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Iterative Construction of the Optimal Bermudan Stopping Time

Anastasia Kolodko[1], John Schoenmakers [1]

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[1] Weierstrass Institute for Applied Analysis and Stochastics,
Mohrenstr. 39, 10117 Berlin;
E-mail: kolodko@wias-berlin.de, schoenma@wias-berlin.de.

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We present an iterative procedure for computing the optimal Bermudan stopping time. We prove convergence and, as a consequence, the method allows for approximation of the Snell envelope from below. By using duality, we then deduce a convergent procedure for approximating the Snell envelope from above as well. We provide numerical examples for Bermudan swaptions in the context of a LIBOR market model.

1 Introduction

Evaluation of American style derivatives on a high dimensional system of underlyings is considered a perennial problem for the last decades. On the one hand such high dimensional options are difficult, if not impossible, to compute by PDE methods for free boundary value problems. On the other hand Monte Carlo simulation, which is for high dimensional European options an almost canonical alternative to PDE solving, is for American options highly non-trivial since the (optimal) exercise boundary is usually unknown. In the past literature, many approaches for Monte Carlo simulation of American options are developed. With respect to Bermudan derivatives, which are in fact American options with a finite number of exercise dates there is, for example, the stochastic mesh method of Broadie & Glasserman (1997, 2000), a cross-sectional regression approach by Longstaff & Schwartz (2001), a dual approach by Rogers (2001) (and independently Haugh & Kogan (2001) for Bermudan style instruments), a multiplicative dual approach by Jamshidian (2003a,b), and for Bermudan swaptions a method by Andersen (1999). Further recent papers on methods for high-dimensional American options include Belomestny & Milstein (2004), Milstein, Reiß & Schoenmakers (2003), Berridge & Schumacher (2004), and for a more detailed and general overview we refer to Glasserman (2003) and the references therein.

The central result in this paper is an iterative construction of the Bermudan Snell envelope by a convergent sequence of stopping times, corresponding lower bounds and (dual) upper bounds, obtained by probabilistic methods. We underline that the presented method is quite general and can in principle be applied to any discrete optimal stopping problem, regardless the nature of the underlying process.

The paper is organized as follows. In Section 2 we give a concise re-cap of the Bermudan pricing problem. In Section 3 we show that a family of stopping times (τ_i) can be improved if this family possesses some natural properties. By using the procedure developed in Section 3 we construct in Section 4 a sequence of stopping times and lower approximations of the Snell envelope which converge to an optimal stopping time and the Snell envelope, respectively. Then, in Section 5, we recall dual upper bound representations for the Snell envelope by Rogers, Haugh & Kogan and Jamshidian and give some extensions. Based on the dual approach, the convergent lower approximations constructed in Section 4, and an approximation theorem by Kolodko & Schoenmakers (2003), we deduce a sequence of upper bounds which converges to the Snell envelope from above. Finally, in Section 6, we apply our method to Bermudan swaptions in the context of a LIBOR market model. We give a numerical comparison with Andersen's lower bound method and its dual considered by Andersen and Broadie (2001). As a result, starting from a trivial stopping family, by two iterations of our procedure we obtain lower approximations of more factor based Bermudan swaptions which are more or less in the middle of Andersen's lower bound and its corresponding dual upper bound.

2 The Bermudan pricing problem

We consider general Bermudan style derivatives with respect to an underlying process $L(t)$, over some finite time interval $[0, T]$ with time horizon $T < \infty$. The process L is assumed to be Markovian with state space $\mathbb{R}^D$. For example, L can be a system of asset prices, but also a not explicitly tradable object such as the term structure of interest rates, or a system of LIBOR rates.

Consider a set of dates $\mathbb{T} := \{\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_k\}$ with $0 = \mathcal{T}_0 < \mathcal{T}_1 < \mathcal{T}_2 < \cdots < \mathcal{T}_k \leq T$. An option issued at time $t = \mathcal{T}_0$, to exercise a cash-flow $C_{\mathcal{T}_\tau} := C(\mathcal{T}_\tau, L(\tau))$ at a date $\mathcal{T}_\tau \in \mathbb{T}$ to be decided by the option holder, is called a Bermudan style derivative. Naturally, we may also consider Bermudan derivatives where the collection of exercise dates is some subset of $\mathbb{T}$. With respect to a pricing measure P connected with some pricing numeraire B, the value of the Bermudan derivative at a future time point t (when the option is not exercised before t) is given by

$$V(t) = B(t) \sup_{\tau \in \{\kappa(t), \ldots, k\}} E^{\mathcal{F}_t} \frac{C_{\mathcal{T}_\tau}}{B(\mathcal{T}_\tau)} \tag{1}$$

with $\kappa(t) := \min\{m : \mathcal{T}_m \geq t\}$. Note that $V(t)$ can also be seen as the price of a Bermudan option newly issued at time t, with exercise opportunities $\mathcal{T}_{\kappa(t)}, \ldots, \mathcal{T}_k$. In (1) it is assumed that for each fixed exercise date the corresponding cash-flow has finite expectation. The fact that (1) can be considered as the fair price for the Bermudan derivative is due to general no-arbitrage principles, e.g. see Duffie (2001). The supremum in (1) is taken over all integer valued $\mathbb{F}$-stopping times τ with values in the set $\{\kappa(t), ..., k\}$, where $\mathbb{F} := \{\mathcal{F}_t,\ 0 \leq t \leq T\}$ denotes the usual filtration generated by the process L.

The $\mathbb{F}$-stopping time τ_t^* is called an *optimal stopping time* in the $[t, T]$, if

$$Y^*(t) := \frac{V(t)}{B(t)} = E^{\mathcal{F}_t} \frac{C_{\mathcal{T}_{\tau_t^*}}}{B(\mathcal{T}_{\tau_t^*})} = \sup_{\tau \in \{\kappa(t), ..., k\}} E^{\mathcal{F}_t} \frac{C_{\mathcal{T}_\tau}}{B(\mathcal{T}_\tau)}.$$

The process (Y^*) is called the *Snell envelope* process.

3 A one step improvement upon a given family of stopping times

In what follows we will consider the process Y^* at the exercise dates $\{\mathcal{T}_0, \mathcal{T}_1, ..., \mathcal{T}_k\}$ and define $Y^{*(j)} := Y^*(\mathcal{T}_j)$, $0 \leq j \leq k$. Further we denote a corresponding optimal stopping family by (τ_i^*), where $\tau_i^* := \tau_{\mathcal{T}_i}^*$.

With respect to the discrete filtration $\left(\mathcal{F}^{(j)}\right)_{0 \leq j \leq k}$ with $\mathcal{F}^{(j)} := \mathcal{F}_{\mathcal{T}_j}$, $0 \leq j \leq k$, we consider a family of integer valued stopping indexes (τ_i), with the following properties,

$$\begin{aligned} & i \leq \tau_i \leq k, \ \tau_k = k, \\ & \tau_i > i \Rightarrow \tau_i = \tau_{i+1}, \qquad 0 \leq i < k, \end{aligned} \tag{2}$$

and the process

$$Y^{(i)} := E^{\mathcal{F}^{(i)}} \frac{C_{\mathcal{T}_{\tau_i}}}{B(\mathcal{T}_{\tau_i})}. \tag{3}$$

For example,

$$\tau_i := \inf\{j \geq i : L(\mathcal{T}_j) \in R\},$$

where R is a certain region in R^D, or, as a more trivial example, the family $\tau_i \equiv i$. Generally, the process $(Y^{(i)})$ is a lower approximation of the Snell envelope process $(Y^{*(i)})$ due to the family of (sub-optimal) stopping times (τ_i). Based on the family (τ_i) we are going to construct a new family $(\widehat{\tau}_i)$ satisfying (2), which induces a new approximation of the Snell envelope.

We first introduce an intermediate process

$$\widetilde{Y}^{(i)} := \max_{p:\, i \leq p \leq k} E^{\mathcal{F}^{(i)}} \frac{C_{\mathcal{T}_{\tau_p}}}{B(\mathcal{T}_{\tau_p})}. \tag{4}$$

Using $\widetilde{Y}^{(i)}$ as a new exercise criterion we define a next family of stopping indexes

$$\begin{aligned}\widehat{\tau}_i : &= \inf\{j : i \le j \le k,\ \widetilde{Y}^{(j)} \le \frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)}\} \\ &= \inf\{j : i \le j \le k,\ \max_{p:\, j\le p\le k} E^{\mathcal{F}(j)} \frac{C_{\mathcal{T}_{\tau_p}}}{B(\mathcal{T}_{\tau_p})} \le \frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)}\}, \quad 0 \le i \le k,\end{aligned} \tag{5}$$

and consider the process

$$\widehat{Y}^{(i)} := E^{\mathcal{F}(i)} \frac{C_{\mathcal{T}_{\widehat{\tau}_i}}}{B(\mathcal{T}_{\widehat{\tau}_i})} \tag{6}$$

as a next approximation of the Snell envelope. Clearly, the family $(\widehat{\tau})$ satisfies the properties (2) as well. As an example, the trivial family $\tau_i \equiv i$ gives for $\widetilde{Y}$ the maximum of still alive Europeans and for $\widehat{Y}$ the second "canonical example" in Kolodko & Schoenmakers (2003). As another example, $\tau_i \equiv k$ gives for $\widetilde{Y}$ the European option process due to the last exercise date k and

$$\widehat{\tau}_i := \inf\{j : i \le j \le k,\ E^{\mathcal{F}(j)} \frac{C_{\mathcal{T}_k}}{B(\mathcal{T}_k)} \le \frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)}\}, \quad 0 \le i \le k.$$

By the next theorem, $(\widehat{Y}^{(i)})$ is generally an improvement of $(Y^{(i)})$.

Theorem 3.1 *Let $\tau^{(i)}$ be a family of stopping times with the property* (2) *and* $(Y^{(i)})$ *be given by* (3). *Let the processes* $(\widetilde{Y}^{(i)})$ *and* $(\widehat{Y}^{(i)})$ *be defined by* (4) and (6), *respectively. Then, it holds*

$$Y^{(i)} \le \widetilde{Y}^{(i)} \le \widehat{Y}^{(i)} \le Y^{*(i)}, \quad 0 \le i \le k.$$

Proof.

The inequalities $Y^{(i)} \le \widetilde{Y}^{(i)}$ and $\widehat{Y}^{(i)} \le Y^{*(i)}$ are trivial. We only need to show the middle inequality. We use induction in i. Due to the definition of $\widetilde{Y}$ and $\widehat{Y}$, we have $\widetilde{Y}^{(k)} = \widehat{Y}^{(k)} = \frac{C_{\mathcal{T}_k}}{B(\mathcal{T}_k)}$. Suppose that $\widetilde{Y}^{(i)} \le \widehat{Y}^{(i)}$ for some i with $0 < i \le k$. We will then show that $\widetilde{Y}^{(i-1)} \le \widehat{Y}^{(i-1)}$. Let us write

$$\begin{aligned}\widehat{Y}^{(i-1)} &= E^{\mathcal{F}(i-1)} \frac{C_{\mathcal{T}_{\widehat{\tau}_{i-1}}}}{B(\mathcal{T}_{\widehat{\tau}_{i-1}})} = 1_{\widehat{\tau}_{i-1}=i-1} \frac{C_{\mathcal{T}_{i-1}}}{B(\mathcal{T}_{i-1})} + 1_{\widehat{\tau}_{i-1}>i-1} E^{\mathcal{F}(i-1)} E^{\mathcal{F}(i)} \frac{C_{\mathcal{T}_{\widehat{\tau}_i}}}{B(\mathcal{T}_{\widehat{\tau}_i})} \\ &= 1_{\widehat{\tau}_{i-1}=i-1} \frac{C_{\mathcal{T}_{i-1}}}{B(\mathcal{T}_{i-1})} + 1_{\widehat{\tau}_{i-1}>i-1} E^{\mathcal{F}(i-1)} \widehat{Y}^{(i)}.\end{aligned}$$

Then, by induction,

$$\begin{aligned}\widehat{Y}^{(i-1)} &\ge 1_{\widehat{\tau}_{i-1}=i-1} \frac{C_{\mathcal{T}_{i-1}}}{B(\mathcal{T}_{i-1})} + 1_{\widehat{\tau}_{i-1}>i-1} E^{\mathcal{F}(i-1)} \widetilde{Y}^{(i)} \\ &= 1_{\widehat{\tau}_{i-1}=i-1} \frac{C_{\mathcal{T}_{i-1}}}{B(\mathcal{T}_{i-1})} + 1_{\widehat{\tau}_{i-1}>i-1} E^{\mathcal{F}(i-1)} \max_{p:\, i\le p\le k} E^{\mathcal{F}(i)} \frac{C_{\mathcal{T}_{\tau_p}}}{B(\mathcal{T}_{\tau_p})} \\ &\ge 1_{\widehat{\tau}_{i-1}=i-1} \widetilde{Y}^{(i-1)} + 1_{\widehat{\tau}_{i-1}>i-1} \max_{p:\, i\le p\le k} E^{\mathcal{F}(i-1)} \frac{C_{\mathcal{T}_{\tau_p}}}{B(\mathcal{T}_{\tau_p})},\end{aligned} \tag{7}$$

since for $\widehat{\tau}_{i-1} = i-1$ we have $i-1 = \inf\{j : i-1 \leq j < k,\ \widetilde{Y}^{(j)} \leq \frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)}\}$, and so

$$\widetilde{Y}^{(i-1)} = \max_{p:\, i-1 \leq p \leq k} E^{\mathcal{F}^{(i-1)}} \frac{C_{\mathcal{T}_{\tau_p}}}{B(\mathcal{T}_{\tau_p})} \leq \frac{C_{\mathcal{T}_{i-1}}}{B(\mathcal{T}_{i-1})}.$$

We may write (7) as

$$\widehat{Y}^{(i-1)} \geq \widetilde{Y}^{(i-1)} + 1_{\widehat{\tau}_{i-1} > i-1} \left(\max_{i \leq p \leq k} E^{\mathcal{F}^{(i-1)}} \frac{C_{\mathcal{T}_{\tau_p}}}{B(\mathcal{T}_{\tau_p})} - \max_{i-1 \leq p \leq k} E^{\mathcal{F}^{(i-1)}} \frac{C_{\mathcal{T}_{\tau_p}}}{B(\mathcal{T}_{\tau_p})} \right).$$

Thus, after showing that $\widehat{\tau}_{i-1} > i-1$ implies

$$E^{\mathcal{F}^{(i-1)}} \frac{C_{\mathcal{T}_{\tau_{i-1}}}}{B(\mathcal{T}_{\tau_{i-1}})} \leq \max_{p: i \leq p \leq k} E^{\mathcal{F}^{(i-1)}} \frac{C_{\mathcal{T}_{\tau_p}}}{B(\mathcal{T}_{\tau_p})},$$

it follows that $\widehat{Y}^{(i-1)} > \widetilde{Y}^{(i-1)}$. It holds,

$$\begin{aligned} E^{\mathcal{F}^{(i-1)}} \frac{C_{\mathcal{T}_{\tau_{i-1}}}}{B(\mathcal{T}_{\tau_{i-1}})} &= 1_{\tau_{i-1} = i-1} \frac{C_{\mathcal{T}_{i-1}}}{B(\mathcal{T}_{i-1})} + 1_{\tau_{i-1} > i-1} E^{\mathcal{F}^{(i-1)}} \frac{C_{\mathcal{T}_{\tau_i}}}{B(\mathcal{T}_{\tau_i})} \\ &\leq 1_{\tau_{i-1} = i-1} \frac{C_{\mathcal{T}_{i-1}}}{B(\mathcal{T}_{i-1})} + 1_{\tau_{i-1} > i-1} \max_{p: i \leq p \leq k} E^{\mathcal{F}^{(i-1)}} \frac{C_{\mathcal{T}_{\tau_p}}}{B(\mathcal{T}_{\tau_p})}. \qquad (8) \end{aligned}$$

Then, on the set $\widehat{\tau}_{i-1} > i-1$ we have

$$\frac{C_{\mathcal{T}_{i-1}}}{B(\mathcal{T}_{i-1})} < \max_{p: i-1 \leq p \leq k} E^{\mathcal{F}^{(i-1)}} \frac{C_{\mathcal{T}_{\tau_p}}}{B(\mathcal{T}_{\tau_p})},$$

so if $(\widehat{\tau}_{i-1} > i-1) \wedge (\ \tau_{i-1} = i-1)$ it follows that

$$\frac{C_{\mathcal{T}_{i-1}}}{B(\mathcal{T}_{i-1})} < \max\left(\frac{C_{\mathcal{T}_{i-1}}}{B(\mathcal{T}_{i-1})}, \max_{p:\, i \leq p \leq k} E^{\mathcal{F}^{(i-1)}} \frac{C_{\mathcal{T}_{\tau_p}}}{B(\mathcal{T}_{\tau_p})}\right).$$

Hence, if $(\widehat{\tau}_{i-1} > i-1) \wedge (\tau_{i-1} = i-1)$,

$$\frac{C_{\mathcal{T}_{i-1}}}{B(\mathcal{T}_{i-1})} < \max_{p:\, i \leq p \leq k} E^{\mathcal{F}^{(i-1)}} \frac{C_{\mathcal{T}_{\tau_p}}}{B(\mathcal{T}_{\tau_p})}.$$

Thus, from (8) we have for $\widehat{\tau}_{i-1} > i-1$,

$$\begin{aligned} E^{\mathcal{F}^{(i-1)}} \frac{C_{\mathcal{T}_{\tau_{i-1}}}}{B(\mathcal{T}_{\tau_{i-1}})} &\leq 1_{\tau_{i-1} = i-1} \max_{p:\, i \leq p \leq k} E^{\mathcal{F}^{(i-1)}} \frac{C_{\mathcal{T}_{\tau_p}}}{B(\mathcal{T}_{\tau_p})} + 1_{\tau_{i-1} > i-1} \max_{p:\, i \leq p \leq k} E^{\mathcal{F}^{(i-1)}} \frac{C_{\mathcal{T}_{\tau_p}}}{B(\mathcal{T}_{\tau_p})} \\ &= \max_{p:\, i \leq p \leq k} E^{\mathcal{F}^{(i-1)}} \frac{C_{\mathcal{T}_{\tau_p}}}{B(\mathcal{T}_{\tau_p})}. \end{aligned}$$

■

4 Iterative construction of the optimal stopping time and the Snell envelope process

Naturally, we may construct by induction via the procedure (5)-(6) a sequence of pairs

$$\left((\tau_i^{(m)})_{0\le i\le k},(Y^{m(i)})_{0\le i\le k}\right)_{m=0,1,2,\dots}$$

in the following way: Start with some family of stopping times $(\tau_i^{(0)})_{0\le i\le k}$, which satisfies (2) and the additional requirement,

$$Y^{0(i)}:=E^{\mathcal{F}_i}\frac{C_{\mathcal{T}_{\tau_i^{(0)}}}}{B(\mathcal{T}_{\tau_i^{(0)}})}\ge\frac{C_{\mathcal{T}_i}}{B(\mathcal{T}_i)},\quad 0\le i\le k. \tag{9}$$

A canonical starting family is obtained, for example, by taking $\tau_i^{(0)}\equiv i$. Suppose that for $m\ge 0$ the pair $\left((\tau_i^{(m)}),(Y^{m(i)})\right)$ is constructed, where

$$Y^{m(i)}:=E^{\mathcal{F}_i}\frac{C_{\mathcal{T}_{\tau_i^{(m)}}}}{B(\mathcal{T}_{\tau_i^{(m)}})}\ge\frac{C_{\mathcal{T}_i}}{B(\mathcal{T}_i)},\quad 0\le i\le k,$$

and the stopping time family $(\tau_i^{(m)})$ satisfies (2). Then define

$$\begin{aligned}\tau_i^{(m+1)}:&=\inf\{j:i\le j\le k,\ \max_{p:\,j\le p\le k}E^{\mathcal{F}(j)}\frac{C_{\mathcal{T}_{\tau_p^{(m)}}}}{B(\mathcal{T}_{\tau_p^{(m)}})}\le\frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)}\}\\&=\inf\{j:i\le j\le k,\ \widetilde{Y}^{m+1(i)}\le\frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)}\},\quad 0\le i\le k,\end{aligned}\tag{10}$$

with

$$\widetilde{Y}^{m+1(i)}:=\max_{p:\,i\le p\le k}E^{\mathcal{F}(i)}\frac{C_{\mathcal{T}_{\tau_p^{(m)}}}}{B(\mathcal{T}_{\tau_p^{(m)}})}$$

being an intermediate dummy process. Clearly, $\tau_i^{(m+1)}$ satisfies (2) as well, and due to Theorem 3.1 we have

$$\frac{C_{\mathcal{T}_i}}{B(\mathcal{T}_i)}\le Y^{0(i)}\le Y^{m(i)}\le\widetilde{Y}^{m+1(i)}\le Y^{m+1(i)}\le Y^{*(i)},\quad 0\le m<\infty,\quad 0\le i\le k. \tag{11}$$

By the following proposition, for each fixed i the sequence $(\tau_i^{(m)})_{m\ge1}$ is nondecreasing in m and bounded by any optimal stopping time τ_i^*.

Proposition 4.1 *Let τ_i^* be an optimal stopping time. For each m: $1\le m<\infty$ and i: $0\le i\le k$, we have*

$$\tau_i^{(m)}\le\tau_i^{(m+1)}\le\tau_i^*.$$

Proof. Suppose that $\tau_i^* < \tau_i^{(m)}$ for some $m \geq 1$ and some i with $0 \leq i \leq k$. Then, by (11) and the definition of $\tau_i^{(m)}$,

$$Y^{*(\tau_i^*)} \geq \widetilde{Y}^{m(\tau_i^*)} > \frac{C_{\mathcal{T}_{\tau_i^*}}}{B(\mathcal{T}_{\tau_i^*})},$$

so τ_i^* is not optimal, hence a contradiction. Thus, the right inequality is proved. Next suppose $\tau_i^{(m+1)} < \tau_i^{(m)}$ for some $m \geq 1$ and some i with $0 \leq i \leq k$. Then, by the definition of $\tau_i^{(m)}$ we have

$$\widetilde{Y}^{m(\tau_i^{(m+1)})} > \frac{C_{\mathcal{T}_{\tau_i^{(m+1)}}}}{B(\mathcal{T}_{\tau_i^{(m+1)}})}.$$

On the other hand, according the definition of $\tau_i^{(m+1)}$, we have

$$\widetilde{Y}^{m+1(\tau_i^{(m+1)})} \leq \frac{C_{\mathcal{T}_{\tau_i^{(m+1)}}}}{B(\mathcal{T}_{\tau_i^{(m+1)}})}.$$

So, we get $\widetilde{Y}^{m(\tau_i^{(m+1)})} > \widetilde{Y}^{m+1(\tau_i^{(m+1)})}$, which contradicts (11). ∎

We now may define a limit lower bound process Y^∞ and a limit family of stopping times (τ_i^∞) by

$$Y^{\infty(i)} := \text{(a.s.)} \lim_{m\uparrow\infty} \uparrow Y^{m(i)} \quad \text{and} \quad \tau_i^\infty := \text{(a.s.)} \lim_{m\uparrow\infty} \uparrow \tau_i^{(m)}, \quad 0 \leq i \leq k, \tag{12}$$

where the uparrows indicate that the respective sequences are non-decreasing. It is clear that the family (τ_i^∞) satisfies (2). Moreover, we have

$$Y^{\infty(i)} = \text{(a.s.)} \lim_{m\uparrow\infty} \uparrow E^{\mathcal{F}(i)} \frac{C_{\mathcal{T}_{\tau_p^{(m)}}}}{B(\mathcal{T}_{\tau_p^{(m)}})} = E^{\mathcal{F}(i)} \frac{C_{\mathcal{T}_{\tau_i^\infty}}}{B(\mathcal{T}_{\tau_i^\infty})}, \quad 0 \leq i \leq k.$$

by dominated convergence.

We are now ready to present our main result.

Theorem 4.2 *The constructed limit process Y^∞ in (12) coincides with the Snell envelope process Y^* and (τ_i^∞) in (12) acts as a family of optimal stopping times. We have*

$$Y^{*(i)} = Y^{\infty(i)} = E^{\mathcal{F}(i)} \frac{C_{\mathcal{T}_{\tau_i^\infty}}}{B(\mathcal{T}_{\tau_i^\infty})}, \quad 0 \leq i \leq k. \tag{13}$$

Proof. The Snell envelope $\left(Y^{*(i)}\right)$ is the smallest supermartingale which dominates the discounted cash-flow process $\left(\frac{C_{\mathcal{T}_i}}{B(\mathcal{T}_i)}\right)$ (see, e.g. Shiryayev (1978), Elliot & Kopp, 1999). So, by (12) and the first inequality in (11) it is enough to show that the process $\left(Y^{\infty(i)}\right)$ is a supermartingale. Note, that for each i: $0 \leq i \leq k$,

$$\begin{aligned}
E^{\mathcal{F}(i)} Y^{\infty(i+1)} &= E^{\mathcal{F}(i)} E^{\mathcal{F}(i+1)} \frac{C_{\mathcal{T}_{\tau_{i+1}^{\infty}}}}{B(\mathcal{T}_{\tau_{i+1}^{\infty}})} \\
&= E^{\mathcal{F}(i)} 1_{\tau_i^{\infty}=i} E^{\mathcal{F}(i+1)} \frac{C_{\mathcal{T}_{\tau_{i+1}^{\infty}}}}{B(\mathcal{T}_{\tau_{i+1}^{\infty}})} + E^{\mathcal{F}(i)} 1_{\tau_i^{\infty}>i} E^{\mathcal{F}(i+1)} \frac{C_{\mathcal{T}_{\tau_{i+1}^{\infty}}}}{B(\mathcal{T}_{\tau_{i+1}^{\infty}})} \\
&= E^{\mathcal{F}(i)} 1_{\tau_i^{\infty}=i} \frac{C_{\mathcal{T}_{\tau_{i+1}^{\infty}}}}{B(\mathcal{T}_{\tau_{i+1}^{\infty}})} + 1_{\tau_i^{\infty}>i} Y^{\infty(i)} \\
&= Y^{\infty(i)} + 1_{\tau_i^{\infty}=i} (E^{\mathcal{F}(i)} \frac{C_{\mathcal{T}_{\tau_{i+1}^{\infty}}}}{B(\mathcal{T}_{\tau_{i+1}^{\infty}})} - Y^{\infty(i)}). \qquad (14)
\end{aligned}$$

Since $\widetilde{Y}^{m(i)}$ is non-decreasing in m, it easily seen that

$$\tau_i^{\infty} = \inf\{j : i \leq j \leq k, \max_{p: j \leq p \leq k} E^{\mathcal{F}(j)} \frac{C_{\mathcal{T}_{\tau_p^{\infty}}}}{B(\mathcal{T}_{\tau_p^{\infty}})} \leq \frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)}\}, \quad 0 \leq i \leq k,$$

by letting $m \uparrow \infty$ in (10) (the definition of $\tau_i^{(m)}$). So, for each j, $0 \leq j \leq k$, with $j < \tau_i^{\infty}$ we have

$$\max_{p: j \leq p \leq k} E^{\mathcal{F}(j)} \frac{C_{\mathcal{T}_{\tau_p^{\infty}}}}{B(\mathcal{T}_{\tau_p^{\infty}})} > \frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)},$$

and

$$\max_{p: \tau_i^{\infty} \leq p \leq k} E^{\mathcal{F}(\tau_i^{\infty})} \frac{C_{\mathcal{T}_{\tau_p^{\infty}}}}{B(\mathcal{T}_{\tau_p^{\infty}})} \leq \frac{C_{\mathcal{T}_{\tau_i^{\infty}}}}{B(\mathcal{T}_{\tau_i^{\infty}})}.$$

Then, in particular,

$$1_{\tau_i^{\infty}=i} E^{\mathcal{F}(i)} \frac{C_{\mathcal{T}_{\tau_{i+1}^{\infty}}}}{B(\mathcal{T}_{\tau_{i+1}^{\infty}})} \leq 1_{\tau_i^{\infty}=i} \max_{p: i \leq p \leq k} E^{\mathcal{F}(i)} \frac{C_{\mathcal{T}_{\tau_p^{\infty}}}}{B(\mathcal{T}_{\tau_p^{\infty}})} \leq 1_{\tau_i^{\infty}=i} \frac{C_{\mathcal{T}_i}}{B(\mathcal{T}_i)}, \quad 0 \leq i \leq k.$$

Therefore, by (11) and (14) it follows that

$$\begin{aligned}
E^{\mathcal{F}(i)} Y^{\infty(i+1)} - Y^{\infty(i)} &= 1_{\tau_i^{\infty}=i} (E^{\mathcal{F}(i)} \frac{C_{\mathcal{T}_{\tau_{i+1}^{\infty}}}}{B(\mathcal{T}_{\tau_{i+1}^{\infty}})} - Y^{\infty(i)}) \qquad (15) \\
&\leq 1_{\tau_i^{\infty}=i} (\frac{C_{\mathcal{T}_i}}{B(\mathcal{T}_i)} - Y^{\infty(i)}) \leq 0, \quad 0 \leq i \leq k,
\end{aligned}$$

and so the process $(Y^{\infty(i)})$ is a supermartingale. ■

Remark 4.3 In addition, we can prove the following expression for the distance between two consequent iterations,

$$Y^{m+1}(i) - Y^{m}(i) = \sum_{p=i}^{\tau_i^{(m+1)}-1} (E^{\mathcal{F}(p)} Y^{m}(p+1) - Y^{m}(p)) \geq 0, \tag{16}$$

see Appendix.

Remark 4.4 *(variance reduced Monte Carlo simulation of Y^m)* Monte Carlo simulation of Y^m requires computation of the vector $(Y^{m-1}(i))_{0\leq i\leq k}$ along each simulated trajectory. Thus, assuming that we can compute European claims in closed form, Y^1 can be computed with a standard (linear) Monte Carlo simulation, but then Y^2 will require a nested (quadratic) Monte Carlo simulation, and so on. So, the cost of the method grows rapidly with each new iteration. Fortunately, we can reduce the number of Monte Carlo simulations for Y^m by using the following variance reduced representation,

$$Y^{m}(i) = E^{\mathcal{F}_i} Z_{\tau_i^{(m)}} = E^{\mathcal{F}_i} Z_{\tau_i^{(m-1)}} + E^{\mathcal{F}_i}(Z_{\tau_i^{(m)}} - Z_{\tau_i^{(m-1)}}), \quad \text{with } Z_i := \frac{C_{\mathcal{T}_i}}{B(\mathcal{T}_i)}. \tag{17}$$

One can expect that $Z_{\tau_i^{(m-1)}}$ and $Z_{\tau_i^{(m)}}$ are strongly correlated and thus the variance of $(Z_{\tau_i^{(m)}} - Z_{\tau_i^{(m-1)}})$ will be less than the variance of $Z_{\tau_i^{(m)}}$. So, the computation of $E^{\mathcal{F}_i}(Z_{\tau_i^{(m)}} - Z_{\tau_i^{(m-1)}})$ for a given accuracy, can usually be done with less Monte Carlo simulations than needed for direct simulation of $E^{\mathcal{F}_i} Z_{\tau_i^{(m)}}$.

5 Iterative upper bounds by the dual approach

Based on the convergent family of lower bound processes Y^m developed in the previous section, we will deduce in this section a convergent family of upper bound processes by a duality approach developed in the works of Davis & Karatzas (1994), Haugh & Kogan (2001), Rogers (2001).

The duality approach is based on the following observation. For any supermartingale $(S^{(j)})$ with $S^{(0)} = 0$ we have,

$$\begin{aligned} Y^{*}(0) &= \sup_{\tau\in\{0,\ldots,k\}} E^{\mathcal{F}_0} \frac{C_{\mathcal{T}_\tau}}{B(\mathcal{T}_\tau)} \leq \sup_{\tau\in\{0,\ldots,k\}} E^{\mathcal{F}_0}\left(\frac{C_{\mathcal{T}_\tau}}{B(\mathcal{T}_\tau)} - S^{(\tau)}\right) \\ &\leq E^{\mathcal{F}_0} \max_{0\leq j\leq k}\left(\frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} - S^{(j)}\right), \end{aligned} \tag{18}$$

hence the right-hand side provides a (dual) upper bound for $Y^{*}(0)$.

Rogers (2001) and independently Haugh & Kogan (2001), provide a representation of $Y^{*(0)}$ as an infimum over a set of supermartingales,

$$Y^{*(0)} = \inf_{S \in \mathcal{S}} E^{\mathcal{F}_0} \max_{0 \le j \le k} \left(\frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} - S^{(j)} \right), \tag{19}$$

where $\mathcal{S}$ denotes the set of supermartingales S with $S^{(0)} = 0$. Moreover, the infimum is attained at the martingale part of the Doob-Meyer decomposition of Y^*,

$$M^{Y^*(0)} = 0; \quad M^{Y^*(j)} = \sum_{l=1}^{j} (Y^{*(l)} - E^{\mathcal{F}_{l-1}} Y^{*(l)}), \tag{20}$$

and also at the shifted Snell envelope process

$$S^{(j)} = Y^{*(j)} - Y^{*(0)}. \tag{21}$$

Recently, Jamshidian (2003a,b) proved a multiplicative analogue of the representation (19),

$$Y^{*(0)} = \inf_{M \in \mathcal{M}} E^{\mathcal{F}_0} \max_{0 \le j \le k} \frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} \frac{M^{(k)}}{M^{(j)}}, \tag{22}$$

where $\mathcal{M}$ is the set of positive martingales. Jamshidian shows that in (22) the infimum is attained at the martingale part of the multiplicative Doob-Meyer decomposition of Y^*,

$$N^{Y^*(0)} = 1; \quad N^{Y^*(j)} = \prod_{l=1}^{j} \frac{Y^{*(l)}}{E^{\mathcal{F}_{l-1}} Y^{*(l)}}. \tag{23}$$

Lemma 5.1 and Lemma 5.3 below give a somewhat more general characterization of the supermartingales and martingales where the infima in (19) and (22) are attained, respectively.

Lemma 5.1 *Let $\mathcal{S}$ be the set of supermartingales S with $S^{(0)} = 0$. Let $S \in \mathcal{S}$ be such that $\frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} - Y^{*(0)} \le S^{(j)}$, $1 \le j \le k$. Then,*

$$Y^{*(0)} = \max_{0 \le j \le k} \left(\frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} - S^{(j)} \right) \quad a.s. \tag{24}$$

Proof. From the assumptions it follows that $\frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} - Y^{*(0)} \le S^{(j)}$, $0 \le j \le k$, then, using (18),

$$0 \le E^{\mathcal{F}_0} \max_{0 \le j \le k} \left(\frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} - S^{(j)} \right) - Y^{*(0)} = E^{\mathcal{F}_0} \max_{0 \le j \le k} \left(\frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} - S^{(j)} - Y^{*(0)} \right) \le 0.$$

Hence, we have $E^{\mathcal{F}_0} \max_{0 \le j \le k} \left(\frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} - S^{(j)} - Y^{*(0)} \right) = 0$ and $\frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} - S^{(j)} - Y^{*(0)} \le 0$, which yields (24). ■

Note that both (20) and (21) satisfy the conditions of Lemma 5.1. Moreover, by Proposition 5.2 it turns out that, somewhat remarkably, the multiplicative martingale part of the Snell envelope which minimizes (22) also provides the infimum in the additive dual representation (19).

Proposition 5.2 *By taking*

$$S^{(j)} = (N^{Y^*(j)} - 1)\, Y^{*(0)},$$

where N^{Y^*} *is given by* (23)*, the equality* (24) *holds.*

Proof. By rearranging (23) and using the supermartingale property of Y^* we get

$$N^{Y^*(j)} Y^{*(0)} = Y^{*(j)} \prod_{l=0}^{j-1} \frac{Y^{*(l)}}{E^{\mathcal{F}_l} Y^{*(l+1)}} \geq Y^{*(j)} \geq \frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} \tag{25}$$

and, consequently,

$$(N^{Y^*(j)} - 1) Y^{*(0)} \geq \frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} - Y^{*(0)}.$$

Now the equality (24) follows Lemma 5.1.

■

Lemma 5.3 *Let* $\mathcal{M}$ *be the set of positive martingales* M*, such that* $M^{(0)} = 1$*. Let* $M \in \mathcal{M}$ *be such that* $\dfrac{C_{\mathcal{T}_j}}{Y^{*(0)} B(\mathcal{T}_j)} \leq M^{(j)}$*,* $1 \leq j \leq k$*. Then,*

$$Y^{*(0)} = E^{\mathcal{F}_0} \max_{0 \leq j \leq k} \frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} \frac{M^{(k)}}{M^{(j)}}. \tag{26}$$

Proof. First of all we observe that

$$\begin{aligned} Y^{*(0)} &= \sup_{\tau \in \{0,\ldots,k\}} E^{\mathcal{F}_0} \frac{C_{\mathcal{T}_\tau}}{B(\mathcal{T}_\tau)} = \sup_{\tau \in \{0,\ldots,k\}} E^{\mathcal{F}_0} \frac{C_{\mathcal{T}_\tau}}{B(\mathcal{T}_\tau)} \frac{E^{\mathcal{F}_\tau} M^{(k)}}{M^{(\tau)}} \\ &= \sup_{\tau \in \{0,\ldots,k\}} E^{\mathcal{F}_0} \frac{C_{\mathcal{T}_\tau}}{B(\mathcal{T}_\tau)} \frac{M^{(k)}}{M^{(\tau)}} \leq E^{\mathcal{F}_0} \max_{0 \leq j \leq k} \frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} \frac{M^{(k)}}{M^{(j)}}. \end{aligned} \tag{27}$$

Then, by the assumptions, it follows that

$$\begin{aligned} Y^{*(0)} &\leq E^{\mathcal{F}_0} \max_{0 \leq j \leq k} \frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} \frac{M^{(k)}}{M^{(j)}} \\ &= Y^{*(0)} E^{\mathcal{F}_0} \max_{0 \leq j \leq k} \frac{C_{\mathcal{T}_j}}{Y^{*(0)} B(\mathcal{T}_j)} \frac{M^{(k)}}{M^{(j)}} \leq Y^{*(0)} E^{\mathcal{F}_0} M^{(k)} = M^{(0)} Y^{*(0)} = Y^{*(0)}, \end{aligned}$$

hence (26).

■

Note, that due to (25), the martingale (23) satisfies the conditions of Lemma 5.3. Moreover, by Proposition 5.4 it turns out, that the martingale part of the Doob-Meyer decomposition of the Snell envelope which minimizes (19) also provides the infimum in the multiplicative dual representation (22).

Proposition 5.4 *By taking*

$$M^{(j)} = \frac{M^{Y^*}(j)}{Y^{*(0)}} + 1,$$

where M^{Y^*} *is given by* (20)*, equality* (26) *holds.*

Proof. By rearranging (20) and using the supermartingale property of Y^* we get

$$M^{Y^*}(j) + Y^{*(0)} = Y^{*(j)} + \sum_{l=0}^{j-1} \left(Y^{*(l)} - E^{\mathcal{F}_l} Y^{*(l+1)}\right) \geq Y^{*(j)} \geq \frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)}$$

and, consequently,

$$\frac{M^{Y^*}(j)}{Y^{*(0)}} + 1 \geq \frac{C_{\mathcal{T}_j}}{Y^{*(0)} B(\mathcal{T}_j)}.$$

Now (26) follows from Lemma 5.3. ∎

The duality representation provides a simple way to estimate the Snell envelope from above, using a lower approximation process denoted by $\overline{Y}$, hence $\overline{Y} \leq Y^*$. Let $\overline{M}$ be the martingale part of the Doob-Meyer decomposition of $\overline{Y}$, satisfying

$$\overline{M}^{(0)} = 0;$$

$$\overline{M}^{(j)} = \overline{M}^{(j-1)} + \overline{Y}^{(j)} - E^{\mathcal{F}(j-1)} \overline{Y}^{(j)} = \sum_{l=i+1}^{j} \overline{Y}^{(l)} - \sum_{l=i+1}^{j} E^{\mathcal{F}(l-1)} \overline{Y}^{(l)}, \; j = 1, \ldots, k.$$

Then, according to (18),

$$Y^{*(0)} \leq E^{\mathcal{F}_0} \max_{0 \leq j \leq k} \left(\frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} - \overline{M}^{(j)}\right) =: \overline{Y}^{(0)}_{up}. \tag{28}$$

By the next theorem (taken from Kolodko & Schoenmakers, (2003)), the gap between $\overline{Y}^{(0)}$ and $\overline{Y}^{(0)}_{up}$ depends, in some sense, on how far the lower bound process $\overline{Y}$ is away from being a supermartingale.

Theorem 5.5 *Suppose, that* $\overline{Y}^{(i)} \geq \frac{C_{\mathcal{T}_i}}{B(\mathcal{T}_i)}$, $i = 0, \ldots, k$. *Then,*

$$0 \leq \overline{Y}^{(0)}_{up} - \overline{Y}^{(0)} \leq E^{\mathcal{F}_0} \sum_{j=0}^{k-1} \max(E^{\mathcal{F}_j} \overline{Y}^{(j+1)} - \overline{Y}^{(j)}, \; 0).$$

Proof. By definition (28), we have

$$\overline{Y}^{(0)}_{up} = E^{\mathcal{F}_0} \max_{0\le j\le k} \Big(\frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} - \sum_{l=1}^{j} \overline{Y}^{(l)} + \sum_{l=1}^{j} E^{\mathcal{F}(l-1)} \overline{Y}^{(l)}\Big) \tag{29}$$

$$= \overline{Y}^{(0)} + E^{\mathcal{F}_0} \max_{0\le j\le k} \Big(\frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} - \overline{Y}^{(j)} + \sum_{l=0}^{j-1} E^{\mathcal{F}(l)} (\overline{Y}^{(l+1)} - \overline{Y}^{(l)})\Big) =: \overline{Y}^{(0)} + \Delta^{(0)}.$$

We thus have the following estimate,

$$\begin{aligned}
\Delta^{(0)} = \overline{Y}^{(0)}_{up} - \overline{Y}^{(0)} &\le E^{\mathcal{F}_0} \max_{0\le j\le k} \sum_{l=0}^{j-1} (E^{\mathcal{F}(l)} \overline{Y}^{(l+1)} - \overline{Y}^{(l)}) \\
&\le E^{\mathcal{F}_0} \max_{0\le j\le k} \sum_{l=0}^{j-1} \max(E^{\mathcal{F}(l)} \overline{Y}^{(l+1)} - \overline{Y}^{(l)}, 0) \\
&\le E^{\mathcal{F}_0} \sum_{j=0}^{k-1} \max(E^{\mathcal{F}(j)} \overline{Y}^{(j+1)} - \overline{Y}^{(j)}, 0).
\end{aligned}$$

■

Let us now consider the sequence of lower bound processes Y^m from the previous section. Analogously to (28) we now deduce a sequence of upper bound processes

$$\begin{aligned}
Y^{m(i)}_{up} &:= E^{\mathcal{F}_i} \max_{i\le j\le k} \Big(\frac{C_{\mathcal{T}_j}}{B(\mathcal{T}_j)} - \sum_{l=i+1}^{j} Y^{m(l)} + \sum_{l=i+1}^{j} E^{\mathcal{F}(l-1)} Y^{m(l)}\Big) \\
&=: Y^{m(i)} + \Delta^{m(i)}.
\end{aligned}$$

By Theorem 5.5 it then follows that

$$0 \le \Delta^{m(i)} \le E^{\mathcal{F}_i} \sum_{j=i}^{k-1} \max\left(E^{\mathcal{F}_j} Y^{m(j+1)} - Y^{m(j)},\ 0\right).$$

By letting $m \uparrow \infty$ on the right-hand side and using Theorem 4.2, we obtain

$$\text{(a.s.)} \lim_{m\to\infty} \Delta^{m(i)} = 0, \quad 0 \le i \le k.$$

Hence, the sequence Y^m_{up} converges to the Snell envelope, i.e.,

$$\text{(a.s.)} \lim_{m\to\infty} Y^{m(i)}_{up} = \text{(a.s.)} \lim_{m\to\infty} Y^{m(i)} = Y^{*(i)}, \quad 0 \le i \le k.$$

6 A numerical example: Bermudan swaptions in the LIBOR market model

Let us first recall the LIBOR Market Model with respect to a tenor structure $0 < T_1 < T_2 < \ldots < T_n$ in the *spot LIBOR measure* P^*, induced by the numeraire

$$B^*(t) := \frac{B_{m(t)}(t)}{B_1(0)} \prod_{i=0}^{m(t)-1} (1 + \delta_i L_i(T_i))$$

with $m(t) := \min\{m : T_m \geq t\}$ denoting the next reset date at time t. The dynamics of the forward LIBOR $L_i(t)$, defined in the interval $[0, T_i]$ for $1 \leq i < n$, is governed by the following system of SDE's (Jamshidian 1997),

$$dL_i = \sum_{j=m(t)}^{i} \frac{\delta_j L_i L_j \, \gamma_i \cdot \gamma_j}{1 + \delta_j L_j} \, dt + L_i \, \gamma_i \cdot dW^*. \tag{30}$$

Here $\delta_i = T_{i+1} - T_i$ are day count fractions, and $t \to \gamma_i(t) = (\gamma_{i,1}(t), \ldots, \gamma_{i,d}(t))$ are deterministic volatility vector functions defined in $[0, T_i]$, called factor loadings. In (30), $(W^*(t) \mid 0 \leq t \leq T_{n-1})$ is a standard d-dimensional Wiener process under the measure P^* with d, $1 \leq d < n$, being the number of driving factors.

A *swaption contract* with maturity T_i and strike θ with principal \$1 gives the right to contract at T_i for paying a fixed coupon θ and receiving floating LIBOR at the settlement dates $T_{i+1}, \ldots, T_n$. So by this definition, its cash-flow at maturity is

$$S_{i,n}(T_i) := \left(\sum_{j=i}^{n-1} B_{j+1}(T_i) \delta_j \left(L_j(T_i) - \theta \right) \right)^+ .$$

We here consider Bermudan swaptions for which the exercise dates coincide with the LIBOR tenor structure. I.e. $k = n$ and $\mathcal{T}_i = T_i$, for $1 \leq i \leq n$.

A *Bermudan swaption*, issued at $t = 0$, gives the right to exercise a cash-flow

$$C_{T_\tau} := S_{\tau,n}(T_\tau)$$

at an exercise date $T_\tau \in \{T_1, \ldots, T_n\}$ to be decided by the option holder. The value of the Bermudan swaption is given by (1).

Remark 6.1 In practice it is more realistic to assume, that the Bermudan swaption cannot be exercised at $t = \mathcal{T}_0 = 0$. This assumption is equivalent to the assumption $C_{\mathcal{T}_0} = 0$ in Section 2.

For simulation experiments we use the following LIBOR volatility structure,

$$\gamma_i(t) = cg(T_i - t)e_i, \quad \text{where} \quad g(s) = g_\infty + (1 - g_\infty + as)e^{-bs}$$

is a parametric volatility function proposed by Rebonato (1999), and e_i are d-dimensional unit vectors, decomposing some input correlation matrix of rank d. For generating LIBOR models with different numbers of factors d, we take as a basis a correlation structure of the form

$$\rho_{ij} = \exp(-\varphi|i-j|); \qquad i, j = 1, \ldots, n-1 \tag{31}$$

which has full-rank for $\varphi > 0$, and then for a particular choice of d we deduce from ρ a rank-d correlation matrix ρ^d with decomposition $\rho^d_{ij} = e_i \cdot e_j$, $1 \le i, j < n$, by principal component analysis. We note that instead of (31) it is possible to use more general and economically more realistic correlation structures. For instance the parametric structures of Schoenmakers & Coffey (2003).

Further we take over the following model parameters used in Kolodko & Schoenmakers (2003): A flat 10% initial LIBOR curve over a 40 period quarterly tenor structure, and the parameters

$$n = 41,\ \delta_i = 0.25,\ c = 0.2,\ a = 1.5,\ b = 3.5,\ g_\infty = 0.5,\ \varphi = 0.0413. \tag{32}$$

For a "practically exact" numerical integration of the SDE (30), we used the log-Euler scheme with $\Delta t = \delta/5$ (e.g., see also Kurbanmuradov, Sabelfeld and Schoenmakers 2002).

In Kolodko & Schoenmakers (2003) we studied lower and upper estimations of different Bermudan swaptions. As lower estimation we considered a lower bound process Y_A, obtained by the Andersen method (Andersen (1999), strategy I). Then, based on Y_A, we computed an upper estimation $Y_{up,A}$ via (28) like in Andersen & Broadie (2001). It turns out that for 1-factor models Andersen's method gets very close to the Snell envelope. In fact, for one factor the relative distance between Y_A and by Y_A induced upper bounds does not exceed 1.5% in the examples considered by Kolodko & Schoenmakers (2003) (see for more examples Andersen & Broadie (2001)). However, in Kolodko & Schoenmakers (2003) it is shown that when the number of factors is larger than 1, this distance increases from ITM to OTM strikes. For OTM strikes and more than 2 factors this distance is even larger than 10% relative.

We now compute, according to the method developed in Section 4, starting from $\tau_i^{(0)} \equiv i$, three successive lower bounds $Y^{m(0)}$: $Y^{0(0)}$, $Y^{1(0)}$ and $Y^{2(0)}$ for OTM cases. The results are compared with the lower and upper estimations due to the Andersen's lower bound process Y_A. For computing the iteration $Y^{2(0)}$, we apply the variance reduction technique (17). We use 5 000 000 Monte Carlo trajectories for $Y^{0}(0)$ and $Y^{1}(0)$ and 20 000 Monte Carlo trajectories (with 100 inner simulations) for computation of $Y^{2}(0) - Y^{1}(0)$. Then, the standard deviations of $Y^{2}(0)$ are roughly equal to the corresponding deviations due to the second term in (17), and are also comparable with the standard deviations of Andersen's lower bounds, reported in Kolodko & Schoenmakers, 2003. The results are given in Table 1.

Table 1.

θ	d	$Y^{0}(0)$ (SD)	$Y^{1}(0)$ (SD)	$Y^{2}(0)$ (SD)	$Y_A^{(0)}$ (SD)†	$Y_{up,A}^{(0)}$ (SD)†
	1	10.2(0.0)	119.3(0.1)	131.0(0.9)	133.5(0.7)	135.4(0.1)
0.12	2	5.2(0.0)	114.5(0.1)	123.4(0.8)	119.7(0.7)	127.4(0.3)
(OTM)	10	3.0(0.0)	104.2(0.1)	110.5(0.6)	102.8(0.6)	113.6(0.3)
	40	2.7(0.0)	101.4(0.1)	106.1(0.6)	98.8(0.5)	110.3(0.3)

It turns out that in all cases, except for the 1-factor case, the secondly iterated lower bound $Y^{2}(0)$ is significantly higher than $Y_A^{(0)}$. Remarkably, for 10 and 40 factors already the first iteration $Y^{1}(0)$ is slightly higher than $Y_A^{(0)}$. In the 1-factor case, where Andersen's lower bound and corresponding dual upper bound are within 1.5% relative, $Y^{2}(0)$ is 1.5% relative below $Y_A^{(0)}$. Note that for more than 1 factor the computed lower bound $Y^{2}(0)$, hence the second iteration, can be found more or less in the middle of $Y_A^{(0)}$ and $Y_{A,up}^{(0)}$.

Remark 6.2 The construction of the new stopping time $\widehat{\tau}$ from τ via (5) provides a general method for improving any given stopping time τ with properties (2). So in principle we can improve Andersen's process Y_A by constructing $\widehat{Y}_A$ via (6). In this respect, we report that preliminary computations for two factor OTM cases yielded comparable values for $\widehat{Y}_A$ and $Y^{2}(0)$.

Concluding remarks

The implementation of the proposed iterative procedure is straightforward, and thus can be done in a generic way for a variety of (not necessarily financial) optimal stopping problems. Although such an implementation gives rise to nested Monte Carlo simulations and therefore may be not too fast, we saw that for $m = 2$ (hence with a quadratic Monte Carlo simulation) practically correct Bermudan swaption prices can be obtained. So the method may serve at least as a Benchmark tool.

†The results are taken from Kolodko & Schoenmakers, 2003

In general, provided that Europeans can be priced analytically, computation of Y^m requires about $O(N^m)$ simulations of the underlying process. However, a person which is only interested in the optimal exercise decision (for instance the buyer of the Bermudan product) can decide the stopping time $\tau^{(m)}$ at a cost of only $O(N^{m-1})$ simulations.

Finally we predict that, when the producers of microprocessor chips keep "riding the exponential", the computation of higher order iterations (hence almost exact prices) will become feasible in the near future.

Appendix.

We now prove the equality (16).

Proof. Let us write

$$Y^{m+1}(i) - Y^m(i) = 1_{\tau_i^{(m+1)}=i}\left(Y^{m+1}(i) - Y^m(i)\right) + 1_{\tau_i^{(m+1)}>i}\left(Y^{m+1}(i) - Y^m(i)\right). \quad (33)$$

Note that the first term in (33) is zero. Indeed, if $\tau_i^{(m+1)} = i$, then $Y^{m+1}(i) = E^{\mathcal{F}(i)} \frac{C_{\mathcal{T}_{\tau_i^{(m+1)}}}}{B(\mathcal{T}_{\tau_i^{(m+1)}})} = \frac{C_{\mathcal{T}_i}}{B(\mathcal{T}_i)}$, and on the other hand, $\frac{C_{\mathcal{T}_i}}{B(\mathcal{T}_i)} \leq Y^m(i) \leq Y^{m+1}(i)$.

Now we consider the second term in (33). If $\tau_i^{(m+1)} > i$, then $\tau_i^{(m+1)} = \tau_{i+1}^{(m+1)}$ and so

$$Y^{m+1}(i) = E^{\mathcal{F}(i)} \frac{C_{\mathcal{T}_{\tau_i^{(m+1)}}}}{B(\mathcal{T}_{\tau_i^{(m+1)}})} = E^{\mathcal{F}(i)} \frac{C_{\mathcal{T}_{\tau_{i+1}^{(m+1)}}}}{B(\mathcal{T}_{\tau_{i+1}^{(m+1)}})} = E^{\mathcal{F}(i)} Y^{m+1}(i+1).$$

We thus obtain,

$$
\begin{aligned}
Y^{m+1}(i) - Y^{m}(i) &= 1_{\tau_i^{(m+1)}>i}\left(E^{\mathcal{F}(i)}Y^{m+1}(i+1) - Y^{m}(i)\right) \\
&= 1_{\tau_i^{(m+1)}>i}E^{\mathcal{F}(i)}(Y^{m+1}(i+1) - Y^{m}(i+1)) + 1_{\tau_i^{(m+1)}>i}E^{\mathcal{F}(i)}(Y^{m}(i+1) - Y^{m}(i)) \\
&= 1_{\tau_i^{(m+1)}>i}E^{\mathcal{F}(i)}\left(1_{\tau_{i+1}^{(m+1)}>i+1}E^{\mathcal{F}(i+1)}(Y^{m+1}(i+2) - Y^{m}(i+2))\right. \\
&\quad \left.+1_{\tau_{i+1}^{(m+1)}>i+1}E^{\mathcal{F}(i+1)}\left(Y^{m}(i+2) - Y^{m}(i+1)\right)\right) \\
&\quad +1_{\tau_i^{(m+1)}>i}E^{\mathcal{F}(i)}\left(Y^{m}(i+1) - Y^{m}(i)\right) \\
&= E^{\mathcal{F}(i)}(1_{\tau_{i+1}^{(m+1)}>i+1}(Y^{m+1}(i+2) - Y^{m}(i+2)) \\
&\quad +1_{\tau_{i+1}^{(m+1)}>i+1}(Y^{m}(i+2) - Y^{m}(i+1))) + 1_{\tau_i^{(m+1)}>i}E^{\mathcal{F}(i)}(Y^{m}(i+1) - Y^{m}(i)) \\
&= E^{\mathcal{F}(i)}1_{\tau_{i+1}^{(m+1)}>i+1}(Y^{m+1}(i+2) - Y^{m}(i+2)) + \\
&\quad +E^{\mathcal{F}(i)}1_{\tau_{i+1}^{(m+1)}>i+1}\left(Y^{m}(i+2) - Y^{m}(i+1)\right) \\
&\quad +1_{\tau_i^{(m+1)}>i}E^{\mathcal{F}(i)}\left(Y^{m}(i+1) - Y^{m}(i)\right),
\end{aligned}
$$

then by induction from i to k and the fact that $Y^{m+1}(k) = Y^{m}(k)$,

$$
\begin{aligned}
Y^{m+1}(i) - Y^{m}(i) &= \sum_{p=i}^{k-1} 1_{\tau_p^{(m+1)}>p}E^{\mathcal{F}(p)}\left(Y^{m}(p+1) - Y^{m}(p)\right) \\
&= \sum_{p=i}^{\tau_i^{(m+1)}-1}\left(E^{\mathcal{F}(p)}Y^{m}(p+1) - Y^{m}(p)\right)
\end{aligned}
$$

■

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