,1,\ldots$. One obtains from (5.10)–(5.11)

$$|\langle\varphi_k,\mu^{(n)}(t)\rangle-\langle\varphi_k,\mu^{(n)}(t_i)\rangle|\le$$
$$c_1\Big[\|A_0(\varphi_k)\|+\|\varphi_k\|\Big](t-t_i)+|M^{(n)}(\varphi_k,t)-M^{(n)}(\varphi_k,t_i)|\,,$$

where $c_1 = 1 + 8\, C_{Q,max}$ and $k \geq 1$. Consequently,

$$
\begin{aligned}
&\sup_{t\in[t_i,t_{i+1})} \tilde{\varrho}(\mu^{(n)}(t), \mu^{(n)}(t_i)) \leq \\
&\leq \quad \sup_{t\in[t_i,t_{i+1})} \sum_{k=1}^{\infty} \frac{1}{2^k} \min\Big(1, c_1\, [\|A_0(\varphi_k)\| + \|\varphi_k\|]\, (t-t_i)\Big) + \\
&\qquad \sup_{t\in[t_i,t_{i+1})} \sum_{k=1}^{\infty} \frac{1}{2^k} \min\Big(1, |M^{(n)}(\varphi_k, t) - M^{(n)}(\varphi_k, t_i)|\Big) \\
&\leq \quad \Delta t\, c_1 \sum_{k=1}^{\infty} \frac{1}{2^k} \min\Big(1, [\|A_0(\varphi_k)\| + \|\varphi_k\|]\Big) + \\
&\qquad \sum_{k=1}^{\infty} \frac{1}{2^k} \min\Big(1, \sup_{t\in[t_i,t_{i+1})} |M^{(n)}(\varphi_k, t) - M^{(n)}(\varphi_k, t_i)|\Big) .
\end{aligned} \tag{5.31}
$$

The first term on the right–hand side of (5.31) does not depend on i, ω, n and, therefore, disappears when $\Delta t \to 0$. Consequently, to establish (5.30), it is sufficient to show

$$
\lim_{n\to\infty} \mathcal{E}^{(n)} \max_{i\,:\,t_i<T} \sum_{k=1}^{\infty} \frac{1}{2^k} \min\Big(1, \sup_{t\in[t_i,t_{i+1})} |M^{(n)}(\varphi_k, t) - M^{(n)}(\varphi_k, t_i)|\Big) = 0\,, \qquad \forall \Delta t > 0\,. \tag{5.32}
$$

Since the maximum is taken over a finite set, (5.32) is fulfilled if

$$
\lim_{n\to\infty} \mathcal{E}^{(n)} \sum_{k=1}^{\infty} \frac{1}{2^k} \min\Big(1, \sup_{t\in[t_i,t_{i+1})} |M^{(n)}(\varphi_k, t) - M^{(n)}(\varphi_k, t_i)|\Big) = 0\,, \qquad \forall i\,, \quad \forall \Delta t > 0\,. \tag{5.33}
$$

To establish (5.33), it is sufficient to show

$$
\lim_{n\to\infty} \mathcal{E}^{(n)} \sup_{t\in[t_i,t_{i+1})} |M^{(n)}(\varphi_k, t) - M^{(n)}(\varphi_k, t_i)| = 0\,, \quad \forall k\,, \quad \forall i\,, \quad \forall \Delta t > 0\,.
$$

Moreover, according to Doob's inequality, it is sufficient if

$$
\lim_{n\to\infty} \mathcal{E}^{(n)} |M^{(n)}(\varphi_k, t_{i+1}) - M^{(n)}(\varphi_k, t_i)|^2 = 0\,, \quad \forall k\,, \quad \forall i\,, \quad \forall \Delta t > 0\,.
$$

This is fulfilled because of (5.12). $\square$

Lemma 5.5 *Suppose A_0 corresponds to a transition function U_0 determined via (2.23) by a deterministic flow F and such that the properties (2.21)–(2.22) are fulfilled.*

Then there exists a dense subset (φ_k) of $\hat{C}(\mathcal{Z})$ such that (5.29) is fulfilled.

Proof. Let (ψ_k) be a dense subset of $\hat{C}(\mathcal{Z})$. Since $T_0(t)$ is strongly continuous on $\hat{C}(\mathcal{Z})$, according to (2.21), (2.22), the set $\mathcal{D}(A_0)$ is dense in $\hat{C}(\mathcal{Z})$. Hence, for any k, there exist functions $\psi_{k,l} \in \mathcal{D}(A_0)$ such that $\psi_k = \lim_{l\to\infty} \psi_{k,l}$. Obviously, the set $\psi_{k,l}$ is dense in $\hat{C}(\mathcal{Z})$.

Thus, it is sufficient to show that $\varphi^2 \in \mathcal{D}(A_0)$ for any $\varphi \in \mathcal{D}(A_0)$. From

$$\frac{1}{t}[\varphi^2(F(t,z)) - \varphi^2(z)] - 2\varphi(z)\, A_0(\varphi)(z) =$$
$$\Big[\frac{1}{t}[\varphi(F(t,z)) - \varphi(z)]\,[\varphi(F(t,z)) + \varphi(z)] - A_0(\varphi)(z)\,[\varphi(F(t,z)) + \varphi(z)]\Big]$$
$$+\Big[A_0(\varphi)(z)\,[\varphi(F(t,z)) + \varphi(z)] - A_0(\varphi)(z)\,[\varphi(z) + \varphi(z)]\Big],$$

one finds the estimate

$$\|\frac{1}{t}[T_0(t)(\varphi^2) - \varphi^2] - 2\varphi\, A_0(\varphi)\| \le$$
$$2\,\|\varphi\|\,\|\frac{1}{t}[T_0(t)(\varphi) - \varphi] - A_0(\varphi)\| + \|A_0(\varphi)\|\,\|T_0(t)(\varphi) - \varphi\|.$$

The right–hand side of the above inequality tends to zero as $t \to 0$. □

6. Proof of the convergence theorem

In this section we prove Theorem 2.3. Assertion (2.36) is equivalent to

$$\mu^{(n)} \Rightarrow \lambda \text{ in } D_{\mathcal{P}(\mathcal{Z})}[0,\infty),$$

according to Lemma 3.6 and Lemma 3.2. This convergence is equivalent to

$$\mu^{(n)} \Rightarrow \lambda \text{ in } D_{\mathcal{M}_1(\mathcal{Z})}[0,\infty),$$

according to Corollary 3.8. Weak convergence of $\mu^{(n)}$ to λ follows from relative compactness and convergence of the finite–dimensional distributions (cf. [8, Ch. 3, Th. 7.8]). Relative compactness has been established in Lemma 5.4

and Lemma 5.5. By Corollary 3.4, it is sufficient to show convergence of the one–dimensional distributions. According to Lemma 3.2 and Corollary 3.3, this is equivalent to

$$\lim_{n\to\infty} \mathcal{E}^{(n)} \varrho_L(\mu^{(n)}(t), \lambda(t)) = 0\,, \quad \forall t \in [0,\infty)\,, \tag{6.1}$$

where ϱ_L is defined in (3.3). Thus, to prove Theorem 2.3, it is sufficient to check condition (6.1).

We introduce an approximation $\tilde{Z}^{(n,N)}(t)$ of the process $Z^{(n)}(t)$, given by the generator

$$\begin{aligned} \tilde{\mathcal{A}}^{(n,N)}(\Phi)(\bar{\zeta}) = \sum_{i=1}^{n} \tilde{A}^{(N)}_{0,\zeta_i}(\Phi)(\bar{\zeta}) + \\ \frac{1}{n} \sum_{1\le i\ne j\le n} \int_{\Theta} \Big[\Phi(J(\bar{\zeta},i,j,\vartheta)) - \Phi(\bar{\zeta})\Big]\, q(\zeta_i,\zeta_j,\vartheta)\, \pi(d\vartheta)\,, \end{aligned} \tag{6.2}$$

where $\bar{\zeta} \in \mathcal{Z}^n$, and J is given in (2.38). Notice that the difference of the generator (6.2) with the generator (2.37) of the process $Z^{(n)}(t)$ is that A_0 is replaced by its approximation $\tilde{A}_0^{(N)}$ defined in (4.21). Let $\tilde{\mu}^{(n,N)}(t)$ denote the empirical measures corresponding to the process $\tilde{Z}^{(n,N)}(t)$.

The processes $Z^{(n)}(t)$ and $\tilde{Z}^{(n,N)}(t)$ are coupled in such a way that their joint generator takes the form

$$\hat{\mathcal{A}}^{(n,N)} = \hat{\mathcal{A}}_0^{(n,N)} + \hat{\mathcal{Q}}^{(n)}\,, \tag{6.3}$$

where

$$\hat{\mathcal{A}}_0^{(n,N)}(\Phi)(\bar{z},\bar{\zeta}) = \sum_{i=1}^{n} \hat{A}^{(N)}_{0,z_i,\zeta_i}(\Phi)(\bar{z},\bar{\zeta})\,, \tag{6.4}$$

$$\hat{A}^{(N)}_{0,z,\zeta} = A_{0,z} + \tilde{A}^{(N)}_{0,\zeta}\,, \tag{6.5}$$

and

$$\begin{aligned} \hat{\mathcal{Q}}^{(n)}(\Phi)(\bar{z},\bar{\zeta}) \;=\; & \frac{1}{n} \sum_{1\le i\ne j\le n} \int_{\Theta}\int_0^1 \Big[\Phi(\hat{J}(\bar{z},i,j,\vartheta,\eta), \hat{J}(\bar{\zeta},i,j,\vartheta,\eta)) \\ & -\Phi(\bar{z},\bar{\zeta})\Big]\, q_{max}(\vartheta)\, d\eta\, \pi(d\vartheta)\,, \end{aligned} \tag{6.6}$$

where

$$\hat{J}(\bar{z},i,j,\vartheta,\eta)=\begin{cases} J(\bar{z},i,j,\vartheta)\,, \text{ if } & \eta\leq\frac{q(z_i,z_j,\vartheta)}{q_{max}(\vartheta)}\,,\\ \bar{z} & , \text{ otherwise}\,,\end{cases} \tag{6.7}$$

$(\bar{z},\bar{\zeta})\in\mathcal{Z}^n\times\mathcal{Z}^n$, and J is given in (2.38).

Lemma 5.1 and Remark 5.2 (with $\mathcal{Y}=\mathcal{Z}^n\times\mathcal{Z}^n$) are applicable to the process $(Z^{(n)}(t)\,,\,\tilde{Z}^{(n,N)}(t))$, since the assumptions (5.1), (5.2), and (5.3) follow from (2.21), (2.22), and (2.34). Note also Lemma 3.9 concerning the properties of the empirical measures. Furthermore, the distribution of the process $Z^{(n)}(t)$ does not depend on N so that

$$\mathcal{E}^{(n,N)}\varrho_L(\mu^{(n)}(t),\lambda(t))=\mathcal{E}^{(n)}\varrho_L(\mu^{(n)}(t),\lambda(t))\,,\quad\forall N\,.$$

The triangle inequality yields

$$\limsup_{n\to\infty}\mathcal{E}^{(n)}\varrho_L(\mu^{(n)}(t),\lambda(t))\leq\varrho_L(\tilde{\lambda}^{(N)}(t),\lambda(t))+$$
$$\limsup_{n\to\infty}\mathcal{E}^{(n,N)}\varrho_L(\tilde{\mu}^{(n,N)}(t),\tilde{\lambda}^{(N)}(t))+\limsup_{n\to\infty}\mathcal{E}^{(n,N)}\varrho_L(\mu^{(n)}(t),\tilde{\mu}^{(n,N)}(t))\,.$$

Thus, (6.1) is a consequence of Lemma 4.8, assumptions (2.29)-(2.34), and the following two assertions.

Lemma 6.1 (approximation of the process) *Let the assumptions of Theorem 2.3 be fulfilled. Then*

$$\lim_{N\to\infty}\limsup_{n\to\infty}\mathcal{E}^{(n,N)}\varrho_L(\mu^{(n)}(t),\tilde{\mu}^{(n,N)}(t))=0\,,\quad\forall t\in[0,\infty)\,.$$

Lemma 6.2 (convergence of the approximate process) *Let the assumptions of Theorem 2.3 be fulfilled. Then*

$$\lim_{n\to\infty}\mathcal{E}^{(n,N)}\varrho_L(\tilde{\mu}^{(n,N)}(t),\tilde{\lambda}^{(N)}(t))=0\,,\quad\forall N\,,\quad\forall t\in[0,\infty)\,.$$

The following three lemmas prepare the proof of Lemma 6.1.

Lemma 6.3 *Let ψ be a function on $[0,\infty)$ such that*

$$\psi(x)>0\,,\quad\psi(y+x)\geq\psi(y)\,,\quad\forall x>0\,,\ y\geq0\,.$$

Let $\varphi\in\bar{C}(\mathcal{Z})$ be such that $\|\varphi\|_L\leq1$ (cf. (3.1)).

Then

$$|\varphi(z)-\varphi(\tilde{z})|\leq\varepsilon+\frac{2}{\psi(\varepsilon)}\,\psi(r(z,\tilde{z}))\,,\quad\forall\varepsilon>0\,,\quad\forall z,\tilde{z}\in\mathcal{Z}\,.$$

Proof. If $r(z,\tilde{z}) \leq \varepsilon$, then $|\varphi(z)-\varphi(\tilde{z})| \leq \varepsilon$.

If $r(z,\tilde{z}) > \varepsilon$, then $\psi(r(z,\tilde{z})) \geq \psi(\varepsilon)$ and consequently

$$|\varphi(z)-\varphi(\tilde{z})| \leq 2 = \frac{2}{\psi(\varepsilon)}\psi(\varepsilon) \leq \frac{2}{\psi(\varepsilon)}\psi(r(z,\tilde{z}))\,. \quad \square$$

Lemma 6.4 *The function* $\psi(x)=\min(1,x)$, $x\in[0,\infty)$, *has the properties*

$$\psi(c\,x) \leq c\,\psi(x)\,, \quad \forall x \in [0,\infty)\,,\ \forall c \geq 1\,, \tag{6.8}$$

$$\psi(x+y) \leq \psi(x)+\psi(y)\,, \quad \forall x,y \in [0,\infty)\,. \tag{6.9}$$

Proof. Elementary. $\square$

Lemma 6.5 *Consider the function*

$$\Phi^{(n,N)}(\bar{z},\bar{\zeta}) = \frac{1}{n}\sum_{i=1}^{n}\Big[\psi(r(z_i,\zeta_i)) + \psi(\alpha_N[1+r_0(z_i)+r_0(\zeta_i)]) + \alpha_N\Big]\,, \tag{6.10}$$

where $\psi(x)=\min(1,x)$, $x\in[0,\infty)$, r_0 *is defined in (2.27), and* $\alpha_N = N^{-\frac{1}{3}}$. *Let* $\hat{\mathcal{T}}^{(n,N)}(t)$ *denote the semigroup corresponding to the generator* $\hat{\mathcal{A}}^{(n,N)}$ *defined in (6.3)–(6.7). Let the assumptions of Theorem 2.3 be fulfilled.*

Then there exists a constant c such that

$$\hat{\mathcal{T}}^{(n,N)}(t)(\Phi^{(n,N)})(\bar{z},\bar{\zeta}) \leq c\,\Phi^{(n,N)}(\bar{z},\bar{\zeta})\,, \quad \forall(\bar{z},\bar{\zeta}) \in \mathcal{Z}^n \times \mathcal{Z}^n\,. \tag{6.11}$$

Proof. We use the series representation established in Lemma 5.1:

$$\begin{aligned}\hat{\mathcal{T}}^{(n,N)}(t) = \hat{\mathcal{T}}_0^{(n,N)}(t) + \sum_{l=1}^{\infty}\int_0^t dt_1 \ldots \int_0^{t_{l-1}} dt_l \\ \hat{\mathcal{T}}_0^{(n,N)}(t-t_1)\,\hat{\mathcal{Q}}^{(n)} \ldots \hat{\mathcal{T}}_0^{(n,N)}(t_{l-1}-t_l)\,\hat{\mathcal{Q}}^{(n)}\hat{\mathcal{T}}_0^{(n,N)}(t_l)\,,\end{aligned} \tag{6.12}$$

where $\hat{\mathcal{T}}_0^{(n,N)}(t)$ denotes the semigroup corresponding to the generator $\hat{\mathcal{A}}_0^{(n,N)}$ defined in (6.4)–(6.5). We proceed in two steps showing

$$\hat{\mathcal{T}}_0^{(n,N)}(t)(\Phi^{(n,N)})(\bar{z},\bar{\zeta}) \leq c_1\,\Phi^{(n,N)}(\bar{z},\bar{\zeta})\,, \quad \forall \bar{z},\bar{\zeta} \in \mathcal{Z}^n \times \mathcal{Z}^n\,, \tag{6.13}$$

and

$$|\hat{\mathcal{Q}}^{(n)}(\Phi^{(n,N)})(\bar{z},\bar{\zeta})| \leq c_2\,\Phi^{(n,N)}(\bar{z},\bar{\zeta})\,, \quad \forall \bar{z},\bar{\zeta} \in \mathcal{Z}^n \times \mathcal{Z}^n\,. \tag{6.14}$$

Then assertion (6.11) follows from (6.12), since

$$\hat{T}^{(n,N)}(t)(\Phi^{(n,N)})(\bar{z},\bar{\zeta}) \le c_1\, \Phi^{(n,N)}(\bar{z},\bar{\zeta}) + \sum_{l=1}^{\infty}\int_0^t dt_1 \dots \int_0^{t_{l-1}} dt_l\, c_1^{l+1}\, c_2^l\, \Phi^{(n,N)}(\bar{z},\bar{\zeta}) = c_1\, e^{t\, c_1\, c_2}\, \Phi^{(n,N)}(\bar{z},\bar{\zeta}) .$$

Step 1. First we show (6.13). We have $\hat{T}_0^{(n,N)}(t) = \prod_{i=1}^n \hat{T}^{(N)}_{0,z_i,\zeta_i}(t)$, where $\hat{T}^{(N)}_{0,z,\zeta}(t)$ denotes the semigroup corresponding to the generator $\hat{A}^{(N)}_{0,z,\zeta}$ defined in (6.5), because of the independence of the components. Consequently, it is sufficient to show

$$\hat{T}^{(N)}_{0,z,\zeta}(t)(\varphi_N)(z,\zeta) \le c_1\, \varphi_N(z,\zeta) , \tag{6.15}$$

where

$$\varphi_N(z,\zeta) = \psi(r(z,\zeta)) + \psi(\alpha_N[1 + r_0(z) + r_0(\zeta)]) + \alpha_N . \tag{6.16}$$

Since the operators $A_{0,z}$ and $\tilde{A}^{(N)}_{0,\zeta}$ act on different variables, we have

$$\hat{T}^{(N)}_{0,z,\zeta}(t) = T_{0,z}(t)\, \tilde{T}^{(N)}_{0,\zeta}(t) . \tag{6.17}$$

It follows from (4.24), (2.23), and (6.17) that

$$\hat{T}_0^{(N)}(t)(\varphi_N)(z,\zeta) = e^{-t\,N} \sum_{k=1}^{\infty} \frac{(t\,N)^k}{k!} \varphi_N(F(t,z), F(\frac{k}{N},\zeta)) .$$

Thus, we have

$$\begin{aligned} \hat{T}_0^{(N)}(t)(\varphi_N)(z,\zeta) &= \alpha_N + \\ &+\; e^{-t\,N} \sum_{k:|k-t\,N|\ge\gamma_N} \frac{(t\,N)^k}{k!} \Big[\psi(r(F(t,z),F(\frac{k}{N},\zeta))) + \\ &\quad \psi(\alpha_N[1 + r_0(F(t,z)) + r_0(F(\frac{k}{N},\zeta))])\Big] \\ &+\; e^{-t\,N} \sum_{k:|k-t\,N|<\gamma_N} \frac{(t\,N)^k}{k!} \Big[\psi(r(F(t,z),F(\frac{k}{N},\zeta))) + \\ &\quad \psi(\alpha_N[1 + r_0(F(t,z)) + r_0(F(\frac{k}{N},\zeta))])\Big] , \end{aligned}$$

where γ_N is a sequence of positive real numbers. Using Chebyshev's inequality, assumption (2.26), and the monotony of the function ψ, we obtain the estimate

$$\hat{T}_0^{(N)}(t)(\varphi_N)(z,\zeta) \le \alpha_N + 2\,\frac{t\,N}{(\gamma_N)^2} + \psi(C_F\,[r(z,\zeta) + \frac{\gamma_N}{N}(1 + r_0(z) + r_0(\zeta))])$$
$$+\psi(\alpha_N(1 + C_F\,[r_0(z) + t(1 + r_0(z))] + C_F\,[r_0(\zeta) + (t + \frac{\gamma_N}{N})(1 + r_0(\zeta))])).$$

Using Lemma 6.4, we obtain

$$\begin{aligned}\hat{T}_0^{(N)}(t)(\varphi_N)(z,\zeta) &\le \alpha_N + \frac{2\,t\,N}{(\gamma_N)^2} + \\ &(C_F + 1)\left[\psi(r(z,\zeta)) + \psi(\frac{\gamma_N}{N}[1 + r_0(z) + r_0(\zeta)])\right] + \\ &+\left[1 + C_F\,(1 + 2\,t + \frac{\gamma_N}{N})\right]\psi(\alpha_N[1 + r_0(z) + r_0(\zeta)]).\end{aligned}$$

Choosing $\frac{\gamma_N}{N} = \alpha_N$ and $\frac{N}{(\gamma_N)^2} = \alpha_N$, i.e. $\gamma_N = N^{\frac{2}{3}}$ and $\alpha_N = N^{-\frac{1}{3}}$, and remembering the definition (6.16), we obtain (6.15).

Step 2. Next we show (6.14). From (6.6) and (6.10) we find that

$$\begin{aligned}&|\hat{Q}^{(n)}(\Phi^{(n,N)})(\bar{z},\bar{\zeta})| = \\ =\ &|\frac{1}{n}\sum_{1\le i\ne j\le n}\int_\Theta\int_0^1 \frac{1}{n}\Big[\varphi_N([\hat{J}(\bar{z},i,j,\vartheta,\eta)]_i,[\hat{J}(\bar{\zeta},i,j,\vartheta,\eta)]_i) + \\ &\varphi_N([\hat{J}(\bar{z},i,j,\vartheta,\eta)]_j,[\hat{J}(\bar{\zeta},i,j,\vartheta,\eta)]_j) - \\ &\varphi_N(z_i,\zeta_i) - \varphi_N(z_j,\zeta_j)\Big]\,q_{max}(\vartheta)\,d\eta\,\pi(d\vartheta)| \\ \le\ &\frac{1}{n}\sum_{1\le i\ne j\le n}\int_\Theta\int_0^1 \frac{1}{n}\Big[\varphi_N([\hat{J}(\bar{z},i,j,\vartheta,\eta)]_i,[\hat{J}(\bar{\zeta},i,j,\vartheta,\eta)]_i) + \\ &\varphi_N([\hat{J}(\bar{z},i,j,\vartheta,\eta)]_j,[\hat{J}(\bar{\zeta},i,j,\vartheta,\eta)]_j)\Big]\,q_{max}(\vartheta)\,d\eta\,\pi(d\vartheta) + \\ &\frac{2}{n}\,C_{Q,max}\sum_{i=1}^n\varphi_N(z_i,\zeta_i).\end{aligned}$$

Thus, it is sufficient to show

$$\begin{aligned}\int_\Theta\int_0^1 &\Big[\psi(r([\hat{J}(\bar{z},i,j,\vartheta,\eta)]_i,[\hat{J}(\bar{\zeta},i,j,\vartheta,\eta)]_i)) + \\ &\psi(r([\hat{J}(\bar{z},i,j,\vartheta,\eta)]_j,[\hat{J}(\bar{\zeta},i,j,\vartheta,\eta)]_j))\Big]\,q_{max}(\vartheta)\,d\eta\,\pi(d\vartheta) \\ \le\ & c_3\left[\psi(r(z_i,\zeta_i)) + \psi(r(z_j,\zeta_j))\right]\end{aligned} \tag{6.18}$$

and

$$\int_\Theta \int_0^1 \Big[\psi(\alpha_N\,[1 + r_0([\hat{J}(\bar{z},i,j,\vartheta,\eta)]_i) + r_0([\hat{J}(\bar{\zeta},i,j,\vartheta,\eta)]_i)]) + \tag{6.19}$$
$$\psi(\alpha_N\,[1 + r_0([\hat{J}(\bar{z},i,j,\vartheta,\eta)]_j) + r_0([\hat{J}(\bar{\zeta},i,j,\vartheta,\eta)]_j)])\Big]\, q_{max}(\vartheta)\, d\eta\, \pi(d\vartheta)$$
$$\le c_4 \Big[\psi(\alpha_N[1 + r_0(z_i) + r_0(\zeta_i)]) + \psi(\alpha_N[1 + r_0(z_j) + r_0(\zeta_j)])\Big]\,.$$

Step 2a. First we show (6.18). We distinguish between the cases

$$\max(r(z_i,\zeta_i), r(z_j,\zeta_j)) > 1 \tag{6.20}$$

and

$$\max(r(z_i,\zeta_i), r(z_j,\zeta_j)) \le 1\,. \tag{6.21}$$

In the case (6.20), we obtain

$$\int_\Theta \int_0^1 \Big[\psi(r([\hat{J}(\bar{z},i,j,\vartheta,\eta)]_i, [\hat{J}(\bar{\zeta},i,j,\vartheta,\eta)]_i)) +$$
$$\psi(r([\hat{J}(\bar{z},i,j,\vartheta,\eta)]_j, [\hat{J}(\bar{\zeta},i,j,\vartheta,\eta)]_j))\Big]\, q_{max}(\vartheta)\, d\eta\, \pi(d\vartheta)$$
$$\le \;\; 2\,C_{Q,max} \le 2\,C_{Q,max}\,[\psi(r(z_i,\zeta_i)) + \psi(r(z_j,\zeta_j))]\,.$$

In the case (6.21), we denote

$$m_1 = \min\left(\frac{q(z_i,z_j,\vartheta)}{q_{max}(\vartheta)}, \frac{q(\zeta_i,\zeta_j,\vartheta)}{q_{max}(\vartheta)}\right)\,, \quad m_2 = \max\left(\frac{q(z_i,z_j,\vartheta)}{q_{max}(\vartheta)}, \frac{q(\zeta_i,\zeta_j,\vartheta)}{q_{max}(\vartheta)}\right)\,.$$

By dividing the integral with respect to η into three parts and using the definition (6.7), we obtain

$$\int_\Theta \int_0^1 \Big[\psi(r([\hat{J}(\bar{z},i,j,\vartheta,\eta)]_i, [\hat{J}(\bar{\zeta},i,j,\vartheta,\eta)]_i)) +$$
$$\psi(r([\hat{J}(\bar{z},i,j,\vartheta,\eta)]_j, [\hat{J}(\bar{\zeta},i,j,\vartheta,\eta)]_j))\Big]\, q_{max}(\vartheta)\, d\eta\, \pi(d\vartheta)$$
$$\le \;\; \int_\Theta \Big[\psi(r([J(\bar{z},i,j,\vartheta)]_i, [J(\bar{\zeta},i,j,\vartheta)]_i)) +$$
$$\psi(r([J(\bar{z},i,j,\vartheta)]_j, [J(\bar{\zeta},i,j,\vartheta)]_j))\Big]\, m_1\, q_{max}(\vartheta)\, \pi(d\vartheta) +$$
$$\int_\Theta 2\,(m_2 - m_1)\, q_{max}(\vartheta)\, \pi(d\vartheta) + C_{Q,max}\Big[\psi(r(z_i,\zeta_i)) + \psi(r(z_j,\zeta_j))\Big]$$

$$
\begin{aligned}
\leq\ & \int_\Theta \Big[\psi(r(f_1(z_i,z_j,\vartheta),f_1(\zeta_i,\zeta_j,\vartheta))) + \\
& \psi(r(f_2(z_i,z_j,\vartheta),f_2(\zeta_i,\zeta_j,\vartheta)))\Big]\, q_{max}(\vartheta)\,\pi(d\vartheta) + \\
& 2\int_\Theta |q(z_i,z_j,\vartheta)-q(\zeta_i,\zeta_j,\vartheta)|\,\pi(d\vartheta) + \\
& C_{Q,max}\Big[\psi(r(z_i,\zeta_i))+\psi(r(z_j,\zeta_j))\Big]\,. \qquad (6.22)
\end{aligned}
$$

Using the assumptions (2.32) and (2.31) concerning the functions f_1, f_2 and q, the right–hand side of (6.22) is estimated by the term

$$
C_{Q,max}\Big[\psi(r(z_i,\zeta_i))+\psi(r(z_j,\zeta_j))\Big] + 2\,(C_{f,L}+C_{q,L})\Big[r(z_i,\zeta_i)+r(z_j,\zeta_j)\Big]\,.
$$

Consequently, (6.18) follows from (6.22) and (6.21).

Step 2b. It remains to show (6.19). Now we distinguish between the cases

$$
\max\Big(\alpha_N\,[1+r_0(z_i)+r_0(\zeta_i)],\alpha_N\,[1+r_0(z_j)+r_0(\zeta_j)]\Big) > 1 \qquad (6.23)
$$

and

$$
\max\Big(\alpha_N\,[1+r_0(z_i)+r_0(\zeta_i)],\alpha_N\,[1+r_0(z_j)+r_0(\zeta_j)]\Big) \leq 1\,. \qquad (6.24)
$$

In the case (6.23), we obtain

$$
\begin{aligned}
& \int_\Theta\int_0^1 \Big[\psi(\alpha_N\,[1+r_0([\hat{J}(\bar{z},i,j,\vartheta,\eta)]_i)+r_0([\hat{J}(\bar{\zeta},i,j,\vartheta,\eta)]_i)])+ \\
& \qquad \psi(\alpha_N\,[1+r_0([\hat{J}(\bar{z},i,j,\vartheta,\eta)]_j)+r_0([\hat{J}(\bar{\zeta},i,j,\vartheta,\eta)]_j)])\Big] \times \\
& \qquad q_{max}(\vartheta)\,d\eta\,\pi(d\vartheta) \leq 2\,C_{Q,max} \\
\leq\ & 2\,C_{Q,max}\Big[\psi(\alpha_N[1+r_0(z_i)+r_0(\zeta_i)])+\psi(\alpha_N[1+r_0(z_j)+r_0(\zeta_j)])\Big]\,.
\end{aligned}
$$

In the case (6.24), we obtain from (6.7) and (2.38) that

$$
\begin{aligned}
& \int_\Theta\int_0^1 \Big[\psi(\alpha_N\,[1+r_0([\hat{J}(\bar{z},i,j,\vartheta,\eta)]_i)+r_0([\hat{J}(\bar{\zeta},i,j,\vartheta,\eta)]_i)])+ \\
& \qquad \psi(\alpha_N\,[1+r_0([\hat{J}(\bar{z},i,j,\vartheta,\eta)]_j)+r_0([\hat{J}(\bar{\zeta},i,j,\vartheta,\eta)]_j)])\Big] \times \\
& \qquad q_{max}(\vartheta)\,d\eta\,\pi(d\vartheta) \\
\leq\ & \int_\Theta \Big[\psi(\alpha_N\,[1+r_0(z_i)+r_0(f_1(z_i,z_j,\vartheta))+r_0(\zeta_i)+r_0(f_1(\zeta_i,\zeta_j,\vartheta))])+
\end{aligned}
$$

$$
\begin{aligned}
&\psi(\alpha_N\,[1+r_0(z_j)+r_0(f_2(z_i,z_j,\vartheta))+r_0(\zeta_j)+r_0(f_2(\zeta_i,\zeta_j,\vartheta))]\big)\big]\times \\
&q_{max}(\vartheta)\,\pi(d\vartheta) \\
\leq\;& C_{Q,max}\Big[\alpha_N\,[1+r_0(z_i)+r_0(\zeta_i)]+\alpha_N\,[1+r_0(z_j)+r_0(\zeta_j)]\Big]+ \qquad (6.25)\\
&\int_\Theta \alpha_N\Big[r_0(f_1(z_i,z_j,\vartheta))+r_0(f_1(\zeta_i,\zeta_j,\vartheta))+ \\
&r_0(f_2(z_i,z_j,\vartheta))+r_0(f_2(\zeta_i,\zeta_j,\vartheta))\Big]\,q_{max}(\vartheta)\,\pi(d\vartheta)\,.
\end{aligned}
$$

Now, one applies (2.32), (2.33), and (6.19) follows from (6.24) and (6.25).
□

Proof of Lemma 6.1. Let $\|\varphi\|_L \leq 1$. It follows from Lemma 6.3 that

$$
\begin{aligned}
|\langle\varphi,\mu^{(n)}(t)\rangle-\langle\varphi,\tilde{\mu}^{(n,N)}(t)\rangle| \;&\leq\; \frac{1}{n}\sum_{i=1}^{n}|\varphi(Z_i^{(n)}(t))-\varphi(\tilde{Z}_i^{(n,N)}(t))| \\
&\leq\; \varepsilon+\frac{2}{\psi(\varepsilon)}\,\frac{1}{n}\sum_{i=1}^{n}\psi(r(Z_i^{(n)}(t),\tilde{Z}_i^{(n,N)}(t)))\,.
\end{aligned}
$$

Thus, we have

$$
\varrho_L(\mu^{(n)}(t),\tilde{\mu}^{(n,N)}(t)) \leq \varepsilon+\frac{2}{\psi(\varepsilon)}\,\frac{1}{n}\sum_{i=1}^{n}\psi(r(Z_i^{(n)}(t),\tilde{Z}_i^{(n,N)}(t)))\,,
$$

and it is sufficient to show that

$$
\lim_{N\to\infty}\limsup_{n\to\infty}\mathcal{E}^{(n,N)}\Phi^{(n,N)}(Z^{(n)}(t),\tilde{Z}^{(n,N)}(t))=0\,, \tag{6.26}
$$

where $\Phi^{(n,N)}$ is defined in (6.10). According to Lemma 6.5, assertion (6.26) follows from

$$
\lim_{N\to\infty}\limsup_{n\to\infty}\mathcal{E}^{(n,N)}\Phi^{(n,N)}(Z^{(n)}(0),\tilde{Z}^{(n,N)}(0))=0\,. \tag{6.27}
$$

Suppose $Z^{(n)}(0)=\tilde{Z}^{(n,N)}(0)$, then (6.27) takes the form

$$
\lim_{N\to\infty}\limsup_{n\to\infty}\mathcal{E}^{(n,N)}\frac{1}{n}\sum_{i=1}^{n}\Big[\psi(\alpha_N[1+2\,r_0(Z_i^{(n)}(0))])+\alpha_N\Big]=0\,. \tag{6.28}
$$

Since the function $\psi_N(z)=\psi(\alpha_N[1+2\,r_0(z)])$ is bounded and Lipschitz–continuous, we obtain from (2.35)

$$
\lim_{n\to\infty}\mathcal{E}^{(n,N)}\int_{\mathcal{Z}}\psi_N(z)\,\mu^{(n)}(0,dz)=\int_{\mathcal{Z}}\psi_N(z)\,\lambda(0,dz)\,.
$$

Now it remains to take the limit $N \to \infty$ and to remember that $\lim_{N\to\infty} \alpha_N = 0$, and (6.28) follows. □

The following lemma prepares the proof of Lemma 6.2.

Lemma 6.6 *Let $\varphi \in \hat{C}(\mathcal{Z})$ be such that $\varphi \in \mathcal{D}(A_0)$ and $A_0(\varphi) \in \hat{C}(\mathcal{Z})$. Let assumptions (2.29)–(2.34) be fulfilled.*

Then the mapping $\Psi_\varphi : D_{\mathcal{M}_1(\mathcal{Z})}[0,\infty) \to D_{\mathcal{R}}[0,\infty)$, defined as

$$\Psi_\varphi(\omega)(t) = \langle\varphi,\omega(t)\rangle - \langle\varphi,\omega(0)\rangle - \int_0^t \langle A_0(\varphi),\omega(s)\rangle\, ds - \int_0^t \langle\varphi, K_{max}(\omega(s),\omega(s))\rangle\, ds + 2\, C_{Q,max} \int_0^t \langle\varphi,\omega(s)\rangle\, ds\,, \tag{6.29}$$

is continuous.

Proof. First we notice that $\lim_{n\to\infty} \omega_n(0) = \omega(0)$ in $\mathcal{M}_1(\mathcal{Z})$, if $\lim_{n\to\infty} \omega_n = \omega$ in $D_{\mathcal{M}_1(\mathcal{Z})}[0,\infty)$, according to [8, Ch. 3, Prop. 5.2], since 0 is a continuity point for any ω. Thus, the mapping

$$\Psi_\varphi^{(1)}(\omega)(t) = \langle\varphi,\omega(0)\rangle \tag{6.30}$$

is continuous.

Next, we use the fact (cf. [8, p. 153]) that the mapping

$$f(x)(t) = \int_0^t x(s)\, ds$$

from $D_{\mathcal{R}}[0,\infty)$ into $D_{\mathcal{R}}[0,\infty)$ is continuous. Therefore, it remains to show that

$$\Psi_\varphi^{(2)}(\omega)(t) = \langle\varphi,\omega(t)\rangle\,, \quad \Psi_\varphi^{(3)}(\omega)(t) = \langle A_0(\varphi),\omega(t)\rangle\,,$$

and

$$\Psi_\varphi^{(4)}(\omega)(t) = \langle\varphi, K_{max}(\omega(t),\omega(t))\rangle$$

are continuous mappings from $D_{\mathcal{M}_1(\mathcal{Z})}[0,\infty)$ into $D_{\mathcal{R}}[0,\infty)$. This is fulfilled (cf. [8, p. 151]), if

$$\hat{\Psi}_\varphi^{(2)}(\nu) = \langle\varphi,\nu\rangle\,, \quad \hat{\Psi}_\varphi^{(3)}(\nu) = \langle A_0(\varphi),\nu\rangle\,, \quad \hat{\Psi}_\varphi^{(4)}(\nu) = \langle\varphi, K_{max}(\nu,\nu)\rangle$$

are continuous mappings from $\mathcal{M}_1(\mathcal{Z})$ into $\mathcal{R}$. These properties are assured by the assumptions concerning the function φ and the properties of the operator K_{max}, which follow from the assumptions (2.29)–(2.34). $\square$

Proof of Lemma 6.2. Notice that $\tilde{\lambda}^{(N)}(t)$ is continuous in t. According to [8, Ch. 3, Th. 7.8], it is sufficient to show that

$$\tilde{\mu}^{(n,N)} \Rightarrow \tilde{\lambda}^{(N)} \text{ in } D_{\mathcal{P}(\mathcal{Z})}[0,\infty), \tag{6.31}$$

when $n \to \infty$ and N is fixed. According to Corollary 3.5, it is sufficient to show this convergence in $D_{\mathcal{M}_1(\mathcal{Z})}[0,\infty)$. The relative compactness follows from Lemma 5.4 (with A_0 replaced by $\tilde{A}_0^{(N)}$). Let $\tilde{P}^{(n,N)}$ denote the measures on $D_{\mathcal{M}_1(\mathcal{Z})}[0,\infty)$ corresponding to $\tilde{\mu}^{(n,N)}$. Let $\tilde{P}^{(\infty,N)}$ be any limiting point of the sequence $\tilde{P}^{(n,N)}$. It remains to prove that $\tilde{P}^{(\infty,N)}$ is concentrated on $\tilde{\lambda}^{(N)}$.

Consider the function

$$\psi(\omega) = d_{\mathcal{R}}(\Psi_\varphi(\omega), 0), \quad \omega \in D_{\mathcal{M}_1(\mathcal{Z})}[0,\infty), \tag{6.32}$$

where Ψ_φ is the mapping defined in (6.29) (with A_0 replaced by $\tilde{A}_0^{(N)}$) and $d_{\mathcal{R}}$ is the Skorokhod metric in $D_{\mathcal{R}}[0,\infty)$. The function (6.32) is bounded and continuous because of Lemma 6.6 and the obvious inequality $|d_{\mathcal{R}}(x,0) - d_{\mathcal{R}}(y,0)| \le d_{\mathcal{R}}(x,y)$. Therefore, we obtain

$$\lim_{n\to\infty} \langle \psi, \tilde{P}^{(n,N)} \rangle = \langle \psi, \tilde{P}^{(\infty,N)} \rangle. \tag{6.33}$$

On the other hand, it follows from Lemma 5.3 (with A_0 replaced by $\tilde{A}_0^{(N)}$) that

$$\Psi_\varphi(\tilde{\mu}^{(n,N)})(t) = \tilde{R}^{(n,N)}(\varphi,t) - \tilde{R}^{(n,N)}(\varphi,0) + \tilde{M}^{(n,N)}(\varphi,t) - \tilde{M}^{(n,N)}(\varphi,0)$$

and

$$\lim_{n\to\infty} \mathcal{E}^{(n,N)} \sup_{0\le s\le t} |\Psi_\varphi(\tilde{\mu}^{(n,N)})(s)| = 0, \quad \forall t > 0. \tag{6.34}$$

According to Lemma 3.6, (6.34) implies $\lim_{n\to\infty} \mathcal{E}^{(n,N)} d_{\mathcal{R}}(\Psi_\varphi(\tilde{\mu}^{(n,N)}),0) = 0$, or

$$\lim_{n\to\infty} \langle \psi, \tilde{P}^{(n,N)} \rangle = 0, \tag{6.35}$$

where the function ψ is defined in (6.32).

From (6.35) and (6.33), one obtains

$$\tilde{P}^{(\infty,N)}(\{\omega\,:\,\Psi_\varphi(\omega)=0\})=1\,. \tag{6.36}$$

Next, we use the assumption (2.35). Since the mapping $\Psi_\varphi^{(1)}$ defined in (6.30) is continuous, we obtain

$$\tilde{P}^{(\infty,N)}(\{\omega\,:\,\langle\varphi,\omega(0)\rangle=\langle\varphi,\lambda_0\rangle\})=1\,. \tag{6.37}$$

Remembering the definition (6.29) of Ψ_φ (note that A_0 is replaced by $\tilde{A}_0^{(N)}$) and denoting

$$\Omega_\varphi=\{\omega\,:\,\Psi_\varphi(\omega)=0 \text{ and } \langle\varphi,\omega(0)\rangle=\langle\varphi,\lambda_0\rangle\}\,, \tag{6.38}$$

we obtain from (6.36) and (6.37) that

$$\langle\varphi,\omega(t)\rangle=\langle\varphi,\lambda_0\rangle+\int_0^t\langle\tilde{A}_0^{(N)}(\varphi),\omega(s)\rangle\,ds-2\,C_{Q,max}\int_0^t\langle\varphi,\omega(s)\rangle\,ds+$$
$$\int_0^t\langle\varphi,K_{max}(\omega(s),\omega(s))\rangle\,ds\,,\quad\forall t\geq 0\,,\quad\forall\omega\in\Omega_\varphi\,, \tag{6.39}$$

and $\tilde{P}^{(\infty,N)}(\Omega_\varphi)=1$.

Let (φ_k) be a dense subset in $\hat{C}(\mathcal{Z})$. Then Eq. (6.39) is fulfilled for any $\omega\in\cap_{k=1}^\infty\Omega_{\varphi_k}$ and for all φ_k. Since the operator $\tilde{A}_0^{(N)}$ is bounded, we obtain that the equation is fulfilled for all functions $\varphi\in\hat{C}(\mathcal{Z})$. Eq. (6.39) is equivalent to Eq. (4.25). From the uniqueness, it follows that $\cap_{k=1}^\infty\Omega_{\varphi_k}=\{\tilde{\lambda}^{(N)}\}$, and $\tilde{P}^{(\infty,N)}(\{\tilde{\lambda}^{(N)}\})=1$. Therefore, there is only one limiting point, namely $\delta_{\tilde{\lambda}^{(N)}}$. □

7. Concluding remarks

The class of stochastic particle systems, for which convergence has been established in Theorem 2.3, includes the Leontovich model (cf. Example 2.5). In this sense, it generalizes the Skorokhod model (cf. Example 2.4). However, the class studied by Skorokhod [23, Ch. 2] contains also Vlasov terms, which have not been considered in this paper.

The assumptions of Theorem 2.3 should be compared with the assumptions of Theorem 1 in [23, Ch. 2, Sect. 4]. Our assumptions (2.21), (2.22), (2.26) concerning the free flow process reduce in the case (2.39) to the assumption (2.44), which corresponds to assumption (1) in [23]. The Lipschitz property (2.32) of the functions f_1 and f_2 is also included in assumption (1) in [23].

Our assumptions (2.33), (2.34) replace assumptions (2a)–(2f) in [23]. Assumptions (2.33) means that two particles being at the same state are not allowed to jump too far away. In models with the Boltzmann collision transformation (2.43), (2.45), such particles will not jump at all so that the left–hand side of (2.33) is simply zero. Assumption (2.34) is fulfilled for the Boltzmann collision transformation (2.43), (2.45). In this sense, it generalizes the property of conservation of momentum and energy. Note that any moment assumptions (like assumption (2e) in [23]) have been avoided. In addition, there are assumptions (2.29)–(2.31) concerning the function q, which is an additional parameter of our model.

Finally, some remarks concerning the relation between the stochastic particle system considered in Theorem 2.3 and the original Boltzmann equation (1.1).

We did not discuss the problem of boundary conditions associated with Eq. (1.1) (cf. [7, Ch. 3]). They enter the stochastic particle system via the free flow transition function U_0. The present results cover periodic boundary conditions, since they are equivalent to considering a torus as the position space. However, in the case of specular reflection at the boundary, the continuity assumption (2.26) is violated, because there is a jump in the velocity. If the boundary conditions are of stochastic nature (e.g., diffuse reflection), then even assumption (2.23) will not be fulfilled. Moreover, in realistic problems from rarefied gas dynamics, there is a flux at some part of the boundary.

To discuss the restrictions concerning the collision kernel Q, we consider the Leontovich model (cf. Example 2.5), with the function α of the form

$$\alpha(x_1, v_1, x_2, v_2, e) = h(x_1, x_2)\, B(v_1, v_2, e)\,.$$

Assumptions (2.29)–(2.31) are fulfilled if the functions h and B are bounded and Lipschitz–continuous. This is a severe restriction, which is not valid for realistic collision kernels B (e.g., $B(v_1, v_2, e) = |(e, v_2 - v_1)|$, in the hard sphere case).

Thus, two problems arise. The first is to weaken the assumptions concerning the function B. There are many results how to do this in the spatially homogeneous case (cf. the references cited in Section 1). It does not seem to be very hard to adapt them to the spatially inhomogeneous case, if the function h remains smooth and bounded. The limiting equation obtained in this case turns out to be a mollified Boltzmann equation (with h called the mollifier) (cf. [7, Ch. 8, Sect. 3]).

The second problem is to remove the mollifier, i.e. to consider a function $h^{(n)}(x_1, x_2)$ tending to the delta–function in the limit $n \to \infty$. This fundamental problem corresponds to the original conjecture by Leontovich (cf. (1.6)).

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