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Optimal Superhedging under Nonconvex Constraints – A BSDE Approach

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Abstract

We apply theoretical results of S. Peng on supersolutions for BSDEs to the problem of finding optimal superhedging strategies in a Black-Scholes market under constraints. Constraints may be imposed simultaneously on wealth process and portfolio. They may be nonconvex, time-dependent, and random. Constraints on the portfolio may e.g. be formulated in terms of the amount of money invested, the portfolio proportion, or the number of shares held.

1 Introduction

The seller of an option ξ with exercise time T faces the following problem: How much money does he need to replicate at least the obligation ξ at exercise time T which he accepts when selling the option? Put in other words, the seller of an option seeks to invest money in the underlying financial market in a way that his wealth at time T exceeds ξ. The minimal initial capital, which has to be invested to achieve this goal, is called the *seller's price* of the option, provided the minimum is attained. The corresponding strategy is said to be the *optimal superhedging strategy*.

It is by now classical that in an arbitrage free, complete, and unconstrained market the seller's price of an option ξ is given by the expectation of the discounted obligation under the risk-neutral probability measure. However, the trading strategies employed by the investor in order to replicate the obligation with this initial capital are usually rather unrealistic. The investor is assumed to trade in a perfect market. So, for example, he may buy and sell (even shortsell), as many shares (even fractions of shares) as he likes. Or he can borrow arbitrarily high amounts of money from the bank. He may also employ very risky strategies, by investing huge amounts of money in one risky asset.

It was therefore suggested to reconsider the problem under constraints on the admissible trading strategies. Cvitanić and Karatzas (1993) imposed convex (time independent and nonrandom) constraints on the portfolio proportion process, i.e. on the ratio of the wealth invested in an asset. In the framework of a Black-Scholes market they derived a dual formulation of the problem and constructed optimal superhedging strategies by means of the optional decomposition. This approach was generalized by Föllmer and Kramkov (1997) to semimartingale models. Note, this procedure heavily relies on the convexity of the constraints, since the dual formulation is basing on properties of the support function of the convex constraint set. Recently, attempts have been made to determine the seller's price in a constrained market without passing to a dual problem. Soner and Touzi (2000, 2003) consider large investors and Gamma constraints. In both situations no obvious dual formulation is available. Instead they establish a dynamic programming approach on the primal problem.

In the present paper we completely deviate from the convexity assumption. Therefore a dual formulation in terms of convex analysis is impossible, and we have to tackle the primal problem directly. The techniques, we apply, are borrowed from the theory of backward stochastic differential equations (BSDEs). Therefore we assume that the financial market is driven by a Brownian motion. The main theorem states that under very general constraints existence of a superhedging strategy implies existence of an optimal one, i.e. one with minimal initial capital. It can be proved by translating the problem into an equivalent problem of finding a minimal supersolution of a constrained BSDE and applying a result by Peng (1999). We also give a direct construction of the optimal superhedging strategy based on a penalization argument. We first consider a sequence of markets, where violation of the constraint is not prohibited, but penalized with increasing weight. Existence of the seller's price process in these markets follows from the Pardoux and Peng (1990) existence theorem for BSDEs. Then the monotonic limit theorem for BSDEs due to Peng (1999) guarantees that the limiting process is again the wealth process of a superhedging strategy in the original market. The limiting process turns out to be the minimal wealth process, and is, thus, the seller's price process for the option in the constrained market.

We believe that the BSDE approach has several advantages over the duality approach: First of all it covers a huge class of constraints. Indeed, the constraint is only supposed to be the zero set of a nonnegative standard generator of a BSDE. This allows for time dependence and randomness of the constraint. Moreover, constraints may be simultaneously imposed on portfolio and wealth process. To demonstrate the power of the BSDE approach we present some classes of admissible constraints in section 4. We show, how constraints can be imposed on the amount of money invested in the stock, the portfolio proportion, or the number of shares held by the investor. For example, we can deal with the constraint that the investor may not hold fractions of the stock. To the best of our knowledge this kind of constraint has not yet been considered. We also show that barrier constraints on the wealth process fit into our framework. So the BSDE approach can even be applied to the optimal superhedging of American options under additional constraints on the portfolio. The second advantage is that the penalization method suggests an approximation procedure. In this procedure all approximating steps are meaningful from an economist's point of view as they are price processes in a penalized market.

The paper is organized as follows: We introduce the market model in section 2 and recall some well known results in the unconstrained market. Section 3 is devoted to the proof of the main result and a construction of the optimal strategy. In section 4 we present some important examples. Section 5 concludes the paper.

2 The Market Model

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space that carries a d-dimensional Brownian motion $(W_t,\ 0 \le t \le T)$. We assume, the filtration is the augmented Brownian filtration. We consider a complete Black-Scholes market consisting of a riskless bond and d-risky assets.

The riskless asset (bond) is given by

$$B_t = \exp\left\{ \int_0^t r_s ds \right\}.$$

Here, r is a progressively measurable bounded process, the *interest rate.*

The d risky assets (stocks) are defined by

$$S_t^i = s_0^i \exp\left\{ \sum_{j=1}^d \int_0^t \sigma_s^{i,j} dW_s^j + \int_0^t \mu_s^i ds - \sum_{j=1}^d \frac{1}{2} \int_0^t |\sigma_s^{i,j}|^2 ds \right\}.$$

We assume that the *drift vector* μ and the *volatility matrix* σ are bounded progressively measurable processes. Moreover, σ is supposed to have a bounded inverse.

Clearly,

$$dS_t^i = S_t^i \mu_t^i dt + S_t^i \sigma_t^i dW_t.$$

An investor may trade in this market as follows.

Definition 2.1 *A portfolio is a triplet (x, π, C), where $x \ge 0$ is the initial wealth of an investor, π is a progressively measurable d-dimensional process, and C is a progressively measurable RCLL increasing process satisfying $C_0 = 0$ and*

$$E\left[\int_0^T |\pi_t^* \sigma_t|^2 dt + C_T^2 \right] < \infty \qquad (1)$$

π_t^i is the amount of money invested in the ith stock at time t and C_t is the cumulated consumption up to time t.

The *wealth process* $X^{x,\pi,C}$ of a portfolio (x, π, C) is the solution of the SDE

$$\begin{aligned} dX_t &= r_t X_t dt + \pi_t^*(\mu_t - r_t \mathbf{1}_d)dt - dC_t + \pi_t^* \sigma_t dW_t \\ X_0 &= x \end{aligned}$$

In the case $C = 0$, this is the standard definition of a *self-financing* portfolio, which means that money is neither added nor withdrawn from the portfolio. Consumption is modeled separately by the process C.

Note, given a progressively measurable process X, there is at most one portfolio (x, π, C) such that $X = X^{x,\pi,C}$. This is ensured by the integrability condition (1) on the portfolio.

We shall next describe the superhedging problem without constraint: Given a contingent claim ξ, i.e. $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, find the portfolios (x, π, C) such that

$$X_T^{x,\pi,C} = \xi. \tag{2}$$

These portfolios are called *superhedging portfolios*, and, when additionally $C = 0$, *hedging portfolios.*

The well-known solution is stated in the following theorem.

Theorem 2.2 *Given a contingent claim ξ and a consumption process C, there exist a unique pair (x, π) such that (x, π, C) is a superhedging portfolio for ξ. Moreover,*

$$X_t^{x,\pi,C} = \Phi_t^{-1} E\left[\Phi_T(\xi + C_T) + \int_t^T \Phi_s r_s C_s ds \,\middle|\, \mathcal{F}_t\right] - C_t$$

with the deflator process given by

$$\Phi_t = \exp\left\{\int_0^t \theta_s^* dW_s - \frac{1}{2}\int_0^t |\theta_s|^2 ds - \int_0^t r_s ds\right\}$$

and the risk premium vector

$$\theta_t = \sigma_t^{-1}(\mu_t - r_t \mathbf{1}_d).$$

Here, $\mathbf{1}_d$ denotes the d-dimensional vector with every component equal to 1.

Notice, the process

$$\Phi_t C_t + \int_0^t \Phi_s r_s C_s ds$$

is a submartingale. Therefore, the wealth process corresponding to the hedging portfolio is minimal among the wealth processes corresponding to superhedging portfolios. Moreover, the deflator process Φ contains both, the discounting of the claim and the density of the risk-neutral probability. So put in classical terms, the (seller's) price in the unconstrained market is the expectation of the discounted contingent claim under the equivalent martingale measure.

3 Construction of Optimal Superhedging Strategies under Constraints

For the remainder of the paper we shall consider the problem of finding minimal superhedging portfolios under market imperfections.

Definition 3.1 *A market is called imperfect, if the portfolio and/or its corresponding wealth process are required to satisfy additional constraints.*

Examples of market imperfections include:

Example 3.2 *(i) American contingent claims: The wealth process of the seller must stay above a given barrier ξ_t.*
(ii) Incomplete market: The investor may only trade in the first k stocks ($k < d$).
(iii) Restrictions on shortselling and/or borrowing
(iv) The investor must not hold fractions of a stock.
(v) Combinations of these constraints.

We will see later that these examples fit into the framework of an admissible constraint described in the following definition.

Definition 3.3 *(i) A mapping $K : \Omega \times [0,T] \to \mathcal{P}(\mathbb{R}^{d+1})$ is called a constraint. Here $\mathcal{P}$ denotes the power set operator.*
(ii) A portfolio (x, π, C) is said to satisfy the constraint K, if

$$\left(X_t^{x,\pi,C}(\omega), \pi_t(\omega)\right) \in K_t(\omega) \quad P - a.s.\ a.e.t$$

(iii) A constraint is admissible, if there exist a mapping $\phi : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ satisfying

1. *For all (ω, t)*

$$K_t(\omega) = \{(x,p);\ \phi(\omega,t,x,p) = 0\};$$

2. $\phi(\cdot, x, p)$ is a progressively measurable process for every $(x,p) \in \mathbb{R}^{d+1}$;

3. ϕ is uniformly (in the first and second variable) Lipschitz continuous in the third and fourth variable;

4. $\phi(\cdot, 0, 0) \in L^2(\Omega \times [0,T])$.

Remark 3.4 *(i) Since by condition 1. an admissble constraint is the zero set of the process ϕ, condition 2. is one way to formulate, that the investor knows the random and time dependent constraint set K_t at time t. Conditions 3. and 4. are imposed for technical reasons.*
(ii) Clearly, the constraint set $K_t(\omega)$ is closed for all (t,ω), since it is the zero set of a continuous function.

The next theorem states that under admissible constraints the mere existence of a superhedging strategy implies existence of an optimal one:

Theorem 3.5 *Suppose K is an admissible constraint and ξ is a contingent claim. Then the following assertions are equivalent:*
(i) There exists a superhedging strategy for ξ, which satisfies the constraint K;
(ii) There exists an optimal superhedging strategy $(\tilde{x}, \tilde{\pi}, \tilde{C})$ for ξ under the constraint K, i.e. $(\tilde{x}, \tilde{\pi}, \tilde{C})$ is a superhedging portfolio for ξ which satisfies K and for every other superhedging portfolio (x, π, C) satisfying K,

$$X^{\tilde{x},\tilde{\pi},\tilde{C}} \leq X^{x,\pi,C}.$$

Moreover, in case of existence, the optimal superhedging strategy is unique.

Proof. Suppose there is a superhedging strategy for ξ which satisfies the constraint. The problem may simply be translated into an equivalent problem of finding an optimal supersolution of a BSDE under constraint, namely

$$\begin{aligned} dY_t &= r_t Y_t dt + Z_t^* \sigma_t^{-1}(\mu_t - r_t \mathbf{1}_d)dt - dC_t + Z_t^* dW_t \\ Y_T &= \xi \end{aligned}$$

under the constraint given by the process

$$\phi_\sigma(t, y, z) = \phi(t, y, (\sigma_t^{-1})^* z).$$

By the boundedness of σ^{-1}, ϕ_σ is also Lipschitz continuous. Hence, theorem 5.1 in Peng (1999) yields existence of an optimal supersolution for the constrained BSDE.

In case of existence, uniqueness follows from the integrability condition (1). ■

Remark 3.6 *Suppose the investor employs the strategy $\tilde{\pi}$ with initial capital $\tilde{x}$, but, instead of consuming, he invests the otherwise consumed money in the bond. It is straightforward, that*

$$X_t^{\tilde{x},\tilde{\pi},0} = X_t^{\tilde{x},\tilde{\pi},\tilde{C}} + \int_0^t e^{\int_u^t r_v dv} d\tilde{C}_u.$$

This strategy is clearly admissible in the unconstrained market, but it will also satisfy most of the reasonable constraints. (The investor modifies a strategy, that fulfills the constraint, just by putting some additional money in the bond.) This modified strategy hedges the contingent claim

$$\tilde{\xi} := \xi + \int_0^T e^{\int_u^T r_v dv} d\tilde{C}_u.$$

with the same initial capital as the original optimal superhedging strategy for ξ under the constraint. Therefore the seller's price for ξ under the constraint equals the price of the face lifted contingent claim $\tilde{\xi}$ in the unconstrained market. Under strong assumptions an explicit expression of the face-lifted contingent claim may be given by a dynamic programming approach. We refer the reader to Soner and Touzi (2003).

Actually, the optimal superhedging portfolio may be constructed via a penalization method. This method has been applied to incomplete markets and to American options by El Karoui and Quenez (1997a) and El Karoui and Quenez (1997b). In the theory of BSDEs existence results for reflected BSDEs (El Karoui et al., 1997a) and the monotonic limit theorem (Peng, 1999) can be proved by penalization.

One advantage of the following penalization construction of the optimal superhedging is, that the approximating steps have a sound economic meaning. Given an admissible constraint K which is the zero set of the process ϕ we consider a sequence of markets $\mathcal{M}_\phi^n$. In these markets the same bond and stocks as in the original market are traded, but the wealth of a portfolio is governed by the SDE

$$\begin{aligned} dX_t &= r_t X_t dt + \pi_t^*(\mu_t - r_t \mathbf{1}_d)dt - dC_t - n\phi(t, X_t, \pi_t)dt + \pi_t^* \sigma_t dW_t \\ X_0 &= x \end{aligned}$$

Denote the solution by $X^{n,x,\pi,C}$. Hence, in the nth market violation of the constraint is penalized by a cumulative payment of

$$n \int_0^t \phi(s, X_s^{n,x,\pi,C}, \pi_s) ds$$

up to time t.

By the Pardoux and Peng (1990) theorem, given a contingent claim ξ and a consumption process C we find a unique pair (x, π) such that the portfolio (x, π, C)

replicates ξ, i.e.

$$X^{n,x,\pi,C} = \xi.$$

Again, as in the unpenalized market, the minimal wealth process of a superhedging strategy for ξ in $\mathcal{M}_\phi^n$ is obtained by the choice $C = 0$. This is an immediate consequence of the comparison theorem for BSDEs (El Karoui et al., 1997b). We denote this hedging strategy in the market $\mathcal{M}_\phi^n$ by (x^n, π^n) and the correponding wealth process $X^{n,x^n,\pi^n,0}$ simply by X^n. The penalization paid for this strategy is

$$C_t^n = n \int_0^t \phi(t, X_t^n, \pi_t^n) dt.$$

We expect that, with increasing n, more money is needed to replicate the contingent claim. An application of the comparison theorem for BSDEs shows that indeed $X^n \leq X^{n+1}$.

We may also consider the triplet (x^n, π^n, C^n) as a superhedging strategy for ξ in the original market $\mathcal{M}_\phi^0$ with wealth process X^n. Since the X^n are dominated by any superhedging strategy for ξ in $\mathcal{M}_\phi^0$ which satisfies the constraint, the sequence X^n is bounded. Thus, the monotonic limit theorem (Peng, 1999) tells us that its limit, say X, is also wealth process of a superhedging strategy for ξ in $\mathcal{M}_\phi^0$. The corresponding portfolio process $\tilde{\pi}$ and the consumption process $\tilde{C}$ are given by

$$\begin{aligned} \tilde{\pi} &= \lim_{n\to\infty} \pi^n \quad \text{(strong limit in } L^1(\Omega \times [0,T])) \\ \tilde{C}_t &= \lim_{n\to\infty} C_t^n \quad \text{(weak limit in } L^2(\Omega, \mathcal{F}_t)). \end{aligned}$$

Since π^n (and, of course, X^n) converge strongly in $L^1(\Omega \times [0,T])$ to $\tilde{\pi}$ (and X), this superhedging strategy satisfies the constraint due to the Lipschitz continuity of ϕ. Finally, optimality again follows from the comparison theorem.

Remark 3.7 *Examining the construction of the optimal superhedging strategy, we see that the existence of a superhedging strategy was only used to guarantee the boundedness of the sequence X^n. Thus conditions (i), (ii) in theorem 3.5 are also equivalent to*
(iii) There is a constant κ such that for all $n \in \mathbb{N}$

$$E\left[\sup_{t\in[0,T]} |X_t^n|^2\right] \leq \kappa.$$

4 Examples of Admissible Constraints

The definition of an admissible constraint as the zero set of an appropriate process is rather abstract. To demonstrate the strength of theorem 3.5, we now present some classes of admissible constraints. We begin with a straightforward proposition, which states that admissible constraints may be combined.

Proposition 4.1 *Given k admissible constraints $K_1, \ldots, K_k$, their intersection is admissible.*

Proof. Suppose the constraint K_i is given as the zero set of ϕ_i satisfying the conditions of an admissible constraint. Then the intersection

$$K = \bigcap_{j=1}^{k} K_j$$

is the zero set of $\phi = \sum_{j=1}^{k} \phi_j$, which inherits Lipschitz continuity,non-negativity, and measurability from the ϕ_j's. ■

Example 4.2 (Barrier Constraints) *A typical example of a constraint on the wealth process is a barrier constraint. Given a progressively measurable process $\xi_t : \Omega \times [0,T] \to \mathbb{R}$ the wealth process X_t of a portfolio must stay above ξ_t. We assume*

$$E \int_0^T (\xi_t)_+^2 dt < \infty$$

and define $\phi(\omega, t, x, p) = (\xi_t(\omega) - x)_+$. Clearly ϕ satisfies the conditions of definition 3.3. The zero set of ϕ is just the barrier constraint

$$K_t(\omega) = \{x,\ x \geq \xi_t(\omega)\} \times \mathbb{R}^d.$$

When $\xi = \xi_T$ and

$$E\left[\sup_{0 \leq t \leq T} (\xi_t)_+^2\right] < \infty,$$

standard estimates (El Karoui et al., 1997a) show that condition (iii) in remark 3.7 is always satisfied. Hence, there exist an optimal superhedging portfolio for ξ_T satisfying the barrier constraint ξ_t. The time 0 value of this portfolio is the price of the American contingent claim ξ_t. It can also be calculated by means of reflected BSDEs (El Karoui and Quenez, 1997b) and optimal stopping (Karatzas and Shreve, 1998).

Example 4.3 (Constraints on the amount of money invested) *Suppose $\tilde{K} \subset \mathbb{R}^d$ is a closed, but not necessarily convex, set. We shall impose the constraint that the portfolio process π_t belongs to $\tilde{K}$. Recall, the kth entry of the vector valued* *process denotes the amount of money invested in the kth stock. The corresponding constraint set is*

$$K_t(\omega) = \mathbb{R} \times \tilde{K}$$

Since it is the zero set of the Lipschitz function

$$\phi(t, x, p) = d_{\tilde{K}}(p) = \inf_{q \in \tilde{K}} |p - q|,$$

this constraint is clearly admissible. Some interesting examples are

1. *Incomplete market:* $\tilde{K} = \mathbb{R}^i \times \{0\} \times \cdots \times \{0\}$, *i.e. no trading in the last* $d - i$ *assets.*

2. *No shortselling:* $\tilde{K} = \mathbb{R}^d_{\geq 0}$

Example 4.4 (Constraints on the portfolio proportion) *Constraints may also be imposed on the portfolio proportion process* P. *Given a portfolio* (x, π, C) *it is defined as*

$$P^i_t = \chi_{\{X^{x,\pi,C}_t \neq 0\}} \frac{\pi^i_t}{X^{x,\pi,C}_t}.$$

Thus, P^i *is the ratio of the total wealth invested in the* i*th stock.*

Let $k \leq d$ *and* $\tilde{K} \subset \mathbb{R}^k$ *compact. The constraint*

$$K_t(\omega) = \{(x, p),\ \left(\frac{p_1}{x}, \dots, \frac{p_d}{x}\right) \in \tilde{K} \times \mathbb{R} \times \cdots \times \mathbb{R}\} \cup \{(0, 0, p_{k+1}, \dots, p_d)\}$$

is admissible. It basically (but in the case of zero wealth when the portfolio proportion is meaningless) states that the first k *components of the portfolio proportion* P *must belong to* $\tilde{K}$. $K_t(\omega)$ *is the zero set of the function*

$$\phi(t, x, p) = \inf_{z \in \tilde{K}} |(p_1, \dots, p_k) - x(z_1, \dots, z_k)|.$$

Since it is independent of ω, *we only have to show that* ϕ *is Lipschitzian. Let* (x, p), $(\tilde{x}, \tilde{p}) \in \mathbb{R}^{1+d}$. *Suppose* $\tilde{z}$ *is a minimizer of*

$$|(\tilde{p}_1, \dots, \tilde{p}_k) - \tilde{x}(z_1, \dots, z_k)|$$

in $\tilde{K}$, *and assume w.l.o.g. that*

$$\inf_{z \in \tilde{K}} |(p_1, \dots, p_k) - x(z_1, \dots, z_k)| \geq |(\tilde{p}_1, \dots, \tilde{p}_k) - \tilde{x}(\tilde{z}_1, \dots, \tilde{z}_k)|.$$

Then,

$$\begin{aligned} & |\phi(t, x, p) - \phi(t, \tilde{x}, \tilde{p})| \\ \leq\ & |(p_1, \dots, p_k) - x(\tilde{z}_1, \dots, \tilde{z}_k)| - |(\tilde{p}_1, \dots, \tilde{p}_k) - \tilde{x}(\tilde{z}_1, \dots, \tilde{z}_k)| \\ \leq\ & |(p_1, \dots, p_k) - (\tilde{p}_1, \dots, \tilde{p}_k)| + \max_{z \in \tilde{K}} |z| \cdot |x - \tilde{x}|. \end{aligned}$$

Note, we again do not assume convexity of $\tilde{K}$. However, the boundedness assumption of $\tilde{K}$ is crucial in the proof of the Lipschitz continuity.

An important application of constraints on the portfolio proportion is to enforce diversification of the portfolio: The investor is not allowed to put an arbitrarily high ratio of his money in the risky assets. From this point of view the compactness assumption on $\tilde{K}$ is not too restrictive. For instance, we can choose $k = d$ and $\tilde{K} = [-b_1, b_1] \times \cdots \times [-b_d, b_d]$ with $b_i > 0$. With this choice there is a superhedging portfolio for $\xi = g(S_T)$, when $g \geq 0$ and $g_{x_i}(x) \leq b_i g(x)$ (Yong, 1999). Thus, theorem 3.5 guarantees existence of a minimal one.

Note, constraints on the portfolio proportion may also be treated by convex analysis in the case of a convex and closed set $\tilde{K}$, see e.g. Cvitanić and Karatzas (1993) and Karatzas and Shreve (1998). There are two important and novel aspects in the BSDE approach: (i) convexity can be skipped; (ii) the penalization methods yields a sequence of auxillary markets with a sound economic interpretation. Contrary, the family of auxillary markets introduced by Cvitanić and Karatzas (1993) via duality theory is rather technical.

Example 4.5 (Constraints on the number of shares held) *We now consider constraints on the number of shares held by an investor. To the best of our knowledge this kind of constraint cannot be tackled with any other known approach to optimal superhedging.*

Suppose $\tilde{K} \subset \mathbb{R}^d$ is closed. Define

$$\phi(\omega, t, x, p) = \min_{i=1,\ldots,d} |S_t^i(\omega)| d_{\tilde{K}} \left(\frac{p}{S_t(\omega)} \right)$$

where,

$$\frac{p}{S_t(\omega)} = \left(\frac{p_1}{S_t^1(\omega)}, \ldots, \frac{p_d}{S_t^d(\omega)} \right)^*$$

Then

$$K_t(\omega) = \mathbb{R} \times \left\{ p,\ \frac{p}{S_t(\omega)} \in \tilde{K} \right\}$$

is the zero set of ϕ. Clearly, this constraints the vector of the numbers of shares held by an investor to belong to $\tilde{K}$. We check that this constraint is admissible. Obviously, $\phi(\cdot, x, p)$ is progressively measurable, since S is. Moreover,

$$E \int_0^T \phi(t, 0, 0)^2 dt \leq d_{\tilde{K}}(0)^2 \cdot E \int_0^T |S_t^1(\omega)|^2 dt < \infty.$$

Lipschitz continuity may be derived from the fact that the distance function is Lipschitzian with constant 1. Thus,

$$|\phi(\omega,t,x,p)-\phi(\omega,t,\tilde{x},\tilde{p})| \leq \min_{i=1,\dots,d} |S_t^i(\omega)| \cdot \left| \frac{p}{S_t(\omega)} - \frac{\tilde{p}}{S_t(\omega)} \right|$$

$$\leq \min_{i=1,\dots,d} |S_t^i(\omega)| \left(\sum_{k=1}^{d} \frac{(p_k-\tilde{p_k})^2}{(S_t^k(\omega))^2} \right)^{1/2} \leq |p-\tilde{p}|$$

Two typical examples are

1. *No fractions of a stock may be held:* $\tilde{K} = \mathbb{Z}^d$.
2. *Illiquid market:* $\tilde{K} = [-l_1, u_1] \times \cdots \times [-l_d, u_d]$, *i.e. only a maximum number of* u_i *(resp.* l_i*) shares of the ith asset can be bought (resp. sold) by an investor.*

Example 4.6 (Borrowing Constraints) *Prohibition of borrowing from the bond can be formulated in terms of the portfolio proportion with the choice*

$$\tilde{K} = \left\{ z, \sum_i z_i \leq 1 \right\},$$

as suggested in Karatzas and Shreve (1998). Note, that the set $\tilde{K}$ *is not compact. So it is not covered by example 4.4.* $\tilde{K}$ *is, however, convex, and the constraint can, thus, be treated by convex analysis.*

We shall now consider a more general constraint on borrowing: The investor must not borrow an arbitrarily large amount from the bond, but a maximum amount of a dollars. Obviuosly, the choice $a = 0$ *covers prohibition of borrowing. For* $a \neq 0$ *this constraint cannot be formulated as a deterministic constraint on the portfolio proportion. So it cannot be treated by the convex analysis approach.*

Define

$$\phi(\omega,t,x,p) = \left(\sum_{i=1}^{d} p_i - a - x \right)_+ .$$

It is straightforward that ϕ *satisfies the conditions of definition 3.3. Its zero set is*

$$K_t(\omega) = \left\{ (y,p),\ x - \sum_i p_i \geq -a \right\}.$$

Recall, given a portfolio (x,π,C), $X^{x,\pi,C} - \sum_i \pi_i$ *is the amount of money invested in the bond. So* $(X_t^{x,\pi,C}, \pi_t) \in K_t$ *just means maximal borrowing of a dollars.*

We conclude this section with the following example: Suppose an investor sells an American put option on the first stock $\xi_t = (q - S_t^1)_+$, i.e. the buyer has the right to sell one stock S^1 to the seller for the strike price q at any time $0 \leq t \leq T$. In order to superhedge his obligation the seller invests in a way that the wealth process stays above the barrier ξ_t. We assume for simplicity that the bond is constant 1. Then, of course, the seller can achieve his goal with initial capital q by investing his q dollars in the bond at time 0 and doing nothing until time T. At time T he consumes $q - (q - S_T^1)_+$. This strategy is also in accordance, for instance, with the borrowing constraints, the no short-selling constraint, incompleteness, and the constraint that no fractions of a risky asset may be held. So, theorem 3.5 ensures the existence of an optimal superhedging strategy for the American put option under each of these constraints on the portfolio (or even under combinations of these constraints).

5 Conclusion

We presented a penalization technique based on BSDEs to prove existence of optimal superhedging strategies under constraints. Since we do not have to pass to a dual problem, the convexity assumption on the constraint set, which is imposed in most papers, can be skipped. Several examples demonstrate the generality and power of the BSDE approach.

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