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Hopf bifurcations and simple structures of periodic solution sets in systems with the Preisach nonlinearity

Martin Brokate[1], Dmitrii Rachinskii[2]

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[1] Zentrum Mathematik
Technische Universität München
Boltzmannstr. 3
85747 Garching b. München
Germany
E-Mail: brokate@ma.tum.de

[2] Weierstrass Institute for
Applied Analysis and Stochastics
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: ratchins@wias-berlin.de

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We survey a number of recent results and suggest some new ones on periodic solutions of systems with hysteresis. The main focus of this work is the situation when simple one-parameter structures of periodic regimes appear. We consider forced oscillations, cycles of autonomous systems and Hopf bifurcations from the equilibrium and from infinity.

1 Introduction

We consider periodic solutions of systems with hysteresis nonlinearities focusing on Hopf bifurcation problems and on the natural situations when periodic regimes form continuum sets. Those situations are defined by the simple condition $\|z\|_C < \rho$ that should be satisfied for the components of periodic solutions in the finite-dimensional space of phase variables z, where ρ is determined by the hysteresis nonlinearity and characterizes the size of the hysteresis region. For example, the estimate $\|z\|_C < \rho$ can be derived from *a priori* estimates of periodic solutions. If it is satisfied, then generically each periodic regime (if any exists) is included in the connected continuum of periodic regimes whose structure can be described in simple terms both for nonautonomous and autonomous systems. Existence of such a continuum was observed already in early applications of the operator theory of hysteresis [24]. In the complementary case $\|z\|_C > \rho$ periodic regimes are typically isolated.

An interest to problems on bifurcations in systems with hysteresis is motivated by phenomena observed in real systems where variation of parameters leads to sudden changes of dynamics. Recent numerical results justified also by theoretical analysis show that systems with hysteresis nonlinearities can exhibit a rich spectrum of bifurcations, in particular bifurcations of periodic regimes, subharmonics, quasi- and almost periodic motions, appearance of regions of complicated and chaotic behavior and others. For example, this is the case for simple models like second order differential equations with a hysteresis term, which is responsible for bifurcations [18, 19, 20]. Here we present a number of results on one type of bifurcations, namely Hopf bifurcations of cycles in autonomous systems, discussing mainly sufficient conditions for some parameter value to be a bifurcation point and structures of the sets of cycles. One should take into account that hysteresis nonlinearities are not differentiable and analysis of bifurcations should be based on approaches alternative to the ones standard in smooth problems.

The paper is organized as follows. In the next section we recall shortly a definition of the Preisach nonlinearity. Section 3 presents statements on continua of periodic solutions and the structure of such continua, which is basically one-parametric. In the last subsection we suggest some results on continua of cycles of autonomous systems analogous to that presented before for problems on forced periodic oscillations. Section 4 contains theorems on Hopf bifurcations, including bifurcations from an equilibrium (subsection 4.2) and from infinity (subsection 4.3) for equations with one scalar parameter. In subsection 4.4 we study systems without parameters that have a continuum of small cycles accumulating at the equilibrium and consider relations of the problem to Hopf bifurcation problems.

The main objective of the paper is to survey a group of recent results on periodic problems with hysteresis that can be approached by common analytic methods. We omit most of the proofs, restricting ourselves with the presentation of some of their sketches and references to the literature.

We consider systems with the Preisach hysteresis nonlinearities, which is an important class in various applications (see, e.g. [17, 23]). Similar results can be obtained for some other classes of hysteresis nonlinearities like, for example, the Prandtl – Ishlinskii, Mayergoyz – Friedman, Mroz models, sometimes straightforwardly, sometimes with a more essential modification of formulations and proofs. It is important to stress that we consider equations that include outputs of the hysteresis nonlinearities rather than time-derivatives of the outputs. If the derivatives only are included, then continua of periodic regimes are generically not observed.

2 Systems with the Preisach hysteresis nonlinearity

2.1 Preisach model

By $\mathcal{R}_{\alpha\beta}[\cdot]$ we denote the input-output operator of the elementary hysteresis nonlinearity called the nonideal relay with the two states $\{0,1\}$, which is defined in a standard way. Here α, β are the thresholds of the relay, they satisfy $\alpha<\beta$. The relay inputs are any continuous scalar functions $x=x(t)$ defined on a semiaxis $t\geq t_0$, its outputs (called also variable states) are functions $\eta=\eta(t)$ with the values 0 and 1 defined on the same semiaxis $t\geq t_0$. Assume that an initial state $\eta_0=\eta(t_0)$ of the relay is admissible for a given input $x=x(t)$, which means that $\eta_0=0$ if $x(t_0)\leq\alpha$, $\eta_0=1$ if $x(t_0)\geq\beta$ and η_0 is any of the numbers $0,1$ if $\alpha<x(t_0)<\beta$. Then the relay output $\eta(t)=(\mathcal{R}_{\alpha\beta}[\eta_0]x)(t)$ is defined by

$$(\mathcal{R}_{\alpha\beta}[\eta_0]x)(\tau)=\begin{cases}\eta_0, & \text{if } \alpha<x(t)<\beta \text{ for all } t\in[t_0,\tau];\\ 1, & \text{if there is a } t_1\in[t_0,\tau] \text{ such that}\\ & x(t_1)\geq\beta \text{ and } x(t)>\alpha \text{ for all } t\in[t_1,\tau];\\ 0, & \text{if there is a } t_1\in[t_0,\tau] \text{ such that}\\ & x(t_1)\leq\alpha \text{ and } x(t)<\beta \text{ for all } t\in[t_1,\tau].\end{cases}$$

This formula implies that the output η has at most finite number of jumps (relay switches) on every segment $[t_0,t]$ and that $\eta(\tau)=1$ whenever $x(\tau)\geq\beta$ as well as $\eta(\tau)=0$ whenever $x(\tau)\leq\alpha$.

The Preisach hysteresis nonlinearity may be described as a collection of relays (with all possible thresholds) that have a common input and function independently. For the strict definition, consider the set W of nonideal relays parameterized by the pairs (α, β), each relay is represented by a point of the half-plane $\Pi = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha < \beta\}$. Let a probabilistic measure μ be defined on Π. Denote by $\Omega(x_0)$ the set of all measurable functions $\eta_0(\alpha, \beta) : \Pi \to \{0, 1\}$ satisfying

$$\eta_0(\alpha, \beta) = 0 \quad \text{for} \quad \alpha \geq x_0; \qquad \eta_0(\alpha, \beta) = 1 \quad \text{for} \quad \beta \leq x_0$$

and define $\mathfrak{E}_* = \cup_{x_0 \in \mathbb{R}} \Omega(x_0)$. The states of the Preisach nonlinearity are all functions $\eta_0 = \eta_0(\alpha, \beta) \in \mathfrak{E}_*$. The further construction is based on the fact that for every continuous input $x : [t_0, \infty) \to \mathbb{R}$ and every initial state $\eta_0 \in \Omega(x(t_0))$ (such initial states are called *admissible* for the input $x = x(t)$) the function

$$\eta(t, \alpha, \beta) = (\mathcal{R}_{\alpha\beta}[\eta_0(\alpha, \beta)]x)(t) \tag{1}$$

satisfies $\eta(t, \cdot, \cdot) \in \Omega(x(t)) \subset \mathfrak{E}_*$ for each $t \geq t_0$. This allows to use formula (1) as the definition of the input-state operator

$$\eta(\cdot) = (\Gamma[\eta_0]x)(\cdot)$$

of the Preisach nonlinearity that assigns the variable state $\eta : [t_0, \infty) \to \mathfrak{E}_*$ to a continuous input x and an admissible initial state η_0. The scalar output $y : [t_0, \infty) \to \mathbb{R}$ of the Preisach nonlinearity is defined by the formula

$$y(\cdot) = (\Phi(\Gamma[\eta_0]x))(\cdot)$$

where the functional $\Phi : \mathfrak{E}_* \to \mathbb{R}$ is given by

$$\Phi(\eta_0(\cdot, \cdot)) = \iint_\Pi \eta_0(\alpha, \beta)\, d\mu.$$

It means that for a given input $x = x(t)$ and an admissible initial state η_0 the output equals

$$y(t) = \iint_\Pi \eta(t, \alpha, \beta)\, d\mu = \iint_\Pi (\mathcal{R}_{\alpha\beta}[\eta_0(\alpha, \beta)]x)(t)\, d\mu, \qquad t \geq t_0,$$

which is interpreted in the sense that the outputs of the individual relays of the set W are averaged over the domain $\Pi \ni (\alpha, \beta)$.

Everywhere below we use measures μ such that all outputs of the Preisach nonlinearity are continuous functions (although the outputs of individual relays have jumps). For example, this is true if μ has a bounded density $h = h(\alpha, \beta)$ with respect to the Lebesgue measure. Since the measure μ is probabilistic, all the outputs satisfy the uniform estimate $0 \leq y(t) \leq 1$ for all t.

Further properties of the Preisach model and its more detailed description can be found for example in [12, 3, 15, 17].

2.2 Lipschitz continuity of the Preisach nonlinearity

Let us metrize the state space $\mathfrak{E}_*$ of the Preisach nonlinearity. A usual way to do it is to use the L_1-type metric

$$\rho_1(\eta_1, \eta_2) = \iint_\Pi |\eta_1(\alpha, \beta) - \eta_2(\alpha, \beta)|\, d\mu. \tag{2}$$

With respect to this metric the state space of the Preisach nonlinearity is partitioned to the natural equivalence classes. The correctness of the partitioning is justified in [12], where it is shown that if the functions $\eta_1(t_0), \eta_2(t_0) \in \mathfrak{E}_*$ satisfy $\rho_1(\eta_1(t_0), \eta_2(t_0)) = 0$ then $\rho_1(\eta_1(t), \eta_2(t)) = 0$ at every moment $t \geq t_0$ for the variable states $\eta_j(t) = (\Gamma[\eta_j(t_0)]x)(t)$, which also implies the equality $y_1 = y_2$ of the outputs $y_j(t) = \Phi(\eta_j(t))$.

For simplicity, assume that the measure μ has a bounded density $h(\alpha, \beta)$ with respect to Lebesgue measure such that $h(\alpha, \beta) \leq h_*$ in Π and $h(\alpha, \beta) = 0$ for $\alpha < \beta - \gamma_*$ with some $\gamma_*, h_* > 0$. As it is proved in [12], these relations imply the global Lipschitz continuity of the input-state and input-output operators of the Preisach nonlinearity. More precisely, for every pair of continuous inputs $x_j : [t_0, \infty) \to \mathbb{R}$ and admissible initial states $\eta_j(t_0)$ the variable states $\eta_j(\cdot) = (\Gamma[\eta_j(t_0)]x_j)(\cdot)$ and outputs $y_j(\cdot) = (\Phi(\eta_j))(\cdot)$ satisfy for each $t \geq t_0$

$$\|y_1 - y_2\|_{C[0,t]} \leq \rho_1(\eta_1(t), \eta_2(t_0)) \leq \rho_1(\eta_1(t_0), \eta_2(t_0)) + 2h_*\gamma_*\|x_1 - x_2\|_{C[0,t]},$$

where $\|x\|_{C[0,t]} = \max\{|x(\tau)| : t_0 \leq \tau \leq t\}$. The proof of these estimates is based on a simple explicit algorithm to construct variable states $\eta_j(\cdot)$, which we do not consider here.

2.3 Closed systems with the Preisach nonlinearity

We shall consider closed systems of the form

$$\begin{array}{rcll} dz/dt & = & F(t, z(t), y(t)), & \quad z \in \mathbb{R}^d,\ t \geq t_0, \\ y(t) & = & \Phi(\eta(t)), & \\ \eta(t) & = & (\Gamma[\eta(t_0)]x)(t), & \\ x(t) & = & \langle c, z(t) \rangle\,, & \end{array} \tag{3}$$

where $\langle \cdot, \cdot \rangle$ is a scalar product in $\mathbb{R}^d$. Here x is a scalar continuous input of the Preisach hysteresis nonlinearity, y and η are the scalar continuous output and the changing state of this nonlinearity.

We assume that the function $F(\cdot, \cdot, \cdot)$ is continuous with respect to the set of its arguments and T-periodic in t. Solutions of (3) are defined in a standard way as pairs $(z, \eta) = (z(t), \eta(t))$ with the continuously differentiable first component. Their values lie in the phase space $\mathbb{R}^d \times \mathfrak{E}_*$ of system (3). Remark that the Volterra property allows to consider inputs, variable states and outputs of the hysteresis nonlinearity on finite intervals. Therefore solutions of system (3) may be defined on finite and infinite intervals like for ordinary differential equations.

Along with system (3), its particular cases will be considered where the first equation has the form $L(d/dt)x = f(t, x, y)$ with a scalar differential polynomial L as well as autonomous systems with F independent of t. In section 4 we consider also autonomous systems depending on a parameter.

3 One-parameter sets of periodic regimes

3.1 Example

In order to start with a simple example of a statement on existence of a continuum of periodic solutions, we consider the system

$$\begin{array}{rcl} L\,(d/dt)\,x &=& f(t,x,y(t)), \\ y(t) &=& \Phi(\eta(t)), \\ \eta(t) &=& (\Gamma[\eta(t_0)]x)(t) \end{array} \tag{4}$$

with

$$L(p) = p^\ell + a_1 p^{\ell-1} + \cdots + a_\ell$$

where $x(t)$ and $y(t)$ are input and output of the Preisach nonlinearity and the function $f(t,x,y)$ is supposed to be continuous and satisfy $f(t,x,y) \equiv f(t+T,x,y)$ for some $T > 0$.

Let us split the set of all T-periodic solutions $(x(t),\eta(t)) = (x(t+T),\eta(t+T))$ $(t \geq t_0)$ of system (4) into disjoint equivalence classes, assigning two periodic solutions to the same class if they have the same first component and their second components $\eta_j(t)$ satisfy $\Phi(\eta_1(t)) \equiv \Phi(\eta_2(t))$, i.e. the variable states $\eta_j(t)$ define the same periodic output of the Preisach nonlinearity. We denote the equivalence class of periodic solutions $(x(t),\eta(t))$ by $[x(t),y(t)]$ where $y(t) = \Phi(\eta(t))$ for all its representatives.

Classes $[x(t),y(t)]$ are natural by several reasons. First, the main first equation of system (4) contains inputs and outputs rather than variable states $\eta(t)$ of the hysteresis nonlinearity. This situation is typical regarding that states of the hysteresis nonlinearity are basically considered as values that one can neither observe nor control. Secondly, the second components of all periodic solutions $(x(t),\eta(t))$ that belong to the same class $[x(t),y(t)]$ are related by simple explicit formulas, which we discuss below.

Let for all $t,x \in \mathbb{R}$

$$|f(t,x,y)| \leq q|x| + b, \qquad y \in [0,1] \tag{5}$$

(here $[0,1]$ is the range of output values of the Preisach nonlinearity). Suppose that all the zeros of the polynomial $L(p)$ are different from the numbers $nw_0 i$ $(n \in \mathbb{Z})$ where $w_0 = 2\pi/T$. Set

$$k = \max_{n\in\mathbb{Z}} |L(nw_0 i)|^{-1}, \qquad k_1 = T^{-1/2}\left(|L(0)|^{-2} + 2\sum_{n=1}^{\infty} |L(nw_0 i)|^{-2}\right)^{1/2}. \tag{6}$$

Actually, these formulas define the norms $k = \|H\|_{L^2\to L^2}$ and $k_1 = \|H\|_{L^2\to C}$ of the solution operator $H = H(T)$ sending a function $u = u(t)$ to the solution $x = Hu$ of the T-periodic problem for the linear equation $L\left(\frac{d}{dt}\right)x = u(t)$ in the spaces $L^2 = L^2[0,T]$ and $C = C[0,T]$.

To be simple, we shall always assume in this section that the measure μ of the Preisach nonlinearity has a bounded density $h = h(\alpha, \beta)$ with respect to the Lebesgue measure. Define the scalar function

$$\overline{\mu}(r) = \mu\big(\{(\alpha,\beta) : \alpha < -r,\, r < \beta\}\big), \qquad r \geq 0, \tag{7}$$

and denote by R_μ the least nonnegative solution of the equation $\overline{\mu}(r) = 0$ if any exists. If $\overline{\mu}(r) > 0$ for all $r \geq 0$ then we set $R_\mu = \infty$. The value R_μ plays an important role in all the statements below.

Proposition 3.1. *Let estimate (5) hold and*

$$qk < 1, \qquad \frac{k_1 b\sqrt{T}}{1 - qk} < R_\mu. \tag{8}$$

Then there exists a one-parameter continuum of classes $[x_\lambda(t), y_\lambda(t)]$ ($0 \leq \lambda \leq 1$) of T-periodic solutions to system (4), which classes are different for different λ. Assume additionally that the density of the measure μ satisfies

$$h(\alpha,\beta) \leq \nu(\beta - \alpha), \qquad \alpha < \beta, \tag{9}$$

where ν is integrable on the positive semiaxis, i.e.

$$\nu_\infty = \int_0^\infty \nu(s)\, ds < \infty. \tag{10}$$

Moreover, assume that the function $f(t,x,y)$ satisfies the Lipschitz condition

$$|f(t,x_1,y_1) - f(t,x_2,y_2)| \leq q|x_1 - x_2| + q_1|y_1 - y_2| \tag{11}$$

for all $t, x_j \in \mathbb{R}$, $0 \leq y_j \leq 1$ and

$$q_1 < \frac{1 - qk}{2\nu_\infty k_1 \sqrt{T}}. \tag{12}$$

Then the set of all T-periodic solutions of system (4) is the join of the classes $[x_\lambda(t), y_\lambda(t)]$ ($0 \leq \lambda \leq 1$) above and there are numbers c_j such that

$$\|x_{\lambda_1}(\cdot) - x_{\lambda_2}(\cdot)\|_C \leq c_1|\lambda_1 - \lambda_2|, \quad \|y_{\lambda_1}(\cdot) - y_{\lambda_2}(\cdot)\|_C \leq c_2|\lambda_1 - \lambda_2|, \quad \lambda_j \in [0,1]. \tag{13}$$

For example, consider equation

$$x'' + x = \sin(\sqrt{2}t) + \rho(2y(t) - 1) \tag{14}$$

where x and y are the input and output of the Preisach nonlinearity, $\rho > 0$. Here $T = \sqrt{2}\pi$, estimate (5) holds with $q = 0$, $b = 1 + \rho$ and quantities (6) equal

$$k = 1, \qquad k_1 = \frac{1}{2}\sqrt{\frac{\sqrt{2}\pi + \sin(\sqrt{2}\pi)}{1 - \cos(\sqrt{2}\pi)}} \approx 0.829.$$

Therefore the first of estimates (8) holds. Assume that the density $h = h(\alpha, \beta)$ of the measure μ is defined by

$$h(\alpha,\beta) = 1/(2r^2) \quad \text{if} \quad -r \le \alpha < \beta \le r; \qquad h(\alpha,\beta) = 0 \quad \text{otherwise.}$$

Then $R_\mu = r$ and the second of estimates (8) has the form

$$\frac{r}{1+\rho} > \sqrt{\frac{2\pi^2 + \sqrt{2}\pi \sin(\sqrt{2}\pi)}{4(1-\cos(\sqrt{2}\pi))}} \approx 1.747. \tag{15}$$

By Proposition 3.1 this estimate implies that equation (14) has a one-parameter continuum of classes $[x_\lambda(t), y_\lambda(t)]$ of periodic solutions of the period $\sqrt{2}\pi$. Furthermore, the Lipschitz estimate (11) holds with $q_1 = 2\rho$ and from

$$h(\alpha,\beta) \le 1/(2r^2) \quad \text{if} \quad 0 < \beta - \alpha \le 2r, \qquad h(\alpha,\beta) = 0 \quad \text{otherwise}$$

it follows that $\nu_\infty = 1/r$, therefore estimate (12) takes the form

$$\frac{r}{\rho} > 4\sqrt{\frac{2\pi^2 + \sqrt{2}\pi \sin(\sqrt{2}\pi)}{4(1-\cos(\sqrt{2}\pi))}} \approx 6.9876. \tag{16}$$

The second part of Proposition 3.1 implies that if estimates (15) and (16) are valid, then the set of all $\sqrt{2}\pi$-periodic solutions of equation (14) is the join of the classes $[x_\lambda(t), y_\lambda(t)]$ $(0 \le \lambda \le 1)$. In fact, these classes are related here by the simple formulas

$$x_\lambda(t) = x_0(t) + 2\rho\delta(\lambda), \qquad y_\lambda(t) = y_0(t) + \delta(\lambda), \qquad \lambda \in [0,1],$$

where the continuous function $\delta = \delta(\lambda)$ strictly increases and $\delta(0) = 0$. Note that estimate (15) implies $r \gtrsim 1.747$.

One can use operator norms different from (6) to conclude that all T-periodic solutions of system (4) constitute a one-parameter set of classes $[x_\lambda(t), y_\lambda(t)]$. For example, those conclusion and estimates (13) are valid if the relation $k_2(q + 2\nu_\infty q_1) < 1$ holds with $k_2 = \|H\|_{C\to C}$. This relation and relation (12) of Proposition 3.1 do not imply each other and give different estimates for the coefficients q and q_1 of the Lipschitz condition (11). Remark that the norm $k = \|H\|_{L^2\to L^2}$ equals the spectral radius of the operator $H = H(T)$ and that if $qk < 1$ (i.e., the first relation of (8) holds) then estimate (5) alone implies that system (4) has at least one T-periodic solution, which may be eventually unique.

3.2 Structure of the set of periodic variable states and outputs

The following proposition describing the set of all possible periodic variable states and outputs of the Preisach nonlinearity for a given periodic input explains why the one-parameter continua of equivalence classes of periodic solutions appear and plays the main role in the proofs of Proposition 3.1 above and further statements of this section on such continua.

Consider a continuous input which is periodic on the semiaxis $t \geq t_0$, i.e. $x(t) \equiv x(t+T)$ $(t \geq t_0)$. For some admissible initial states $\eta_0 = \eta_0(\alpha, \beta)$ the variable state $\eta(t) = (\Gamma[\eta_0]x)(t)$ and the output $y(t) = \Phi(\eta(t))$ of the Preisach nonlinearity are also periodic on the semiaxis $t \geq t_0$. To describe the class of all such variable states and outputs for a given input $x(t)$, let us define the numbers $x_m = \min x(t)$, $x_M = \max x(t)$ and the sets

$$G(x_m, x_M) = \{(\alpha, \beta) : \alpha < x_m \leq x_M < \beta\}, \quad G_c(x_m, x_M) = \Pi \backslash G(x_m, x_M)$$

on the half-plane $\Pi = \{(\alpha, \beta) : \alpha < \beta\}$. From the relay definition it follows that if $(\alpha, \beta) \in G_c(x_m, x_M)$ then the relay output $(\mathcal{R}_{\alpha,\beta}[\cdot]x)(t)$ is periodic on the semiaxis $t \geq t_0$ for exactly one of the two initial relay states 0 and 1, for the other initial state the output is either not periodic or not defined. A unique (for a given $x(t)$) periodic relay output we denote by $\eta_{per}(t, \alpha, \beta)$ and set

$$(\mathcal{J}_\lambda x)(t) = \iint\limits_{G_c(x_m, x_M)} \eta_{per}(t, \alpha, \beta)\, d\mu(\alpha, \beta) + \lambda\, \mu(G(x_m, x_M)), \qquad t \geq t_0, \tag{17}$$

where $0 \leq \lambda \leq 1$. For each λ, this operator is defined on the class of all continuous periodic functions $x : [t_0, \infty) \to \mathbb{R}$, acts in this class and sends a function $x = x(t)$ with a period T to the function $\mathcal{J}_\lambda x$ with the same period.

Denote by $\mathfrak{G}$ the class of all functions $g : \mathbb{R}_+ \to \mathbb{R}$ such that $|g(y_1) - g(y_2)| \leq |y_1 - y_2|$ $(y_1, y_2 \geq 0)$ and define the function

$$\psi(\varepsilon) = \sup_{g = g(y) \in \mathfrak{G}} \mu\,\{(\alpha, \beta) \in \Pi : |\alpha + \beta - g(\beta - \alpha)| \leq \varepsilon\}, \qquad \varepsilon \geq 0, \tag{18}$$

which is nondecreasing, nonnegative and satisfies $\psi(\varepsilon) \to 0$ as $\varepsilon \to 0$, since we assume that the measure μ has a bounded density.

Proposition 3.2. *The following statements hold:*

(i) *For any periodic input $x(t) \equiv x(t+T)$ $(t \geq t_0)$ the formula $y(t) = (\mathcal{J}_\lambda x)(t)$ with λ ranging over the segment $0 \leq \lambda \leq 1$ defines the class of all outputs of the Preisach nonlinearity that are periodic on $t \geq t_0$.*

(ii) *For any periodic input $x(t) \equiv x(t+T)$ $(t \geq t_0)$ the formula*

$$\eta(t, \alpha, \beta) = \begin{cases} \eta_{per}(t, \alpha, \beta) & \text{if either } \beta > \alpha \geq x_m \text{ or } x_M \geq \beta > \alpha \text{ or both,} \\ \eta_*(\alpha, \beta) & \text{if } \alpha < x_m \leq x_M < \beta, \end{cases} \tag{19}$$

where $\eta_*(\alpha, \beta)$ is an arbitrary measurable (w.r.t. the measure μ) function with the values $0, 1$, defines the class of all periodic variable states $\eta(t) = \eta(t, \alpha, \beta)$ of the Preisach nonlinearity. Their periods equal the period of the input, i.e. $\eta(t) = \eta(t+T)$ $(t \geq t_0)$, and the periodic output $y(t) = \Phi(\eta(t))$ for variable state (19) is defined by $y(t) = (\mathcal{J}_\lambda x)(t)$ with

$$\lambda\, \mu(G(x_m, x_M)) = \mu(\{(\alpha, \beta) \in G(x_m, x_M) : \ \eta_*(\alpha, \beta) = 1\}). \tag{20}$$

(iii) *Each variable state $\eta(t) = (\Gamma[\cdot]x)(t)$ $(t \geq t_0)$ equals one of variable states (19) on the semiaxis $t \geq t_0 + T$.*

(iv) *For every pair of periodic inputs $x_j = x_j(t)$ $(t \geq t_0)$ with a period T and every $\lambda \in [0,1]$*

$$\|\mathcal{J}_\lambda x_1 - \mathcal{J}_\lambda x_2\|_{C[t_0,t_0+T]} \leq 2\psi\big(\|x_1 - x_2\|_{C[t_0,t_0+T]}\big). \tag{21}$$

From conclusion (iv) it follows that operator (17) is uniformly continuous in the space $C[t_0, t_0+T]$ for each λ (we identify periodic inputs and outputs with their restrictions to the period). If relations (9) and (10) hold, then $|\psi(\varepsilon)| \leq \nu_\infty \varepsilon$, which implies the global Lipschitz estimate

$$\|\mathcal{J}_\lambda x_1 - \mathcal{J}_\lambda x_2\|_{C[t_0,t_0+T]} \leq 2\nu_\infty \|x_1 - x_2\|_{C[t_0,t_0+T]}. \tag{22}$$

By Proposition 3.2 for every class $[x(t), y(t)]$ of T-periodic solutions of system (4) the relation $y(t) = (\mathcal{J}_\lambda x)(t)$ with some $\lambda \in [0,1]$ is valid. Under the conditions of Proposition 3.1 such classes $[x(t), (\mathcal{J}_\lambda x)(t)]$ exist for all $0 \leq \lambda \leq 1$ and are pairwise disjoint. Actually, dependence of the periodic problem on the parameter λ rather than any special behavior of the function $f(t,x,y)$ is the reason for existence of a one-parameter continuum of classes of periodic solutions. Simple assumptions like estimates of the measure μ density and Lipschitz continuity of the nonlinearity $f(t,x,y)$ used in Proposition 3.1 may guarantee that the set of all classes $[x(t), y(t)]$ of T-periodic solutions is the image of the segment $0 \leq \lambda \leq 1$ under a one-to-one continuous mapping. The easiest way to construct examples of periodic solution sets with any number of connected components is to consider open loop systems consisting of the differential equation $L(d/dt)x = f(t,x)$ without y and the relations $\eta(t) = (\Gamma[\eta(t_0)]x)(t)$, $y(t) = \Phi(x(t))$ that define variable states and outputs of the hysteresis nonlinearity. Here every periodic solution $x_j(t)$ of the differential equation (if any exists) defines the one-parameter set of classes $[x_j(t), (\mathcal{J}_\lambda x_j)(t)]$ $(0 \leq \lambda \leq 1)$ of periodic solutions for the system.

3.3 Classes of periodic solutions

Now let us consider individual T-periodic solutions $(x(t), \eta(t))$ of system (4) from any class $[x(t), y(t)]$. Simple relations between their second components $\eta(t)$ follow from Proposition 3.2. Recall that by definition $(x(t), \eta(t))$ belongs to the class $[x(t), y(t)]$ if and only if $\eta(t) = (\Gamma[\eta(t_0)]x)(t)$, $y(t) = \Phi(\eta(t))$ and $\eta(t) = \eta(t+T)$ for all $t \geq t_0$.

Statements (i), (ii) of Proposition 3.2 imply that for every periodic solution $(x(t), \eta(t)) \in [x(t), y(t)]$ the component $\eta(t) = \eta(t, \alpha, \beta)$ coincides on the domain $(\alpha, \beta) \in G_c(x_m, x_M)$ for all $t \geq t_0$ with the function $\eta_{per}(t, \alpha, \beta)$ uniquely defined by the input $x(t)$. Thus, the components $\eta(t) = \eta(t, \alpha, \beta)$ of the representatives $(x(t), \eta(t))$ of the class $[x(t), y(t)]$ differ from each other on the domain $G(x_m, x_M)$ only, where these components do not change with time, which means that $\eta(t, \alpha, \beta) \equiv \eta_*(\alpha, \beta)$ on $G(x_m, x_M)$ for each individual solution $(x(t), \eta(t))$. If $\mu(G(x_m, x_M)) > 0$, then one obtains all representatives of the class $[x(t), y(t)]$ inserting here characteristic functions $\eta_*(\alpha, \beta)$ of all measurable subsets of $G(x_m, x_M)$ that satisfy a unique restriction, namely (20) should hold with the $\lambda \in [0,1]$ defined by the equality

$$y(t) = \iint\limits_{G_c(x_m, x_M)} \eta_{per}(t, \alpha, \beta)\, d\mu(\alpha, \beta) + \lambda\, \mu(G(x_m, x_M)),$$

which is nothing else as $y = \mathcal{J}_\lambda x$. Since there is a continuum of such functions $\eta_* = \eta_*(\alpha, \beta)$, each class $[x(t), y(t)]$ consists of the continuum of periodic solutions $(x(t), \eta(t))$ if $\mu(G(x_m, x_M)) > 0$. For example, under the conditions of Proposition 3.1 system (4) has a one-parameter continuum of classes $[x_\lambda(t), y_\lambda(t)]$ of T-periodic solutions and each of those classes is a continuum parameterized by the functional parameter $\eta_* = \eta_*(\alpha, \beta)$.

Formula (17) implies that $\mathcal{J}_\lambda x = \mathcal{J}_0 x$ for all λ and the Preisach nonlinearity assigns a unique periodic output to a given periodic input $x = x(t)$ if and only if $\mu(G(x_m, x_M)) = 0$. In contrast to the situation considered above, it means that the class $[x(t), y(t)]$ of periodic solutions is not included in the one-parameter set of such classes $[x_\lambda(t), y_\lambda(t)]$ if $\mu(G(x_m, x_M)) = 0$. Moreover, formula (20) implies that the class $[x(t), y(t)]$ contains a unique element $(x(t), \eta_{per}(t))$ in this case, since we consider functions $\eta = \eta(\alpha, \beta)$ equal almost everywhere with respect to the measure μ as the same state of the Preisach nonlinearity. Thus, periodic solutions $(x(t), \eta(t))$ such that $\mu(G(x_m, x_M)) = 0$ are basically isolated (if the nonlinearity f itself is not as complicated that it 'produces' nonisolated solutions). For example, the relation $\mu(G(x_m, x_M)) = 0$ holds if $x_m < -R_\mu$ and $x_M > R_\mu$.

3.4 Application of guiding functions

It is instructive to modify standard statements that guarantee the existence of at least one T-periodic solution for systems with the Preisach nonlinearity like that using dissipativity properties of the system, two-sided and one-sided estimates of nonlinearities, the Schauder principle and degree theory (see, for example, [1, 10, 11]) in order to formulate sufficient conditions for the existence of the continuum of such solutions. Under the hypotheses of such statements obtained for different types of systems the periodic solutions admit an *a priori* estimate $\|z(t)\|_C \le \rho$ of the z-component. This *a priori* estimate implies the similar estimate $\|x(t)\|_C \le \rho_0$ for the corresponding periodic inputs $x = \langle c, z\rangle$ of the Preisach nonlinearity, for example with $\rho_0 = \rho|c|$. If we assume that $\rho_0 < R_\mu$ in addition to the conditions of those propositions, then the set of periodic solutions will include the one-parameter continuum of disjoint classes $[z_\lambda(t), y_\lambda(t)]$, $0 \le \lambda \le 1$. Here and henceforth we denote by $[z(t), y(t)]$ the class of all periodic solutions $(z(t), \eta(t))$ satisfying $y(t) = \Phi(\eta(t))$ for systems (3), similarly to the notation $[x(t), y(t)]$ used for system (4).

Let us consider in more detail how this approach works in an application based on the method of guiding functions.

A scalar valued continuously differentiable function $V(z)$ is called a guiding function for the ODE system $z' = F(t, z)$ with $z \in \mathbb{R}^d$ if the gradient $\bigtriangledown V(z)$ of this function satisfies for some $\rho > 0$

$$\langle \bigtriangledown V(z), F(t, z)\rangle < 0 \quad \text{if} \quad |z| > \rho,\ t \in \mathbb{R}. \tag{23}$$

Let the continuous function $F(t, z)$ be T-periodic with respect to t. The simplest fact illustrating the role of guiding functions in the study periodic problems is that estimate (23) and the relation $|V(z)| \to \infty$ as $|z| \to \infty$ imply the existence of at least one T-periodic solution of system $z' = F(t, z)$. An important idea that leads to less restrictive existence conditions is to use several guiding functions. From (23) it follows that outside some ball in $\mathbb{R}^d$ the vector fields $\bigtriangledown V(z)$ and $-F(t_0, z)$ are linearly homotopic and do not have zeros for every t_0. Therefore if $V_j(z)$ with $j = 1, \dots, k$ are guiding functions for the system $z' = F(t,z)$, then the Kronecker index at infinity is defined for each gradient vector field $\bigtriangledown V_j(z)$ and all these indices coincide. The common value of these Kronecker indices is called the topological index of the finite set of guiding functions $V_1(z), \ldots, V_k(z)$. Such a set is called complete if

$$|V_1(z)| + \cdots + |V_k(z)| \to \infty \quad \text{as} \quad |z| \to \infty. \tag{24}$$

The most classical theorem is that if there exists a complete finite set of guiding functions with a nonzero topological index, then the system has at least one T-periodic solution (see, e.g. [14]).

If one uses a proper straightforward modification of the guiding function definition, then the same statement holds for systems with hysteresis nonlinearities. Namely, consider the system

$$\begin{array}{rcll} dz/dt & = & F(t, z(t), y(t)), & \qquad z \in \mathbb{R}^d, \\ y(t) & = & \Phi(\eta(t)), & \\ \eta(t) & = & (\Gamma[\eta(t_0)]x)(t), & \\ x(t) & = & \langle c, z(t) \rangle & \end{array} \tag{25}$$

with the Preisach nonlinearity, where $F(t,z,y) \equiv F(t+T,z,y)$ is a continuous function. Assume that for $j = 1, \ldots, k$

$$\langle \bigtriangledown V_j(z), F(t,z,y) \rangle < 0 \quad \text{if} \quad |z| > \rho_j, \; 0 \le y \le 1, \; t \in \mathbb{R}, \tag{26}$$

relation (24) holds and the common value of the Kronecker index at infinity of the gradient vector fields $\bigtriangledown V_j(z)$ is nonzero. Then system (25) has at least one T-periodic solution (see, e.g. [10]).

Combining this statement with Proposition 3.2 according to our basic approach, we arrive at the following conditions for the existence of a continuum of periodic solutions. We use the notations m_j and M_j for the smallest and the largest values of the guiding function $V_j(z)$ on the ball $|z| \le \rho_j$ and the notation

$$G_j = \{z \in \mathbb{R}^d : m_j \le V_j(z) \le M_j\}.$$

Proposition 3.3. *Let scalar valued continuously differentiable functions $V_1(z)$, …, $V_k(z)$ satisfy (24) and (26) and the Kronecker index at infinity of the gradient vector fields $\bigtriangledown V_j(z)$ in $\mathbb{R}^d$ be nonzero. Let a ball $\{z \in \mathbb{R}^\ell : |z| \le \rho_*\}$ contain the set $G_1 \cap \cdots \cap G_k$ and let $R_\mu > \rho_*|c|$. Then the set of T-periodic solutions of system (25) includes a one-parameter continuum of disjoint classes $[z_\lambda(t), y_\lambda(t)]$ $(0 \le \lambda \le 1)$.*

Remark that relation (24) implies that the set $G_1 \cap \cdots \cap G_k$ is bounded. The conditions of Proposition 3.3 imply the inclusion $z(t) \in G_1 \cap \cdots \cap G_k$, $t \ge t_0$ for the z-component of each periodic solution of system (25) and therefore the *a priori* estimate $\|z(t)\|_C \le \rho_*$ for all such solutions.

If $|V(z)| \to \infty$ as $|z| \to \infty$ (i.e., $k = 1$) then the Kronecker index at infinity of the vector field $\bigtriangledown V(z)$ is either 1 or -1. There are many other simple conditions, which guarantee that the topological index of the complete set of guiding functions $V_1(z)$, …,

$V_k(z)$ is different from zero. For example, if one of these guiding functions is even, then the topological index is odd.

The proofs of Propositions 3.1 and 3.3 are based on the same idea. Using operator (17), one constructs an operator equation depending on λ, whose solutions define the classes $[x(t), y(t)]$ of T-periodic solutions of system (4) (the classes $[z(t), y(t)]$ for system (25), respectively), then proves the existence of a solution for each $\lambda \in [0,1]$ and shows that the solutions are different for different λ. For example, one can use the operator equation

$$\xi(t) = f(t, (H\xi)(t), (\mathcal{J}_\lambda H\xi)(t)) \tag{27}$$

in the space $L^2[0,T]$ to prove Proposition 3.1. From the definition of the linear operator $H = H(T)$ and Proposition 3.2 it follows that each solution $\xi = \xi(t)$ of (27) defines the class $[x(t), y(t)]$ of T-periodic solutions of system (4) with $x(t) = (H\xi)(t)$, $y(t) = (\mathcal{J}_\lambda x)(t)$. The existence of a solution ξ_λ of (27) for each $\lambda \in [0,1]$ follows from (5) and the first of estimates (8) by the Schauder principle. The second of estimates (8) and the *a priori* estimate $\|x\|_C \leq k_1 b\sqrt{T}/(1-qk)$ of all T-periodic solutions of (4) imply that all the classes $[x_\lambda(t), y_\lambda(t)]$ of such solutions defined by the solutions ξ_λ of (27) are disjoint. If the additional assumptions of the second part of Proposition 3.1 hold, then equation (27) satisfies the conditions of the contracting mapping principle and therefore has a unique solution ξ_λ for each $\lambda \in [0,1]$; moreover, the map $\lambda \to \xi_\lambda$ is Lipschitz continuous.

For the proof of Proposition 3.3 one can use the operator equation

$$z(t) = z(0) + \int_0^t F(s, z(s), (\mathcal{J}_\lambda x)(s))\, ds \qquad \text{with} \qquad x(t) = \langle c, z(t)\rangle$$

in the space $C_T([0,T];\mathbb{R}^d)$ of continuous functions $z = z(t)$ satisfying $z(0) = z(T)$ with the uniform norm. The proof of the existence of a solution z_λ of this operator equation for each $\lambda \in [0,1]$ follows basically standard lines of the guiding functions method. For details of the proofs of Propositions 3.1 and 3.3 and for the proof of Proposition 3.2 we refer to [22].

3.5 Autonomous systems

Our approach works similarly in periodic problems for autonomous systems. Yet, here it leads to statements on existence of continua of cycles. By the same reason as in problems on forced periodic oscillations, the continua of cycles have basically the similar type of one-parametric structure as described above if the *a priori* estimate $\|x\|_C < R_\mu$ holds for the cycles. We present one particular statement.

Consider the system

$$\begin{aligned} L\,(d/dt)\,x &= f(x, y(t)), \\ y(t) &= \Phi(\eta(t)), \\ \eta(t) &= (\Gamma[\eta(t_0)]x)(t) \end{aligned} \tag{28}$$

with the Preisach nonlinearity. Its solutions of the form $(x(t), \eta(t)) \equiv (x_0, \eta_0)$ are called stationary, for such solutions also $y(t) \equiv y_0$ with $y_0 = \Phi(\eta_0)$. We are interested in conditions that guarantee the existence of nonstationary periodic solutions $(x(t), \eta(t)) \equiv$

$(x(t+T), \eta(t+T))$ of some period $T>0$. Since system (28) is autonomous, those periods are *a priori* unknown. Furthermore, one should take into account that the set of nonstationary periodic solutions contains together with any solution $(x(t), \eta(t))$ all its time shifts $(x_\tau(t), \eta_\tau(t)) = (x(t+\tau), \eta(t+\tau))$ with $0<\tau<T$. Passing to the equivalence classes $[x(t), y(t)] = \{(x(t), \eta(t)) : \ y(t) = \Phi(\eta(t))\}$ of periodic solutions of system (28), we see that the time shifts of solutions from a class $[x(t), y(t)]$ by the time τ form the class $[x(t+\tau), y(t+\tau)]$. In order to simplify the description of shift invariant structures of periodic solution sets, we introduce closed curves $\gamma = \{(x(t), y(t)) \in \mathbb{R}^2 : \ 0 \le t < T\}$ on the input-output plane of the Preisach nonlinearity and say that the curve γ represents the class of all periodic solutions from the join $\bigcup_{0\le\tau<T}[x(t+\tau), y(t+\tau)]$. The curve γ is called degenerate if $(x(t), y(t)) \equiv (x_0, y_0)$ and nondegenerate otherwise.

Let us consider stationary solutions in more detail. For $0 \le \lambda \le 1$, $\rho \in \mathbb{R}$ set

$$\phi_\lambda(\rho) = \mu(\{(\alpha,\beta) : \ \alpha<\beta<\rho\}) + \lambda\, \mu(\{(\alpha,\beta) : \ \alpha<\rho<\beta\}). \tag{29}$$

This function is continuous with respect to the set of arguments λ, ρ. As we know, the Preisach nonlinearity assigns stationary outputs to stationary inputs for any admissible initial states. Moreover, from conclusion (i) of Proposition 3.2 it follows that the values of all the outputs $y(t) \equiv y_0$ corresponding to the input $x(t) \equiv x_0$ form the segment $\{y_0 = \phi_\lambda(x_0) : 0 \le \lambda \le 1\} = [\phi_0(x_0), \phi_1(x_0)]$. Therefore the point (x_0, y_0) of the plane (x, y) represents a nonempty class of stationary solutions $(x(t), \eta(t)) \equiv (x_0, \eta_0)$ of system (28) with $y_0 = \Phi(\eta_0)$ if and only if $(x_0, y_0) = (\rho, \phi_\lambda(\rho))$ for some $\lambda \in [0,1]$ where ρ is a solution of the equation

$$L(0)\rho = f(\rho, \phi_\lambda(\rho)). \tag{30}$$

If $L(0) \ne 0$ and the continuous function $f(x,y)$ is bounded, then equation (30) has at least one solution $\rho = \rho^*(\lambda)$ for each $\lambda \in [0,1]$ with $|\rho^*(\lambda)| \le \rho_1 = \sup|f(x,y)||L(0)|^{-1}$ and therefore system (28) has stationary solutions. In particular, if there is a unique solution $\rho = \rho^*(\lambda)$ of (30) for each λ and $\mu(\{(\alpha,\beta) : \alpha<\rho<\beta\}) > 0$ whenever $\rho \in [-\rho_1, \rho_1]$, then the points representing all the classes of stationary solutions of system (28) on the plane (x,y) form the continuous curve $\Gamma = \{(x_0, y_0) = (\rho^*(\lambda), \phi_\lambda(\rho^*(\lambda))) : \ 0 \le \lambda \le 1\}$ without self-intersections.

Now we are ready to formulate a statement on existence of nonstationary solutions. Let $f(x,y)$ be a bounded continuously differentiable function satisfying the global Lipschitz condition

$$|f(x_1,y_1) - f(x_2,y_2)| \le q|x_1-x_2| + q_1|y_1-y_2|, \qquad x_j, y_j \in \mathbb{R}. \tag{31}$$

We assume that relations (9), (10) are satisfied for the Preisach nonlinearity.

It is easy to see that if there are no zeros of the polynomial $L(p)$ on the imaginary axis and the coefficients q, q_1 in (31) are small enough, then system (28) does not have nonstationary periodic solutions.

Let us assume that the polynomial $L(p)$ has a pair of the conjugate imaginary zeros $\pm w_0 i$ and the so-called nonresonance condition

$$L(nw_0 i) \ne 0 \qquad \text{for} \qquad n = 0, \pm2, \pm3, \ldots \tag{32}$$

holds. Define on the positive semiaxis $r > 0$ the function

$$d(r) = \int_0^\pi \sin t \ f(r \sin t, 1) \, dt - \int_0^\pi \sin t \ f(-r \sin t, 0) \, dt. \tag{33}$$

Proposition 3.4. *Let the polynomials $L(wi)$ and $\Im m L(wi)$ of the variable w have a pair of common zeros $\pm w_0$ ($w_0 > 0$) of the same odd multiplicity. Let nonresonance condition (32) hold. Suppose that estimates (9) and (10) are valid, the bounded continuously differentiable function $f(x, y)$ satisfies the Lipschitz condition (31) and function (33) satisfies $|d(r)| \geq c > 0$ for $r \geq r_*$. Then there are numbers $q_0 > 0$, $w_1 \in (w_0/2, w_0)$ and $w_2 > w_0$, such that if $q + \nu_\infty q_1 < q_0$, then the following statements are valid.*

(i) *The set of all stationary solutions of system (28) is represented on the plane (x, y) by a continuous curve Γ, which may eventually degenerate to a unique point.*

(ii) *If there is a point (x^*, y^*) of Γ such that*

$$d(r_*) \frac{\partial f}{\partial x}(x^*, y^*) < 0, \tag{34}$$

then system (28) has at least one class of nonstationary periodic solutions represented by a nondegenerate closed curve $\gamma = \{(x, y) \in \mathbb{R}^2 : \ x = x(t), \ y = y(t), \ t \in [0, T)\}$ with a period satisfying $2\pi/w_2 \leq T \leq 2\pi/w_1$.

(iii) *If estimate (34) is valid for all points of the curve Γ, then there exist numbers $r_1, r_0 > 0$ such that the x-components of all periodic solutions of system (28) with periods $2\pi/w_2 \leq T \leq 2\pi/w_1$ satisfy*

$$\|x(t) - \bar{x}\|_{L_2[0,T]} \geq r_1, \qquad \|x(t)\|_{C[0,T]} \leq r_0 \tag{35}$$

where $\bar{x}$ denotes the average value of the function $x(t)$ over its period. If in addition $R_\mu > r_0$, then the set of all nonstationary periodic solutions of system (28) includes a one-parameter continuum of disjoint classes represented on the plane (x, y) by nondegenerate closed curves γ_λ where λ ranges over the segment $0 \leq \lambda \leq 1$, those curves are different for different λ and the periods of all periodic solutions represented by them satisfy $2\pi/w_2 \leq T \leq 2\pi/w_1$.

By conclusion (iii), the x-components of all periodic solutions represented by the curves γ_λ satisfy estimates (35) and consequently are separated from the one-dimensional subspace of constant functions and from infinity in L^2, C and other functional spaces, which implies particularly that such periodic solutions are separated from stationary ones. Additional assumptions about the function f and the measure μ of the Preisach hysteresis nonlinearity guarantee that the one-parameter set of curves γ_λ represents all periodic solutions of system (28) and these curves depend continuously on λ.

Formulas for the numbers q_0, w_1, w_2 (they are defined by the polynomial $L(p)$) and estimates (35) can be given explicitly. We do not present cumbersome explicit expressions[1], but consider as an example the equation

$$x''' + x'' + x' + x = f(x, y(t)). \tag{36}$$

[1] Further details, statements and proofs can be found in [9].

Here the polynomial $L(p) = p^3 + p^2 + p + 1$ has the pair of simple roots $\pm i$,

$$L(wi) = (wi + 1)(1 - w^2), \qquad \Im m L(wi) = w(1 - w^2), \qquad w_0 = 1$$

and $1/2 < w_1 < 1 < w_2$. For example, if the Lipschitz estimate (31) holds with $q = 0.305$ and $\nu_\infty q_1 \leq 0.295$, the equation $\rho = f(\rho, \phi_\lambda(\rho))$ has a unique solution $\rho = \rho^*_\lambda$ for some $\lambda \in [0, 1]$ and for all sufficiently large r

$$d(r) \frac{\partial f}{\partial x}\big(\rho^*_\lambda, \phi_\lambda(\rho^*_\lambda)\big) < 0, \qquad\qquad |d(r)| \geq c > 0,$$

then equation (36) with the Preisach nonlinearity has nonstationary T-periodic solutions with $5.575 \leq T \leq 8.913$.

The so-called describing function (33) in particular cases has simple form. For example, let the first equation of system (28) be

$$L\Big(\frac{d}{dt}\Big)x = g(x) + q_1 y(t).$$

Then

$$d(r) = 2q_1 + 4 \int_0^{\pi/2} \sin t \; g_{odd}(r \sin t)\, dt$$

where $g_{odd}(x) = (g(x) - g(-x))/2$ is the odd part of g. If $g_{odd}(x) \to g_0$ as $x \to +\infty$, then $d(r) \to d_0 = 2q_1 + 4g_0$ as $r \to +\infty$. If $d_0 \neq 0$, then the condition $|d(r)| \geq c > 0$ of Proposition 3.4 is satisfied for all sufficiently large r.

4 Hopf bifurcations

4.1 Preliminaries

Let some autonomous system depending on a scalar parameter b have an equilibrium $z_*(b)$ for each parameter value from some interval (b_-, b_+) and let those equilibria form a smooth curve in the phase space Z of the system. We say that Hopf bifurcation occurs and a parameter value b_0 is a Hopf bifurcation point if for any sufficiently small $r > 0$ one can find a parameter value $b_r \in (b_0 - r, b_0 + r)$ such that the system with $b = b_r$ has a cycle which lies in the ball of radius r centered at the equilibrium $z_*(b_r)$ (this type of a weak definition of Hopf bifurcation was introduced in [5], where nonsmooth problems were studied for the first time). Freely speaking, we consider bifurcation of a small cycle from an equilibrium point of the autonomous system.

The simplest example is the equation

$$x'' + bx' + x = f(x, x', b) \tag{37}$$

with f satisfying $f(z_1, z_2, b) = o(|z_1| + |z_2|)$ as $z_1, z_2 \to 0$ uniformly with respect to b. Here the origin is an equilibrium of the equivalent planar first order system for every b;

the value $b = 0$ is a unique Hopf bifurcation point. Figure 1 presents a typical structure of the set of cycles in the product of the phase space $\mathbb{R}^2$ of the equivalent system and the axis of parameter values b in case of a sufficiently smooth f. The cycles form a smooth two-dimensional surface Σ (a 'cup') passing through the origin (note that the picture is local both in b and $z = (z_1, z_2)$, i.e. we consider cycles lying in a small vicinity of the zero equilibrium and b small in absolute value only). Appropriate non-degeneracy conditions guarantee that Σ is tangent to the plane $b = 0$ and lies on one side of it. In Fig. 1, Σ lies to the right of this plane, which means that the system has no small cycles for $b < 0$ and has such cycles for small $b > 0$; for $b = b'$ the cycle is a cross section of Σ with the plane $b = b'$.

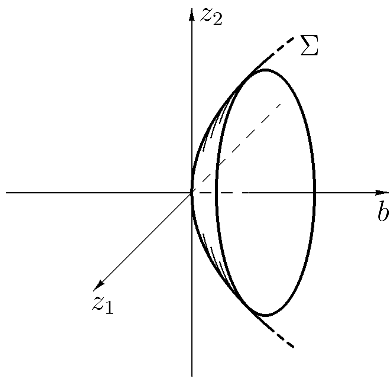

FIGURE 1: Hopf bifurcation

Under the conditions of the classical Hopf Theorem (see, e.g. [16]) a similar picture is observed for autonomous systems

$$dz/dt = F(z, b), \qquad z \in \mathbb{R}^d, \tag{38}$$

of ODE of any dimension in some vicinity of the equilibrium $z_*(b)$. Let us recall more precisely the algorithm to determine bifurcation points by linearization of (38). Since the change of variables $z \mapsto z + z_*(b)$ places the equilibrium at the origin, let us assume without loss of generality that $z_*(b) \equiv 0$, i.e. system (38) has the zero equilibrium for all b. In this situation, we call b_0 a Hopf bifurcation point for system (38) with a limit period $T_0 > 0$ if for any sufficiently small $r > 0$ the system has a periodic solution $z_r(t)$ of a minimal period $T_r > 0$ for some $b = b_r$ and

$$\max_{t \in \mathbb{R}} |z_r(t)| \to 0, \quad b_r \to b_0, \quad T_r \to T_0 \quad \text{as} \quad r \to 0.$$

Remark that periods T_r are not known *a priori* and that we use an auxiliary parameter r. The reason why the parameter b is not good for parametrization of the set of small cycles becomes clear if one considers linear equations, for example $x'' + bx' + x = 0$, i.e. equation (37) with $f \equiv 0$. Here all the cycles exist for one parameter value $b_0 = 0$, which is a unique Hopf bifurcation point (the surface Σ in Fig. 1 becomes a piece of the plane $b = 0$).

Suppose F is differentiable with respect to z at the point $z = 0$ uniformly with respect to the parameter b, i.e. $F(z, b) = A(b)z + g(z, b)$ where $\sup_b |g(z, b)|/|z| \to 0$ as $|z| \to 0$, and let the matrix-valued function $A(b)$ and the vector-valued function $g(z,b)$ be continuous. Assume that the matrix $A(b_0)$ has a pair of simple imaginary eigenvalues[2] $\pm iw_0$ ($w_0 > 0$) for some b_0, which implies that $A(b)$ has a unique pair of simple eigenvalues $\sigma(b) \pm iw(b)$ close to $\pm iw_0$ for b close to b_0. Let the derivative $\sigma'(b_0)$ of the real part of these eigenvalues at the point $b = b_0$ exist and the so-call transversality condition

$$\sigma'(b_0) \neq 0 \tag{39}$$

and nonresonance condition

$$iw_0 k \notin \mathrm{Sp}(A(b_0)) \quad \text{for all} \quad k \in \mathbb{Z},\ k \neq \pm 1 \tag{40}$$

hold, where $\mathrm{Sp}(A)$ is the spectrum of A. Then b_0 is a Hopf bifurcation point for system (38) with the limit period $2\pi/w_0$ (for example, this follows from the results of [5]). If in addition F is sufficiently smooth, then according to the Hopf Theorem the small cycles form a two-dimensional smooth surface in the product of the phase space $\mathbb{R}^d$ of (38) and the axis b. Moreover, information about the Taylor expansion of F at the origin allows to study further important problems including stability of the cycles, but we do not consider those problems here.

4.2 Bifurcation in systems with Preisach hysteresis

Consider the autonomous system

$$\begin{array}{rcll} dz/dt & = & F(z(t), y(t), b), & \qquad z \in \mathbb{R}^d, \\ y(t) & = & \Phi(\eta(t)), & \\ \eta(t) & = & (\Gamma[\eta(t_0)]x)(t), & \\ x(t) & = & \langle c, z(t) \rangle & \end{array} \tag{41}$$

with the Preisach nonlinearity, where the first equation depends on the scalar parameter b; the function F is supposed to be continuous. We use the notation similar to that introduced above in subsection 3.5. Solutions of the form $(z(t), \eta(t)) \equiv (z_0, \eta_0)$ are called stationary, the point (z_0, η_0) in the product $\mathbb{R}^d \times \mathfrak{E}_*$ is called an equilibrium of the system. By a cycle we mean the trajectory of any non-stationary periodic solution $(z(t), \eta(t)) \equiv (z(t+T), \eta(t+T))$ in $\mathbb{R}^d \times \mathfrak{E}_*$, with any $T > 0$; actually, the cycle is a common trajectory for all the time shifted periodic solutions $(z_\tau(t), \eta_\tau(t)) = (z(t+\tau), \eta(t+\tau))$, $0 \leq \tau < T$.

By definition, the pair (z_0, η_0) is an equilibrium for some b if and only if $F(z_0, y_0, b) = 0$ with $y_0 = \Phi(\eta_0)$ and the state $\eta_0 = \eta_0(\alpha, \beta)$ is admissible for the stationary input $x(t) \equiv \langle c, z_0 \rangle$ of the Preisach nonlinearity, which means that $\eta_0(\alpha, \beta) = 1$ whenever $\beta < \langle c, z_0 \rangle$ and $\eta_0(\alpha, \beta) = 0$ whenever $\alpha > \langle c, z_0 \rangle$. Since the equation $y_0 = \Phi(\eta_0)$ with a given y_0 has a solution η_0 in this class of admissible states if and only if

$$\mu(\{(\alpha, \beta) : \alpha < \beta < \langle c, z_0 \rangle\}) \leq y_0 \leq \mu(\{(\alpha, \beta) : \alpha < \beta, \alpha < \langle c, z_0 \rangle\}), \tag{42}$$

[2] The existence of imaginary eigenvalues of $A(b_0)$ is a necessary condition for b_0 to be a Hopf bifurcation point.

we conclude that system (41) has an equilibrium if and only if the equation $F(z_0, y_0, b) = 0$ has a solution (z_0, y_0) satisfying (42). Similarly to the definition for ODE above, we call b_0 a Hopf bifurcation point for system (41) with a limit period T_0 if for any sufficiently small $r > 0$ the system has a nonstationary periodic solution $(z_r(t), \eta_r(t))$ with a period $T_r > 0$ for some $b = b_r$ and

$$\max_t |z_r(t) - z_0| \to 0, \quad \max_t \rho_1(\eta_r(t), \eta_0) \to 0, \quad b_r \to b_0, \quad T_r \to T_0 \quad \text{as} \quad r \to 0,$$

where (z_0, η_0) is an equilibrium for $b = b_0$ and $\rho_1(\cdot, \cdot)$ is the metric (2) in the state space $\mathfrak{E}_*$ of the Preisach nonlinearity.

To formulate the simplest statement on Hopf bifurcation for system (41), suppose that there are z_0, y_0 satisfying (42) such that

$$F(z_0, y_0, b) = 0 \quad \text{for all} \quad b.$$

Therefore system (41) has an equilibrium at the same point $(z_0, \eta_0) \in \mathbb{R}^d \times \mathfrak{E}_*$ for all b. Assume that the function F can be linearized at the point (z_0, y_0), more precisely

$$F(z, y, b) = A(b)(z - z_0) + a(b)(y - y_0) + g(z, y, b) \tag{43}$$

with the matrix $A = A(b)$ and the vector $a = a(b)$ depending continuously on b and

$$\lim_{z \to z_0,\ y \to y_0} \frac{\max_{|b-b_0| \le R} |g(z, y, b)|}{|z - z_0| + |y - y_0|} = 0 \tag{44}$$

for some $R > 0$. In this situation, Hopf bifurcation points for system (41) are determined by the matrix $A(b)$ in the same way as for ODE.

Proposition 4.1. *Let relations (43), (44) be valid and strict estimates*

$$\mu(\{(\alpha, \beta) : \alpha < \beta < \langle c, z_0 \rangle\}) < y_0 < \mu(\{(\alpha, \beta) : \alpha < \beta, \alpha < \langle c, z_0 \rangle\}) \tag{45}$$

hold in place of (42). Let the measure μ have a bounded density with respect to the Lebesgue measure. Suppose that the matrix $A(b_0)$ has a pair of simple eigenvalues $\pm i w_0$, the transversality condition (39) and the nonresonance condition (40) hold. Then b_0 is a Hopf bifurcation point for system (41) with a limit period $2\pi / w_0$.

As an example, consider the equation

$$x'' + bx' + x = 2y(t) - 1$$

where $x(t)$ and $y(t)$ are the input and output of the Preisach nonlinearity. If

$$\mu(\{(\alpha, \beta) : \alpha < \beta < 0\}) < 1/2 < \mu(\{(\alpha, \beta) : \alpha < \beta,\ \alpha < 0\})$$

then $b = 0$ is a Hopf bifurcation point for this equation with the limit period 2π. Here $x \equiv x' \equiv 0$, $\eta \equiv \eta_0$ is a stationary solution for all b whenever η_0 is an admissible initial state for the zero input and $\Phi(\eta_0) = 1/2$.

4.3 Bifurcations from infinity

A similar approach to find bifurcation points works in problems on Hopf bifurcation from infinity (bifurcation of large cycles).

Let us call b_0 an *asymptotic* Hopf bifurcation point for system (41) with a limit period T_0 if for any sufficiently large r the system has a nonstationary periodic solution $(z_r(t), \eta_r(t))$ with a period $T_r > 0$ for some $b = b_r$ and

$$\max_{t,\tau} |z_r(t) - z_r(\tau)| \to \infty, \quad b_r \to b_0, \quad T_r \to T_0 \quad \text{as} \quad r \to \infty. \tag{46}$$

In other words, the system has cycles with arbitrarily large amplitudes of the z-component and periods close to T_0 for some parameter values arbitrarily close to b_0.

Suppose that the function F can be represented as

$$F(z, y, b) = A(b)z + g(z, y, b), \tag{47}$$

where

$$\lim_{|z|\to\infty} \max_{|b-b_0|\le R,\ |y|\le R_1} \frac{|g(z, y, b)|}{|z|} = 0 \tag{48}$$

for some $R > 0$ and every $R_1 > 0$. Then the derivative $A(b)$ of F with respect to z at infinity plays the same role in problems on Hopf bifurcation of large cycles as the derivative at the equilibrium point in local problems above.

Proposition 4.2. *Suppose that relations (47), (48) hold, the matrix $A(b_0)$ has a pair of simple eigenvalues $\pm iw_0$, the transversality condition (39) and the nonresonance condition (40) are valid. Then b_0 is an asymptotic Hopf bifurcation point for system (41) with a limit period $2\pi/w_0$.*

This proposition is a consequence of a more general statement of [6]. Proposition 4.1 and some further statements on Hopf bifurcation from an equilibrium can be found in [13].

4.4 Bifurcation with respect to internal parameter

As in section 3, let $[z(t), y(t)]$ denote the class of periodic solutions $(z(t), \eta(t)) \equiv (z(t + T), \eta(t + T))$ $(t \ge t_0)$ such that $y(t) \equiv \Phi(\eta(t))$ and let us say that the closed curve $\gamma = \{(z(t), y(t)) : 0 \le t < T\}$ in the space $\mathbb{R}^{d+1}$ of pairs (z, y) represents the periodic solutions of the join $\bigcup_{0\le\tau<T}[z_\tau(t), y_\tau(t)]$ and the cycles defined by these solutions.

Consider the autonomous system

$$\begin{array}{rcll} dz/dt &=& F(z(t), y(t)), & \qquad z \in \mathbb{R}^d, \\ y(t) &=& \Phi(\eta(t)), & \\ \eta(t) &=& (\Gamma[\eta(t_0)]x)(t), & \\ x(t) &=& \langle c, z(t)\rangle & \end{array} \tag{49}$$

that does not depend explicitly on parameters. Nevertheless, as we already know from the previous section, problems on cycles of this system contain the parameter λ, which we call ‘internal’. This results in the fact that each closed curve $\gamma \subset \mathbb{R}^{d+1}$ representing a class of cycles (if any exists) such that the amplitude of their z-component is smaller than the characteristic size of the Preisach nonlinearity is generically included in the one-parameter continuum such curves γ_λ; a typical example is presented by conclusion (iii) of Proposition 3.4.

It turns out that the curves γ representing small cycles of system (49) can accumulate at its equilibrium and form the structure similar to that observed for Hopf bifurcation problems for ODE with a scalar parameter. Moreover, sufficient conditions for the existence of those structure for system (49) with hysteresis are close to the classical assumptions of the Hopf Theorem. Let us consider one example.

Let the equation $F(z, \lambda) = 0$ define a continuous branch $z = \varphi(\lambda)$ of the implicit function on some open interval Λ^*. Denote by Γ the curve $(z, y) = (\varphi(\lambda), \lambda)$, $\lambda \in \Lambda^*$ in $\mathbb{R}^{d+1}$ and assume that for each point $(z_0, y_0) \in \Gamma$ relations (45) hold, which implies that this point represents the class $[z_0, y_0]$ of equilibria of system (49). We shall suggest sufficient conditions that guarantee the existence of a one-parameter set of the closed curves γ with vanishing diameters that accumulate at one of the points $\mathcal{M}_0 = (z_0, y_0) \in \Gamma$. The density of the measure μ of the Preisach nonlinearity will be everywhere assumed to be bounded.

Let us suppose that the function $F(z, y)$ is continuously differentiable with respect to z and Lipschitz continuous with respect to y in some neighborhood of the curve Γ. As usual, set

$$A(\lambda) = \partial F/\partial z\big(\varphi(\lambda), \lambda\big), \qquad \lambda \in \Lambda^*.$$

Suppose that at some point $\mathcal{M}_0 = (\varphi(\lambda_0), \lambda_0) \in \Gamma$ the spectrum $\mathrm{Sp}(A(\lambda_0))$ of the $d \times d$ matrix $A(\lambda_0) = \partial F/\partial z\big(\varphi(\lambda_0), \lambda_0\big)$ contains the pair of simple eigenvalues $\pm i w_0$ $(w_0 > 0)$ and the nonresonance condition $i w_0 k \notin \mathrm{Sp}(A(\lambda_0))$ holds for all integer $k \neq \pm 1$. Consider the pair of the simple eigenvalues $\sigma(\lambda) \pm i w(\lambda)$ of $A(\lambda)$ satisfying $\sigma(\lambda_0) \pm i w(\lambda_0) = \pm i w_0$, where the real continuous functions $\sigma(\lambda)$, $w(\lambda)$ are defined in some neighborhood of λ_0.

Proposition 4.3. *Let the function $\sigma(\lambda)$ take values of both sign in each neighborhood of $\lambda = \lambda_0$. Then for every small $r > 0$ system (49) has a class $[z_r(t), y_r(t)]$ of nonstationary periodic solutions of a period T_r, all the closed curves $\gamma_r = \{(z, y) \in \mathbb{R}^{d+1} : z = z_r(t), y = y_r(t), t \in [0, T_r)\}$ are different for different r and*

$$\max_t |z_r(t) - \varphi(\lambda_0)| \to 0, \quad \max_t |y_r(t) - \lambda_0| \to 0, \quad T_r \to T_0 = 2\pi/w_0 \tag{50}$$

as $r \to 0$, i.e. the curves γ_r shrink to the point $\mathcal{M}_0$ and periods approach T_0.

Particularly, the conditions of this propositions are satisfied if the function $\sigma(\lambda)$ is differentiable at the point $\lambda = \lambda_0$ and the transversality condition $\sigma'(\lambda_0) \neq 0$ holds, which one can consider as the main case.

The simplest example of system (49) satisfying the conditions of Proposition 4.3 is

$$x'' + (y(t) - \kappa)x' + x = g(x, y),$$

where x and y are the input and output of the Preisach nonlinearity and $g(x, y) = o(|x| + |y|)$ as $x, y \to 0$. If

$$\mu(\{(\alpha, \beta) : \alpha < \beta < 0\}) < \kappa < \mu(\{(\alpha, \beta) : \alpha < \beta, \alpha < 0\})$$

then this second order equation has the continuum of disjoint classes $[x_r(t), y_r(t)]$ $(0 < r < r_0)$ of nonstationary periodic solutions with periods T_r such that $\max_t |x_r(t)| \to 0$, $\max_t |y_r(t) - \kappa| \to 0$ and $T_r \to 2\pi$ as $r \to 0$.

If the function $F(z, y)$ is sufficiently smooth in a neighborhood of the point $\mathcal{M}_0$, then the functions $\sigma(\lambda)$, $w(\lambda)$ are continuously differentiable and

$$F(\varphi(\lambda) + \zeta, \lambda + \nu) = A(\lambda)\zeta + a(\lambda)\nu + g(\lambda, \zeta, \nu) \tag{51}$$

for λ close to λ_0 and $\zeta \in \mathbb{R}^d, \nu \in \mathbb{R}$ small in absolute value. Let the matrix-valued function $A(\lambda)$ and the vector-valued function $a(\lambda)$ be Lipschitz continuous and let the residual term g satisfy uniformly in λ

$$|g(\lambda, \zeta_1, \nu_1)| \le o(\rho), \tag{52}$$

$$|g(\lambda_1, \zeta_1, \nu_1) - g(\lambda_2, \zeta_2, \nu_2)| \le o(\rho)|\lambda_1 - \lambda_2| + \varepsilon(\rho)\big(|\zeta_1 - \zeta_2| + |\nu_1 - \nu_2|\big) \tag{53}$$

whenever $|\zeta_j|, |\nu_j| \le \rho$ with $\varepsilon(\rho) = o(1)$, $\rho \to 0$. In this situation, Proposition 4.3 is supplemented by the following statement.

Proposition 4.4. *Let relations (51) – (53) hold and $\sigma'(\lambda_0) \ne 0$. Let the density of the measure μ of the Preisach nonlinearity be Lipschitz continuous with respect to both the arguments α, β. Then there is a $r_1 > 0$ such that the curves $\gamma_r = \{(z, y) \in \mathbb{R}^{d+1} : z = z_r(t), y = y_r(t), t \in [0, T_r)\}$ representing cycles of (49) in $\mathbb{R}^{d+1}$ and satisfying (50) as well as their periods T_r depend Lipschitz continuously on r for $0 < r < r_1$. Moreover, there is a neighborhood $U \subset \mathbb{R}^{d+1}$ of the point $\mathcal{M}_0 \in \Gamma$ such that each closed curve γ lying in U and representing a class of cycles of (49) with a period sufficiently close to T_0 belongs to the set of curves γ_r, $0 < r < r_1$, i.e. there are no other such curves in U.*

We stress the difference between the conclusion of this proposition and that of the statements of subsection 4.1 on Hopf bifurcations in systems without hysteresis. System (49) with the Preisach nonlinearity has a continuum of small cycles in the phase space (this continuum is the join of the classes of cycles represented by the curves γ_r with $0 < r < r_1$), while system (38) has typically at most one small cycle in the phase space for each value of the parameter, the surface like that shown in Figure 1 in the product $\mathbb{R}^{d+1}$ of the phase space and the parameter axis consists of cycles existing for different parameter values.

Another remarkable difference is seen if we consider the asymptotics of the cycles. Small cycles of system (38) form a smooth surface[3] in the space $\mathbb{R}^{d+1}$ of pairs (z, b). The curves γ_r representing cycles of system (49) in the space $\mathbb{R}^{d+1}$ of pairs (z, y) form a structure close to a cone, like it is shown in Figure 2. More precisely, the curves γ_r are asymptotically close to the ellipses $\mathcal{E}_r = \mathcal{M}_0 + r(z_*, y_*) + r(\mathcal{C}, 0)$ for small $r > 0$, where $\mathcal{M}_0 + r(z_*, y_*)$ is a ray tangent to the curve Γ at the point $\mathcal{M}_0$ and $\mathcal{C}$ is an appropriate cycle of the linear system $z' = A(\lambda_0)z$ (the cycles $r\mathcal{C}$ form a plane in the phase space $\mathbb{R}^d$ of $z' = A(\lambda_0)z$). The Hausdorf distance between γ_r and $\mathcal{E}_r$ is $o(r)$ as $r \to 0$; Figure 2 shows the curves γ_r and the two-dimensional cone surface formed by the ellipses $\mathcal{E}_r$ with the vertex at the point $\mathcal{M}_0$. Some further details can be found in [7].

[3] We consider sufficiently smooth nonlinearities $F(\cdot, \cdot)$.

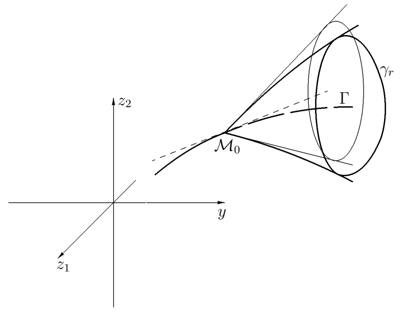

FIGURE 2: Curves γ_r

4.5 Dimension of the set of cycles of system with a parameter

Let us consider again small cycles of system (41) in case of Hopf bifurcation from an equilibrium (z_0, η_0) at the Hopf bifurcation point $b = b_0$. Here the problem includes the 'external' parameter b and the 'internal' parameter λ. So, one might expect that the set of closed curves $\gamma = \{(z(t), y(t), b) \in \mathbb{R}^{d+2} : t \in \mathbb{R}\}$ representing small cycles of (41) in the space $\mathbb{R}^{d+2}$ of triples (z, y, b) in a vicinity of the point $\mathcal{N}_0 = (z_0, y_0, b_0)$ with $y_0 = \Phi(\eta_0)$ is generically two-parametric and that these curves accumulate at the points of some curve $\mathcal{S}$ passing through the point $\mathcal{N}_0$ and representing equilibria of (41). Namely, if $F(z_0, y_0, b_0) = 0$ with $y_0 = \Phi(\eta_0)$ satisfying (45), the matrix $A(\cdot, \cdot, \cdot) = \partial F/\partial z(\cdot, \cdot, \cdot)$ has a pair of simple eigenvalues $\pm i w_0$ $(w_0 > 0)$ at the point $\mathcal{N}_0$ and proper smoothness and nondegeneracy conditions are satisfied, then the equality $F(z, y, b) = 0$ defines a two-dimensional surface Θ in some vicinity of the point $\mathcal{N}_0$ in $\mathbb{R}^{d+2}$, each point $(z_*, y_*, b_*) \in \Theta$ defines the class $[z_*, y_*]$ of stationary solutions of (41) for $b = b_*$ and the curve $\mathcal{S} \subset \Theta$ is defined by the relations $\mathrm{Sp}(A(z, y, b)) \ni iw$, $|w - w_0| < \delta$, $w > 0$ with a sufficiently small δ. Yet, we do not know exact statements of this type.

In problems on Hopf bifurcation from infinity for system (41) one should distinguish between the cases $R_\mu = \infty$ and $R_\mu < \infty$. For example, suppose that the conditions of Proposition 4.2 are satisfied. If $R_\mu = \infty$ then the set of closed curves $\gamma = \{(z(t), y(t), b) \in \mathbb{R}^{d+2} : t \in \mathbb{R}\}$ representing large cycles of (41) for b close to the asymptotic Hopf bifurcation point b_0 is generically two-parametric; if $R_\mu < \infty$ then this set is generically (for 'nonexotic' nonlinearities F) one-parametric. One can see it more precisely from operator equations equivalent to the problem. We outline the approach roughly, not presenting any particular operator equation, which can be constructed in different ways in the form $W_r(z, y, b, T) = 0$. Here r is an auxiliary parameter; the unknowns are the functions $z = z(t)$, $y = y(t)$, the parameter value b for which the class $\bigcup_{0 \le \tau < T} [z(t + \tau), y(t + \tau)]$ of cycles exists and the period T of these cycles; the operator W_r acts in the appropriate Banach space of vectors (z, y, b, T). Suppose $R_\mu = \infty$. Since Proposition 3.2 implies $y(t) = (\mathcal{J}_\lambda x)(t)$, the problem reduces to the system

$$W_r(z, y, b, T) = 0, \quad y = \mathcal{J}_\lambda x, \quad x = \langle c, z \rangle \tag{54}$$

with the two parameters r and λ in this case. After those reduction, one proves that (54) has a solution $(z_{r,\lambda}, y_{r,\lambda}, b_{r,\lambda}, T_{r,\lambda})$ for every $r > r_0$ and every $0 \le \lambda \le 1$, the relations

$$\max_{t,\tau} |z_{r,\lambda}(t) - z_{r,\lambda}(\tau)| \to \infty, \quad b_{r,\lambda} \to b_0, \quad T_{r,\lambda} \to T_0 = 2\pi/w_0 \quad \text{as} \quad r \to \infty$$

similar to (46) hold uniformly with respect to $\lambda \in [0,1]$, and $z_{r,\lambda} = z_{r',\lambda'}$, $y_{r,\lambda} = y_{r',\lambda'}$ if and only if $r = r'$, $\lambda = \lambda'$. This implies that there is the two-parameter set of curves $\gamma_{r,\lambda} = \{(z_{r,\lambda}(t), y_{r,\lambda}(t), b_{r,\lambda}) \in \mathbb{R}^{d+2} : t \in [0, T_{r,\lambda})\}$ ($r > r_0$, $0 \le \lambda \le 1$) which are different for different (r, λ) and that the diameter of $\gamma_{r,\lambda}$ goes to infinity as r increases. Moreover, it turns out that the maximum $\max\{d(\gamma_{r,\lambda}, \gamma_{r,\lambda'}) : 0 \le \lambda, \lambda' \le 1\}$ of the Hausdorf distance $d(\cdot,\cdot)$ between the curves $\gamma_{r,\lambda}$ and $\gamma_{r,\lambda'}$ with the same r vanishes as r increases, i.e. the set $\bigcup_{0\le\lambda\le1} \gamma_{r,\lambda}$ shrinks to the individual curve $\gamma_{r,0}$ as $r \to \infty$.

If the vector c does not belong to the invariant plane of the matrix $A(b_0)$ corresponding to the pair of its simple eigenvalues $\pm i w_0$, then the relations $\min_t\langle c, z(t)\rangle \to -\infty$, $\max_t\langle c, z(t)\rangle \to \infty$ as $b \to b_0$ hold *a priori* for the cycles bifurcating from infinity. In case $R_\mu < \infty$ this implies $(\mathcal{J}_\lambda x)(t) \equiv (\mathcal{J}_0 x)(t)$ with $x(t) = \langle c, z(t)\rangle$ for all $0 \le \lambda \le 1$ and all the large cycles. Therefore in place of (54) one arrives at the system $W_r(z, y, b, T) = 0$, $y = \mathcal{J}_0 x$, $x = \langle c, z\rangle$ with the only parameter r. One proves that for all $r > r_0$ this system has solutions (z_r, y_r, b_r, T_r) satisfying (46). In this case, large cycles of system (41) form the one-parameter set of classes represented by the curves γ_r ($r > r_0$) with the diameters going to infinity as r increases.

The auxiliary parameter r can be chosen in different ways. For example, a 'good' choice (which allows to construct topologically nondegenerate systems (54)) is $r = \|z\|_C$. With this choice, the above statements guarantee the existence of cycles with all sufficiently large amplitudes of the z-component.

Statements of this type on Hopf bifurcations from infinity for systems with the Prandtl – Ishlinskii hysteresis nonlinearities (a class of nonlinearities close to a particular class of the Preisach models) in case $R_\mu = \infty$ was presented in [8]. Case $R_\mu < \infty$ for systems with the stop hysteresis nonlinearity was studied in [4], including stability analysis of large cycles. A similar approach or that of [1, 21] can be applied here. Stability of the continuum of periodic regimes in case of small hysteresis perturbations of systems without hysteresis was studied in [2].

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