Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Tortuosity and objective relative acceleration in the theory of porous materials

Krzysztof Wilmanski

submitted: 22nd April 2004

Weierstrass Institute
for Applied Analysis
and Stochastics
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: wilmansk@wias-berlin.de

No. 922
Berlin 2004

2000 *Mathematics Subject Classification.* 80A17, 74A20, 74F10, 74J10.

Key words and phrases. Thermodynamics, Biot's model, poroelastic materials, acoustic waves.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

The aim of this work is twofold. We show the construction of an objective relative acceleration for a two-component mixture and prove that its incorporation in the momentum source requires additional terms in partial stresses and in the energy. This may be interpreted as an influence of tortuosity in the theory of saturated poroelastic materials and a connection of tortuosity with fluctuations of the kinetic energy on a mesoscopic level of observation. The linearization of such a model yields Biot's equations used in poroacoustics.

We demonstrate as well that results for the propagation of acoustic waves in saturated poroelastic media are qualitatively similar for Biot's model and for the simple mixture model in which both the tortuosity and the Biot's coupling between partial stresses are neglected. It is also indicated that the coupling constant of Biot's model obtained by means of the Gassmann relation may be too large as it leads to very small differences in the speed of propagation of the P1-wave for small and large frequencies which contradicts the data for soils.

1 Introduction

A celebrated property of the Biot's model of two-component porous materials is related to equations of motion containing a contribution of the relative acceleration. Linear equations describing such a model in a chosen inertial frame of reference have the following form (e.g. [1] [2])

$$\rho_0^S \frac{\partial \mathbf{v}^S}{\partial t} = \lambda^S \operatorname{grad} \operatorname{tr} \mathbf{e}^S + 2\mu^S \operatorname{div} \mathbf{e}^S + Q \operatorname{grad} \varepsilon + \pi \left(\mathbf{v}^F - \mathbf{v}^S\right) - \rho_{12} \left(\frac{\partial \mathbf{v}^F}{\partial t} - \frac{\partial \mathbf{v}^S}{\partial t}\right), \quad (1)$$

$$\rho_0^F \frac{\partial \mathbf{v}^F}{\partial t} = \kappa \rho_0^F \operatorname{grad} \varepsilon + Q \operatorname{grad} \operatorname{tr} \mathbf{e}^S - \pi \left(\mathbf{v}^F - \mathbf{v}^S\right) + \rho_{12} \left(\frac{\partial \mathbf{v}^F}{\partial t} - \frac{\partial \mathbf{v}^S}{\partial t}\right),$$

where

$$\frac{\partial \mathbf{e}^S}{\partial t} = \operatorname{sym} \operatorname{grad} \mathbf{v}^S, \quad \frac{\partial \varepsilon}{\partial t} = \operatorname{div} \mathbf{v}^F, \quad \varepsilon := \frac{\rho_0^F - \rho^F}{\rho_0^F} \equiv \operatorname{tr} \mathbf{e}^S - \frac{\zeta}{n_0}, \quad (2)$$

and $\mathbf{e}^S$ denotes the macroscopical Almansi-Hamel **deformation tensor of the skeleton**, its trace, $\operatorname{tr} \mathbf{e}^S$, is the **volume change** (small deformations!) **of the skeleton**, ε is the **volume change of the fluid** and this is related to the **increment of fluid content**, ζ, by the relation $(2)_3$. ρ_0^S, ρ_0^F are constant initial mass densities connected to the true mass densities ρ_0^{SR}, ρ_0^{FR} in the following way

$$\rho_0^S = (1 - n_0)\, \rho_0^{SR}, \quad \rho_0^F = n_0 \rho_0^{FR}, \quad (3)$$

where n_0 is the initial porosity. $\mathbf{v}^S, \mathbf{v}^F$ are **macroscopic velocities** of both components, i.e. $\mathbf{v}^F - \mathbf{v}^S$ is the seepage velocity. The material parameters $\lambda^S, \mu^S, \kappa, Q, \pi, \rho_{12}$ are constant.

The literature on Biot's model is far from being unique in relation to the notation and this creates a lot of confusion. The above material parameters which we use further in this work are characteristic for the formulation of a two-component mixture. Usually in soil mechanics use is being made of the total bulk stress $\mathbf{T} = \mathbf{T}^S + \mathbf{T}^F$, and the fluid partial stress is related solely to the pore pressure p. Namely $\mathbf{T}^F = -n_0 p \mathbf{1}$.

For this reason the material parameters are introduced, for instance, in the following way [3]

$$K := \lambda^S + \frac{2}{3}\mu^S + \rho_0^F \kappa + 2Q, \quad G := \mu^S, \tag{4}$$
$$C := \frac{1}{n_0}\left(Q + \rho_0^F \kappa\right), \quad M := \frac{\rho_0^F \kappa}{n_0^2}.$$

On the other hand in the standard reference book on linear acoustics of porous materials [4] the following form of the set (1) is used

$$\rho \frac{\partial \mathbf{v}^S}{\partial t} + \rho_{uw} \frac{\partial^2 \mathbf{w}}{\partial t^2} = \operatorname{div} \mathbf{T},$$
$$\mathbf{T} = \mathbf{T}^S + \mathbf{T}^F, \quad \mathbf{T}^F = -n_0 p \mathbf{1}, \quad \frac{\partial \mathbf{w}}{\partial t} = n_0 \left(\frac{\partial \mathbf{v}^F}{\partial t} - \frac{\partial \mathbf{v}^S}{\partial t}\right),$$
$$\mathbf{T} = \lambda_f \operatorname{tr} \mathbf{e}^S \mathbf{1} + 2\mu \mathbf{e}^S - \beta M \zeta \mathbf{1},$$

$$\rho_{uw} \frac{\partial \mathbf{v}^S}{\partial t} + \rho_w \frac{\partial^2 \mathbf{w}}{\partial t^2} = -\operatorname{grad} p - \frac{1}{\mathcal{K}} \mathbf{w}, \tag{5}$$
$$p = M\left(-\beta \operatorname{tr} \mathbf{e}^S + \zeta\right),$$

with the following relations among parameters

$$\rho = \rho_0^S + \rho_0^F, \quad \rho_{uw} = \rho_0^{FR}, \quad \rho_w = \frac{1}{n_0^2}\left(\rho_0^F - \rho_{12}\right), \tag{6}$$
$$\lambda_f = \lambda^S + \kappa \rho_0^F + 2Q, \quad \mu = \mu^S, \quad \beta M = \frac{1}{n_0}\left(Q + \kappa \rho_0^F\right), \quad \frac{1}{\mathcal{K}} = \frac{\pi}{n_0^2}.$$

Still another set of parameters is used by Allard [5], where $\rho_{12} = -\rho_a$, $\rho_0^{FR} = \rho_0$, $\rho_0^S = \rho_1$, etc.

Let us return to the set (1). The parameter ρ_{12} describing the contribution of the relative acceleration is usually related to the **tortuosity** of the porous material. For example, in the works [6], [7] the following approximate relation between this parameter, the porosity n_0, and the tortuosity parameter $a \in [1, \infty)$, is proposed

$$\rho_{12} = \rho_0^F (1 - a), \quad a = \frac{1}{2}\left(\frac{1}{n_0} + 1\right). \tag{7}$$

It is easy to show that the model (1) is nonobjective. This means that the change of the reference frame to a noninertial system (a time dependent change of observer)

$$\mathbf{x}^* = \mathbf{O}(t)\,\mathbf{x} + \mathbf{c}(t), \quad \mathbf{O}^T = \mathbf{O}^{-1}, \tag{8}$$

yields constitutive contributions in these equations following from the presence of the relative acceleration. These contributions appear additionally to the usual centrifugal, Coriolis, Euler and translational accelerations which are characteristic for the continuum mechanics in noninertial frames (e.g. I-Shih Liu [8]).

This violation of objectivity is a bothering feature because solutions of practical problems must then depend on a chosen reference: we obtain entirely different solutions in, say, a laboratory reference, and in a moving reference on a turntable. This problem has been investigated in the paper [9].

The question arises if one could overcome this difficulty by assuming that the nonobjectivity follows from the linearization of some objective nonlinear equations. If this was the case one would have to describe porous materials by Biot's equations solely in inertial reference systems and a time dependent change of reference would require solely an addition of classical acceleration terms and ignoring contributions from the relative acceleration. We investigate some aspects of this question in the present work.

In the next section we introduce the notion of an objective relative acceleration. We follow here the same line as D. Drew *et al* [10] who have considered a problem of suspensions of bubbles in a fluid. In contrast to that work, we apply a Lagrangian description (e.g. see [11]). Then we show that indeed a nonlinear poroelastic two-component model yields the Biot's model by linearization.

In the third section we provide some thermodynamical arguments to show that a nonlinear objective model with a contribution of relative accelerations is thermodynamically admissible if we add some nonlinear contributions to partial stresses and to the free energy. They reflect in the simplest manner the existence of fluctuations of the microstructural kinetic energy caused by the heterogeneity of momentum in the representative elementary volume. The existence of such fluctuations as a result of tortuosity of porous materials has been indicated by O. Coussy [12]. There exist some attempts to derive the Biot's model with the contribution of relative acceleration by means of Hamilton's principle based on the fluctuation kinetic energy. As the true variational principle does not hold for dissipative systems the dissipation through fluctuation and diffusion is accounted for by a pseudo-potential and a pseudo-variational principle. This does not seem to be the right way of handling irreversible processes. For this reason we rely rather on the nonequilibrium thermodynamics in our considerations.

For completeness we show in the fourth section the conditions for propagation of acoustic waves (hyperbolicity) and, in the fifth section, differences in the behavior of bulk monochromatic waves in porous materials within the linear Biot's model and a simplified model (the so-called "simple mixture model") where the coupling through relative accelerations is left out.

Let us mention in passing that the lack of relative accelerations in the model does not mean that the influence of tortuosity is neglected. Certainly, the permeability of the material described by the parameter π in our notation contains an influence of the morphology of the porous materials and this includes an influence of tortuosity.

2 Objective relative acceleration

We consider a two-component continuum consisting of a solid skeleton and of a fluid. The motion of the skeleton is assumed to be described by the following twice continuously differentiable function

$$\mathbf{x} = \mathbf{f}^S(\mathbf{X}, t), \quad \mathbf{X} \in \mathcal{B}, \quad t \in \mathcal{T}, \tag{9}$$

where $\mathcal{B}$ denotes the reference configuration of the skeleton and $\mathcal{T}$ is the time interval. The velocity, acceleration and the deformation gradient of the skeleton are defined by the relations

$$\acute{\mathbf{x}}^S := \frac{\partial \mathbf{f}^S}{\partial t}, \quad \acute{\acute{\mathbf{x}}}^S := \frac{\partial \acute{\mathbf{x}}}{\partial t}, \quad \mathbf{F}^S := \operatorname{Grad} \mathbf{f}^S. \tag{10}$$

Certainly, the value $\mathbf{F}^S = \mathbf{1}$ corresponds to the reference configuration for, say, $t = t_0$ in which $\mathbf{f}^S(\mathbf{X}, t_0) = \mathbf{X}$.

The motion of the fluid is described by the transformation of the Eulerian velocity field $\mathbf{v}^F = \mathbf{v}^F(\mathbf{x}, t)$ defined on the space of current configurations $\mathbf{f}^S(\mathcal{B}, t)$ of the skeleton. We have

$$\mathbf{v}^F = \mathbf{v}^F\left(\mathbf{f}^S(\mathbf{X}, t), t\right) =: \acute{\mathbf{x}}^F(\mathbf{X}, t). \tag{11}$$

The acceleration of the fluid is then given by

$$\acute{\acute{\mathbf{x}}}^F = \frac{\partial \acute{\mathbf{x}}^F}{\partial t} + \acute{\mathbf{X}}^F \cdot \operatorname{Grad} \acute{\mathbf{x}}^F, \quad \acute{\mathbf{X}}^F := \mathbf{F}^{S-1}\left(\acute{\mathbf{x}}^F - \acute{\mathbf{x}}^S\right), \tag{12}$$

where $\acute{\mathbf{X}}^F$ is the so-called Lagrangian velocity of the fluid with respect to the skeleton.

We proceed to determine the transformation rules for the above quantities specified by the relation (8). The relations (10) and the time differentiation of the relation (8) yield the following quantities in the new reference system

$$\mathbf{F}^{S*} = \mathbf{O}\mathbf{F}^S, \quad \acute{\mathbf{x}}^{S*} = \mathbf{O}\acute{\mathbf{x}}^S + \dot{\mathbf{O}}\mathbf{x} + \dot{\mathbf{c}}, \quad \acute{\acute{\mathbf{x}}}^{S*} = \mathbf{O}\mathbf{x}^S + 2\dot{\mathbf{O}}\acute{\mathbf{x}}^S + \ddot{\mathbf{O}}\mathbf{x} + \ddot{\mathbf{c}}, \tag{13}$$

where the dot denotes the time derivative.

We assume that the transformation rule for the velocity field of the fluid component has the same form as it does for the skeleton

$$\acute{\mathbf{x}}^{F*} = \mathbf{O}\acute{\mathbf{x}}^F + \dot{\mathbf{O}}\mathbf{x} + \dot{\mathbf{c}}. \tag{14}$$

Consequently

$$\acute{\mathbf{x}}^{F*} - \acute{\mathbf{x}}^{S*} = \mathbf{O}\left(\acute{\mathbf{x}}^F - \acute{\mathbf{x}}^S\right) \quad \Rightarrow \quad \acute{\mathbf{X}}^{F*} = \acute{\mathbf{X}}^F. \tag{15}$$

Bearing these relations in mind we can now easily derive the transformation of the acceleration of the fluid. We obtain immediately

$$\begin{aligned} \acute{\acute{\mathbf{x}}}^{F*} &= \frac{\partial}{\partial t}\left(\mathbf{O}\acute{\mathbf{x}}^F + \dot{\mathbf{O}}\mathbf{x} + \dot{\mathbf{c}}\right) + \acute{\mathbf{X}}^F \cdot \operatorname{Grad}\left(\mathbf{O}\acute{\mathbf{x}}^F + \dot{\mathbf{O}}\mathbf{x} + \dot{\mathbf{c}}\right) = \\ &= \mathbf{O}\acute{\acute{\mathbf{x}}}^F + 2\dot{\mathbf{O}}\acute{\mathbf{x}}^F + \ddot{\mathbf{O}}\mathbf{x} + \ddot{\mathbf{c}}, \end{aligned} \tag{16}$$

where the definition of the Lagrangian velocity has been used.

Due to the presence of contributions dependent solely on the choice of the frame we say that velocities $\acute{\mathbf{x}}^S, \acute{\mathbf{x}}^F$ and accelerations $\ddot{\mathbf{x}}^S, \ddot{\mathbf{x}}^F$ are nonobjective. Consequently, their difference is also nonobjective. We have

$$\ddot{\mathbf{x}}^{F*} - \ddot{\mathbf{x}}^{S*} = \mathbf{O}\left(\ddot{\mathbf{x}}^F - \ddot{\mathbf{x}}^S\right) + 2\dot{\mathbf{O}}\left(\acute{\mathbf{x}}^F - \acute{\mathbf{x}}^S\right). \tag{17}$$

For this reason the difference of accelerations cannot be used as a constitutive variable in a construction of the macroscopic model of a two-component system.

In the paper [10] a method has been proposed to overcome these difficulties in the Eulerian description of suspensions. We shall use a similar way in the Lagrangian description. If we take the gradient of the transformation relations for velocities we obtain

$$\begin{aligned} &\operatorname{Grad}\acute{\mathbf{x}}^{S*} = \mathbf{O}\operatorname{Grad}\acute{\mathbf{x}}^S + \dot{\mathbf{O}}\mathbf{F}^S \quad \Rightarrow \quad \dot{\mathbf{O}} = \operatorname{Grad}\left(\acute{\mathbf{x}}^{S*} - \mathbf{O}\acute{\mathbf{x}}^S\right)\mathbf{F}^{S-1}, \\ &\operatorname{Grad}\acute{\mathbf{x}}^{F*} = \mathbf{O}\operatorname{Grad}\acute{\mathbf{x}}^F + \dot{\mathbf{O}}\mathbf{F}^S \quad \Rightarrow \quad \dot{\mathbf{O}} = \operatorname{Grad}\left(\acute{\mathbf{x}}^{F*} - \mathbf{O}\acute{\mathbf{x}}^F\right)\mathbf{F}^{S-1}. \end{aligned} \tag{18}$$

Consequently, we can write

$$\begin{aligned} &2\dot{\mathbf{O}}\left(\acute{\mathbf{x}}^F - \acute{\mathbf{x}}^S\right) = (2-\mathfrak{z})\,\dot{\mathbf{O}}\left(\acute{\mathbf{x}}^F - \acute{\mathbf{x}}^S\right) + \mathfrak{z}\dot{\mathbf{O}}\left(\acute{\mathbf{x}}^F - \acute{\mathbf{x}}^S\right) = \\ &= (2-\mathfrak{z})\operatorname{Grad}\left(\acute{\mathbf{x}}^{F*} - \mathbf{O}\acute{\mathbf{x}}^F\right)\acute{\mathbf{X}}^F + \mathfrak{z}\operatorname{Grad}\left(\acute{\mathbf{x}}^{S*} - \mathbf{O}\acute{\mathbf{x}}^S\right)\acute{\mathbf{X}}^F, \end{aligned} \tag{19}$$

where $\mathfrak{z}$ is arbitrary.

Substitution of this relation in (17) yields

$$\begin{aligned} &\ddot{\mathbf{x}}^{F*} - \ddot{\mathbf{x}}^{S*} - (2-\mathfrak{z})\,\acute{\mathbf{X}}^F \cdot \operatorname{Grad}\acute{\mathbf{x}}^{F*} - \mathfrak{z}\acute{\mathbf{X}}^F \cdot \operatorname{Grad}\acute{\mathbf{x}}^{S*} = \\ &= \mathbf{O}\left(\ddot{\mathbf{x}}^F - \ddot{\mathbf{x}}^S - (2-\mathfrak{z})\,\acute{\mathbf{X}}^F \cdot \operatorname{Grad}\acute{\mathbf{x}}^F - \mathfrak{z}\acute{\mathbf{X}}^F \cdot \operatorname{Grad}\acute{\mathbf{x}}^S\right). \end{aligned} \tag{20}$$

It means that the quantity

$$\begin{aligned} &\mathbf{a}_r := \frac{\partial}{\partial t}\left(\acute{\mathbf{x}}^F - \acute{\mathbf{x}}^S\right) + \acute{\mathbf{X}}^F \cdot \operatorname{Grad}\acute{\mathbf{x}}^F - (2-\mathfrak{z})\,\acute{\mathbf{X}}^F \cdot \operatorname{Grad}\acute{\mathbf{x}}^F - \mathfrak{z}\acute{\mathbf{X}}^F \cdot \operatorname{Grad}\acute{\mathbf{x}}^S \equiv \\ &\equiv \frac{\partial}{\partial t}\left(\acute{\mathbf{x}}^F - \acute{\mathbf{x}}^S\right) - (1-\mathfrak{z})\,\acute{\mathbf{X}}^F \cdot \operatorname{Grad}\acute{\mathbf{x}}^F - \mathfrak{z}\acute{\mathbf{X}}^F \cdot \operatorname{Grad}\acute{\mathbf{x}}^S, \end{aligned} \tag{21}$$

is objective, i.e.

$$\mathbf{a}_r^* = \mathbf{O}\mathbf{a}_r. \tag{22}$$

We call this quantity an **objective relative acceleration**. As an objective variable it can be incorporated into the set of constitutive variables. Obviously, there exists a class of such accelerations specified by the constitutive coefficient $\mathfrak{z}$.

It is easy to see that a linear momentum source $\hat{\mathbf{p}}$ in an isotropic material would contain a term $\rho_{12}^0\mathbf{a}_r \approx \rho_{12}^0\frac{\partial}{\partial t}\left(\acute{\mathbf{x}}^F - \acute{\mathbf{x}}^S\right)$ as required by the relations (1) of the Biot's model. The open question is if the second law of thermodynamics admits this type of contribution in a fully nonlinear model. We address the next section to this problem.

3 Thermodynamical admissibility

A nonlinear poroelastic two-component model requires the formulation of field equations for the following fields

$$\mathcal{F} := \left\{ \rho^F, \acute{\mathbf{x}}^S, \acute{\mathbf{x}}^F, \mathbf{F}^S, T, n \right\} \quad \text{for} \quad (\mathbf{X}, t) \in \mathcal{B} \times \mathcal{T}, \tag{23}$$

where ρ^F is the partial mass density of the fluid per unit volume in the reference configuration of the skeleton, i.e. in the current configuration it is given by the relation $\rho_t^F = \rho^F J^{S-1}$, $J^S := \det \mathbf{F}^S$. T is the absolute temperature of the medium common for both components, and n is the current porosity. Other symbols have the same meaning as before.

The partial mass density of the skeleton in the reference configuration, ρ^S, does not appear among the fields because it is constant in a homogeneous material without mass exchange between components.

These fields are assumed to fulfil the following set of balance equations (e.g. see: [14])

$$R^F := \frac{\partial \rho^F}{\partial t} + \operatorname{Div}\left(\rho^F \acute{\mathbf{X}}^F\right) = 0, \tag{24}$$

$$\mathbf{M}^S := \rho^S \frac{\partial \acute{\mathbf{x}}^S}{\partial t} - \operatorname{Div} \mathbf{P}^S - \hat{\mathbf{p}} = 0, \tag{25}$$

$$\mathbf{M}^F := \rho^F \left(\frac{\partial \acute{\mathbf{x}}^F}{\partial t} + \acute{\mathbf{X}}^F \cdot \operatorname{Grad} \acute{\mathbf{x}}^F \right) - \operatorname{Div} \mathbf{P}^F + \hat{\mathbf{p}} = 0, \tag{26}$$

$$E := \frac{\partial \rho \varepsilon}{\partial t} + \operatorname{Div} \mathbf{Q} - \mathbf{P}^S \cdot \operatorname{Grad} \acute{\mathbf{x}}^S - \mathbf{P}^F \cdot \operatorname{Grad} \acute{\mathbf{x}}^F - \left(\mathbf{F}^{ST} \hat{\mathbf{p}}\right) \cdot \acute{\mathbf{X}}^F = 0, \qquad \rho := \rho^S + \rho^F, \tag{27}$$

$$\mathbf{F} := \frac{\partial \mathbf{F}^S}{\partial t} - \operatorname{Grad} \acute{\mathbf{x}}^S = 0, \tag{28}$$

$$N := \frac{\partial \Delta_n}{\partial t} + \operatorname{Div} \mathbf{J} - \hat{n} = 0, \quad \Delta_n := n - n_E, \tag{29}$$

where $\mathbf{P}^S, \mathbf{P}^F$ denote the first Piola-Kirchhoff partial stress tensors, $\hat{\mathbf{p}}$ is the momentum source, ε is the specific internal energy per unit mass of the mixture, $\mathbf{Q}$ is the heat flux vector, n_E describes the so-called equilibrium porosity, $\mathbf{J}$ is the porosity flux, and $\hat{n}$ is the porosity source.

The porosity balance equation (29) yields the model essentially beyond the frame of Biot's model due to the contribution of relaxation source $\hat{n}$. It has been introduced some years ago [15] and analyzed in numerous papers. For instance, the applicability in the theory of abrasion has been discussed by N. Kirchner (e.g. [16]).

In order to obtain field equations from the above balance equations we have to specify constitutive relations for these quantities, i.e

$$\mathcal{C} := \left\{ \mathbf{P}^S, \mathbf{P}^F, \hat{\mathbf{p}}, \varepsilon, \mathbf{Q}, n_E, \mathbf{J}, \hat{n} \right\}, \tag{30}$$

must be functions of constitutive variables. In this work the set of constitutive variables is chosen as follows

$$\mathcal{V} := \left\{ \rho^F, \mathbf{F}^S, \acute{\mathbf{X}}^F, \Delta_n, T, \mathbf{G}, \mathbf{a}_r \right\}, \quad \mathbf{G} := \operatorname{Grad} T. \tag{31}$$

Once the function

$$\mathcal{C} = \mathcal{C}(\mathcal{V}), \tag{32}$$

is given, we obtain a closed system of differential equations for fields $\mathcal{F}$.

It has been shown earlier ([13], [14]) that the existence of coupling between partial stresses requires a constitutive dependence on some gradients of fields. Analysis has been performed for the model with a dependence on the porosity gradient. It was shown that within a linearized model one obtains the Biot's coupling described by the constant Q.

We shall not include this point in the thermodynamical analysis of this work. It may be shown that the existence of such a dependence yields possibilities of additional couplings but it has no influence on the thermodynamical admissibility of a dependence on the relative acceleration. Simultaneously the analysis is much simpler without these additional gradient constitutive variables. Consequently, the constitutive variable $\mathbf{N} := \operatorname{Grad} n$ does not appear in the list (31).

We say that constitutive relations (32) satisfy the **second law of thermodynamics** if the following entropy inequality

$$\frac{\partial \rho\eta}{\partial t} + \operatorname{Div} \mathbf{H} \geq 0, \quad \eta = \eta(\mathcal{V}), \quad \mathbf{H} = \mathbf{H}(\mathcal{V}), \tag{33}$$

is satisfied by all solutions of field equations. In this inequality η is the specific entropy and $\mathbf{H}$ its flux.

This requirement is equivalent to the following inequality which must hold for **all fields**

$$\frac{\partial \rho\eta}{\partial t} + \operatorname{Div} \mathbf{H} - \Lambda^{\rho^F} R^F - \mathbf{\Lambda}^{v^S} \cdot \mathbf{M}^S - \mathbf{\Lambda}^{v^F} \cdot \mathbf{M}^F - \Lambda^{\varepsilon} E - \mathbf{\Lambda}^F \cdot \mathbf{F} - \Lambda^n N \geq 0, \tag{34}$$

where

$$\Lambda^{\rho^F}, \mathbf{\Lambda}^{v^S}, \mathbf{\Lambda}^{v^F}, \Lambda^{\varepsilon}, \mathbf{\Lambda}^F, \Lambda^n \tag{35}$$

are Lagrange multipliers dependent on constitutive variables $\mathcal{V}$.

The exploitation of the second law of thermodynamics in the general case is technically impossible. Therefore we make simplifying assumptions sufficient for the second law to be satisfied and yielding explicit limitations on constitutive relations. They are as follows:

1° The material is isotropic. Consequently, scalar constitutive functions, for instance, depend on vector and tensor variables solely through invariants. This assumption will be applied in some steps of our proofs. Some relations are simpler in a general form and then we do not introduce this limitation.

2° The dependence on the relative velocity $\acute{\mathbf{X}}^F$ is at most quadratic. This assumption is related to the structure of the nonlinear contribution to the objective relative acceleration. We motivate its form further.

3° The dependence on the temperature gradient $\mathbf{G}$ is linear. We could skip this assumption on the cost of some technicalities but the experience with the thermodynamical construction of poroelastic models shows that it does not yield any undesired results.

4° The dependence on the deviation of porosity Δ_n from its equilibrium value n_E is quadratic.

5° The dependence on the relative acceleration $\mathbf{a}_r$ is linear.

6° Higher order combinations of variables $\mathbf{G}, \acute{\mathbf{X}}^F, \Delta_n, \mathbf{a}_r$ can be neglected.

As we see further these assumptions limit thermodynamical considerations to a vicinity of the thermodynamical equilibrium similar to this appearing in the classical Onsager thermodynamics.

Bearing these assumptions in mind we can write the following representations of constitutive functions

– partial stresses

$$\mathbf{P}^S = \mathbf{P}_0^S\left(\mathcal{V}_E, \Delta_n\right) + \frac{1}{2}\sigma^S\left(\mathcal{V}_E\right)\mathbf{F}^S\acute{\mathbf{X}}^F \otimes \acute{\mathbf{X}}^F, \quad \mathcal{V}_E := \left\{\rho^F, \mathbf{F}^S, T\right\},$$

$$\mathbf{P}^F = \mathbf{P}_0^F\left(\mathcal{V}_E, \Delta_n\right) + \frac{1}{2}\sigma^F\left(\mathcal{V}_E\right)\mathbf{F}^S\acute{\mathbf{X}}^F \otimes \acute{\mathbf{X}}^F, \quad n_E = n_E\left(\mathcal{V}_E\right), \tag{36}$$

– internal energy and entropy

$$\rho\varepsilon = \rho\varepsilon_0\left(\mathcal{V}_E, \Delta_n\right) + \frac{1}{2}\varepsilon_d\left(\mathcal{V}_E\right)\left(\mathbf{F}^S\acute{\mathbf{X}}^F\right)\cdot\left(\mathbf{F}^S\acute{\mathbf{X}}^F\right),$$

$$\rho\eta = \rho\eta_0\left(\mathcal{V}_E, \Delta_n\right) + \frac{1}{2}\eta_d\left(\mathcal{V}_E\right)\left(\mathbf{F}^S\acute{\mathbf{X}}^F\right)\cdot\left(\mathbf{F}^S\acute{\mathbf{X}}^F\right), \tag{37}$$

– fluxes of energy, entropy, porosity

$$\begin{aligned}
\mathbf{Q} &= Q_V\acute{\mathbf{X}}^F - K\mathbf{G} + Q_a\mathbf{F}^{ST}\mathbf{a}_r, \\
\mathbf{H} &= H_V\acute{\mathbf{X}}^F + H_T\mathbf{G} + H_a\mathbf{F}^{ST}\mathbf{a}_r, \\
\mathbf{J} &= \Phi\acute{\mathbf{X}}^F + J_T\mathbf{G} + J_a\mathbf{F}^{ST}\mathbf{a}_r,
\end{aligned} \tag{38}$$

where all coefficients are functions of variables $\mathcal{V}_E$,

– momentum source

$$\mathbf{F}^{ST}\hat{\mathbf{p}} = \Pi_V\acute{\mathbf{X}}^F + \Pi_T\mathbf{G} - \rho_{12}^0\mathbf{F}^{ST}\mathbf{a}_r, \tag{39}$$

with coefficients dependent again on variables $\mathcal{V}_E$.

The above simplifying assumption yields an additional structure of these relations which we do not need to specify at this stage.

The notation of some coefficients in the above relations corresponds to this which is customary in the literature.

The contributions with the coefficients ε_d, η_d to the energy and entropy are motivated by fluctuations of the microstructural kinetic energy caused by the tortuosity. We do not introduce any additional microstructural variable describing changes of tortuosity. For this reason a macroscopic influence of tortuosity can be solely reflected by the seepage velocity which in our model corresponds to the Lagrangian velocity $\acute{\mathbf{X}}^F$. The classical kinetic energy in this model is given by $\frac{1}{2}\left(\rho^S\acute{\mathbf{x}}^S\cdot\acute{\mathbf{x}}^S + \rho^F\acute{\mathbf{x}}^F\cdot\acute{\mathbf{x}}^F\right)$. Consequently, the correction

$\frac{1}{2}\varepsilon_d\left(\acute{\mathbf{x}}^F-\acute{\mathbf{x}}^S\right)\cdot\left(\acute{\mathbf{x}}^F-\acute{\mathbf{x}}^S\right)$ may be considered as an **added mass effect** resulting from tortuosity.

As we see further, the dependence of partial stresses on this velocity is then required by the consistency of the model with the second law of thermodynamics.

The exploitation of the second law of thermodynamics (34) is standard. We apply the chain rule of differentiation to constitutive laws. We skip here rather cumbersome technical details.

Linearity of the second law of thermodynamics with respect to time derivatives

$$\left\{\frac{\partial\rho^F}{\partial t},\frac{\partial\mathbf{F}^S}{\partial t},\frac{\partial\Delta_n}{\partial t},\frac{\partial\acute{\mathbf{x}}^S}{\partial t},\frac{\partial\acute{\mathbf{x}}^F}{\partial t},\frac{\partial T}{\partial t},\frac{\partial\mathbf{G}}{\partial t}\right\}$$

yields

$$\Lambda^{\rho^F}=\Lambda_0^{\rho^F}+\frac{1}{2}\left(\frac{\partial\eta_d}{\partial\rho^F}-\Lambda^\varepsilon\frac{\partial\varepsilon_d}{\partial\rho^F}\right)\left(\mathbf{F}^S\acute{\mathbf{X}}^F\right)\cdot\left(\mathbf{F}^S\acute{\mathbf{X}}^F\right), \tag{40}$$

$$\Lambda_0^{\rho^F}:=\frac{\partial\rho\eta_0}{\partial\rho^F}-\Lambda^\varepsilon\frac{\partial\rho\varepsilon_0}{\partial\rho^F}$$

$$\mathbf{\Lambda}^F=\frac{\partial\rho\eta_0}{\partial\mathbf{F}^F}-\Lambda^\varepsilon\frac{\partial\rho\varepsilon_0}{\partial\mathbf{F}^F}+\frac{1}{2}\left(\frac{\partial\eta_d}{\partial\mathbf{F}^F}-\Lambda^\varepsilon\frac{\partial\varepsilon_d}{\partial\mathbf{F}^F}\right)\left(\mathbf{F}^S\acute{\mathbf{X}}^F\right)\cdot\left(\mathbf{F}^S\acute{\mathbf{X}}^F\right), \tag{41}$$

$$\Lambda^n=\frac{\partial\rho\eta_0}{\partial\Delta_n}-\Lambda^\varepsilon\frac{\partial\rho\varepsilon_0}{\partial\Delta_n}, \tag{42}$$

$$\begin{aligned}\left(\rho^S-\rho_{12}^0\right)\mathbf{\Lambda}^{v^S}+\rho_{12}^0\mathbf{\Lambda}^{v^F}&=-\left(\eta_d-\Lambda^\varepsilon\varepsilon_d\right)\mathbf{F}^S\acute{\mathbf{X}}^F+\rho_{12}^0\Lambda^\varepsilon\mathbf{F}^S\acute{\mathbf{X}}^F-\\&-\operatorname{Div}\left(H_a\mathbf{F}^S\right)+\Lambda^\varepsilon\operatorname{Div}\left(Q_a\mathbf{F}^S\right)+\Lambda^n\operatorname{Div}\left(J_a\mathbf{F}^S\right)=0,\end{aligned} \tag{43}$$

$$\begin{aligned}\left(\rho^F-\rho_{12}^0\right)\mathbf{\Lambda}^{v^F}+\rho_{12}^0\mathbf{\Lambda}^{v^S}&=\left(\eta_d-\Lambda^\varepsilon\varepsilon_d\right)\mathbf{F}^S\acute{\mathbf{X}}^F-\rho_{12}^0\Lambda^\varepsilon\mathbf{F}^S\acute{\mathbf{X}}^F+\\&+\operatorname{Div}\left(H_a\mathbf{F}^S\right)-\Lambda^\varepsilon\operatorname{Div}\left(Q_a\mathbf{F}^S\right)-\Lambda^n\operatorname{Div}\left(J_a\mathbf{F}^S\right)=0,\end{aligned} \tag{44}$$

$$\frac{\partial\rho\eta_0}{\partial T}-\Lambda^\varepsilon\frac{\partial\rho\varepsilon_0}{\partial T}+\frac{1}{2}\left(\frac{\partial\eta_d}{\partial T}-\Lambda^\varepsilon\frac{\partial\varepsilon_d}{\partial T}\right)\left(\mathbf{F}^S\acute{\mathbf{X}}^F\right)\cdot\left(\mathbf{F}^S\acute{\mathbf{X}}^F\right)=0. \tag{45}$$

These identities still contain linear contributions of $\operatorname{Grad}\mathbf{F}^S$, Δ_n, quadratic contributions of the latter as well as quadratic contributions of Lagrangian velocity. As they should hold for arbitrary fields coefficients of these contributions must vanish separately. After easy analysis we obtain

$$H_a=0,\quad Q_a=0,\quad J_a=0, \tag{46}$$

$$\rho^S\mathbf{\Lambda}^{v^S}=-\rho^F\mathbf{\Lambda}^{v^F}=\mathfrak{y}\mathbf{F}^S\acute{\mathbf{X}}^F,\quad \mathfrak{y}:=-\frac{\eta_d-\Lambda^\varepsilon\varepsilon_d-\Lambda^\varepsilon\rho_{12}^0}{\rho^S-\rho_{12}^0\left(1+\frac{\rho^S}{\rho^F}\right)}. \tag{47}$$

The second law of thermodynamics is also linear with respect to the following spatial derivatives

$$\operatorname{Grad}\acute{\mathbf{x}}^S,\operatorname{Grad}\acute{\mathbf{x}}^F,\operatorname{Grad}\rho^F,\operatorname{Grad}\mathbf{F}^S,\operatorname{Grad}\mathbf{G},\operatorname{Grad}\Delta_n. \tag{48}$$

We have listed them in the order of the further analysis. This yields a set of identities and leaves a residual inequality which is essentially nonlinear. It defines the **dissipation** in the system and has the following form

$$\begin{aligned}\mathcal{D} := & \left(\frac{\partial H_V}{\partial T} - \Lambda^\varepsilon \frac{\partial Q_V}{\partial T} - \Lambda^n \frac{\partial \Phi}{\partial T} + \Pi_T\right) \acute{\mathbf{X}}^F \cdot \mathbf{G} + \\ & + \left(\frac{\partial H_T}{\partial T} + \Lambda^\varepsilon \frac{\partial K}{\partial T} - \Lambda^n \frac{\partial J_T}{\partial T}\right) \mathbf{G} \cdot \mathbf{G} + \\ & + \Lambda^\varepsilon \Pi_V \acute{\mathbf{X}}^F \cdot \acute{\mathbf{X}}^F + \Lambda^n \hat{n} \geq 0.\end{aligned} \tag{49}$$

Hence the state of thermodynamical equilibrium defined by $\mathcal{D} = 0$ appears if

$$\mathbf{G} = 0, \quad \acute{\mathbf{X}}^F = 0, \quad \hat{n} = 0, \tag{50}$$

i.e. the temperature gradient, relative motion (diffusion), and the relaxation of porosity cause the deviation from the equilibrium.

Clearly the assumption 4° yields the linearity of $\hat{n}$ and Λ^n with respect to Δ_n. In addition, the above inequality yields homogeneity of these functions, i.e. we can write

$$\hat{n} = -\frac{\Delta_n}{\tau}, \quad \Lambda^n = \lambda^n \Delta_n, \tag{51}$$

where τ, λ^n can be solely functions of variables $\mathcal{V}_E$. Consequently, we obtain as well

$$\frac{\partial \Phi}{\partial T} = 0, \quad \frac{\partial J_T}{\partial T} = 0. \tag{52}$$

It is worth mentioning that due to (46) the relative acceleration does not contribute to the dissipation. This property of the model follows from the fact that the model does not possess any independent field of tortuosity which could relax to the thermodynamical equilibrium.

Now we return to the coefficients of spatial derivatives of fields. The vanishing coefficient of $\operatorname{Grad} \acute{\mathbf{x}}^S$ yields the following results

$$\Lambda^\varepsilon \mathbf{P}_0^S \mathbf{F}^{ST} + \left(\frac{\partial \rho\eta_0}{\partial \mathbf{F}^S} - \Lambda^\varepsilon \frac{\partial \rho\varepsilon_0}{\partial \mathbf{F}^S}\right) \mathbf{F}^{ST} + \left(-H_V + \Lambda^\varepsilon Q_V + \Lambda^n \Phi + \rho^F \Lambda_0^{\rho^F}\right) \mathbf{1} = 0, \tag{53}$$

$$\frac{\partial \eta_d}{\partial I} - \Lambda^\varepsilon \frac{\partial \varepsilon_d}{\partial I} = 0, \quad \frac{\partial \eta_d}{\partial II} - \Lambda^\varepsilon \frac{\partial \varepsilon_d}{\partial II} = 0, \tag{54}$$

$$2\left(\frac{\partial \eta_d}{\partial III} - \Lambda^\varepsilon \frac{\partial \varepsilon_d}{\partial III}\right) III + \rho^F \left(\frac{\partial \eta_d}{\partial \rho^F} - \Lambda^\varepsilon \frac{\partial \varepsilon_d}{\partial \rho^F}\right) = \mathfrak{y}\left(\sigma^S + \frac{\rho^S}{\rho^F}\sigma^F\right), \tag{55}$$

$$\mathfrak{z} = -\frac{1}{2\rho_{12}^0} \frac{\sigma^S \Lambda^\varepsilon - \mathfrak{y}\left(\sigma^S + \frac{\rho^S}{\rho^F}\sigma^F\right)}{\mathfrak{y}\left(1 + \frac{\rho^S}{\rho^F}\right) + \Lambda^\varepsilon}, \tag{56}$$

where

$$I := \operatorname{tr} \mathbf{C}^S, \quad II := \frac{1}{2}\left(I^2 - \operatorname{tr} \mathbf{C}^S\right), \quad III := \det \mathbf{C}^S, \quad \mathbf{C}^S := \mathbf{F}^{ST} \mathbf{F}^S, \tag{57}$$

are main invariants of the Cauchy-Green deformation tensor $\mathbf{C}^S$.

The coefficient of Grad $\acute{\mathbf{x}}^F$ yields

$$\Lambda^\varepsilon \left(\mathbf{P}_0^S + \mathbf{P}_0^F\right) = -\left(\frac{\partial \rho\eta_0}{\partial \mathbf{F}^S} - \Lambda^\varepsilon \frac{\partial \rho\varepsilon_0}{\partial \mathbf{F}^S}\right), \tag{58}$$

$$\frac{\partial \eta_d}{\partial III} - \Lambda^\varepsilon \frac{\partial \varepsilon_d}{\partial III} = 0, \tag{59}$$

$$\frac{1}{2}\Lambda^\varepsilon \left(\sigma^S + \sigma^F\right) = -\rho_{12}^0 \left(1 + \frac{\rho^S}{\rho^F}\right)\mathfrak{y} - \rho^S \mathfrak{y} - \rho_{12}^0 \Lambda^\varepsilon. \tag{60}$$

Consequently, bearing (54) and (59) in mind,

$$\frac{\partial \eta_d}{\partial \mathbf{F}^S} - \Lambda^\varepsilon \frac{\partial \varepsilon_d}{\partial \mathbf{F}^S} = 0. \tag{61}$$

Next we consider the coefficient of Grad ρ^F. We have

$$\left(\frac{\partial H_V}{\partial \rho^F} - \Lambda^\varepsilon \frac{\partial Q_V}{\partial \rho^F} - \Lambda^n \frac{\partial \Phi}{\partial \rho^F} - \Lambda^{\rho^F}\right)\acute{\mathbf{X}}^F + \mathfrak{y}\frac{\partial \mathbf{P}^{ST}}{\partial \rho^F}\mathbf{F}^S \acute{\mathbf{X}}^F - \mathfrak{y}\frac{\rho^F}{\rho^S}\frac{\partial \mathbf{P}^{FT}}{\partial \rho^F}\mathbf{F}^S \acute{\mathbf{X}}^F = 0, \tag{62}$$

$$\frac{\partial H_T}{\partial \rho^F} + \Lambda^\varepsilon \frac{\partial K}{\partial \rho^F} = 0, \quad \frac{\partial J_T}{\partial \rho^F} = 0. \tag{63}$$

Similarly, the coefficient of Grad $\mathbf{F}^S$ yields

$$\operatorname{sym}\left\{\left(\frac{\partial H_V}{\partial \mathbf{F}^S} - \Lambda^\varepsilon \frac{\partial Q_V}{\partial \mathbf{F}^S} - \Lambda^n \frac{\partial \Phi}{\partial \mathbf{F}^S}\right) \otimes \acute{\mathbf{X}}^F + \rho^F \Lambda^{\rho^F} \mathbf{F}^{S-T} \otimes \acute{\mathbf{X}}^F + \mathfrak{y}\boldsymbol{\Xi}^S + \mathfrak{y}\boldsymbol{\Xi}^F\right\} -$$

$$-\operatorname{sym}\left\{\left(H_V - \Lambda^\varepsilon - \Lambda^n \Phi\right)\mathbf{F}^{S-T} \otimes \acute{\mathbf{X}}^F\right\} = 0, \tag{64}$$

$$\frac{\partial H_T}{\partial \mathbf{F}^S} + \Lambda^\varepsilon \frac{\partial K}{\partial \mathbf{F}^S} = 0, \quad \frac{\partial J_T}{\partial \mathbf{F}^S} = 0, \tag{65}$$

where the components of tensors $\boldsymbol{\Xi}^S, \boldsymbol{\Xi}^F$ in Cartesian coordinates are given by the relations

$$\Xi_{kKL}^S = \frac{\partial P_{0lL}^S}{\partial F_{kK}^S} F_{lM}^S \acute{X}_M^F, \quad \Xi_{kKL}^F = -\frac{\rho^S}{\rho^F}\frac{\partial P_{0lL}^F}{\partial F_{kK}^S} F_{lM}^S \acute{X}_M^F. \tag{66}$$

Under our assumptions the contribution of Grad Δ_n does not yield any restrictions.

Finally, the last condition follows from the vanishing coefficient of Grad $\mathbf{G}$ and it has the form

$$H_T + \Lambda^\varepsilon K = 0, \quad J_T = 0. \tag{67}$$

Inspection of the results (63), (65) for thermal coefficients yields

$$\Lambda^\varepsilon = \Lambda^\varepsilon (T) \quad \Rightarrow \quad \Lambda^\varepsilon = \frac{1}{T}, \quad \text{i.e.} \quad H_T = -\frac{K}{T}, \tag{68}$$

where the standard argument (e.g. [14]) has been used.

It is not quite clear what limitations on partial stress tensors are imposed by conditions (62), (64). Derivatives of partial stresses with respect to the mass density ρ^F as well as with respect to the deformation gradient $\mathbf{F}^S$ seem to restrict elastic properties of the system in equilibrium. This does not seem very plausible. Hence we assume that the coefficient $\mathfrak{y}$ vanishes, i.e.

$$\mathfrak{y} = 0. \tag{69}$$

Then the multipliers of momentum equations vanish as well. As the consequence of (45), (47), (55), (61) we obtain

$$-\rho_{12}^0 = \varepsilon_d - T\eta_d = const. \quad \Rightarrow \quad \varepsilon_d = const., \ \eta_d = 0. \tag{70}$$

It is convenient to introduce the following notation

$$\begin{aligned} \psi &:= \varepsilon - T\eta, \\ \rho\psi_0 &:= \rho\psi - \tfrac{1}{2}\varepsilon_d \left(\mathbf{F}^S \acute{\mathbf{X}}^F\right) \cdot \left(\mathbf{F}^S \acute{\mathbf{X}}^F\right). \end{aligned} \tag{71}$$

Then, for Lagrange multipliers we have

$$\Lambda^{\rho^F} = -\frac{1}{T}\frac{\partial \rho\psi_0}{\partial \rho^F} = \Lambda_0^{\rho^F}, \quad \mathbf{\Lambda}^F = -\frac{1}{T}\frac{\partial \rho\psi_0}{\partial \mathbf{F}^S}, \quad \Lambda^n = -\frac{1}{T}\frac{\partial \rho\psi_0}{\partial \Delta_n} = \lambda^n \Delta_n, \tag{72}$$

and the relation (45) implies the following classical formula for the internal energy

$$\varepsilon = \psi - T\frac{\partial \psi}{\partial T}. \tag{73}$$

Simultaneously the relations (62), (64) yield

$$\begin{aligned} &\frac{\partial H_V}{\partial \rho^F} - \Lambda^\varepsilon \frac{\partial Q_V}{\partial \rho^F} - \Lambda^{\rho^F} = 0, \quad \frac{\partial \Phi}{\partial \rho^F} = 0, \\ &2III\left(\frac{\partial H_V}{\partial III} - \Lambda^\varepsilon \frac{\partial Q_V}{\partial III}\right) + \rho^F \Lambda^{\rho^F} - \left(H_V - \Lambda^\varepsilon Q_V\right) = 0, \end{aligned} \tag{74}$$

$$2III\frac{\partial \Phi}{\partial III} - \Phi = 0 \quad \Rightarrow \quad \Phi = J^S \Phi_0, \quad \Phi_0 = const. \tag{75}$$

These relations yield the following integrability condition for the multiplier Λ^{ρ^F}

$$\rho^F \frac{\partial \Lambda^{\rho^F}}{\partial \rho^F} + J^S \frac{\partial \Lambda^{\rho^F}}{\partial J^S} = 0 \quad \Rightarrow \quad \Lambda^{\rho^F} = \Lambda^{\rho^F}\left(T, \rho_t^F\right), \quad \rho_t^F := J^{S-1}\rho^F. \tag{76}$$

Consequently, integration of $(72)_1$ leads to the following additive splitting of the free energy ψ

$$\begin{aligned} \rho\psi &= \rho^F\psi^F + \rho^S\psi^S - \frac{1}{2}\lambda^n T \Delta_n^2 + \frac{1}{2}\varepsilon_d \left(\mathbf{F}^S \acute{\mathbf{X}}^F\right) \cdot \left(\mathbf{F}^S \acute{\mathbf{X}}^F\right), \\ \psi^F &= \psi^F\left(T, \rho_t^F\right), \quad \psi^S = \psi^S\left(T, \mathbf{F}^S\right). \end{aligned} \tag{77}$$

The above separation property is characteristic for the so-called **simple mixtures**. In addition, integration in $(74)_1$ yields

$$H_V - \frac{Q_V}{T} = -\frac{\rho^F \psi^F}{T}, \quad \text{i.e.} \quad \mathbf{H} = \frac{1}{T}\left(\mathbf{Q} - \rho^F \psi^F \acute{\mathbf{X}}^F\right), \tag{78}$$

where we have accounted for the relations (67) and (68).

Inspection of relations (70), (56) and (60) leads immediately to the following identification of constitutive coefficients coupled to the relative acceleration

$$\varepsilon_d = -\rho_{12}^0, \quad \sigma^S = -2\mathfrak{z}\rho_{12}^0, \quad \sigma^F = -2\left(1-\mathfrak{z}\right)\rho_{12}^0. \tag{79}$$

Simultaneously, relation (53) with (71), (72), (77) and (78) for partial stresses $\mathbf{P}_0^S$ and relation (58) for partial stresses $\mathbf{P}_0^F$ yield

$$\begin{aligned} \mathbf{P}^S &= \frac{\partial \rho^S \psi^S}{\partial \mathbf{F}^S} + \beta \Delta_n J^S \mathbf{F}^{S-T} - \mathfrak{z}\rho_{12}^0 \mathbf{F}^S \acute{\mathbf{X}}^S \otimes \acute{\mathbf{X}}^S, \quad \beta := T\lambda^n \Phi_0 J^{S-1}, \\ \mathbf{P}^F &= -\rho_t^{F2} \frac{\partial \psi^F}{\partial \rho_t^F} J^S \mathbf{F}^{S-T} - \beta \Delta_n J^S \mathbf{F}^{S-T} - \left(1-\mathfrak{z}\right)\rho_{12}^0 \mathbf{F}^S \acute{\mathbf{X}}^S \otimes \acute{\mathbf{X}}^S. \end{aligned} \tag{80}$$

Hence, as mentioned in the introduction, the partial stresses do not possess a coupling term characteristic for the Biot's model and this fallacy of the model can be removed by additional constitutive variables.

For practical purposes it is convenient to transform equations of the model to **Eulerian coordinates**. We write them in an arbitrary **noninertial** reference system. The set of balance equations (24) has then the form

– mass balance for the fluid component

$$\frac{\partial \rho_t^F}{\partial t} + \operatorname{div}\left(\rho_t^F \mathbf{v}^F\right) = 0, \tag{81}$$

– momentum balance for the skeleton

$$\begin{aligned} &\rho_t^S \left(\frac{\partial \mathbf{v}^S}{\partial t} + \mathbf{v}^S \cdot \operatorname{grad} \mathbf{v}^S\right) = \operatorname{div} \mathbf{T}^S + \rho_t^S \mathbf{b}^S + J^{S-1}\Pi_T \operatorname{grad} \mathbf{T} + \boldsymbol{\pi}\left(\mathbf{v}^F - \mathbf{v}^S\right) - \\ &-\rho_{12}\left[\frac{\partial}{\partial t}\left(\mathbf{v}^F - \mathbf{v}^S\right) - \left(1-\mathfrak{z}\right)\left(\mathbf{v}^F - \mathbf{v}^S\right)\cdot \operatorname{grad} \mathbf{v}^F - \mathfrak{z}\left(\mathbf{v}^F - \mathbf{v}^S\right)\cdot \operatorname{grad} \mathbf{v}^S\right], \end{aligned} \tag{82}$$

– momentum balance for the fluid

$$\begin{aligned} &\rho_t^F \left(\frac{\partial \mathbf{v}^F}{\partial t} + \mathbf{v}^F \cdot \operatorname{grad} \mathbf{v}^F\right) = \operatorname{div} \mathbf{T}^F + \rho_t^F \mathbf{b}^F - J^{S-1}\Pi_T \operatorname{grad} \mathbf{T} - \boldsymbol{\pi}\left(\mathbf{v}^F - \mathbf{v}^S\right) + \\ &+\rho_{12}\left[\frac{\partial}{\partial t}\left(\mathbf{v}^F - \mathbf{v}^S\right) - \left(1-\mathfrak{z}\right)\left(\mathbf{v}^F - \mathbf{v}^S\right)\cdot \operatorname{grad} \mathbf{v}^F - \mathfrak{z}\left(\mathbf{v}^F - \mathbf{v}^S\right)\cdot \operatorname{grad} \mathbf{v}^S\right], \end{aligned} \tag{83}$$

– energy balance

$$\frac{\partial \rho_t \varepsilon}{\partial t} + \operatorname{div}\left(\rho_t \varepsilon \mathbf{v}^S + \mathbf{q}\right) - \mathbf{T}^S \cdot \operatorname{grad} \mathbf{v}^S - \mathbf{T}^F \cdot \operatorname{grad} \mathbf{v}^F - \tag{84}$$

$$- \left(\mathbf{v}^F - \mathbf{v}^S\right) \cdot \left\{\pi \left(\mathbf{v}^F - \mathbf{v}^S\right) + J^{S-1} \Pi_T \operatorname{grad} T - \right.$$

$$\left. - \rho_{12} \left[\frac{\partial}{\partial t}\left(\mathbf{v}^F - \mathbf{v}^S\right) - (1 - \mathfrak{z})\left(\mathbf{v}^F - \mathbf{v}^S\right) \cdot \operatorname{grad} \mathbf{v}^F - \mathfrak{z}\left(\mathbf{v}^F - \mathbf{v}^S\right) \cdot \operatorname{grad} \mathbf{v}^S\right]\right\},$$

– porosity balance

$$\frac{\partial J^{S-1} \Delta_n}{\partial t} + \operatorname{div}\left(J^{S-1} \Delta_n \mathbf{v}^S + \mathbf{j}\right) + \frac{J^{S-1} \Delta_n}{\tau} = 0. \tag{85}$$

The external forces $\rho_t^S \mathbf{b}^S, \rho_t^F \mathbf{b}_t^F$, called **apparent body forces**, contributing to momentum balance equations have the following structure

$$\begin{aligned}
&\rho_t^S \mathbf{b}^S = \rho_t^S \left(\mathbf{b}_b^S + \mathbf{i}^S\right), \quad \rho_t^F \mathbf{b}^F = \rho_t^F \left(\mathbf{b}_b^F + \mathbf{i}^F\right), \\
&\mathbf{i}^S := \ddot{\mathbf{c}} + 2\mathbf{\Omega}\left(\mathbf{v}^S - \dot{\mathbf{c}}\right) + \left(\dot{\mathbf{\Omega}} - \mathbf{\Omega}^2\right)(\mathbf{x} - \mathbf{c}), \\
&\mathbf{i}^F := \ddot{\mathbf{c}} + 2\mathbf{\Omega}\left(\mathbf{v}^F - \dot{\mathbf{c}}\right) + \left(\dot{\mathbf{\Omega}} - \mathbf{\Omega}^2\right)(\mathbf{x} - \mathbf{c}), \quad \mathbf{\Omega} := \dot{\mathbf{O}}\mathbf{O}^T \equiv -\mathbf{\Omega}^T,
\end{aligned} \tag{86}$$

where $\rho_t^S \mathbf{b}_b^S, \rho_t^F \mathbf{b}_b^F$ are true (e.g. gravitational) body forces, and $\rho_t^S \mathbf{i}^S, \rho_t^F \mathbf{i}^F$ are called **inertial body forces**. In order of appearance in the above relations, they consist of the inertial force of relative translation, Coriolis force, Euler force, centrifugal force. They depend on the matrix of angular velocity $\mathbf{\Omega}$ of the noninertial system with respect to an inertial one. Certainly, the inertial body forces vanish in an inertial reference system. It should be mentioned that the partial accelerations appearing in the above partial momentum balance equations combined with apparent body forces are objective, i.e. invariant with respect to the transformation (8).

The remaining notation used above is as follows

$$\rho_t^S = \rho^S J^{S-1}, \quad \rho_t = \rho_t^S + \rho_t^F, \quad \rho_{12} = \rho_{12}^0 J^{S-1}, \quad \pi = \Pi_V J^{S-1}, \tag{87}$$

while the Cauchy stress tensors $\mathbf{T}^S, \mathbf{T}^F$ are given by the following constitutive relations

$$\mathbf{T}^S = J^{S-1} \mathbf{P}^S \mathbf{F}^{ST} = 2\rho_t^S \left[\frac{\partial \psi^S}{\partial I} \mathbf{B}^S + \frac{\partial \psi^S}{\partial II}\left(I\mathbf{1} - \mathbf{B}^S\right)\mathbf{B}^S + \frac{\partial \psi^S}{\partial III} III\mathbf{1}\right] + \tag{88}$$

$$+ \beta \Delta_n \mathbf{1} - \mathfrak{z}\rho_{12}\left(\mathbf{v}^F - \mathbf{v}^S\right) \otimes \left(\mathbf{v}^F - \mathbf{v}^S\right), \quad \mathbf{B}^S = \mathbf{F}^S \mathbf{F}^{ST},$$

$$\begin{aligned}
\mathbf{T}^F &= J^{S-1} \mathbf{P}^F \mathbf{F}^{ST} = -p^F \mathbf{1} - \beta \Delta_n \mathbf{1} - (1 - \mathfrak{z})\rho_{12}\left(\mathbf{v}^F - \mathbf{v}^S\right) \otimes \left(\mathbf{v}^F - \mathbf{v}^S\right), \\
p^F &= \rho_t^{F2} \frac{\partial \psi^F}{\partial \rho_t^F},
\end{aligned} \tag{89}$$

with the free energy given by

$$\rho_t \psi = \rho_t^S \psi^S (T, I, II, III) + \rho_t^F \psi^F \left(T, \rho_t^F\right) - \rho_{12} \left(\mathbf{v}^F - \mathbf{v}^S\right) \cdot \left(\mathbf{v}^F - \mathbf{v}^S\right). \tag{90}$$

The energy ε and the energy flux vector $\mathbf{q}$ are given by

$$\varepsilon = \psi - T \frac{\partial \psi}{\partial T}, \quad \mathbf{q} = J^{S-1} \mathbf{F}^S \mathbf{Q} = J^{S-1} Q_V \left(\mathbf{v}^F - \mathbf{v}^S\right) - J^{S-1} K \operatorname{grad} T, \tag{91}$$

and the porosity flux has the form

$$\mathbf{j} = J^{S-1} \mathbf{F}^S \mathbf{J} = \Phi_0 \left(\mathbf{v}^F - \mathbf{v}^S\right). \tag{92}$$

It is easy to see that the linearization of the above set for isothermal processes without the source of porosity leads to Biot's equations (1) without the coupling constant Q.

There remains the question of practical estimation of additional parameters ρ_{12}^0 and $\mathfrak{z}$. The added mass coefficient ρ_{12}^0 has been extensively studied in literature concerning Biot's model. The parameter $\mathfrak{z}$ is new. There seem to exist various possibilities for its estimation connected to the fact that it appears in contributions which may be time independent. As an example let us consider a stationary isothermal process in which, in a chosen inertial frame of reference, the skeleton does not move (i.e. $\mathbf{v}^S = 0$). Such a flow of the fluid through a porous material is described by the mass balance and by the momentum balance for the fluid. For simplicity we neglect changes of porosity. Then we have

$$\operatorname{div} \left(\rho_t^F \mathbf{v}^F\right) = 0, \tag{93}$$

$$\begin{aligned} \left[\rho_t^F + 2(1 - \mathfrak{z}) \rho_{12}\right] \mathbf{v}^F \cdot \operatorname{grad} \mathbf{v}^F &= -\operatorname{grad} p^F - \left[\pi + (1 - \mathfrak{z}) \rho_{12} \operatorname{div} \mathbf{v}^F\right] \mathbf{v}^F, \\ p^F &= p^F \left(\rho_t^F\right). \end{aligned}$$

The correction of the permeability coefficient π driven by volume changes of the fluid $\operatorname{div} \mathbf{v}^F$ seems to be very small. However the correction of mass density appearing on the left-hand side of this equation may be essential and measurable. For instance, in an irrotational flow ($\operatorname{rot} \mathbf{v}^F = 0$) we have approximately

$$\operatorname{grad} \left[n_0 (p - p_0) + \frac{1}{2} \left(\rho_t^F + 2(1 - \mathfrak{z}) \rho_{12}\right) \mathbf{v}^F \cdot \mathbf{v}^F\right] + \pi \mathbf{v}^F = 0. \tag{94}$$

where $p = p^F / n$ is the pore pressure and p_0 its constant reference value. If the pressure increment is of the order of, say, 10 kPa the velocity of the fluid must be of the order of 1 m/s to make both contributions of the similar order. Practically measurable would be the influence of $\mathfrak{z}$ for much smaller velocities which seem to be plausible at least for rocks.

4 Propagation of fronts of acoustic waves in Biot's model

As already mentioned, the linearization of the above presented model yields the contribution of the difference of accelerations in Biot's model written in a chosen inertial frame of reference. Consequently, one can ask the question if such a contribution as well as the contribution of the coupling of stresses reflected by the material parameter Q essentially influence the results for the propagation of acoustic waves in porous materials. There exist even claims in the literature that the added mass effect is necessary for the existence of the so-called Biot's wave.

In the next two sections, we present an example of analysis of weak discontinuity (acoustic) waves for Biot's model as well as the "simple mixture" model in which both the coupling Q and the tortuosity coefficient $(a-1)$ are assumed to be zero. Similarly to Biot's model the latter model has already a rather extensive literature (for the review of results, see: [17], [18]).

The main aim of this analysis is to show that differences between these two models are solely quantitative. This has a particular bearing in applications to such complex problems as the propagation of surface waves which play an important role in nondestructive testing of porous materials.

Let us repeat the set of equations of the Biot's model (1), (2) with a small modification of the notation. For the fields $\mathbf{v}^S, \mathbf{v}^F, \mathbf{e}^S, \varepsilon$, we have the field equations

$$\rho_{11}\frac{\partial \mathbf{v}^S}{\partial t} + \rho_{12}\frac{\partial \mathbf{v}^F}{\partial t} = \lambda^S \operatorname{grad} \operatorname{tr} \mathbf{e}^S + 2\mu^S \operatorname{div} \mathbf{e}^S + Q \operatorname{grad} \varepsilon + \pi\left(\mathbf{v}^F - \mathbf{v}^S\right), \tag{95}$$

$$\rho_{22}\frac{\partial \mathbf{v}^F}{\partial t} + \rho_{12}\frac{\partial \mathbf{v}^S}{\partial t} = \kappa\rho_0^F \operatorname{grad} \varepsilon + Q \operatorname{grad} \operatorname{tr} \mathbf{e}^S - \pi\left(\mathbf{v}^F - \mathbf{v}^S\right),$$

where

$$\begin{aligned} \frac{\partial \mathbf{e}^S}{\partial t} &= \operatorname{sym} \operatorname{grad} \mathbf{v}^S, \quad \frac{\partial \varepsilon}{\partial t} = \operatorname{div} \mathbf{v}^F, \\ \rho_{11} &= \rho_0^S\left[1 - r\left(1-a\right)\right], \quad \rho_{12} = r\left(1-a\right)\rho_0^S, \quad \rho_{22} = ra\rho_0^S, \\ a &= \frac{1}{2}\left(\frac{1}{n_0}+1\right), \quad r = \frac{\rho_0^F}{\rho_0^S}. \end{aligned} \tag{96}$$

We begin the analysis of this system by proving its hyperbolicity. To this aim we consider the propagation of the front $\mathcal{S}$ of the weak discontinuity wave, i.e. of a singular surface on which

$$\left[\left[\mathbf{v}^S\right]\right] = 0, \quad \left[\left[\mathbf{v}^F\right]\right] = 0, \tag{97}$$

where [[...]] denotes the jump of the quantity. On such a surface the accelerations may be discontinuous and we call their jumps the **amplitudes of discontinuity**

$$\mathbf{a}^S := \left[\left[\frac{\partial \mathbf{v}^S}{\partial t}\right]\right], \quad \mathbf{a}^F := \left[\left[\frac{\partial \mathbf{v}^F}{\partial t}\right]\right]. \tag{98}$$

Then the following compatibility conditions hold

$$\left[\left[\operatorname{grad}\mathbf{v}^S\right]\right] = -\frac{1}{c}\mathbf{a}^S \otimes \mathbf{n}, \quad \left[\left[\operatorname{grad}\mathbf{v}^F\right]\right] = -\frac{1}{c}\mathbf{a}^F \otimes \mathbf{n}, \tag{99}$$

$$\left[\left[\operatorname{grad}\mathbf{e}^S\right]\right] = -\frac{1}{c}\left[\left[\frac{\partial \mathbf{e}^S}{\partial t}\right]\right] \otimes \mathbf{n}, \quad \left[\left[\operatorname{grad}\varepsilon\right]\right] = -\frac{1}{c}\left[\left[\frac{\partial \varepsilon}{\partial t}\right]\right]\mathbf{n},$$

where c is the speed of propagation of the surface $\mathcal{S}$ and $\mathbf{n}$ its unit normal vector. The latter gives, of course, the direction of propagation of the wave.

Bearing (96) in mind we obtain immediately

$$\begin{aligned} \left[\left[\operatorname{grad}\mathbf{e}^S\right]\right] &= \frac{1}{2c^2}\left(\mathbf{a}^S \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{a}^S\right) \otimes \mathbf{n}, \\ \left[\left[\operatorname{grad}\varepsilon\right]\right] &= \frac{1}{c^2}\mathbf{a}^S \cdot \mathbf{nn}. \end{aligned} \tag{100}$$

We evaluate the jump of field equations (95) on the surface $\mathcal{S}$. It follows immediately

$$\left[\rho_{11}c^2\mathbf{1} - \lambda^S \mathbf{n} \otimes \mathbf{n} - \mu^S\left(\mathbf{1} + \mathbf{n} \otimes \mathbf{n}\right)\right]\mathbf{a}^S + \left[\rho_{12}c^2\mathbf{1} - Q\mathbf{n} \otimes \mathbf{n}\right]\mathbf{a}^F = 0,$$

$$\left[\rho_{12}c^2\mathbf{1} - Q\mathbf{n} \otimes \mathbf{n}\right]\mathbf{a}^S + \left[\rho_{22}c^2\mathbf{1} - \kappa\rho_0^F\mathbf{n} \otimes \mathbf{n}\right]\mathbf{a}^F = 0. \tag{101}$$

This is clearly an eigenvalue problem. We say that the system (95) is **hyperbolic** if the eigenvalues c are real and the corresponding eigenvectors $\left[\mathbf{a}^S, \mathbf{a}^F\right]$ linearly independent. We prove that this is indeed the case.

It is convenient to separate the transversal and longitudinal parts of the problem (101). The **transversal** part follows if we take the scalar product of the equations with a vector $\mathbf{n}_\perp$ perpendicular to $\mathbf{n}$. We obtain

$$\begin{aligned} \left(\rho_{11}c^2 - \mu^S\right)a_\perp^S + \rho_{12}c^2 a_\perp^F &= 0, \\ \rho_{12}a_\perp^S + \rho_{22}a_\perp^F &= 0, \end{aligned}$$

$$a_\perp^S := \mathbf{a}^S \cdot \mathbf{n}_\perp, \quad a_\perp^F := \mathbf{a}^F \cdot \mathbf{n}_\perp. \tag{102}$$

Hence we have for the speed of the front

$$c^2 = \frac{\rho_{22}}{\rho_{11}\rho_{22} - \rho_{12}^2}\mu^S. \tag{103}$$

As $\rho_{22} > 0,\ \mu^S > 0$ it follows the **first condition for hyperbolicity** of the set (95)

$$a - r\left(1 - a\right) > 0. \tag{104}$$

This condition is obviously fulfilled because a is not smaller than 1.

The speed of propagation (103) describes the shear wave. It is easy to see that in the particular case without the influence of tortuosity $a = 1$ this relation reduces to the classical formula $c = \sqrt{\mu^S/\rho_0^S}$. In this case, according to $(102)_2$, the amplitude in the fluid $a_\perp^F$ is zero, i.e. the shear wave is carried solely by the skeleton.

We proceed to the **longitudinal** part. To this aim we take the scalar product of the relations (101) with the vector $\mathbf{n}$. It follows

$$\begin{aligned} \left[\rho_{11}c^2 - \left(\lambda^S + 2\mu^S\right)\right]\mathbf{a}^S\cdot\mathbf{n} + \left[\rho_{12}c^2 - Q\right]\mathbf{a}^F\cdot\mathbf{n} &= 0, \\ \left[\rho_{12}c^2 - Q\right]\mathbf{a}^S\cdot\mathbf{n} + \left[\rho_{22}c^2 - \kappa\rho_0^F\right]\mathbf{a}^F\cdot\mathbf{n} &= 0, \end{aligned} \tag{105}$$

and the dispersion relation is as follows

$$r\left[\left(1 - r\left(1-a\right)\right)c^2 - c_{P1}^2\right]\left[ac^2 - c_{P2}^2\right] - \left[r\left(1-a\right)c^2 - \frac{Q}{\rho_0^S}\right]^2 = 0, \tag{106}$$

where

$$c_{P1}^2 := \frac{\lambda^S + 2\mu^S}{\rho_0^S}, \quad c_{P2}^2 := \kappa. \tag{107}$$

The eigenvalues of this problem have the form

$$c^2 = \frac{1}{2r\left[a - r\left(1-a\right)\right]}\left[A \pm \sqrt{B}\right], \tag{108}$$

where

$$\begin{aligned} A &:= rac_{P1}^2 + \left[1 - r\left(1-a\right)\right]rc_{P2}^2 - 2\frac{Q}{\rho_0^S}r\left(1-a\right), \\ B &:= A^2 - 4r\left[a - r\left(1-a\right)\right]\left[c_{P1}^2c_{P2}^2r - \frac{Q^2}{\rho_0^{S2}}\right]. \end{aligned} \tag{109}$$

It can be easily shown that under the condition (104) $B > 0$ for all $a \geq 1$, $Q \geq 0$. However, c^2 defined by (108) is positive solely if the additional condition on Q is satisfied

$$Q \leq \rho_0^S\sqrt{r}c_{P1}c_{P2} \equiv \sqrt{\rho_0^F\kappa\left(\lambda^S + 2\mu^S\right)}. \tag{110}$$

This is the **second condition for hyperbolicity**.

In the particular case $a = 1$, $Q = 0$ we have c equal to either c_{P1} or c_{P2} which means that the set is unconditionally hyperbolic.

The two solutions for c^2 define two longitudinal modes of propagation, P1 and P2. The P2-mode, called the Biot's wave or the **slow wave** in the theory of porous materials, is also known as the **second sound** and it appears in all two-component systems described by hyperbolic field equations. For instance, it is known in the theory of binary mixtures of fluids in which it is applied to describe dynamical properties of liquid helium as discovered by L. Tisza in 1938 [19]. For porous materials, it has been discovered by Ya. Frenkel in 1944 [20].

5 Biot's model vs. the simple mixture model on example of monochromatic acoustic waves

The above analysis yields solely the propagation properties of the wave front $\mathcal{S}$. We do not learn anything about, for instance, the attenuation of the waves. For this reason we proceed to analyze monochromatic waves. As we see the speeds of propagation obtained above follow in the limit of frequency $\omega \to \infty$.

We seek solutions of equations (95) which have the form of the following monochromatic waves

$$\mathbf{v}^S = \mathbf{V}^S \mathcal{E}, \quad \mathbf{v}^F = \mathbf{V}^F \mathcal{E}, \quad \mathbf{e}^S = \mathbf{E}^S \mathcal{E}, \quad \varepsilon = E^F \mathcal{E}, \tag{111}$$

$$\mathcal{E} := \exp\left[i\left(\mathbf{k}\cdot\mathbf{x} - \omega t\right)\right],$$

where $\mathbf{V}^S, \mathbf{V}^F, \mathbf{E}^S, E^F$ are constant amplitudes, $\mathbf{k}$ is the wave vector, ω real frequency.

Substitution of this ansatz in field equations yields the following compatibility conditions

$$\begin{aligned}
&\left[\rho_{11}\omega^2\mathbf{1} - \lambda^S \mathbf{k}\otimes\mathbf{k} - \mu^S\left(k^2\mathbf{1} + \mathbf{k}\otimes\mathbf{k}\right) + i\pi\omega\mathbf{1}\right]\mathbf{V}^S + \\
&+\left[\rho_{12}\omega^2\mathbf{1} - Q\mathbf{k}\otimes\mathbf{k} - i\pi\omega\mathbf{1}\right]\mathbf{V}^F = 0, \\
&\left[\rho_{12}\omega^2\mathbf{1} - Q\mathbf{k}\otimes\mathbf{k} - i\pi\omega\mathbf{1}\right]\mathbf{V}^S + \left[\rho_{22}\omega^2\mathbf{1} - \kappa\rho_0^F\mathbf{k}\otimes\mathbf{k} + i\pi\omega\mathbf{1}\right]\mathbf{V}^F = 0.
\end{aligned} \tag{112}$$

As usual, the problem of existence of such waves reduces to the eigenvalue problem with the eigenvector $\left[\mathbf{V}^S, \mathbf{V}^F\right]$. As before we split the problem into two parts: in the direction $\mathbf{k}_\perp$ perpendicular to $\mathbf{k}$ (transversal modes) and in the direction of the wave vector $\mathbf{k}$ (longitudinal modes).

For transversal modes (monochromatic shear waves) we have

$$\begin{aligned}
\left[\rho_{11}\omega^2 - \mu^S k^2 + i\pi\omega\right]V_\perp^S + \left[\rho_{12}\omega^2 - i\pi\omega\right]V_\perp^F &= 0, \quad k^2 = \mathbf{k}\cdot\mathbf{k}, \\
\left[\rho_{12}\omega^2 - i\pi\omega\right]V_\perp^S + \left[\rho_{22}\omega^2 + i\pi\omega\right]V_\perp^F &= 0,
\end{aligned} \tag{113}$$

$$V_\perp^S = \mathbf{V}^S\cdot\mathbf{k}_\perp, \quad V_\perp^F = \mathbf{V}^F\cdot\mathbf{k}_\perp.$$

The dispersion relation can be written in this case in the following form

$$\begin{aligned}
&\omega\left\{\left(\rho_{11}\rho_{22} - \rho_{12}^2\right)\left(\frac{\omega}{k}\right)^2 - \mu^S\rho_{22}\right\} + \\
&+i\pi\left\{\left(\rho_{11} + \rho_{22} + 2\rho_{12}\right)\left(\frac{\omega}{k}\right)^2 - \mu^S\right\} = 0,
\end{aligned} \tag{114}$$

i.e.

$$\left(\frac{\omega}{k}\right)^2 = \frac{\omega r a + i\frac{\pi}{\rho_0^S}}{\omega r\left[a - r\left(1-a\right)\right] + i\frac{\pi}{\rho_0^S}\left(1+r\right)}c_S^2, \quad c_S^2 = \frac{\mu^S}{\rho_0^S}. \tag{115}$$

Consequently, neither the phase speed $\omega/\operatorname{Re}k$ nor the attenuation $\operatorname{Im}k$ of monochromatic shear waves is dependent on the coupling coefficient Q.

In the two limits of frequencies we have then the following solutions

$$\underline{\omega \to 0}: \quad \lim_{\omega\to 0}\left(\frac{\omega}{\operatorname{Re}k}\right)^2 = \frac{\mu^S}{\rho_0^S + \rho_0^F}, \quad \lim_{\omega\to 0}\left(\operatorname{Im}k\right) = 0,$$

$$\underline{\omega \to \infty} \quad : \quad \lim_{\omega\to\infty}\left(\frac{\omega}{\operatorname{Re} k}\right)^2 = \frac{\rho_{22}}{\rho_{11}\rho_{22}-\rho_{12}^2}\mu^S, \tag{116}$$

$$\lim_{\omega\to\infty}(\operatorname{Im} k) = \frac{\pi}{2\sqrt{\rho_0^S\mu^S}}\frac{1}{a^2}\sqrt{\frac{a}{a-r(1-a)}}.$$

The first result checks with the results of the classical one-component model commonly used in soil mechanics. The speed in the second one is identical with this of formula (103). Hence the propagation of the front of shear waves is identical with the propagation of monochromatic waves of infinite frequency. Let us notice that the attenuation in this limit is finite.

We demonstrate further properties of these monochromatic waves on a numerical example.

For longitudinal modes we obtain the dispersion relation

$$\left[\rho_{11}\omega^2 - \left(\lambda^S + 2\mu^S\right)k^2 + i\pi\omega\right]\left[\rho_{22}\omega^2 - \kappa\rho_0^F k^2 + i\pi\omega\right] -$$
$$-\left(\rho_{12}\omega^2 - Qk^2 - i\pi\omega\right)^2 = 0, \tag{117}$$

or, after easy manipulations,

$$\omega\left\{[1-r(1-a)]\left(\frac{\omega}{k}\right)^2 - c_{P1}^2\right\}\left\{a\left(\frac{\omega}{k}\right)^2 - c_{P2}^2\right\} +$$
$$+\frac{1}{r}i\frac{\pi}{\rho_0^S}\left(\frac{\omega}{k}\right)^2\left\{(1+r)\left(\frac{\omega}{k}\right)^2 - rc_{P2}^2 - c_{P1}^2 - 2\frac{Q}{\rho_0^S}\right\} - \tag{118}$$
$$-\frac{1}{r}\omega\left\{r(1-a)\left(\frac{\omega}{k}\right)^2 - \frac{Q}{\rho_0^S}\right\}^2 = 0.$$

Let us check again two limits of frequencies: $\omega \to 0$, and $\omega \to \infty$.

In the first case we obtain

$$\underline{\omega \to 0}: \quad c_0 := \lim_{\omega\to 0}\left(\frac{\omega}{\operatorname{Re} k}\right),$$

$$c_0^2\left\{(1+r)c_0^2 - rc_{P2}^2 - c_{P1}^2 + 2\frac{Q}{\rho_0^S}\right\} = 0, \quad \lim_{\omega\to 0}(\operatorname{Im} k) = 0. \tag{119}$$

Obviously, we obtain two real solutions of this equation

$$\lim_{\omega\to 0}\left(\frac{\omega}{\operatorname{Re} k}\right)^2\Bigg|_1 \; : = c_{oP1}^2 = \frac{c_{P1}^2 + rc_{P2}^2 + 2\frac{Q}{\rho_0^S}}{1+r} \equiv \frac{\lambda^S + 2\mu^S + \rho_0^F\kappa + 2\frac{Q}{\rho_0^S}}{\rho_0^S + \rho_0^F}, \tag{120}$$

$$\lim_{\omega\to 0}\left(\frac{\omega}{\operatorname{Re} k}\right)^2\Bigg|_2 \; : = c_{oP2}^2 = 0.$$

These are squares of speeds of propagation of two longitudinal modes in the limit of zero frequency. Clearly, the second mode, P2-wave, does not propagate in this limit. Both limits are independent of tortuosity. The result (120) checks with the relation for the speed of longitudinal waves used in the classical one-component model of soil mechanics provided $Q = 0$.

In the second case we have

$$\underline{\omega \to \infty}: \quad c_\infty := \lim_{\omega\to\infty} \left(\frac{\omega}{\operatorname{Re} k}\right),$$

$$r\left\{[1 - r(1-a)]\, c_\infty^2 - c_{P1}^2\right\}\left\{a c_\infty^2 - c_{P2}^2\right\} - \left\{r(1-a)\, c_\infty^2 - \frac{Q}{\rho_0^S}\right\}^2 = 0. \tag{121}$$

This coincides with the relation (106). Consequently, the limit $\omega \to \infty$ gives indeed the properties of the front of acoustic longitudinal waves in the system.

Simultaneously we obtain the following attenuation in the limit of infinite frequencies

$$\lim_{\omega\to\infty} (\operatorname{Im} k) = \frac{\pi \Gamma_1}{2\rho_0^S r \Gamma_2}, \tag{122}$$

$$\begin{aligned} \Gamma_1 &= c_\infty \left[1 + r - \frac{1}{c_\infty^2}\left(c_{P1}^2 + r c_{P2}^2 + 2\frac{Q}{\rho_0^S}\right)\right], \\ \Gamma_2 &= c_{P1}^2\left(a - \frac{c_{P2}^2}{c_\infty^2}\right) + c_{P2}^2\left(1 - r(1-a) - \frac{c_{P1}^2}{c_\infty^2}\right) + 2\frac{Q}{\rho_0^S}\left(1 - a - \frac{Q}{r\rho_0^S c_\infty^2}\right). \end{aligned}$$

Hence both limits of attenuation for the P1-wave and P2-wave are finite.

We proceed to the presentation of a numerical result in the whole range of frequencies $\omega \in [0, \infty)$. We use the following numerical data

$$\begin{aligned} c_{P1} &= 2500\ \tfrac{m}{s}, \quad c_{P2} = 1000\ \tfrac{m}{s}, \quad c_S = 1500\ \tfrac{m}{s}, \\ \rho_0^S &= 2500\ \tfrac{kg}{m^3}, \quad r = 0.1, \quad \pi = 10^8\ \tfrac{kg}{m^3 s}, \\ Q &= 0.8\, GPa, \quad n_0 = 0.4, \quad a = 1.75. \end{aligned} \tag{123}$$

Speeds c_{P1}, c_{P2}, c_S, the mass density ρ_0^S (i.e $\rho_0^{SR} = 4167\ \frac{kg}{m^3}$ for the porosity $n_0 = 0.4$) and the fraction $r = \rho_0^F/\rho_0^S$ possess values typical for many granular materials under a confining pressure of a few atmospheres and saturated by water. In units standard for soil mechanics the permeability π corresponds to app. 0.1 Darcy. The coupling coefficient Q has been estimated by means of the Gassmann relation (e.g. [21]). The tortuosity coefficient $a = 1.75$ follows from Berryman formula $(7)_2$.

Transversal waves described by the relation (115) are characterized by the following distribution of speeds and attenuation in function of frequency (Fig.1). The solid lines correspond to the solution of Biot's model and the dashed lines to the solution of the simple mixture model.

It is clear that the qualitative behavior of the speed of propagation is the same in both models. It is a few percent smaller in Biot's model than this in the simple mixture model in the range of high frequencies. A large quantitative difference between these models appears for the attenuation. In the range of higher frequencies it is much smaller in the Biot's model, i.e. tortuosity decreases the dissipation of shear waves.

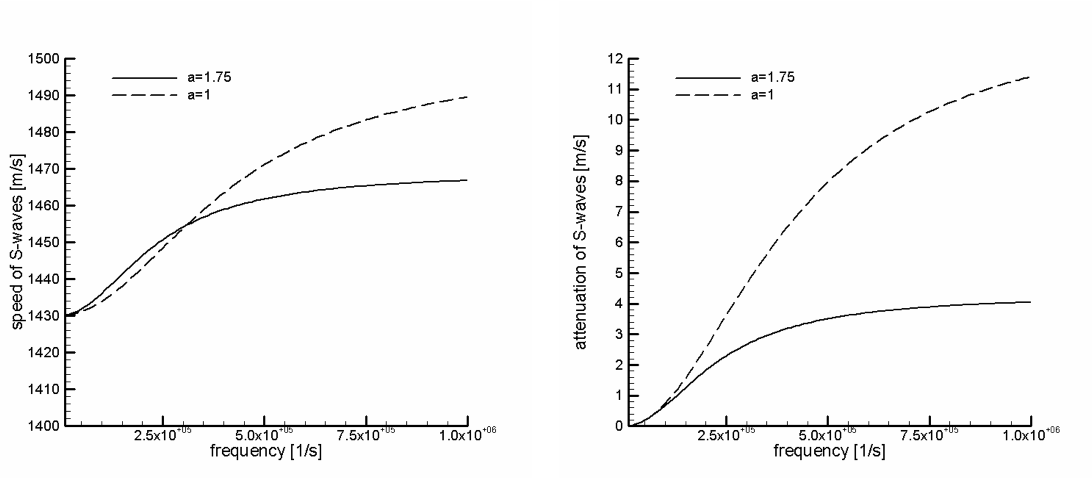

Figure 1: *Speed of propagation and attenuation of monochromatic S-waves for two values of the tortuosity coefficient* a : 1.75 *(Biot)*, 1.00 *(simple mixture).*

The latter property is illustrated in Fig.2 where we plot the attenuation of the front of shear waves, i.e. $\lim_{\omega \to \infty} \operatorname{Im} k$, as a function of the tortuosity coefficient a. This behavior of attenuation indicates that damping of waves created by the tortuosity, which is connected in the macroscopic model to the relative velocity of components, is not related to scattering of waves on the microstructure. It is rather related to the decrease of the macroscopic diffusion velocity in comparison with the difference of velocities on the microscopic level due to the curvature of channels and volume averaging. Fluctuations are related solely to this averaging and not to temporal deviation from time averages (lack of ergodicity!).

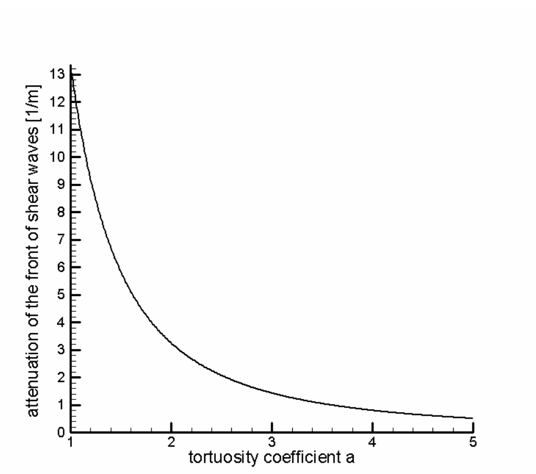

Figure 2: *Attenuation of the front of shear waves in function of the tortuosity coefficient a.*

We proceed to longitudinal waves. The solid lines on the following Figures correspond again to Biot's model, the dashed lines to the simple mixture model. In order to show separately the influence of tortuosity a and of the coupling Q we plot as well the solutions with $a = 1$ (dashed dotted lines) and the solutions with $Q = 0$ (dashed double dotted lines).

Even though similar again the quantitative differences are much more substantial for P1-waves (Fig.3). This is primarily an influence of the coupling through partial stresses described by the parameter Q. The simple mixture model ($Q = 0, a = 1$) as well as Biot's model with $Q = 0$ yield speeds of these waves different only a few percent (lower curves in the left diagram). The coupling Q shifts the curves to higher values and reduces the difference caused by the tortuosity. This result does not seem to be very realistic because the real differences between low frequency and high frequency speeds were measured in soils to be rather as big as indicated by the simple mixture model. This may be an indication that Gassmann relations give much too big values of the coupling parameter Q with respect to these indeed appearing in real granular materials.

Both the tortuosity a and the coupling Q reduce the attenuation quite considerably as indicated in the right Figure.

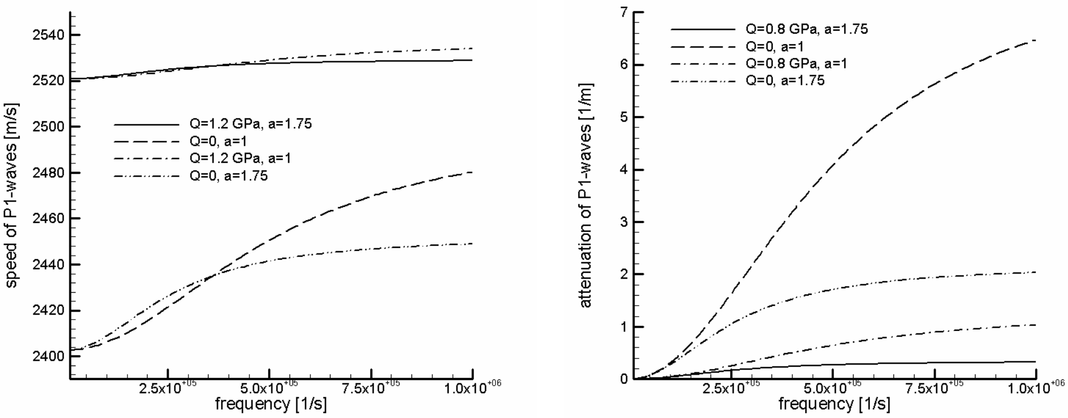

Figure 3: *Speed of propagation and attenuation of monochromatic P1-waves for various coupling parameters Q and tortuosity coefficients a.*

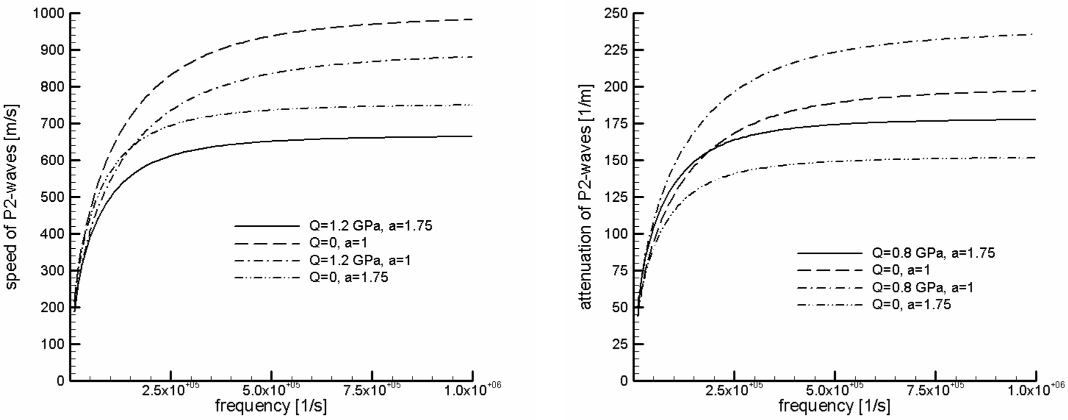

Figure 4: *Speed of propagation and attenuation of monochromatic P2-waves for various coupling parameters Q and tortuosity coefficients a.*

In spite of some claims in the literature the tortuosity a does not influence the existence of the slow (P2-) wave (Fig. 4). Speeds of this wave are again qualitatively similar in Biot's model and in the simple mixture model. The maximum differences appear in the range of high frequencies and reach some 35 percent. The same concerns the attenuation even though quantitative differences are not so big (app. 8 percent).

6 Conclusions

The analysis presented in this work yields the following conclusions.

1° As demonstrated in the second section, it is possible to construct a relative acceleration for a two-component model of a poroelastic material in such a way that it transforms as an objective quantity. Additional contributions to the difference of partial accelerations are nonlinear and contain a single scalar constitutive parameter $\mathfrak{z}$.

As shown in the third section the linear dependence of the source of momentum on such a relative acceleration is thermodynamically admissible provided the following conditions are fulfilled. The partial stress tensors contain additional contributions quadratic in the relative velocity with material coefficients determined by the combination of two parameters appearing in the contribution of the relative acceleration: the tortuosity coefficient a and the parameter $\mathfrak{z}$. The internal energy contains an additional contribution of the kinetic energy of relative motion with the constitutive coefficient dependent solely on tortuosity a. Such a model fulfils the second law of thermodynamics and the principle of material objectivity.

Linearization of the above described model yields Biot's contribution of relative accelerations. This is, of course, not objective anymore. Consequently, Biot's model can be used solely in inertial frames of reference. In noninertial frames the transition from the nonlinear model yields apparent body forces but not additional terms which would follow by the transformation of the system of Biot's equations.

2° We have demonstrated on the example of acoustic waves that tortuosity a and the coupling parameter Q have a quantitative but not qualitative influence on results. We have compared results for Biot's model with these for the simple mixture model in which the tortuosity $a = 1$ and the coupling parameter $Q = 0$. We have proven that both models are hyperbolic provided the parameter Q satisfies a condition bounding this parameter from above. In particular, both models predict the existence of the P2-wave. Speeds and attenuations of monochromatic P1-, P2- and S-waves are qualitatively the same but there are quantitative discrepancies which we discuss below.

3° Tortuosity introduced to the model through the relative acceleration yields dissipation solely due to the modification of the relative motion. Namely if we assume the permeability coefficient $\pi = 0$ the dissipation in isothermal processes without relaxation of porosity vanishes. This is due to the fact that tortuosity, in contrast to porosity, is not introduced as a field described by its own field equation. This is an explanation of a rather unexpected behavior of attenuation of monochromatic waves. Inspection of figures shown in this work makes clear that the presence of tortuosity $a \neq 1$ yields a smaller attenuation rather than bigger as it would be in the case of a dissipative field. This may be explained by the fact that tortuosity reduces the relative velocity $\mathbf{v}^F - \mathbf{v}^S$ and, consequently, it reduces the contribution to dissipation $\pi \left(\mathbf{v}^F - \mathbf{v}^S\right) \cdot \left(\mathbf{v}^F - \mathbf{v}^S\right)$.

4° We have demonstrated that a rather moderate value of the parameter Q suggested by the classical Gassmann relation for granular materials leads to an unreasonable increment of speeds of propagation and reduction of attenuation. In addition, the speed of propagation of monochromatic P1-waves becomes very flat as a function of frequency. This contradicts observations in soil mechanics and geotechnics and indicates that the Gassmann relation predicts too big values of this parameter. The situation would improve if we used the model proposed in [13]. This model contains a constitutive dependence on the porosity gradient which yields a modification of Gassmann relations [21] and a considerable reduction of the parameter Q.

References

[1] M. A. Biot; *Theory of Propagation of Elastic Waves in a Fluid-Saturated Porous Solid. I. Low-Frequency Range*, *JASA*, **28**, 168-178, 1956.

[2] I. Tolstoy; *Acoustics, Elasticity and Thermodynamics of Porous Media: Twenty-One Papers by M. A. Biot, Acoustical Society of America*, 1991.

[3] R. D. Stoll; *Sediment Acoustics*, Lecture Notes in Earth Sciences, #26, Springer, New York, 1989.

[4] T. Bourbie, O. Coussy, B. Zinszner; *Acoustics of Porous Media,* Edition Technip, Paris, 1987.

[5] J. F. Allard; *Propagation of Sound in Porous Media. Modelling Sound Absorbing Materials*, Elsevier, Essex, 1993.

[6] J. G. Berryman; *Confirmation of Biot's Theory*, *Appl. Phys. Lett.*, **37**, 382-384, 1980.

[7] D. L. Johnson, J. Koplik, R. Dashen; *Theory of Dynamic Permeability and Tortuosity in Fluid-Saturated Porous Media*, *J. Fluid Mechanics,* **176**, 379-402, 1987.

[8] I-Shih Liu; *Continuum Mechanics*, Springer, Berlin, 2002.

[9] K. Wilmanski; *Some Questions on Material Objectivity Arising in Models of Porous Materials*, in: P. Podio-Guidugli, M. Brocato (Eds.), *Rational Continua, Classical and New*, Springer-Italy, Milan, 149-161, 2001.

[10] D. Drew, L. Cheng, R. T. Lahey, Jr.; *The Analysis of Virtual Mass Effects in Two-Phase Flow, Int. J. Multiphase Flow*, **5**, 233-242, 1979.

[11] K. Wilmanski; *Thermomechanics of Continua*, Springer, Berlin, 1998.

[12] O. Coussy; *Mécanique des Milieux Poreux*, Editions Technip, Paris, 1991.

[13] K. Wilmanski; *Thermodynamical Admissibility of Biot's Model of Poroelastic Saturated Materials*, *Arch. Mech.*, **54**, 5-6, 709-736, 2002.

[14] K. Wilmanski; *Thermodynamics of Simple Two-Component Thermo-Poroelastic Media*, WIAS-Preprint #901, 2004; to appear in: G. Romano, S. Rionero (Eds.), *Recent Trends in the Applications of Mathematics to Mechanics*, STAMM 2002, Springer-Italia, Milan, 2004.

[15] K. Wilmanski; *A Thermodynamic Model of Compressible Porous Materials with the Balance Equation of Porosity*, *Transport in Porous Media*, **32**:, 21-47, 1998.

[16] N. P. Kirchner; *Thermodynamically Consistent Modelling of Abrasive Granular Materials. Part I: Nonequilibrium Theory*, *Proc. Royal Soc. (Mathematical, Physical and Engineering Sciences) Lond. A*, **458**, 2153-2176, 2002.

[17] K. Wilmanski; *Waves in Porous and Granular Materials*, in: K. Hutter, K. Wilmanski (Eds.), *Kinetic and Continuum Theories of Granular and Porous Media*, SpringerWienNewYork, 131-186, 1999.

[18] K. Wilmanski, B. Albers; *Acoustic Waves in Porous Solid-Fluid Mixtures*, in: K. Hutter, N. Kirchner (Eds.), *Dynamic Response of Granular and Porous Materials under Large and Catastrophic Deformations*, Lecture Notes in Applied and Computational Mechanics, Vol. 11, Springer, Berlin, 285-314, 2003.

[19] L. Tisza; *Transport Phenomena in He II, Nature,* 141, 1938.

[20] Ya. Frenkel; *On the Theory of Seismic and Seismoelectric Phenomena in Moist Soils*, *J. Phys.*, **8**, 4, 230-241, 1944.

[21] K. Wilmanski; *On a Micro-Macro Transition for Poroelastic Biot's Model and Corresponding Gassmann-type Relations*, submitted to *Géotechnique*; see also: WIAS-Preprint #868, 2003.