Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Some limit theorems for a particle system of single point catalytic branching random walks

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submitted: April 6, 2004

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No. 919
Berlin 2004

2000 *Mathematics Subject Classification.* 60J80, 60K25.

Key words and phrases. Renewal equation, branching particle system, scaling limit .

[1] Research supported partially by DFG and grants RFBR 02-01-00266 and Russian Scientific School 1758.2003.1

[2] Research supported partially by NSA and by Alexander von Humboldt Foundation. Thanks go also to the Weierstrass Institute for Applied Analysis and Stochastics.

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Abstract

We study the scaling limit for a catalytic branching particle system whose particles performs random walks on $\mathbb{Z}$ and can branch at 0 only. Varying the initial (finite) number of particles we get for this system different limiting distributions. To be more specific, suppose that initially there are n^{β} particles and consider the scaled process $Z_t^n(\bullet) = Z_{nt}(\sqrt{n}\,\bullet)$ where Z_t is the measure-valued process representing the original particle system. We prove that Z_t^n converges to 0 when $\beta < \frac{1}{4}$ and to a nondegenerate discrete distribution when $\beta = \frac{1}{4}$. In addition, if $\frac{1}{4} < \beta < \frac{1}{2}$ then $n^{-(2\beta-\frac{1}{2})}Z_t^n$ converges to a random limit while if $\beta > \frac{1}{2}$ then $n^{-\beta}Z_t^n$ converges to a deterministic limit. Note that according to Kaj and Sagitov [13] $n^{-\frac{1}{2}}Z_t^n$ converges to a random limit if $\beta = \frac{1}{2}$.

1 Introduction

Since the early work of Dawson and Fleischmann, catalytic superprocesses have been studied by many authors. We refer the reader to the survey papers of Dawson and Fleischmann [6] and Klenke [14] for an account of this development. Inspired by the continuous case, many authors have studied the catalytic super random walks (CSRW) (cf. Greven et al [11] and the references therein).

In [7], Dawson and Fleischmann introduced and studied the catalytic super-Brownian motion (CSBM) with a single point catalyst. This process is also studied by Fleischmann and Le Gall [10] from another point of view. Kaj and Sagitov [13] considered the discrete counterpart of this process, i.e. the CSRW with a single point catalyst. They proved that if the system starts with $\sqrt{n}$ number of particles at 0 (each with mass $n^{-1/2}$) then under Brownian scaling, the CSRW converges to a CSBM with a single point catalyst. The *aim* of this article is to study various other limiting behavior of the CSRW under Brownian scaling.

Now we introduce the CSRW model in more details. Consider a system of particles performing independent standard continuous-time random walks on $\mathbb{Z}$. Namely, each particle stays at a site other than 0 for an exponential time and then moves with probability $\frac{1}{2}$ to the left and with probability $\frac{1}{2}$ to the right. At its new position, it waits for another exponential time and moves on. The behavior of a particle at state 0 is a bit different. It stays here for an exponential time and than either moves to the left or to the right or dies or splits into two particles. The particle selects each of these four options with probability $\frac{1}{4}$. All the exponential waiting times at each site for each particles are of parameter 1 and are independent of each other. In this case the offspring generating function is $f(s) = \frac{1}{2}(1+s^2)$. The newborn particles appear at point 0 and move and branch in the same fashion as their parent.

The model described is a particular case of the branching random walk in catalytic medium. The longterm behavior of such processes with various types of catalytic media were studied by many authors (cf. Greven et al [11] and Wakolbinger [19] and the references therein). The longtime limit of the moments of the population size process for the current model of single point catalyst was considered by Albeverio and Bogachev [1], Albeverio et al [2], Bogachev and Yarovaya [4],[5]. Topchii and Vatutin [15], [16] and Topchii et al [17] studied the limiting behavior for the population size of the process as well as the subpopulation size at the catalyst position. In this paper, we consider the longterm behavior of this system as evolution of measures.

Let $Z_t(A)$ be the number of particles in the region $A \subset \bar{\mathbb{R}}$ at time t, where $\bar{\mathbb{R}}$ is the one-point compactification of $\mathbb{R}$. The *aim* of this article is to study the long term behavior of Z_t. Namely, we consider the scaling limit of Z_t. We scale the time by a factor n and the space by $\sqrt{n}$ so that the scaled spatial motions of particles are approximated by Brownian motions and define the random measures $Z_t^n(\bullet) = Z_{nt}(\sqrt{n}\,\bullet)$. Let $\mathcal{M}_F(\bar{\mathbb{R}})$ be the space of finite Borel measures on $\bar{\mathbb{R}}$. Then Z^n is an $\mathcal{M}_F(\bar{\mathbb{R}})$-valued process.

Denote the Dirac measure at 0 by δ_0. Let τ be the branching time for a random walk particle started at 0. Let G be the distribution of τ. Let $\tau_0, \tau_1, \tau_2, \cdots$, be i.i.d. with common exponential distribution of parameter 1. Let η_0 be the first time for a standard continuous-time random walk starting at 1 or -1 to hit 0. Let $\eta_1, \eta_2, \cdots$, be independent copies of η_0. Here $\tau_1, \tau_2, \cdots$, are the exponential times a particle spent at 0 while $\eta_1, \eta_2, \cdots$, are the time intervals which this particle spent away from 0. Then

$$\tau = \tau_0 + \eta_1 + \tau_1 + \eta_2 + \cdots + \eta_N + \tau_N,$$

where N is a random variable independent of τ_i, η_i, $i = 1, 2, \ldots$, and having geometric distribution with parameter $\frac{1}{2}$:

$$\mathbb{P}(N = k) = \frac{1}{2^{k+1}}, \ k = 0, 1, \ldots.$$

The following fact (cf. [17]) will be used frequently in this article: As $t \to \infty$,

$$1 - G_0(t) := \mathbb{P}(\eta_0 > t) = \frac{d^2}{2\sqrt{t}} + o\left(\frac{1}{\sqrt{t}}\right) \tag{1.1}$$

where $d > 0$ is a constant known explicitly.

First, we investigate how many initial individuals are needed for Z_t^n to have a nontrivial limit without extra scaling on the mass of particles. Throughout the paper we denote by $\phi^1, \cdots, \phi^k$ nonnegative Lipschitz continuous functions on $\mathbb{R}$ and for a measure $\mu \in \mathcal{M}_F(\bar{\mathbb{R}})$ write $\langle \mu, \phi \rangle := \int \phi d\mu$.

The Brownian meander plays an important role in the description of the limits in this paper. Let W_t be a standard Brownian motion defined on $[0, 1]$ and let s_0 be the first time W reaches 0. Then

$$W_t^+ := (1 - s_0)^{-1/2} |W(s_0 + t(1 - s_0))|, \qquad 0 \le t \le 1$$

is called the Brownian meander (cf. Iglehart [12] and Durrett et al [8] for its properties). Let $\{\hat{W}_t, 0 \le t \le 1\}$ be the process which equals $\{W_t^+, 0 \le t \le 1\}$ with probability $\frac{1}{2}$ and $\{-W_t^+, 0 \le t \le 1\}$ with probability $\frac{1}{2}$.

Theorem 1.1 *i) If $\beta < \frac{1}{4}$ and $Z_0^n = n^\beta \delta_0$, then Z_t^n converges in probability to 0 as $n \to \infty$, for any fixed $t > 0$.*

ii) If $Z_0^n = n^{1/4}\delta_0$, then for any $0 < t_1 < \cdots < t_k < \infty$, $(Z_{t_1}^n, \cdots, Z_{t_k}^n)$ converges in law on $\mathcal{M}_F(\bar{\mathbb{R}})^k$ as $n \to \infty$ to a tuple of measures $(Z_{t_1}^\infty, \cdots, Z_{t_k}^\infty)$ whose law is determined by

$$\mathbb{E}\left[\exp\left(-\sum_{i=1}^k \langle Z_{t_i}^\infty, \phi^i \rangle\right)\right] = \exp\left(-\sqrt{2} g(t_1, \cdots, t_k)\right)$$

with

$$g^2(t_1,\cdots,t_k) := d^2\sum_{i=1}^k t_i^{-1/2}\mathbb{E}\left[\exp\left(-\sum_{j=1}^{i-1}\phi^j(\sqrt{t_i}\hat{W}_{t_j/t_i})\right)\left(1-e^{-\phi^i(\sqrt{t_i}\hat{W}_1)}\right)\right]. \quad (1.2)$$

Let us give a heuristic commentary to this result. Setting $\phi^j = \lambda_j$, $j = 1,\dots,k$ with $\lambda_j \ge 0$ we get

$$\begin{aligned} H(\lambda_1,\dots,\lambda_k) &:= \mathbb{E}\left[\exp\left(-\sum_{i=1}^k \lambda_i Z_{t_i}^\infty(\bar{\mathbb{R}})\right)\right] \\ &= \exp\left(-\sqrt{2d^2\sum_{i=1}^k t_i^{-1/2}e^{-\sum_{j=1}^{i-1}\lambda_j}(1-e^{-\lambda_i})}\right). \end{aligned}$$

This shows that the limiting measures $Z_{t_i}^\infty(\cdot)$, $i = 1,\dots,k$ are *discrete*. Moreover, one can deduce from the results of [18] that in the limit only offsprings of a Poisson number of the initial particles survive. Besides, associating with the branching random walk we consider here a Bellman-Harris branching process (see, for instance, [15], [16] or [17]) and referring to a statement from [20] one can conclude that if the population of our branching random walk generated by a individual at time zero is nonempty at a large moment T then the surviving members of this population had a chance to visit the origin only at moments $o(T)$ (see Lemma 2.5 below for analogous arguments). Since the total size of the population in the limit is finite with probability 1, this means that all the individuals in the limiting population described, say, by the measure $Z_{t_i}^\infty$ were “born” at moment $t = 0$ and then never returned to the origin. Hence in the limit individuals surviving up to a (scaled) moment t_k perform a motion according to a Brownian meander.

Put now $\lambda_j = 0$, $j < k$ and let $\lambda_k \to 0$. Then

$$1 - H(0,\dots,0,\lambda_k) = 1 - \exp\left(-\sqrt{2d^2t_k^{-1/2}\lambda_k}\right) \sim d\sqrt{2t_k^{-1/2}}\sqrt{\lambda_k}.$$

Hence (see, for instance, [9]) the distribution of $Z_{t_k}^\infty(\bar{\mathbb{R}})$ belongs to the normal domain of attraction of a one-sided stable law with index 1/2. Clearly, the same conclusion is valid for any $Z_{t_i}^\infty(\bar{\mathbb{R}})$, $i = 1,\dots,k$. In particular, this means that if we have a collection of i.i.d. random variables of $Z_{t_k,r}^\infty(\bar{\mathbb{R}})$, $r = 1,2,\dots,N$, distributed like $Z_{t_k}^\infty(\bar{\mathbb{R}})$ then, as $N \to \infty$

$$\frac{\sum_{r=1}^N Z_{t_k,r}^\infty(\bar{\mathbb{R}})}{N^2} \xrightarrow{d} \mathcal{Z} \quad (1.3)$$

where $\mathcal{Z}$ is a random variable obeying a one-sided stable law with index 1/2. It is natural to guess that if N grows with n not too fast then something like (1.3) has to be true for the prelimiting variables in point ii) of Theorem 1.1 as well. Our results confirm this hypothesis.

Indeed, take $\beta > \frac{1}{4}$. Then n^β initial particles can be divided into $N = n^{\beta-\frac{1}{4}}$ groups of independent copies of branching particle systems each of which with starting measure $Z_0^n = n^{1/4}\delta_0$. The following two theorems say that if β is not too large, then $N^{-2}Z_t^n$ has, as $n \to \infty$, a random limit. In particular, the (scaled) total size of the population converges to a random variable having a stable law with index 1/2 (compare with [18]). If, however, β is large, then $n^{-\beta}Z_t^n$ has a deterministic limit in complete agreement with the law of large numbers.

Theorem 1.2 *Suppose that* $\frac{1}{4} < \beta < \frac{1}{2}$. *If* $Z_0^n = n^\beta \delta_0$ *and* $\alpha = 2\beta - \frac{1}{2}$, *then for any* $0 < t_1 < \cdots < t_k < \infty$, $(n^{-\alpha} Z_{t_1}^n, \cdots, n^{-\alpha} Z_{t_k}^n)$ *converges in law on* $\mathcal{M}_F(\bar{\mathbb{R}})^k$ *as* $n \to \infty$ *to a tuple of measures* $(Z_{t_1}^\infty, \cdots, Z_{t_k}^\infty)$ *whose law is determined by*

$$\mathbb{E}\left[\exp\left(-\sum_{i=1}^k \langle Z_{t_i}^\infty, \phi^i \rangle\right)\right] = \exp\left(-\sqrt{2} g_0(t_1, \cdots, t_k)\right)$$

with

$$g_0^2(t_1, \cdots, t_k) := d^2 \sum_{i=1}^k t_i^{-1/2} \mathbb{E}\left[\phi^i(\sqrt{t_i}\hat{W}_1)\right].$$

Note that Kaj and Sagitov [13] proved that if $Z_0^n = n^{1/2}\delta_0$, then $n^{-1/2} Z^n$ converges as $n \to \infty$ to a catalytic super Brownian motion with single point catalyst at 0. The following theorem says that when the number of initial points is large enough a law of large number behavior holds for our model.

Theorem 1.3 *If* $Z_0^n = n^\beta \delta_0$ *with* $\beta > \frac{1}{2}$, *then for any nonnegative Lipschitz continuous function* ϕ *on* $\mathbb{R}$

$$n^{-\beta} \langle Z_t^n, \phi \rangle \to \frac{1}{\pi} \int_0^t \frac{\mathbb{E}[\phi(\sqrt{t-u}\hat{W}_1)]}{\sqrt{u(t-u)}}\, du, \quad n \to \infty,$$

in probability.

These three theorems will be proved in Sections 2, 3, 4 respectively. We shall use c for a constant which can vary from place to place.

2 Limit theorem for Z_t^n without mass scaling

In this section we investigate how many initial particles are needed in order to get a nontrivial limit without scaling the mass of the particles. It turns out that when the initial number of points is of order $n^{1/4}$, a nontrivial limit is obtained. When the initial number of points is of order n^β with $\beta < \frac{1}{4}$, the random measure Z_t^n tends to 0. However, first we establish a number of simple results related to properties of the random variables τ_i, η_i and the process ξ_t, $0 \le t < \infty$, the standard continuous time random walk on $\mathbb{Z}$.

Set $G_1(t) := \mathbb{P}(\tau_1 + \eta_0 \le t)$, denote

$$G_2(t) := \sum_{k=1}^\infty \frac{1}{2^k} G_1^{*k}(t),$$

where G_1^{*k} is the kth convolution of G_1 with itself. Clearly, $G_2(t)$ is a distribution function and we let ζ be a random variable such that $G_2(t) = \mathbb{P}(\zeta \le t)$.

Lemma 2.1 *As* $t \to \infty$

$$1 - G_1(t) = \mathbb{P}(\tau_1 + \eta_0 > t) \sim \mathbb{P}(\eta_0 > t) \sim \frac{d^2}{2\sqrt{t}}.$$

Proof. It is easy to see by (1.1) that for any $\varepsilon > 0$ there exists $t_0 = t_0(\varepsilon)$ such that for all $t \geq t_0$

$$\begin{aligned} \frac{(1-\varepsilon)d^2}{2\sqrt{t}} &\leq \mathbb{P}(\eta_0 > t) \leq \mathbb{P}(\tau_1 + \eta_0 > t) \\ &= \mathbb{P}(\tau_1 + \eta_0 > t;\ \tau_1 \leq 3\ln t) + \mathbb{P}(\tau_1 + \eta_0 > t;\ \tau_1 > 3\ln t) \\ &\leq \mathbb{P}(\eta_0 > t - 3\ln t) + \mathbb{P}(\tau_1 > 3\ln t) \\ &\leq \frac{(1+\varepsilon)d^2}{2\sqrt{t - 3\ln t}} + \frac{1}{t^3}. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the lemma follows. ∎

Lemma 2.2 *As* $t \to \infty$

$$1 - G_2(t) = \mathbb{P}(\zeta > t) \sim 2\mathbb{P}(\eta_0 > t) = 2(1 - G_0(t)) \sim \frac{d^2}{\sqrt{t}}.$$

Proof. This statement is a particular case of Theorem 3, Section 4, Chapter IV in [3] if one takes $\gamma = 1/2$ in the mentioned theorem. ∎

Let $G(t) := \mathbb{P}(\tau \leq t)$ be the lifelength distribution of particles of the process.

Corollary 2.3 *As* $t \to \infty$

$$1 - G(t) = \frac{d^2}{\sqrt{t}} + o\left(\frac{1}{\sqrt{t}}\right). \tag{2.1}$$

Proof. It is not difficult to check that

$$\mathbb{P}(\tau > t) = \mathbb{P}(\tau_0 + \zeta > t) \tag{2.2}$$

and, to complete the proof one should repeat the arguments used in the proof of Lemma 2.1. ∎

The following two lemmas relate the lifelength of a particle started from point 0 at moment $t = 0$ with its position ξ_t at moment $t > 0$.

Lemma 2.4 *As* $t \to \infty$

$$\mathbb{P}(\tau > t,\, \xi_t = 0) = o(\mathbb{P}(\tau > t)).$$

Proof. Clearly,

$$\begin{aligned} \mathbb{P}(\tau > t,\, \xi_t = 0) &= \sum_{k=0}^{\infty} \mathbb{P}(\tau > t,\, \xi_t = 0,\, N = k) \\ &= \mathbb{P}(\tau_0 > t) + \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \mathbb{P}(S_k < t, S_k + \tau_k > t) \end{aligned}$$

where $S_0 := 0$, $S_k := \tau_0 + \eta_1 + \tau_1 + \eta_2 + \ldots + \eta_k$, $k = 1, 2, \cdots$. Therefore,

$$\begin{aligned}
\mathbb{P}(\tau > t,\, \xi_t = 0) &= \mathbb{P}(\tau_0 > t) + \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \int_0^t \mathbb{P}(\tau_k > t-u) d\mathbb{P}(S_k \le u) \\
&= e^{-t} + \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} \int_0^t e^{-(t-u)} dG_1^{*k}(u) \\
&= e^{-t} + \frac{1}{2} \int_0^t e^{-(t-u)} dG_2(u).
\end{aligned}$$

Splitting the integral into two parts and recalling Lemma 2.2 we get

$$\begin{aligned}
\int_0^t e^{-(t-u)} dG_2(u) &= \int_0^{t-3\ln t} e^{-(t-u)} dG_2(u) + \int_{t-3\ln t}^t e^{-(t-u)} dG_2(u) \\
&\le \frac{1}{t^3} + G_2(t) - G_2(t - 3\ln t) = o(1 - G_2(t)).
\end{aligned}$$

This estimate, clearly, implies the statement of the lemma. ∎

Let $\theta_t := t - \sup\{0 \le u \le t : \xi_u = 0\}$ denote the time passed after the last visit of a particle to zero. The next lemma shows that given $\{\tau > t\}$ the last visit occurred "very long time" ago.

Lemma 2.5 *As* $t \to \infty$

$$\mathbb{P}(\tau > t,\, \theta_t < t - \sqrt{t}) = o(\mathbb{P}(\tau > t)),\ t \to \infty. \tag{2.3}$$

Proof. In view of the previous lemma it is enough to demonstrate that

$$\mathbb{P}(\tau > t,\, 0 < \theta_t < t - \sqrt{t}) = o(\mathbb{P}(\tau > t)),\ t \to \infty.$$

Similar to Lemma 2.4

$$\begin{aligned}
\mathbb{P}(\tau > t,\, 0 < \theta_t < t - \sqrt{t}) &= \sum_{k=0}^{\infty} \mathbb{P}(\tau > t,\, 0 < \theta_t < t - \sqrt{t},\, N = k) \\
&= \sum_{k=1}^{\infty} \frac{1}{2^k} \mathbb{P}(S_{k-1} + \tau_{k-1} < t, S_k > t, \theta_t < t - \sqrt{t}) \\
&= \sum_{k=1}^{\infty} \frac{1}{2^k} \int_{\sqrt{t}}^t \mathbb{P}(S_{k-1} + \tau_{k-1} \in du, \eta_k > t-u)
\end{aligned}$$

Denoting for brevity $G_3(t) := 1 - e^{-t}, t \ge 0$ we have

$$\begin{aligned}
\mathbb{P}(\tau > t,\, 0 < \theta_t < t - \sqrt{t}) &= \frac{1}{2} \int_{\sqrt{t}}^t (1 - G_0(t-u)) e^{-u} du \\
&\quad + \sum_{k=2}^{\infty} \frac{1}{2^k} \int_{\sqrt{t}}^t (1 - G_0(t-u)) d(G_3 * G_1^{(k-1)*}(u)) \\
&= \frac{1}{2} \int_{\sqrt{t}}^t (1 - G_0(t-u)) e^{-u} du \\
&\quad + \frac{1}{2} \int_{\sqrt{t}}^t (1 - G_0(t-u)) d(G_3 * G_2(u))
\end{aligned}$$

Clearly,

$$\int_{\sqrt{t}}^{t}(1-G_0(t-u))e^{-u}du \le e^{-\sqrt{t}} = o(\mathbb{P}(\tau > t)),\ t\to\infty.$$

Recalling (2.2) that $G(t) = G_3 * G_2(t)$ we have for any $\varepsilon \in (0,1)$:

$$\begin{aligned}
& \int_{\sqrt{t}}^{t}(1-G_0(t-u))d(G_3 * G_2(u)) \\
= & \int_{\sqrt{t}}^{(1-\varepsilon)t}(1-G_0(t-u))dG(u) + \int_{(1-\varepsilon)t}^{t}(1-G_0(t-u))dG(u) \\
\le & (1-G_0(t\varepsilon))(1-G(\sqrt{t})) + G(t) - G(t(1-\varepsilon)).
\end{aligned}$$

By (1.1) and (2.1) we see that

$$\limsup_{\varepsilon\to+0}\limsup_{t\to\infty}\frac{(1-G_0(t\varepsilon))(1-G(\sqrt{t}))+G(t)-G(t(1-\varepsilon))}{1-G(t)} = 0.$$

Combining this fact with the preceding estimates we easily get (2.3). ∎

We have finished the preliminary estimates and are ready now to pass to the proofs of our first main statement, Theorem 1.1.

For $0 \le t_1 < \cdots < t_k < \infty$, define

$$Q_{\bar{t}}(\bar{\phi}) := \mathbb{E}_{\delta_0}\left[1-\exp\left(-\sum_{i=1}^{k}\langle Z_{t_i},\phi^i\rangle\right)\right].$$

Then

$$\mathbb{E}_{\delta_0}\left[1-\exp\left(-\sum_{i=1}^{k}\langle Z_{t_i}^n,\phi^i\rangle\right)\right] = Q_{n\bar{t}}(\bar{\phi}_n)$$

where $\phi_n^i(x) = \phi^i(n^{-1/2}x)$. To prove Theorem 1.1, we first prove

$$\lim_{n\to\infty} n^{1/4}Q_{n\bar{t}}(\bar{\phi}_n) = \sqrt{t}g(t_1,\cdots,t_k). \tag{2.4}$$

Consider the case $k=1$ and take a nonnegative Lipschitz continuous test function ϕ with $\phi(0)>0$. Let

$$Q_t(\phi) = \mathbb{E}_{\delta_0}\left(1-\exp\left\{-\langle Z_t,\phi\rangle\right\}\right).$$

The following renewal equation plays a pivotal role in the proofs of all results in this article:

$$Q_t(\phi) = \mathbb{E}_0\left[1-\exp\{-\phi(\xi_t)\}\,|\,\tau>t\right](1-G(t)) + \int_0^t\left(1-f\left(1-Q_{t-u}(\phi)\right)\right)dG(u). \tag{2.5}$$

Set

$$\phi_n(x) = \phi\left(n^{-1/2}x\right)$$

and

$$h(\phi_n, nt) = \mathbb{E}_0\left[1-\exp\{-\phi_n(\xi_{nt})\}\,|\,\tau>nt\right].$$

It is easy to check that

$$\mathbb{E}_{\delta_0}\left[1-\exp\left\{-\langle Z_t^n,\phi\rangle\right\}\right] = Q_{nt}(\phi_n)$$

satisfies the following scaled renewal equation

$$Q_{nt}(\phi_n) = h(\phi_n, nt)(1 - G(nt)) + \int_0^{nt} (1 - f(1 - Q_{nt-u}(\phi_n)))\, dG(u). \tag{2.6}$$

Lemma 2.6 *For $t > 0$, we have*

$$h(\phi_n, nt) \to \mathbb{E}_0\left[1 - \exp\left\{-\phi\left(\sqrt{t}\hat{W}_1\right)\right\}\right], \quad n \to \infty. \tag{2.7}$$

Proof. Note that there is a Poisson random variable $M = M(n,t)$ of parameter θ_{nt} such that

$$\xi_{nt} - \xi_{nt-\theta_{nt}} = \tilde{S}_M,$$

where $\tilde{S}_k$ is the partial sum of an i.i.d. sequence of random variables of mean 0. The condition

$$\min_{nt-\theta_{nt} \le u \le nt} \xi_u > 0$$

is equivalent to that the hitting time of $\tilde{S}_k$ for the set $(-\infty, 0]$ is greater than M. By a proposition in Iglehart [12], we have

$$\frac{\xi_{nt} - \xi_{nt-\theta_{nt}}}{\sqrt{M}} \to W_1^+, \qquad \text{as } n \to \infty.$$

Under the condition $nt - \theta_{nt} \le \sqrt{nt}$, we have

$$\frac{M}{n} = \frac{M}{\theta_{nt}} \frac{\theta_{nt}}{n} \to t, \qquad a.s.$$

Therefore, the conditional probability measure

$$\mathbb{P}\left(\frac{\xi_{nt} - \xi_{nt-\theta_{nt}}}{\sqrt{n}} \in \cdot \,\middle|\, nt - \theta_{nt} \le \sqrt{nt}, \min_{nt-\theta_{nt} \le u \le nt} \xi_u > 0\right)$$

converges to the probability measure induced by $\sqrt{t}W_1^+$ as $n \to \infty$. Evidently,

$$\left(\frac{\xi_{nt-\theta_{nt}}}{\sqrt{n}} \,\middle|\, nt - \theta_{nt} \le \sqrt{nt}\right) \to 0, \quad n \to \infty,$$

in probability. As

$$\phi_n(\xi_{nt}) = \phi\left(\frac{\xi_{nt} - \xi_{nt-\theta_{nt}}}{\sqrt{n}} + \frac{\xi_{nt-\theta_{nt}}}{\sqrt{n}}\right),$$

we have

$$\begin{aligned} &\mathbb{E}_0\left[1 - \exp\{-\phi_n(\xi_{nt})\} \,\middle|\, \theta_{nt} \ge nt - \sqrt{nt}, \min_{nt-\theta_{nt} \le u \le nt} \xi_u > 0\right] \\ \to\ &\mathbb{E}_0\left[1 - \exp\left\{-\phi\left(\sqrt{t}W_1^+\right)\right\}\right]. \end{aligned}$$

Similar arguments are valid for the case $\max_{nt-\theta_{nt} \le u \le nt} \xi_u < 0$ (with replacement of W_1^+ by $-W_1^+$). Since each of these possibilities happens with probability $1/2$, we see that

$$\mathbb{E}_0\left[1 - \exp\{-\phi_n(\xi_{nt})\} \,\middle|\, \theta_{nt} \ge nt - \sqrt{nt},\ \tau > nt\right] \to \mathbb{E}_0\left[1 - \exp\left\{-\phi\left(\sqrt{t}\hat{W}_1\right)\right\}\right].$$

(2.7) then follows from Lemma 2.5. ∎

Our next lemma states that the main contribution of $Q_{nt-u}(\phi_n)$ to the integral in (2.6) comes from those with small u.

Lemma 2.7 *Let $a(nt,n)$ be defined by the equality*

$$\begin{aligned} Q_{nt}(\phi_n) &= g^2(t)\frac{1}{\sqrt{n}} + \frac{a(nt,n)}{\sqrt{nt}} \\ &\quad + \int_0^{n^{5/8}t} (1 - f(1 - Q_{nt-u}(\phi_n)))\, dG(u) \end{aligned} \tag{2.8}$$

and for fixed $0 < t_0 < T < \infty$ let

$$a(n) := \sup_{t_0 \le t \le T} |\, a(nt,n)\, |.$$

Then

$$\lim_{n\to\infty} a(n) = 0.$$

Proof. Recalling relation (2.7) we get

$$\begin{aligned} h(\phi_n, nt)(1 - G(nt)) &= \left(\mathbb{E}\left[1 - \exp\left\{-\phi\left(\sqrt{t}\hat{W}_1\right)\right\}\right] + o(1)\right)\frac{d^2}{\sqrt{nt}} + o\left(\frac{1}{\sqrt{n}}\right) \\ &= g^2(t)\frac{1}{\sqrt{n}} + o\left(\frac{1}{\sqrt{n}}\right),\ n \to \infty. \end{aligned} \tag{2.9}$$

By Theorem 1 of [17], we have

$$Q_{nt}(\phi_n) \le P(Z_{nt} > 0) \le \frac{c}{(nt+1)^{1/4}}. \tag{2.10}$$

Hence

$$1 - f(1 - Q_{nt-u}(\phi_n)) \le Q_{nt-u}(\phi_n) \le \frac{c}{(nt-u+1)^{1/4}}.$$

Consequently, the integral term in (2.6) (taken over the range $[n^{5/8}t, nt]$) does not exceed

$$c\int_{n^{5/8}t}^{nt} \frac{dG(u)}{(nt-u+1)^{1/4}} = c\int_{nt-n^{7/8}t}^{nt} \frac{dG(u)}{(nt-u+1)^{1/4}} + c\int_{n^{5/8}t}^{nt-n^{7/8}t} \frac{dG(u)}{(nt-u+1)^{1/4}}. \tag{2.11}$$

Since the integrand is bounded by 1, the first term on the right hand side of (2.11) is bounded by

$$\begin{aligned} 1 - G\left(nt - n^{7/8}t\right) - (1 - G(nt)) &= \frac{d^2}{\left(nt - n^{7/8}t\right)^{1/2}} - \frac{d^2}{(nt)^{1/2}} + o\left(\frac{1}{(nt)^{1/2}}\right) \\ &= o\left(\frac{1}{n^{1/2}}\right), \quad \text{as } n \to \infty. \end{aligned}$$

On the other hand, the integrand for the second term on the right hand side of (2.11) is bounded by $(n^{7/8}t)^{-1/4}$ and hence, the respective term is estimated from above by

$$\begin{aligned} \frac{1}{n^{7/32}t^{1/4}}\int_{n^{5/8}t}^{nt-n^{7/8}t} dG(u) &\le \frac{1}{n^{7/32}t^{1/4}}\left(1 - G\left(n^{5/8}t\right)\right) \\ &= \frac{1}{n^{7/32}t^{1/4}}\left(\frac{c}{n^{5/16}t^{1/2}} + o(\frac{1}{n^{5/16}})\right) \\ &= o\left(\frac{1}{n^{1/2}}\right). \end{aligned}$$

Plugging back to (2.11) we find that the integral term in (2.6) over the range $[n^{5/8}t, nt]$ is $o(n^{-1/2})$ as $n \to \infty$. The conclusion of the lemma then follows from (2.9) and (2.6). $\blacksquare$

Now we define a function $\varepsilon(nt, n)$ by the equality

$$Q_{nt}(\phi_n) = g(t)\frac{\sqrt{2}}{n^{1/4}}\left(1 + \frac{\varepsilon(nt,n)}{\ln(nt+1)}\right). \tag{2.12}$$

Our next lemma says that $Q_{nt-u}(\phi_n)$ in (2.8) is close to $Q_{nt}(\phi_n)$ for $0 \le u \le n^{5/8}t$.

Lemma 2.8 *Let*

$$\varepsilon^+(n) := \sup_{t_0 \le t \le T} \varepsilon(nt,n) \ \ \text{and} \ \ \varepsilon^-(n) := \inf_{t_0 \le t \le T} \varepsilon(nt,n).$$

Then $\varepsilon^+(n)$ and $\varepsilon^-(n)$ are finite. Further, there exist functions $r^{\pm}(n,u)$ such that for fixed $0 < t_0 < T < \infty$

$$\limsup_{n\to\infty} \sup_{t_0 \le t \le T} \sup_{0 \le u \le n^{5/8}t} \left(|r^-(n,u)| + |r^+(n,u)|\right) = 0 \tag{2.13}$$

and there exists a constant $n_0(t_0,T)$ such that for all $n > n_0(t_0, T)$ and $t_0 < t < T$

$$R^-(nt,u;n) \le Q_{nt-u}(\phi_n) \le R^+(nt,u;n)$$

where

$$R^{\pm}(nt,u;n) := g(t)\frac{\sqrt{2}}{n^{1/4}}\left(1 + \frac{\varepsilon^{\pm}(n)}{\ln(nt+1)}\right) + \frac{r^{\pm}(n,u)}{n^{1/2}}. \tag{2.14}$$

Proof. Since $\phi(0) > 0$ and ϕ is continuous it follows that

$$\inf_{t_0 \le t \le T} g(t) > 0. \tag{2.15}$$

By (2.12) and (2.10) we have

$$\begin{aligned} 1 + \frac{\varepsilon(nt,n)}{\ln(nt+1)} &= \frac{Q_{nt}(\phi_n)}{\sqrt{2}g(t)} n^{1/4} \\ &\le \mathbb{P}(Z_{nt} > 0)\frac{1}{\sqrt{2}g(t)} n^{1/4} \le c. \end{aligned} \tag{2.16}$$

It follows from (2.15) and (2.16) that $\varepsilon^+(n) < \infty$ for each $n > 0$. Since $Q_{nt}(\phi_n) > 0$, we have $\varepsilon(nt,n) > -\ln(1+nt)$. Thus, $\varepsilon^-(n) > -\infty$ for each $n > 0$. Recall that $g(t)$ is defined in (1.2) such that

$$t^{1/4}g(t) = d\sqrt{\mathbb{E}\left(1 - e^{-\phi(\sqrt{t}\hat{W}_1)}\right)}.$$

Note that the density of the Brownian meander W_t^+ (cf. [12]) at $t = 1$ is given by

$$p(x) = x\exp\left(-\frac{x^2}{2}\right), \qquad x > 0.$$

It is easy to show that for $t_0 \le s < t \le T$,

$$\left|\mathbb{E}e^{-\phi(\sqrt{t}W_1^+)} - \mathbb{E}e^{-\phi(\sqrt{s}W_1^+)}\right| \le c(t_0,T)|t-s|$$

where $c(t_0, T)$ is a finite constant. Then

$$\left|t^{1/4}g(t) - s^{1/4}g(s)\right| \le c_1(t_0, T)|t-s|. \tag{2.17}$$

By (2.15) and (2.17) we have for $0 \le u \le n^{5/8}t$,

$$\left|t^{1/4}g\left(t\right) - \left(\frac{nt-u}{n}\right)^{1/4} g\left(\frac{nt-u}{n}\right)\right| \le c\left|t - \frac{nt-u}{n}\right| = o\left(\frac{1}{n^{1/4}}\right). \tag{2.18}$$

Clearly,

$$\frac{1}{(nt-u)^{1/4}} = \frac{1}{(nt)^{1/4}} + o\left(\frac{1}{n^{1/2}}\right) \tag{2.19}$$

and

$$\frac{1}{\ln\left(nt-u+1\right)} = \frac{1}{\ln\left(nt+1\right)}\left(1 + o\left(\frac{1}{n^{1/2}}\right)\right). \tag{2.20}$$

By (2.12) we have

$$Q_{nt-u}(\phi_n) = g\left(\frac{nt-u}{n}\right)\frac{\sqrt{2}}{n^{1/4}}\left(1 + \frac{\varepsilon(nt-u,n)}{\ln(nt-u+1)}\right). \tag{2.21}$$

Using (2.18)-(2.20) in (2.21) we get

$$Q_{nt-u}(\phi_n) = g(t)\frac{\sqrt{2}}{n^{1/4}}\left(1 + \frac{\varepsilon(nt-u,n)}{\ln(nt+1)}\right) + o\left(\frac{1}{n^{1/2}}\right).$$

Hence the conclusions of the lemma follows easily. ∎

Remarks 2.9 *Since $Q_{nt}(\phi_n) \le c(nt)^{-1/4}$, we have*

$$1 + \frac{\varepsilon(nt,n)}{\ln(nt+1)} \le c$$

and, therefore, $\varepsilon^+(n) \le c\ln n$. This, in view of (2.14) implies $R^+(nt,u;n) \to 0$, $n \to \infty$.

Clearly,

$$\begin{aligned} g\left(t\right)\frac{\sqrt{2}}{n^{1/4}}\left(1 + \frac{\varepsilon^-\left(n\right)}{\ln\left(nt+1\right)}\right) &\le Q_{nt}\left(\phi_n\right) \\ &\le g\left(t\right)\frac{\sqrt{2}}{n^{1/4}}\left(1 + \frac{\varepsilon^+\left(n\right)}{\ln\left(nt+1\right)}\right). \end{aligned} \tag{2.22}$$

Our aim is to show that

$$\limsup_{n\to\infty}\frac{\varepsilon^+\left(n\right)}{\ln(nt+1)} \le 0 \tag{2.23}$$

and

$$\liminf_{n\to\infty}\frac{\varepsilon^-\left(n\right)}{\ln(nt+1)} \ge 0. \tag{2.24}$$

Since the function

$$1 - f(1-x) = x - \frac{1}{2}x^2$$

is monotone increasing in $x \in (-\infty, 1)$, we have by Lemmas 2.7 and 2.8, for all sufficiently large n:

$$\begin{aligned} & g^2(t)\frac{1}{\sqrt{n}} + \frac{a(nt,n)}{\sqrt{nt}} + \int_0^{n^{5/8}t}\left(1 - f\left(1 - R^-(tn,u;n)\right)\right)dG(u) \\ \le\ & Q_{nt}(\phi_n) \\ \le\ & g^2(t)\frac{1}{\sqrt{n}} + \frac{a(nt,n)}{\sqrt{nt}} + \int_0^{n^{5/8}t}\left(1 - f\left(1 - R^+(tn,u;n)\right)\right)dG(u). \end{aligned}$$

Lemma 2.10

$$\limsup_{n\to\infty} n^{1/4} Q_{nt}(\phi_n) \le \sqrt{2}g(t).$$

Proof. Clearly, if $\limsup_{n\to\infty}\varepsilon^+(n) < \infty$ then (2.23) is valid and hence, Lemma 2.10 follows from (2.22). Thus, we assume that

$$\limsup_{n\to\infty}\varepsilon^+(n) = \infty.$$

Let $n_0 = 1$ and $n_{j+1} = \min\{n > n_j : \varepsilon^+(n) > \varepsilon^+(n_j)\}$. Under our assumption

$$\lim_{j\to\infty}\varepsilon^+(n_j) = \infty.$$

Evidently, for each $j = 1, 2, 3, ...$ there exists $t_j \in [t_0, T]$ such that

$$\varepsilon(t_j n_j, n_j) \ge \varepsilon^+(n_j) - \frac{1}{n_j}. \tag{2.25}$$

It is easy to see that

$$\begin{aligned} \int_0^{n_j^{5/8}t_j} R^+(n_j t_j, u; n_j)dG(u) &= g(t_j)\frac{\sqrt{2}}{n_j^{1/4}}\left(1 + \frac{\varepsilon^+(n_j)}{\ln(n_j t_j + 1)}\right)G\left(n_j^{5/8}t_j\right) \\ &\quad + \frac{1}{n_j^{1/2}}\int_0^{n_j^{5/8}t_j} r^+(n_j, u)dG(u) \qquad (2.26) \\ &= g(t_j)\frac{\sqrt{2}}{n_j^{1/4}}\left(1 + \frac{\varepsilon^+(n_j)}{\ln(n_j t_j + 1)}\right) + o\left(\frac{1}{n_j^{1/2}}\right), \end{aligned}$$

where we took into account (2.13), (2.14) and (2.1). By the same argument, we have

$$\frac{1}{2}\int_0^{n_j^{5/8}t_j}\left(R^+(n_j t_j, u; n_j)\right)^2 dG(u) = g^2(t_j)\frac{1}{n_j^{1/2}}\left(1 + \frac{\varepsilon^+(n_j)}{\ln(n_j t_j + 1)}\right)^2 + o\left(\frac{1}{n_j^{1/2}}\right). \tag{2.27}$$

Recalling (2.25) we get

$$\begin{aligned} Q_{n_j t_j}(\phi_{n_j}) &= g(t_j)\frac{\sqrt{2}}{n_j^{1/4}}\left(1 + \frac{\varepsilon(n_j t_j, n_j)}{\ln(n_j t_j + 1)}\right) \\ &\ge g(t_j)\frac{\sqrt{2}}{n_j^{1/4}}\left(1 + \frac{\varepsilon^+(n_j)}{\ln(n_j t_j + 1)}\right) - \frac{c}{n_j}. \end{aligned}$$

Therefore, by (2.8), (2.26) and (2.27), we have

$$
\begin{aligned}
& g\left(t_j\right) \frac{\sqrt{2}}{n_j^{1/4}}\left(1+\frac{\varepsilon^{+}(n_j)}{\ln\left(n_j t_j+1\right)}\right)-\frac{c}{n_j} \\
\leq\; & g^2\left(t_j\right) \frac{1}{n_j^{1/2}}+\frac{a\left(n_j t_j, n_j\right)}{\left(n_j t_j\right)^{1/2}}+g\left(t_j\right) \frac{\sqrt{2}}{n_j^{1/4}}\left(1+\frac{\varepsilon^{+}(n_j)}{\ln\left(n_j t_j+1\right)}\right)+o\left(\frac{1}{n_j^{1/2}}\right) \\
& -g^2\left(t_j\right) \frac{1}{n_j^{1/2}}\left(1+\frac{\varepsilon^{+}(n_j)}{\ln\left(n_j t_j+1\right)}\right)^2 .
\end{aligned}
$$

After cancellations and evident transformations, Lemma 2.7 implies

$$
g^2\left(t_j\right)\left(\frac{2 \varepsilon^{+}(n_j)}{\ln\left(n_j t_j+1\right)}+\left(\frac{\varepsilon^{+}(n_j)}{\ln\left(n_j t_j+1\right)}\right)^2\right)=o(1), \quad k \rightarrow \infty.
$$

Since

$$
\inf_{t_0 \leq t \leq T} g^2(t) \geq c>0,
$$

we get

$$
\varepsilon^{+}(n_j)=o\left(\ln\left(n_j t_j+1\right)\right)=o\left(\ln n_j\right), \quad j \rightarrow \infty.
$$

For $n_j \leq n<n_{j+1}$, we have $\varepsilon^{+}(n) \leq \varepsilon^{+}(n_j)$ and consequently,

$$
\frac{\varepsilon^{+}(n)}{\ln n} \leq \frac{\varepsilon^{+}(n_j)}{\ln n_j}.
$$

This implies

$$
\limsup_{n \rightarrow \infty} \frac{\varepsilon^{+}(n)}{\ln n}=0
$$

and, in particular, (2.23) is valid. Hence the statement of the lemma follows. ∎

Note that

$$
0 \leq 1+\frac{\varepsilon^{-}(n)}{\ln(nt+1)} \leq c.
$$

Now similar to Lemma 2.10 we have

Lemma 2.11

$$
\liminf_{n \rightarrow \infty} n^{1/4} Q_{nt}\left(\phi_n\right) \geq \sqrt{2} g(t).
$$

From Lemmas 2.10 and 2.11 it follows that

$$
Q_{nt}\left(\phi_n\right) \sim \sqrt{2} n^{-1/4} g(t). \tag{2.28}
$$

Thus, we have finished the proof of (2.4) for $k=1$.

Next, we consider finite-dimensional distributions. For simplicity of notation we treat the case $k=2$ only. The general case is similar to that at the end of the next section where the notation is relatively simpler, so we have treated the general k there.

Let

$$
Q_{t_1, t_2}(\phi^1, \phi^2)=\mathbb{E}_{\delta_0}\left[1-\exp\left(-\left\langle Z_{t_1}, \phi^1\right\rangle-\left\langle Z_{t_2}, \phi^2\right\rangle\right)\right].
$$

Then

$$
\begin{aligned}
Q_{t_1,t_2}(\phi^1,\phi^2) &= 1-\mathbb{E}\left[e^{-\phi^1(\xi_{t_1})-\phi^2(\xi_{t_2})}1_{\{\tau>t_2\}}\right]\\
&\quad -\mathbb{E}\left[e^{-\phi^1(\xi_{t_1})}f(1-Q_{t_2-\tau}(\phi^2))1_{\{t_1<\tau\le t_2\}}\right]\\
&\quad -\mathbb{E}\Big[f(1-Q_{t_1-\tau,t_2-\tau}(\phi^1,\phi^2))1_{\{\tau\le t_1\}}\Big].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
Q_{nt_1,nt_2}(\phi^1_n,\phi^2_n) &= 1-\mathbb{E}\left[e^{-\phi^1_n(\xi_{nt_1})-\phi^2_n(\xi_{nt_2})}1_{\{\tau>nt_2\}}\right] \qquad (2.29)\\
&\quad -\mathbb{E}\left[e^{-\phi^1_n(\xi_{nt_1})}f(1-Q_{nt_2-\tau}(\phi^2_n))1_{\{nt_1<\tau\le nt_2\}}\right]\\
&\quad -\mathbb{E}\Big(f(1-Q_{nt_1-\tau,nt_2-\tau}(\phi^1_n,\phi^2_n))1_{\{\tau\le nt_1\}}\Big)\\
&= \mathbb{E}\Big[\Big(1-e^{-\phi^1_n(\xi_{nt_1})-\phi^2_n(\xi_{nt_2})}\Big)1_{\{\tau>nt_2\}}\Big]\\
&\quad +\mathbb{E}\left[\Big(1-e^{-\phi^1_n(\xi_{nt_1})}\Big)1_{\{nt_1<\tau\le nt_2\}}\right]\\
&\quad +\mathbb{E}\Big[e^{-\phi^1_n(\xi_{nt_1})}\big(1-f(1-Q_{nt_2-\tau}(\phi^2_n))\big)1_{\{nt_1<\tau\le nt_2\}}\Big]\\
&\quad +\mathbb{E}\Big[\big(1-f(1-Q_{nt_1-\tau,nt_2-\tau}(\phi^1_n,\phi^2_n))\big)1_{\{\tau\le nt_1\}}\Big]\\
&= \mathbb{E}\Big[e^{-\phi^1_n(\xi_{nt_1})}\Big(1-e^{-\phi^2_n(\xi_{nt_2})}\Big)1_{\{\tau>nt_2\}}\Big]\\
&\quad +\mathbb{E}\left[\Big(1-e^{-\phi^1_n(\xi_{nt_1})}\Big)1_{\{\tau>nt_1\}}\right]\\
&\quad +\mathbb{E}\Big[e^{-\phi^1_n(\xi_{nt_1})}\big(1-f(1-Q_{nt_2-\tau}(\phi^2_n))\big)1_{\{nt_1<\tau\le nt_2\}}\Big]\\
&\quad +\mathbb{E}\Big[\big(1-f(1-Q_{nt_1-\tau,nt_2-\tau}(\phi^1_n,\phi^2_n))\big)1_{\{\tau\le nt_1\}}\Big]
\end{aligned}
$$

As for the case $k=1$ we have by Lemma 2.5 and Iglehart [12], as $n\to\infty$:

$$
\mathbb{E}\Big[e^{-\phi^1_n(\xi_{nt_1})}\Big(1-e^{-\phi^2_n(\xi_{nt_2})}\Big)\Big|\tau>nt_2\Big]\to\mathbb{E}\Big[e^{-\phi^1(\sqrt{t_2}\hat W_{t_1/t_2})}\Big(1-e^{-\phi^2(\sqrt{t_2}\hat W_1)}\Big)\Big]
$$

and

$$
\mathbb{E}\left[1-e^{-\phi^1_n(\xi_{nt_1})}\middle|\tau>nt_1\right]\to\mathbb{E}\left[1-e^{-\phi^1(\sqrt{t_1}\hat W_1)}\right].
$$

Similar to the proof of Lemma 2.7, we get

$$
\mathbb{E}\Big[e^{-\phi^1_n(\xi_{nt_1})}\big(1-f(1-Q_{nt_2-\tau}(\phi^2_n))\big)1_{\{nt_1<\tau\le nt_2\}}\Big]=o\left(\frac{1}{\sqrt{n}}\right).
$$

Therefore, (2.29) yields

$$
Q_{nt_1,nt_2}(\phi^1_n,\phi^2_n)=g^2(t_1,t_2)\frac{1+o(1)}{\sqrt{n}}+\int_0^{nt_1}\big(1-f\left(1-Q_{nt_1-u,nt_2-u}(\phi^1_n,\phi^2_n)\right)\big)\,dG(u).
$$

By arguments similar to those leading from (2.8) to (2.28) one can show that

$$
n^{1/4}Q_{nt_1,nt_2}(\phi^1_n,\phi^2_n)\to\sqrt{2}g(t_1,t_2),\ n\to\infty.
$$

Thus, we have finished the proof of the convergence of the Laplace transforms of the measures in question. To pass to the convergence of the finite dimensional distributions of these measures themselves, we need the following two lemmas.

Lemma 2.12 *Suppose that E is a Polish space with compactification $\bar{E}$. Suppose that K_m is a sequence of compact subsets of E increasing to E and $K_m \subset K_{m+1}^o$ for all m, where K^o denotes the interior of K. If $\mu_n \to \mu$ in $\mathcal{P}(\bar{E})$ and $\mu(E) = 1$, then $\mu_n \to \mu$ in $\mathcal{P}(E)$.*

Proof. According to the conditions of the lemma for all $\epsilon > 0$, there exists m such that $\mu(K_m) > 1 - \epsilon$. In particular, $\mu(K_{m+1}^o) > 1 - \epsilon$ for the same m. Since

$$\limsup_{n\to\infty} \mu_n(K_{m+1}^o) \geq \mu(K_{m+1}^o) > 1 - \epsilon,$$

there exists N such that for any $n \geq N$,

$$\mu_n(K_{m+1}) \geq \mu_n(K_{m+1}^o) > 1 - \epsilon.$$

Therefore, $\{\mu_n\}$ is tight in $\mathcal{P}(E)$. Clearly, μ is the only limit for this sequence. ∎

Lemma 2.13 $E := \mathcal{M}_F(\bar{\mathbb{R}})$ *satisfies the conditions of Lemma 2.12.*

Proof. Clearly, $\mathcal{M}_F(\bar{\mathbb{R}}) \sim \mathcal{P}(\bar{\mathbb{R}}) \times \mathbb{R}_+$ under the map $\nu \mapsto \left(\frac{\nu}{\nu(\bar{\mathbb{R}})}, \nu(\bar{\mathbb{R}})\right)$. Therefore, $\mathcal{M}_F(\bar{\mathbb{R}})$ has a compactification $\mathcal{P}(\bar{\mathbb{R}}) \times \overline{\mathbb{R}}_+$. Let

$$K_m = \{\mu \in \mathcal{M}_F(\bar{\mathbb{R}}) : \ \mu(\bar{\mathbb{R}}) \leq m\}.$$

The conditions in Lemma 2.12 can be verified easily. ∎

Completion of the proof of Theorem 1.1. Recall that $Q_{n\bar{t}}(\bar{\phi}_n)$ is given at the beginning of this section. Note that

$$\begin{aligned} \lim_{n\to\infty} \mathbb{E}_{n^{1/4}\delta_0}\left[\exp\left(-\sum_{i=1}^k \left\langle Z_{t_i}^n, \phi^i\right\rangle\right)\right] &= \lim_{n\to\infty} \left(1 - Q_{n\bar{t}}(\bar{\phi}_n)\right)^{n^{1/4}} \qquad (2.30)\\ &= \lim_{n\to\infty} \exp\left(n^{1/4}\log\left(1 - Q_{n\bar{t}}(\bar{\phi}_n)\right)\right)\\ &= \lim_{n\to\infty} \exp\left(-n^{1/4}Q_{n\bar{t}}(\bar{\phi}_n)\right)\\ &= \exp\left(-\sqrt{2}g(t_1, \cdots, t_k)\right) \end{aligned}$$

It is trivial that $(Z_{t_1}^n, \cdots, Z_{t_k}^n)$ is tight in the compactification of $\mathcal{M}_F(\bar{\mathbb{R}})^k$. Thus, there exists a subsequence $n_j \to \infty$, $j \to \infty$, such that $(Z_{t_1}^{n_j}, \cdots, Z_{t_k}^{n_j})$ converges to a limit $(Z_{t_1}^\infty, \cdots, Z_{t_k}^\infty)$. By (2.30) the limit is unique and hence $(Z_{t_1}^n, \cdots, Z_{t_k}^n) \to (Z_{t_1}^\infty, \cdots, Z_{t_k}^\infty)$ in distribution on the compactification of $\mathcal{M}_F(\bar{\mathbb{R}})^k$. Applying Lemmas 2.12 and 2.13, we see that $(Z_{t_1}^n, \cdots, Z_{t_k}^n) \to (Z_{t_1}^\infty, \cdots, Z_{t_k}^\infty)$ in distribution on $\mathcal{M}_F(\bar{\mathbb{R}})^k$. This proves point ii) of Theorem 1.1. Point i) of Theorem 1.1 then follows easily. ∎

3 Random limit for rescaled processes

This section is devoted to the proof of Theorem 1.2. The major steps of the subsequent proofs are similar to those of Section 2 and for this reason we only indicate those parts of the proofs which are essentially different. First, we give an estimate for the population size. Let

$$R_t(\lambda) := \mathbb{E}_{\delta_0}\left[1 - \exp\left\{-\lambda \left\langle Z_t, 1\right\rangle\right\}\right], \qquad \lambda \in [0, \infty).$$

Lemma 3.1 *There exists a constant c such that for all $\lambda \in [0,\infty)$*

$$R_t(\lambda) \le c\sqrt{1-e^{-\lambda}}\sqrt{1-G(t)}.$$

Proof. By (2.5), we have

$$R_t(\lambda) = (1-e^{-\lambda})(1-G(t)) + \int_0^t (1-f(1-R_{t-u}(\lambda))dG(u)$$

Applying the renewal theorem gives

$$R_t(\lambda) = 1-e^{-\lambda} - \frac{1}{2}\int_0^t R_{t-u}^2(\lambda))dU^1(u),$$

where

$$U^1(t) = \sum_{k=1}^{\infty} G^{*k}(t),$$

and G^{*k} is the k-multiple convolution of G with itself. It is known (see, for instance, [3], Ch. IV, Section 3, Theorem 1) that $R_t(\lambda)$ is monotone decreasing in t for each fixed $\lambda > 0$. Hence

$$\int_0^t R_{t-u}^2(\lambda)dU^1(u) \ge R_t^2(\lambda)\, U^1(t)$$

and, therefore,

$$R_t(\lambda) \le \sqrt{2(1-e^{-\lambda})}\frac{1}{\sqrt{U^1(t)}}.$$

By (2.1) and a statement in Section 14.3 of Feller [9],

$$U^1(t)(1-G(t)) \to \frac{2}{\pi}, \quad t\to\infty.$$

From this we get

$$R_t(\lambda) \le c_1\sqrt{2(1-e^{-\lambda})}\sqrt{1-G(t)}$$

as needed. ∎

Now we proceed to proving the convergence of the Laplace transform:

$$\lim_{n\to\infty} \mathbb{E}_{n^{1/4}\delta_0} \exp\left(-\sum_{i=1}^{k}\left\langle n^{-\alpha}Z_{t_i}^n, \phi^i\right\rangle\right) = \exp\left(-\sqrt{2}g_0(t_1,\cdots,t_k)\right). \tag{3.1}$$

We start with the case $k=1$. Let Q be the same as in Section 2. Then

$$\mathbb{E}_{\delta_0}\left(1-\exp\left\{-\left\langle n^{-\alpha}Z_t^n,\phi\right\rangle\right\}\right) = Q_{nt}(n^{-\alpha}\phi_n).$$

Lemma 3.2 *Let $\gamma := \frac{\alpha}{2}+\frac{3}{4}$. Let $b(nt,n)$ be defined by equality*

$$\begin{aligned} Q_{nt}\left(n^{-\alpha}\phi_n\right) &= n^{-2\beta}g_0^2(t) + b(nt,n)\,n^{-2\beta} \\ &\quad + \int_0^{n^{\gamma}t}\left(1-f\left(1-Q_{nt-u}\left(n^{-\alpha}\phi_n\right)\right)\right)dG(u). \end{aligned} \tag{3.2}$$

For fixed $0<t_0<T<\infty$ let

$$b(n) := \sup_{t_0\le t\le T} |\,b(nt,n)\,|.$$

Then

$$\lim_{n\to\infty} b(n) = 0.$$

Proof. Let h, G be as in the previous section. By (2.6) we have

$$Q_{nt}\left(n^{-\alpha}\phi_n\right) = h\left(n^{-\alpha}\phi_n, nt\right)\left(1 - G\left(nt\right)\right) + \int_0^{nt}\left(1 - f\left(1 - Q_{nt-u}\left(n^{-\alpha}\phi_n\right)\right)\right) dG\left(u\right). \tag{3.3}$$

Similar to (2.7) we have by means of Lemma 2.5 and Iglehart [12])

$$n^{\alpha} h\left(n^{-\alpha}\phi_n, nt\right) \to \mathbb{E}\left[\phi\left(\sqrt{t}\hat{W}_1\right)\right], \quad n \to \infty. \tag{3.4}$$

Observe that

$$Q_s(n^{-\alpha}\phi_n) \le \|\phi\|_{\infty} n^{-\alpha}.$$

Therefore, the integral term in (3.3) (taken over the range $[nt(1-\epsilon), nt]$) does not exceed

$$\begin{aligned} c\int_{nt(1-\epsilon)}^{nt} n^{-\alpha} dG\left(u\right) &= cn^{-\alpha}((1 - G(nt(1-\epsilon))) - (1 - G(nt)) \\ &= cn^{-\alpha}\left(\frac{1}{\sqrt{nt(1-\epsilon)}} - \frac{1}{\sqrt{nt}} + o(\frac{1}{\sqrt{n}})\right) \\ &= \frac{c}{n^{\alpha}\sqrt{nt}}\frac{\epsilon}{\sqrt{1-\epsilon} + 1 - \epsilon} + o(n^{-(\alpha+1/2)}). \end{aligned}$$

By Lemma 3.1, we have

$$Q_u(n^{-\alpha}\phi_n) \le c\sqrt{n^{-\alpha}(1 - G(u))}. \tag{3.5}$$

Then the integral term in (3.3) (taken over the range $[n^{\gamma}t, nt(1-\epsilon)]$) does not exceed

$$\begin{aligned} \int_{n^{\gamma}t}^{nt(1-\epsilon)} Q_{nt-u}(n^{-\alpha}\phi_n) dG(u) &\le c\sqrt{n^{-\alpha}(1 - G(nt\epsilon))}(1 - G(n^{\gamma}t)) \\ &= cn^{-\alpha/2}(nt\epsilon)^{-1/4}(n^{\gamma}t)^{-1/2} \\ &= cn^{-(\alpha/2+1/4+\gamma/2)} \\ &= o(n^{-(\alpha+1/2)}). \end{aligned}$$

The conclusion of the lemma then follows from (2.1) and (3.2). ∎

Now we define a function $\varepsilon_0\left(nt, n\right)$ by the equality

$$Q_{nt}\left(n^{-\alpha}\phi_n\right) = g_0\left(t\right)\sqrt{2}n^{-\beta}\left(1 + \frac{\varepsilon_0\left(nt, n\right)}{\ln\left(nt + 1\right)}\right). \tag{3.6}$$

The following lemma says that $Q_{nt-u}(n^{-\alpha}\phi_n)$ in (3.3) is close to $Q_{nt}(n^{-\alpha}\phi_n)$ for $0 \le u \le n^{\gamma}t$.

Lemma 3.3 *Let*

$$\varepsilon_0^{+}\left(n\right) := \sup_{t_0 \le t \le T} \varepsilon_0\left(nt, n\right) \;\; \textit{and} \; \varepsilon_0^{-}\left(n\right) := \inf_{t_0 \le t \le T} \varepsilon_0\left(nt, n\right).$$

Then $\varepsilon_0^{+}(n)$ and $\varepsilon_0^{-}(n)$ are finite. Further, there exist functions $r_0^{\pm}(n,u)$ such that for all $t \in [t_0, T]$, we have

$$\limsup_{n\to\infty} \sup_{0 \le u \le n^{\gamma}t} \left(|r_0^{-}(n,u)| + |r_0^{+}(n,u)|\right) = 0 \tag{3.7}$$

and there exists a constant $n_0(t_0, T)$ *such that for all* $n > n_0(t_0, T)$

$$R_0^-(nt, u, n) \le Q_{nt-u}\left(n^{-\alpha}\phi_n\right) \le R_0^+(nt, u, n),$$

where

$$R_0^{\pm}(tn, u, n) = g_0(t)\sqrt{2}n^{-\beta}\left(1 + \frac{\varepsilon_0^{\pm}(n)}{\ln(nt+1)}\right) + \frac{r_0^{\pm}(n, u)}{n^{\beta}}.$$

Proof. Since $\phi(0) > 0$ and ϕ is continuous it follows that

$$\inf_{t_0 \le t \le T} g_0(t) > 0$$

and, in view of the inequality

$$1 + \frac{\varepsilon_0(nt, n)}{\ln(nt+1)} \le \frac{Q_{nt}(n^{-\alpha}\phi_n)}{\sqrt{2}g_0(t)}n^{\beta}$$

we conclude that $\varepsilon_0^+(n) < \infty$ for each $n > 0$. Similarly, $\varepsilon_0^-(n) > -\infty$ for each $n > 0$. Similar to (2.18), for $0 \le u \le n^{\gamma}t$ we have

$$\left| t^{1/4}g_0(t) - \left(\frac{nt-u}{n}\right)^{1/4} g_0\left(\frac{nt-u}{n}\right)\right| = o(n^{\gamma-1}). \tag{3.8}$$

Note that as $n \to \infty$

$$\frac{1}{(nt-u)^{\beta}} = \frac{1}{(nt)^{\beta}} + o\left(n^{-\beta}\right) \tag{3.9}$$

and

$$\frac{1}{\ln(nt-u+1)} = \frac{1}{\ln(nt+1)}\left(1 + o\left(n^{\gamma-1}\right)\right). \tag{3.10}$$

The conclusions of the lemma then follow easily. ∎

With these preparations, the proof of the following lemma is similar to that of Lemma 2.3. We omit the details.

Lemma 3.4

$$\lim_{n\to\infty} n^{\beta}Q_{nt}(n^{\alpha}\phi_n) = \sqrt{2}g_0(t).$$

Now we consider the multidimensional case and introduce additional notations. For a fixed tuple $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} = \infty$ and Lipschitz continuous functions $\phi^1, \cdots, \phi^k$ set

$$Q_{\bar{t}_{j,k}}(\bar{\phi}) := Q_{t_j,\ldots,t_k}(\phi^j, \cdots, \phi^k),\ 1 \le j \le k, \quad \text{and} \quad Q_{\bar{t}_{k,k+1}}(\bar{\phi}) :\equiv 0.$$

For a fixed $u \in [0, t_j)$ we use the notation $Q_{\bar{t}_{j,\,k}-u}(\bar{\phi}) := Q_{t_j-u,\ldots,t_k-u}(\phi^j, \cdots, \phi^k)$. Note that

$$\begin{aligned}
& Q_{t_1,\cdots,t_k}(\phi^1, \cdots, \phi^k) \\
= \; & Q_{\bar{t}_{1,k}}(\bar{\phi}) = \mathbb{E}\left[1 - \exp\left(-\sum_{i=1}^{k}\langle Z_{t_i}, \phi^i\rangle\right)\right] \\
= \; & 1 - \sum_{j=1}^{k+1}\mathbb{E}\left[\exp\left(-\sum_{i=1}^{k}\langle Z_{t_i}, \phi^i\rangle\right)1_{\{t_{j-1}<\tau\le t_j\}}\right] \\
= \; & 1 - \sum_{j=1}^{k+1}\mathbb{E}\left[\left(\exp\left(-\sum_{i=1}^{j-1}\phi^i(\xi_{t_i})\right)\right) f\left(1 - Q_{\bar{t}_{j,k}-\tau}(\bar{\phi})\right) 1_{\{t_{j-1}<\tau\le t_j\}})\right].
\end{aligned}$$

Then

$$\begin{aligned}
&Q_{n\bar{t}_{1,k}}(\bar{\phi}_n) \\
=\;& Q_{nt_1,\cdots,nt_k}(n^{-\alpha}\phi_n^1,\cdots,n^{-\alpha}\phi_n^k) \\
=\;& 1-\sum_{j=1}^{k+1}\mathbb{E}\left[\left(\exp\left(-n^{-\alpha}\sum_{i=1}^{j-1}\phi_n^i(\xi_{nt_i})\right)\right)f\left(1-Q_{n\bar{t}_{j,k}-\tau}(n^{-\alpha}\bar{\phi}_n)\right)1_{\{nt_{j-1}<\tau\le nt_j\}}\right] \\
=\;& \sum_{j=1}^{k+1}\mathbb{E}\left[\left(1-\exp\left(-n^{-\alpha}\sum_{i=1}^{j-1}\phi_n^i(\xi_{nt_i})\right)\right)f\left(1-Q_{n\bar{t}_{j,k}-\tau}(n^{-\alpha}\bar{\phi}_n)\right)1_{\{nt_{j-1}<\tau\le nt_j\}}\right] \\
=\;& \sum_{j=2}^{k+1}\mathbb{E}\left[\left(1-\exp\left(-n^{-\alpha}\sum_{i=1}^{j-1}\phi_n^i(\xi_{nt_i})\right)\right)1_{\{nt_{j-1}<\tau\le nt_j\}}\right] \\
&-\sum_{j=2}^{k+1}\mathbb{E}\left[\left(1-f\left(1-Q_{n\bar{t}_{j,k}-\tau}(n^{-\alpha}\bar{\phi}_n)\right)\right)1_{\{nt_{j-1}<\tau\le nt_j\}}\right] \\
&+\mathbb{E}\left[\left(1-f\left(1-Q_{n\bar{t}_{1,k}-\tau}(n^{-\alpha}\bar{\phi}_n,)\right)\right)1_{\{\tau\le nt_1\}}\right].
\end{aligned}$$

Similar to the proof of Lemma 3.2, we have as $n\to\infty$

$$\sum_{j=2}^{k+1}\mathbb{E}\left[\left(1-f\left(1-Q_{n\bar{t}_{j,k}-\tau}(n^{-\alpha}\bar{\phi}_n)\right)\right)1_{\{nt_{j-1}<\tau\le nt_j\}}\right]=o(n^{-2\beta})$$

and

$$\begin{aligned}
&\sum_{j=2}^{k+1}\mathbb{E}\left[\left(1-\exp\left(-n^{-\alpha}\sum_{i=1}^{j-1}\phi_n^i(\xi_{nt_i})\right)\right)1_{\{nt_{j-1}<\tau\le nt_j\}}\right] \\
&\quad=\sum_{j=2}^{k+1}\mathbb{E}\left[\left(1-\exp\left(-n^{-\alpha}\sum_{i=1}^{j-1}\phi_n^i(\xi_{nt_i})\right)\right)\left(1_{\{\tau>nt_{j-1}\}}-1_{\{\tau>nt_j\}}\right)\right] \\
&\quad=\sum_{j=1}^{k}\mathbb{E}\left[\exp\left(-n^{-\alpha}\sum_{i=1}^{j-1}\phi_n^i(\xi_{nt_i})\right)(1-e^{-n^{-\alpha}\phi_n^j(\xi_{nt_j})})1_{\{\tau>nt_j\}}\right].
\end{aligned}$$

The same as before, we have

$$n^{\alpha}\mathbb{E}\left[\exp\left(-n^{-\alpha}\sum_{i=1}^{j-1}\phi_n^i(\xi_{nt_i})\right)(1-e^{-n^{-\alpha}\phi_n^j(\xi_{nt_j})})\,\middle|\,\tau>nt_j\right]\to\mathbb{E}\left[\phi^j(\sqrt{t_j}\hat{W}_1)\right].$$

Therefore,

$$\begin{aligned}
Q_{n\bar{t}_{1,k}}(n^{-\alpha}\bar{\phi}_n) \;&=\; n^{-2\beta}g_0^2(t_1,\cdots,t_k)+o(n^{-2\beta}) \\
&\quad+\int_0^{nt_1}\left(1-f\left(1-Q_{n\bar{t}_{1,k}-u}(n^{-\alpha}\bar{\phi}_n)\right)\right)dG(u).
\end{aligned}$$

Similar to the arguments leading from (3.3) to Lemma 3.4, we see that (3.1) holds. Then, proceed as at the end of the last section, we finish the proof of Theorem 1.2.

4 A law of large number type theorem

In this section, we show that when the initial system is rich enough, a law of large number type theorem holds. Denote $Q_{nt}(n^{-\beta}\phi_n)$ by $q_n(t)$. By (2.6), we have

$$q_n(t) = h\left(n^{-\beta}\phi_n, nt\right)(1 - G(nt)) + \int_0^t \left(q_n(t-u) - \frac{1}{2}q_n(t-u)^2\right) dG^n(u)$$

where $G^n(u) := G(nu)$. By renewal theorem (cf. Feller [9]), we then have

$$q_n(t) = \int_0^t h\left(n^{-\beta}\phi_n, n(t-u)\right)(1 - G(n(t-u)))\, dU^n(u) - \frac{1}{2}\int_0^t q_n(t-u)^2 dU_1^n(u) \quad (4.1)$$

where

$$U^n = \sum_{k=0}^{\infty}(G^n)^{*k} \qquad \text{and} \qquad U_1^n = U^n * G.$$

We know that

$$h\left(n^{-\beta}\phi_n, nt\right) \le \|\phi\|_\infty n^{-\beta}.$$

This estimate combined with (4.1) gives

$$q_n(t) \le \|\phi\|_\infty n^{-\beta}\int_0^t (1 - G(n(t-u)))\, dU^n(u) = \|\phi\|_\infty n^{-\beta}.$$

Therefore,

$$\int_0^t q_n(t-u)^2 dU_1^n(u) \le (\|\phi\|_\infty n^{-\beta})^2 U_1^n(t) \le \frac{c}{n^{2\beta-1/2}}.$$

We know from (2.1) that as $n \to \infty$

$$\sqrt{n}(1 - G(nt)) \to \frac{d^2}{\sqrt{t}}$$

and that (cf. [9])

$$\frac{1}{\sqrt{n}}U^n(t) \to \frac{2\sqrt{t}}{\pi d^2}.$$

Hence, recalling (3.4) we get as $n \to \infty$

$$n^\beta \int_0^t h\left(n^{-\beta}\phi_n, n(t-u)\right)(1 - G(n(t-u)))\, dU^n(u) \to \frac{2}{\pi}\int_0^t \frac{\mathbb{E}[\phi(\sqrt{t-u}\hat{W}_1)]}{\sqrt{t-u}}\, d\sqrt{u}.$$

This proves Theorem 1.3.

Acknowledgement: Most of this work was done when both authors visited WIAS. We would like to thank WIAS for the financial support and hospitality. We would like to thank Klaus Fleischmann for helpful suggestions.

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