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A simple method to study the transitional dynamics in endogenous growth models

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Abstract

We introduce a simple method of analyzing the transitional dynamics of the Uzawa-Lucas endogenous growth model with human capital externalities. We use the value function approach to solve both the social planner's optimization problem in the centralized economy and the representative agent's optimization problem in the decentralized economy. The complexity of the Hamilton-Jacobi-Bellman equations is significantly reduced to an initial value problem for one ordinary differential equation. This approach allows us to find the optimal controls for the non-concave Hamiltonian in the centralized case and to identify the symmetric Nash equilibrium of the agents' optimal strategies in the decentralized case. For a wide range of the degree of the human capital externality we calculate the global transitional dynamics towards the balanced growth path. The U-shaped course of output growth rates is explained in detail.

JEL Classifications: C61, O41, C72

1 Introduction

We introduce a simple solution method for the analysis of endogenous growth models. We demonstrate this method by studying the transitional dynamics of the Uzawa (1965) and Lucas (1988) model. Our method is of global character and is closest to the time-elimination method by Mulligan and Sala-i-Martin (1991). In his seminal paper, Lucas (1988) argues that the economy's average level of human capital contributes to total factor productivity in goods production thereby causing an externality. Since our method is based on the value function approach, it is generally applicable to the centralized, possibly non-concave, case as well as to the decentralized case. Furthermore, the value function is deterministic even in a model with uncertainty such that this approach is easier to generalize in this direction. We apply our method first to the social planner's optimization problem and second to the representative agent's problem in a decentralized economy. We are able to give analytically explicit expressions for functions of the resulting highly non-linear Hamilton-Jacobi-Bellman (HJB) equations, which to our knowledge have not yet been obtained before. This allows us to follow the value function approach rather than the Pontryagin maximum principle (Kamien and Schwartz, 1991). Using the model's homogeneity (Caballé and Santos, 1993) we reduce the dimension of the optimal decision rules and thereby simplify our analysis. The knowledge of an explicit functional form that solves the HJB equation facilitates our efforts. This 'candidate' function, however, is not the value function except for one specific initial value which corresponds to the balanced growth path solution. We show that at this particular value a saddle point behavior occurs.

The candidate function outside the balanced growth path yields the unstable solution branch in the phase diagram and thus non-feasible controls. Finding an analytic expression for the stable solution branch, which gives the true value function, seems to be a daunting task. We therefore transform the HJB equation suitably and finally arrive at an explicit one-dimensional ordinary differential equation, which can be solved by standard numerical schemes. In the centralized case, the usual transformation for implicit differential equations applies and the candidate function provides the correct initial conditions. In the decentralized case, our analysis is restricted to non-cooperative symmetric Nash equilibria and we first arrive at a partial differential equation. The additional variable enters because the path of the average level of human capital is treated as exogenously given by the agents. However, by exploiting the imposed symmetric Nash equilibrium condition we again obtain an ordinary differential equation. Finally, the global character of our method allows to analyze the centralized as well as the decentralized economy far away from the balanced growth path.

A main feature of the Uzawa-Lucas model constitutes that the agents have to 'learn or to do' (Chamley, 1993), i.e. they have to allocate their human capital between two production sectors. The first sector is a goods sector where a single good usable for consumption and physical capital investment is produced. This sector exhibits a production technology that uses human as well as physical capital. The second sector is a schooling sector where agents augment their stock of human capital. Here, human capital is the only input factor. Since the average level of human capital influences the productivity of each individual's stock of human capital, there is a clear linkage between the average stock of human capital and the opportunity costs of schooling. In the decentralized economy, agents are only compensated for their respective private factor supplies but not for their influence on the economy-wide average stock of human capital. Also, they do not coordinate their actions which leads to the situation where agents treat the opportunity costs of schooling as exogenously given. In other words: the average level of human capital causes a cost externality in the schooling decision. As a result, the agents' schooling decisions are not Pareto-optimal and the incentive structure described above leads to non-efficient equilibria. By contrast, the central planner, who internalizes all relevant incentives, reaches the social optimum by choosing the efficient level of schooling activities.

The theoretical model considered here differs from that studied by Lucas (1988) only in the choice of the utility function. We assume logarithmic preferences, implying that the constant intertemporal elasticity of substitution is equal to one. This assumption reduces the number of parameters by one and simplifies the calculations. Nevertheless, the balanced growth path implications are analogous to those in the more general case. Only in some very restrictive cases explicit solutions are known. Xie (1994) studies the special constellation where the inverse of the intertemporal elasticity of substitution and the output elasticity of physical capital in the goods sector are equal and focusses on the decentralized case. Hartley and Rogers (2003) solve an Arrow and Kurz (1970) type of a two sector growth model in closed form after introducing a stochastic disturbance in the law of capital accumulation.

The allocation of human capital is the mechanism that causes the differences between the outcomes of the centralized and the decentralized case. Our numerical results reveal the underlying incentive structure and confirm that the model implies U-shaped adjustment of output growth rates (Mulligan and Sala-i-Martin, 1993). When physical capital is relatively scarce, the growth rate of output is very high but declining. It even falls below the balanced growth path value before it rises again and finally converges to the balanced growth rate.

The paper is organized as follows. Section 2 introduces the model. Section 3 presents our strategy of solving the social planner's problem. Based on our results of the third section, we treat in Section 4 the decentralized case, which requires a higher analytical and numerical effort. Section 5 discusses our solution method and compares it to other approaches. Section 6 presents numerical results explaining the U-shaped course of output growth rates. Section 7 concludes. The Appendix contains proofs of statements omitted in the paper.

2 The model

We assume a closed economy populated by a large number of identical infinitely-lived agents. Firms are producing a single good and there is a schooling sector providing educational services. Population is constant and normalized to one. The representative agent has logarithmic preferences over consumption streams

$$U = \int_{t=0}^{\infty} e^{-\rho t} \ln c_t \, dt, \tag{1}$$

where c_t is the level of consumption at time t and $\rho > 0$ is the subjective discount rate. The logarithmic utility function implies that the intertemporal elasticity of substitution is equal to one. Agents have a fixed endowment of time, which is normalized as a constant flow of one unit. The variable u_t denotes the fraction of time allocated to goods production at time t. As agents do not benefit from leisure, the whole time budget is allocated to the two sectors. The fraction $1 - u_t$ of time is spent in the schooling sector. Hence, in any solution the condition

$$u_t \in [0, 1] \tag{2}$$

has to be fulfilled. The variables c_t and u_t are the agent's control variables. Human capital production is determined by a linear technology in human capital

$$\dot{h}_t = B\,(1 - u_t)\,h_t, \tag{3}$$

where we assume that B is positive. This schooling technology together with condition (2) implies that human capital will never shrink, i.e. the growth rate

$\dot{h}$ must be non-negative. It also implies that the realized marginal and average product are equal to $B(1-u_t)$. Note that we abstract from depreciation. We assume an infinitely large number of profit-maximizing firms producing a single good. They are using a Cobb-Douglas technology in the two inputs physical and human capital. The level of human capital utilized in goods production equals the total level of the stock of human capital multiplied by the fraction of time spent in the goods sector at time t. Total factor productivity A is enhanced by the external effect γ of the economy's average stock of human capital, $h_{a,t}$. The output y_t is determined by

$$y_t = Ak_t^{\alpha}\left(u_t h_t\right)^{1-\alpha} h_{a,t}^{\gamma}.$$

The parameter α is the output elasticity of physical capital and we assume $\alpha \in (0,1)$. We further assume that the exponent γ is nonnegative. If we set u_t equal to one, we get the potential output in the goods sector. Since all agents are homogeneous, the economy's average level of human capital must equal the representative agent's level of human capital at any point in time

$$h_t = h_{a,t}, \quad \forall t \geq 0. \tag{4}$$

In the decentralized economy the representative firm is taking $h_{a,t}$ as given and rents physical and human capital on complete factor markets. Market clearing factor prices and the zero profit condition allow to state the agent's budget constraint as

$$y_t = c_t + \dot{k}_t, \qquad \forall t \geq 0.$$

The right-hand side describes the spending of the agent's earnings, where $\dot{k}_t$ is the rate of change of the agent's physical capital stock k_t. Since we abstract from depreciation, this rate corresponds to the agent's net investment in physical capital. The left-hand side collects the streams of income stemming from the agent's physical capital stock and from his work effort $u_t h_t$. We assume that the initial values k_0 and h_0 are strictly positive. Note that by consuming more than current production it is possible to disinvest in physical capital, i.e. the growth rate of physical capital turns negative.

Informational structure

We want to analyze the model from two perspectives. The first perspective is the one taken by the benevolent planer in a centralized economy. In order to derive the social optimum, all information is used and all incentives are internalized. The second refers to the representative agent in a decentralized economy. At time 0 he has to choose optimal time dependent consumption and schooling decision paths. Then he is committed to these paths for the whole future. The non-cooperative symmetric Nash equilibrium condition implies that he is trying to find the best response given the path of the economy-wide average schooling decision. Furthermore, he takes into account that every single agent does also play the best response and so forth. Perfect foresight implies that we are looking for fixed points in the space of optimal time dependent decision rules.

3 The centralized economy

This section presents our strategy of solving the social planner's problem. The planner exploits the equality condition (4) and his dynamic optimization problem (DOP) is given by:

$$U = \max_{c_t, u_t} \int_{t=0}^{\infty} e^{-\rho t} \ln c_t \, dt,$$

with respect to the state dynamics

$$\begin{aligned}
\dot{k}_t &= A k_t^{\alpha} u_t^{1-\alpha} h_t^{1-\alpha+\gamma} - c_t, && \forall\, t \geq 0, \\
\dot{h}_t &= B\,(1 - u_t)\, h_t, && \forall\, t \geq 0, \\
k_t &\geq 0 \quad \text{and} \quad h_t \geq 0 && \forall\, t \geq 0.
\end{aligned}$$

The initial values $k_0, h_0 > 0$ are assumed to be given. Requiring these endowments to be strictly positive ensures an interior solution and rules out trivial solutions. Since we assume a Cobb Douglas production technology and logarithmic utility, this restriction will be satisfied automatically under optimal controls. In Section 3.1 we analyze the above DOP. Using homogeneity in the initial conditions, we reduce the HJB equation to only one implicit ordinary differential equation and give an explicit solution of the HJB equation. This 'candidate' is indeed the planner's value function at one point, which corresponds to the balanced growth path of the economy. An application of the candidate function outside this point yields non-feasible controls. The implied allocation of human capital leads to high schooling efforts when the opportunity costs of schooling in goods production are low and it leads to little schooling efforts when these costs are high. Surely, this affords an opportunity for arbitrage. As a consequence this inefficiency is accelerating and finally implies non-feasible controls. In Section 3.2, we transform the problem of determining the value function into an initial value problem for an explicit one-dimensional ordinary differential equation. The linear approximation at the saddle point is given in terms of the parameters. Moreover, the explicit form allows it to apply classical numerical methods in order to determine the value function globally: In our simulations the standard Mathematica procedure NDSolve worked very efficiently.

3.1 The social planner's optimization problem

In the DOP the two control functions c_t and u_t are chosen by the social planner given the set of admissible controls

$$(c_t, u_t)_{t \geq 0} \in \mathcal{X} := \{(f, g) : [0, \infty) \to X \mid f,\, g \text{ locally bounded and measurable}\}$$

with $X := [0, \infty) \times [0, 1]$. Using the logarithmic preferences and the exponential discount rate, the planner defines the representative agent's value function:

$$V(k_0, h_0) := \max_{(c,u) \in \mathcal{X}} \begin{cases} \int_0^{\infty} e^{-\rho t} \ln c_t \, dt, & \tau = \infty \\ -\infty, & \tau < \infty, \end{cases}$$

where τ denotes the stopping time $\tau := \inf\{t \geq 0 \,|\, k_t = 0\}$. This is a classical optimal control problem with infinite horizon (Fleming and Soner, 1995, Section I.7). However, the results derived there are not directly applicable because $x \mapsto x^p$ for $p \in (0,1)$ and $x > 0$ is only locally Lipschitz continuous and we allow $V = -\infty$. Nevertheless, it turns out that the optimal controls imply dynamics where the state variables are bounded away from zero so that $\tau = \infty$ holds and the above-mentioned conditions are satisfied. In order to determine the value function, we write down the HJB equation for the value function V evaluated at $k, h > 0$ and $t \geq 0$:

$$\rho V = \max_{(c,u)\in X} \left\{ \ln c + V_k (A k^\alpha u^{1-\alpha} h^{1-\alpha+\gamma} - c) + V_h B(1-u)h \right\}.$$

Here, V_k and V_h denote the partial derivatives with respect to k and h, which can be interpreted as the shadow prices of relaxing the corresponding constraints. Recall that in the case of an infinite time horizon, autonomous equations, and an exponential discount rate the HJB equation simplifies to a differential equation independent of time. We determine the maximum by looking at the first order necessary conditions. The implied optimal controls are given by:

$$c^* = V_k^{-1}, \tag{5}$$

$$u^* = \left(\frac{A(1-\alpha)V_k}{BV_h}\right)^{\frac{1}{\alpha}} \frac{k}{h^{\frac{\alpha-\gamma}{\alpha}}}. \tag{6}$$

The planner chooses the consumption stream such that the marginal utility is equal to the marginal change of wealth with respect to physical capital. The optimal allocation of human capital between the two sectors is determined by the weighted ratio of the marginal changes in goods production and schooling due to a marginal shifting of the human capital allocation. The respective weights are the planner's shadow prices of the corresponding state variable. Since the value function V is obviously increasing in its arguments, the relation (5) ensures that the consumption rate is positive. Equally, $u^* \in (0, \infty)$ holds, but $u^* > 1$ may well occur. For the moment, let us suppose that the controls (u^*, c^*) found above are feasible. Then the HJB equation becomes:

$$\rho V + 1 = -\ln V_k + \alpha k \left(A V_k h^\gamma\right)^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{BV_h}\right)^{\frac{1-\alpha}{\alpha}} + BV_h h. \tag{7}$$

In fact, the HJB equation is homogeneous in the initial conditions. This allows us to follow Mulligan and Sala-i-Martin (1991) in defining a so-called state-like variable $x_t := k_t h_t^{-(1-\alpha+\gamma)/(1-\alpha)}$. The introduction of x_t reduces the complexity of the problem by one dimension. Its dynamics are given by

$$\dot{x}_t = A x_t^\alpha u_t^{1-\alpha} - c_t x_t k_t^{-1} - \tfrac{1-\alpha+\gamma}{1-\alpha} B(1-u_t)x_t. \tag{8}$$

Introducing the control-like variable $q_t := c_t x_t k_t^{-1}$, we see that the evolution of x_t is completely described by x_t, u_t and q_t. For any initial state $(\tilde{k}_0, \tilde{h}_0)$ with $\tilde{x}_0 := \tilde{k}_0 \tilde{h}_0^{-(1-\alpha+\gamma)/(1-\alpha)} = x_0$ we are led to apply the same controls $\tilde{u}_t = u_t$ and

$\tilde{q}_t = q_t$. The only difference is that the consumption rate $\tilde{c}_t$ differs from c_t by the factor $(\tilde{h}_0/h_0)^{(1-\alpha+\gamma)/(1-\alpha)}$. Any solution $V(k,h)$ can thus be deduced from $V(x,1) =: f(x)$ via

$$V(k,h) = f\big(kh^{-\frac{1-\alpha+\gamma}{1-\alpha}}\big) + \tfrac{1-\alpha+\gamma}{\rho(1-\alpha)} \ln h.$$

The HJB equation in terms of f can be derived from

$$V_k = \tfrac{f'(x)x}{k} \quad \text{and} \quad V_h = \tfrac{1-\alpha+\gamma}{1-\alpha}\big(\tfrac{1}{\rho h} - \tfrac{f'(x)x}{h}\big).$$

Note that $q^* = 1/f'$ holds. Furthermore, the optimal human capital allocation (6) can be stated in terms of the optimal choice of the control-like variable q^*:

$$u^* = \left(\frac{A(1-\alpha)^2}{B(1-\alpha+\gamma)\big(\frac{1}{\rho}q^* - x\big)}\right)^{\frac{1}{\alpha}} x. \tag{9}$$

After simplifying and collecting terms, the HJB equation reads as follows

$$\rho f(x) + 1 - \tfrac{B(1-\alpha+\gamma)}{\rho(1-\alpha)} + \ln f'(x) = \tfrac{B(1-\alpha+\gamma)}{1-\alpha}\, x \Big(\frac{\varphi^{\frac{1-\alpha}{\alpha}} f'(x)^{\frac{1}{\alpha}}}{(\frac{1}{\rho} - f'(x)x)^{\frac{1-\alpha}{\alpha}}} - f'(x)\Big), \tag{10}$$

where the constant φ is given by

$$\varphi := \left(\tfrac{A(1-\alpha)^{2-\alpha}\alpha^{\alpha}}{B(1-\alpha+\gamma)}\right)^{\frac{1}{1-\alpha}} > 0.$$

We claim that a solution to this equation is given by

$$f(x) := \frac{B\frac{1-\alpha+\gamma}{1-\alpha} + \rho \ln \rho - \rho}{\rho^2} + \tfrac{1}{\rho} \ln(x + \varphi). \tag{11}$$

Indeed we have $f'(x) = 1/(\rho x + \rho\varphi)$ as well as $\frac{1}{\rho} - f'(x)x = \varphi f'(x)$ such that (10) is satisfied. We infer that a candidate for the value function is given by:

$$W(k,h) = \frac{B\frac{1-\alpha+\gamma}{1-\alpha} + \rho \ln \rho - \rho}{\rho^2} + \tfrac{1}{\rho} \ln\big(k + \varphi h^{\frac{1-\alpha+\gamma}{1-\alpha}}\big), \quad k,\, h > 0. \tag{12}$$

However, note that $\lim_{k\to 0} W(k,h) = W(0,h) > -\infty = V(0,h)$ holds, which is contradictory to $\tau = 0$. Thus, using W in the first order necessary condition along the optimal consumption path (5) implies positive consumption rates even if $k = 0$ holds such that the physical capital stock turns negative. Similarly, the insertion of W into the first order necessary condition along the optimal human capital allocation path (6) implies that the planner chooses a high level of schooling activities if physical capital is relatively scarce. On the other hand, if human capital is relatively scarce, his schooling efforts are low. In any case, the function W is an upper bound for the true value function V:

$$V(k,h) \le W(k,h), \quad \forall\, k,h > 0.$$

The Appendix presents a proof of this fact. Moreover, if for some (k_0, h_0) the pair (c_t^*, u_t^*), derived from the first order necessary conditions, is in $\mathcal{X}$ and $\tau = \infty$

holds, then all inequalities in the proof become equalities, (c_t^*, u_t^*) is the pair of optimal controls and $V(k_0, h_0) = W(k_0, h_0)$ holds. We insert the controls derived from W

$$c^* = \rho(x+\varphi)\tfrac{k}{x}, \qquad u^* = \left(\tfrac{A(1-\alpha)^2}{B\varphi(1-\alpha+\gamma)}\right)^{\frac{1}{\alpha}} x = \left(\tfrac{B(1-\alpha+\gamma)}{A\alpha(1-\alpha)}\right)^{\frac{1}{1-\alpha}} x \tag{13}$$

into the dynamics equation (8) for x_t:

$$\dot{x}_t = \left(\tfrac{B(1-\alpha+\gamma)}{(A\alpha)^{\frac{1}{2-\alpha}}(1-\alpha)}\right)^{\frac{2-\alpha}{1-\alpha}} x_t^2 + \left(\tfrac{B(1-\alpha+\gamma)}{\alpha} - \rho\right)x_t - \rho\left(\tfrac{A(1-\alpha)^{2-\alpha}\alpha^\alpha}{B(1-\alpha+\gamma)}\right)^{\frac{1}{1-\alpha}}. \tag{14}$$

A search for steady states of x_t shows that on the positive axis $\dot{x}_t$ only vanishes for the value

$$x^{ss} := \rho\left(\tfrac{A(1-\alpha)^{2-\alpha}\alpha}{B^{2-\alpha}(1-\alpha+\gamma)^{2-\alpha}}\right)^{1/(1-\alpha)} = \tfrac{\rho\alpha\varphi}{B(1-\alpha+\gamma)}.$$

This steady state x^{ss} leads to the balanced growth path, for which the controls $q^{ss} = c_t^{ss} x^{ss}/k_t$ and u^{ss} derived from W remain constant and are thus admissible as long as $u^* \leq 1$ holds. Linearizing the right hand-side of equation (14) at $x = x^{ss}$ shows that x^{ss} is locally unstable and we infer that W yields the unstable solution branch in the phase diagram (see Figure 1).

Proposition 1. *If* $x^{ss} := \frac{k(0)}{h(0)} = \frac{\rho\alpha\varphi}{B(1-\alpha+\gamma)}$ *and* $\rho\frac{1-\alpha}{1-\alpha+\gamma} \leq B$ *hold, then the controls*

$$c_t^{ss} = \rho\tfrac{x^{ss}+\varphi}{x^{ss}}k_t = \rho\big(k(0) + \varphi h(0)^{\frac{1-\alpha+\gamma}{1-\alpha}}\big)\exp\big((B\tfrac{1-\alpha+\gamma}{1-\alpha} - \rho)t\big), \tag{15}$$

$$u_t^{ss} = u^{ss} = \tfrac{\rho(1-\alpha)}{B(1-\alpha+\gamma)} \tag{16}$$

are indeed optimal in $\mathcal{X}$. *In addition to the control* u, *the control-like variable* q *remains constant as well:*

$$q_t^{ss} = q^{ss} = \rho(x^{ss}+\varphi) = \rho\varphi\tfrac{\rho\alpha + B(1-\alpha+\gamma)}{B(1-\alpha+\gamma)}.$$

The brief proof of this proposition is given in the Appendix. The fixed point derived above is the unique balanced growth path equilibrium of the centralized economy. For $\gamma = 0$ we recover the findings of Benhabib and Perli (1994). Since the social planner's Hamiltonian is convex for $\gamma > 0$ their approach does not offhand furnish a solution in the decentralized case. The above results show that the balanced growth path allocation of human capital is split over both production sectors and remains constant along this path. Furthermore, we learn that it is negatively related to the degree of the external effect of human capital in goods production captured by the parameter γ.

3.2 Determining the centralized solution

The main problem in solving the reduced HJB equation (7) stems from the fact that it is not explicit in f'. For this implicit differential equation standard techniques (Bronshtein and Semendyayew, 1997) are used to establish an explicit differential equation for f'. Since we can always add suitable constants to f solving (10), we restrict our attention to the homogeneous form $f(x) = G(x, f'(x))$ of (10) where the function G is given by

$$G(x,p) := -\tfrac{\ln p}{\rho} + \frac{x}{u^{ss}}\left(\psi p^{\frac{1}{\alpha}}\left(\tfrac{1}{\rho} - px\right)^{\frac{\alpha-1}{\alpha}} - p\right) \tag{17}$$

with $\psi := \varphi^{(1-\alpha)/\alpha}$. The function G equals up to an additive constant and up to the factor ρ the Hamiltonian of the transformed DOP. We find for the derivatives

$$G_x(x,p) = \tfrac{p}{u^{ss}}\left(\psi\left(\tfrac{1}{\rho p} - x\right)^{\frac{-1}{\alpha}}\left(\tfrac{1}{\rho p} + \tfrac{1-2\alpha}{\alpha}x\right) - 1\right), \tag{18}$$

$$G_p(x,p) = -\tfrac{1}{\rho p} + \tfrac{\psi x}{u^{ss}}\left(\tfrac{1}{\rho p} - x\right)^{\frac{-1}{\alpha}}\left(\tfrac{1}{\alpha\rho p} - x\right) - \tfrac{x}{u^{ss}} \tag{19}$$

By the relationship of G with the Hamiltonian, the Pontryagin maximum principle states for $p_t := f'(x_t)$

$$\dot{x}_t = -p_t + \tfrac{\psi x_t(\alpha^{-1}p_t^{-1} - \rho x_t)}{u^{ss}(\rho^{-1}p_t^{-1} - x_t)^{\frac{1}{\alpha}}} - \tfrac{\rho x_t}{u^{ss}}, \tag{20}$$

which can also be easily verified from equation (8) directly . Due to $f = G(x, f')$ we get $f' = G_x(x, f') + G_p(x, f')f''$. Thus, setting $p(x) := f'(x)$, we arrive at the explicit differential equation in p

$$p'(x) = \tfrac{p(x) - G_x(x,p(x))}{G_p(x,p(x))},$$

which in our case yields

$$p' = p\frac{u^{ss} + 1 - \psi(\rho^{-1}p^{-1} - x)^{\frac{-1}{\alpha}}(\rho^{-1}p^{-1} + \frac{1-2\alpha}{\alpha}x)}{-x - u^{ss}\rho^{-1}p^{-1} + \psi x(\rho^{-1}p^{-1} - x)^{\frac{-1}{\alpha}}(\alpha^{-1}\rho^{-1}p^{-1} - x)}.$$

The optimal consumption rate satisfies $c^* = V_k^{-1} = k/(xf')$, such that considering $q(x) = cx/k = f'(x)^{-1} = p(x)^{-1}$, the rescaled consumption rate, which we have already encountered in (8), we obtain a differential equation for this control-like variable in terms of the state-like variable x:

$$q' = \frac{-p'}{p^2} = q\frac{u^{ss} + 1 - \psi(\rho^{-1}q - x)^{\frac{-1}{\alpha}}(\rho^{-1}q + \frac{1-2\alpha}{\alpha}x)}{x + u^{ss}\rho^{-1}q - \psi x(\rho^{-1}q - x)^{\frac{-1}{\alpha}}(\alpha^{-1}\rho^{-1}q - x)}. \tag{21}$$

This equation is now explicit in q' and standard analytical and numerical methods can be used for its study. Since we know the value of q at the steady-state initial condition $x = x^{ss}$, we face a classical initial value problem where the solution is usually unique. Here, however, uniqueness fails because the candidate function W as well as the true value function both solve the initial value problem. This is due to the fact that at $(x^*, q(x^*))$ in the fraction appearing in equation (21) both numerator and denominator vanish and the right-hand side is indeterminate. We proceed as follows. The differential equation can be written as

$$q'(x) = \frac{K(x, q(x))}{L(x, q(x))} \text{ with } K(x^{ss}, q(x^{ss})) = L(x^{ss}, q(x^{ss})) = 0.$$

In order to obtain determinacy at x^{ss} we use L'Hôpital's rule, which gives

$$q'(x^{ss}) = \frac{K_x(x^{ss}, q(x^{ss})) + K_q(x^{ss}, q(x^{ss}))q'(x^{ss})}{L_x(x^{ss}, q(x^{ss})) + L_q(x^{ss}, q(x^{ss}))q'(x^{ss})}.$$

This leads us to a quadratic equation in $q'(x^{ss})$, one solution of which we already know from W, namely $q'(x^{ss}) = \rho$. Therefore, there exists exactly one other possible solution of $q'(x^{ss})$ which is given by

$$q'(x^{ss}) = \frac{-K_x\left(x^{ss}, q\left(x^{ss}\right)\right)}{\rho L_q(x^{ss}, q(x^{ss}))}.$$

This fraction is now determinate and

$$K_x = \tfrac{-(1-\alpha)\psi}{u^{ss}\alpha^2} q(\rho^{-1}q - x)^{-(1+\alpha)/\alpha}(2\alpha\rho^{-1}q + (1-2\alpha)x),$$

$$L_q = \rho^{-1} + \tfrac{\psi(1-\alpha)}{u^{ss}\alpha^2\rho^2} xq(\rho^{-1}q - x)^{-(1+\alpha)/\alpha},$$

$$\tfrac{-K_x}{\rho L_q} = \tfrac{(1-\alpha)\psi q(2\alpha\rho^{-1}q + (1-2\alpha)x)}{u^{ss}\alpha^2(\rho^{-1}q - x)^{(1+\alpha)/\alpha} + \psi(1-\alpha)\rho^{-1}xq}$$

implies

$$q'(x^{ss}) = \rho\Big(1 + \tfrac{2(u^{ss})^{-1}(1-\alpha)^2 + \alpha(1-\alpha)}{1-\alpha^2 + u^{ss}\alpha}\Big). \tag{22}$$

Note that this value is always larger than the other root ρ. This is explained by the fact that this solution, corresponding to the true value function, will run through the origin and thus has to be smaller than the first solution on the interval $[0, x^{ss})$. Though uniquely determined, this stable solution branch can only be approximated locally by the linearization given in (22) or globally by a numerical solver. In sum, using the differential equation (21) and the steady-state values found in Proposition 1, we can determine the values of the control-like variable q at the state-like values x. From this we deduce the corresponding values of f', c and u as well as of f and V for specified initial values x_0 or h_0 and k_0, respectively. For small values of x, our method yields solutions with $u^* > 1$. In view of the first order conditions we have to set $u^* = 1$ and solve for the optimal c^* in the remaining one sector growth model. Mathematically, this corresponds to a free boundary problem.

4 The decentralized economy

The decentralized economy is characterized by the fact that, although all agents could benefit from a collusive agreement, they are not able to enforce such a deal. Existing firms face a strong shirking incentive, but even if they cooperate, new entrants may benefit from free-riding. The following considerations are based on Romer (1986) and lead us to the representative agent's maximization problem. The production technology is linear homogeneous with respect to the factors that receive compensation in the decentralized economy. This ensures that the firms' profits will be zero and consequently their number is indeterminate. Therefore we restrict our attention to an equilibrium path along which the number of firms equals the number of agents. This means that per firm and per capita values coincide and we focus on a single representative agent running a competitive firm. It remains to take care of the impact of the average level of human capital on total factor productivity. In order to analyze the law of motion of the average stock of human capital the representative agent needs three ingredients: First, the infinite number of agents implies that the representative agent's allocation decision has no measurable influence on the average decision in the economy. Second, note that at time $t = 0$ all agents are identically endowed with the per capita stocks h_0 and k_0. Third, since the agents are assumed to be identical, we restrict our solution to symmetric equilibria. Hence, we can think of the representative agent's maximization problem as the problem of finding symmetric time-dependent strategies for the solutions of an infinite number of interdependent DOP's indexed by $n \in \mathbb{N}$. Although the agent has no measurable influence on the economy's average decision of allocating human capital the (yet unknown) path of u_a does influence the agents' optimal decisions on c and u. This is rational because h_a affects the evolution of the productivity in the goods sector such that the opportunity costs of schooling are directly linked with u_a. In Section 4.1 we interpret the interdependent utility maximization problems of agents in a decentralized economy as a non-cooperative game and we define the symmetric Nash equilibrium. In Section 4.2 we use again the homogeneity in the initial conditions of the HJB equation and determine an implicit partial differential equation. The structure of this equation is similar to the planner's case, but it now also depends on time. This occurs because the time dependent function $u_{a,t}$ enters. A parameter depending version of the unstable solution branch helps us to determine the steady state of the decentralized economy. In Section 4.3 we transform the problem into solving one explicit partial differential equation assuming $u_{a,t}$ to be given. Using the Nash condition we again reduce to an ordinary differential equation.

4.1 Nash equilibrium

We suppose infinitely many homogeneous agents $A^{(n)}$, $n = 1, 2, ..$ each of them equipped with his own physical capital $k^{(n)}$ and human capital $h^{(n)}$. Assuming that the average human capital $h_a = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} h^{(n)}$ exists in a mathematical sense, each agent faces his own dynamical optimization problem to attain the maximum lifetime utility

$$U^{(n)} = \max_{c_t^{(n)}, u_t^{(n)}} \int_{t=0}^{\infty} e^{-\rho t} \ln c_t^{(n)} dt,$$

subject to the state dynamics

$$\begin{aligned}
\dot{k}_t^{(n)} &= A(k_t^{(n)})^\alpha (u_t^{(n)})^{1-\alpha} (h_t^{(n)})^{1-\alpha} h_{a,t}^\gamma - c_t^{(n)}, && \forall\, t \geq 0,\\
\dot{h}_t^{(n)} &= B\big(1 - u_t^{(n)}\big) h_t^{(n)}, && \forall\, t \geq 0,\\
k_t^{(n)} &\geq 0 \quad \text{and} \quad h_t^{(n)} \geq 0 && \forall\, t \geq 0.
\end{aligned}$$

The constraints on the control variables $(u_t^{(n)})_{t\geq 0}$ and $(c_t^{(n)})_{t\geq 0}$ are as before in the social planner's problem. We interpret this optimization problem as a multiple players' non-cooperative game where the different optimization problems are connected via the value of the average human capital stock. Since all agents are identical, their initial endowments $k_0^{(n)}$ and $h_0^{(n)}$ are the same and it is sensible to seek symmetric Nash equilibria. Applying the definition (e.g., Dockner et al. (2000), Chapter 4) to our setup, the controls $(u_t^{(n*)})_{t\geq 0}$ and $(c_t^{(n*)})_{t\geq 0}$ form a Nash equilibrium if

$$\begin{aligned}
&U^{(n)}\big((u_t^{(1*)})_{t\geq 0}, (c_t^{(1*)})_{t\geq 0}; (u_t^{(2*)})_{t\geq 0}, (c_t^{(2*)})_{t\geq 0}; \dots; (u_t^{(n*)})_{t\geq 0}, (c_t^{(n*)})_{t\geq 0}; \dots\big)\\
&\geq U^{(n)}\big((u_t^{(1*)})_{t\geq 0}, (c_t^{(1*)})_{t\geq 0}; (u_t^{(2*)})_{t\geq 0}, (c_t^{(2*)})_{t\geq 0}; \dots; (u_t^{(n)})_{t\geq 0}, (c_t^{(n)})_{t\geq 0}; \dots\big)
\end{aligned}$$

holds for all feasible controls $(u_t^{(n)})_{t\geq 0}$ and $(c_t^{(n)})_{t\geq 0}$ and for all n. By symmetry, $u^{(n*)} = u^*$ and $c^{(n*)} = c^*$ do not depend on the agent $A^{(n)}$ and in particular the average human capital satisfies $h_a = h^{(n*)}$. Hence, the agent's lifetime utility $U^{(n)}$ only depends on his own controls $u^{(n)}$ and $c^{(n)}$ and on the average stock of human capital h_a the value of which does not change under different decisions of merely one agent. Thus, u^* and c^* satisfy the Nash condition if

$$U\big((u_t^{(*)})_{t\geq 0}, (c_t^{(*)})_{t\geq 0}, (h_{a,t})_{t\geq 0}\big) \geq U\big((u_t)_{t\geq 0}, (c_t)_{t\geq 0}, (h_{a,t})_{t\geq 0}\big)$$

holds for all feasible controls $(u_t)_{t\geq 0}$ and $(c_t)_{t\geq 0}$. Note that from now on we drop the superscript (n) in the notation.

4.2 The representative agent's optimization problem

Our first step is again to define the value function for the DOP at hand. The two controls c_t and u_t are chosen by the representative agent such that they are in X and maximize his discounted utility while taking the economy's average level of human capital $h_{a,t}$ as given. In fact, if a symmetric Nash equilibrium exists then

$$\dot{h}_{a,t} = B(1 - u_t^*) h_{a,t}$$

holds. This implies that $h_{a,t}$ cannot grow faster than exponentially such that we can apply Proposition 1. In the sequel, we shall express the formulae for the single agent in terms of $u_{a,t} = u_t^*$ and only use that $(u_{a,t})_{t\geq 0}$ is a continuous function of time with values in $[0, 1]$. Observe that we thus introduce a time dependence of the optimization problem and obtain a non-autonomous HJB-equation. Indeed, unless $u_{a,t}$ remains constant, the value function will change in time because of differing future productivity in the goods sector. As an alternative to our time inhomogeneous DOP in the three state variables (k, h, h_a), we could have considered the four state variables (k, h, k_a, h_a) and used the time-autonomous character of the game to model the representative agent's DOP with some exogenous control functions $u_a(k_a, h_a)$ and $c_a(k_a, h_a)$. This approach, though, yields less intuitive formulae and cannot be simplified as easily as the one proposed here. In order to remain close to the notation in Section 3, the discounting of the value function over time is cancelled by the factor $e^{\rho t}$. Thus, the dynamic optimization problem reads as follows[1]:

$$\tilde{V}(k_t, h_t, h_{a,t}, t; u_{a,t}) := \max_{(c,u)\in\mathcal{X}} \begin{cases} e^{\rho t}\int_t^\infty e^{-\rho s}\ln c_s\, ds, & \tau_t = \infty \\ -\infty, & \tau_t < \infty. \end{cases}$$

The stopping time τ_t at point t in time is defined as $\tau_t := \inf\{s \geq t \,|\, k_s = 0\}$ and the set $\mathcal{X}$ of admissible controls is the same as in Section 3. The corresponding HJB equation now also depends on the function $(u_{a,t})_{t\geq 0}$:

$$\rho\tilde{V}(\cdot; u_a) = \max_{(c,u)\in X}\left\{\ln c + \tilde{V}_k(\cdot; u_a)\dot{k}_t + \tilde{V}_h(\cdot; u_a)\dot{h}_t + \tilde{V}_{h_a}(\cdot; u_a)\dot{h}_a + \tilde{V}_t(\cdot; u_a)\right\} \quad (23)$$

The agent has no influence on the average allocation u_a and it makes no difference whether the braces include the last two terms or not. As in the centralized case, we show that a solution $\tilde{W}(k, h, h_a, t; u_a)$ of this equation is always an upper bound for the true value function $\tilde{V}(k, h, h_a, t; u_a)$, if $\tilde{W}$ is concave in k and h, see the Appendix. We proceed as for the social planner's DOP. The first order necessary conditions are:

$$c^* = \tilde{V}_k^{-1}, \quad (24)$$

$$u^* = \left(\frac{A(1-\alpha)\tilde{V}_k}{B\tilde{V}_h}\right)^{\frac{1}{\alpha}} \frac{k h_a^{\frac{\gamma}{\alpha}}}{h}. \quad (25)$$

The interpretation of these conditions is almost the same as in the centralized case, the only difference is that the shadow values of the capital stocks now depend on the path of the economy's average human capital allocation decision u_a. We continue with the insertion of these findings into the HJB equation (23). We obtain

$$\rho\tilde{V} + 1 + \ln\tilde{V}_k = \alpha k\left(A\tilde{V}_k h_a^\gamma\right)^{\frac{1}{\alpha}}\left(\frac{1-\alpha}{B\tilde{V}_h}\right)^{\frac{1-\alpha}{\alpha}} + B\tilde{V}_h h + \tilde{V}_{h_a} B\left(1-u_a\right)h_a + \tilde{V}_t. \quad (26)$$

The redefinition of the state-like variable $x_t := k_t h_t^{-1} h_{a,t}^{-\gamma/(1-\alpha)}$ and the control-like variable $q_t := c_t h_t^{-1} h_{a,t}^{-\gamma/(1-\alpha)}$ implies in analogy to the centralized case

$$\dot{x}_t = A x_t^\alpha u_t^{1-\alpha} - q_t - B(1-u_t)x_t - \tfrac{\gamma}{1-\alpha}B(1-u_{a,t})x_t. \quad (27)$$

[1]The tilde stresses that we consider the value function of the representative agent and not the social planner's value function.

Again we exploit the inherent homogeneity and write

$$\tilde{V}(k,h,h_a,t;u_a) = \tilde{f}\big(kh^{-1}h_a^{\frac{-\gamma}{1-\alpha}},t;u_a\big) + \tfrac{1}{\rho}\ln\big(hh_a^{\frac{\gamma}{1-\alpha}}\big).$$

The derivatives of $\tilde{V}(k,h,h_a,t;u_a)$, expressed in terms of the state-like variable x and the parameter depending redefined function $\tilde{f} := \tilde{f}(x,t;u_a)$, are

$$\tilde{V}_k = \tfrac{\tilde{f}'x}{k}, \quad \tilde{V}_h = \tfrac{1}{\rho h} - \tfrac{\tilde{f}'x}{h}, \quad \tilde{V}_{h_a} = \tfrac{\gamma}{1-\alpha}\big(\tfrac{1}{\rho h_a} - \tfrac{\tilde{f}'x}{h_a}\big), \quad \text{and} \quad \tilde{V}_t = \dot{\tilde{f}},$$

where $\tilde{f}' := \frac{d}{dx}\tilde{f}(x,t;u_a)$ and $\dot{\tilde{f}} := \frac{d}{dt}\tilde{f}(x,t;u_a)$. Thus the parameter depending function $\tilde{f}$ determines decisively the shadow values of the three production factors. The introduction of $\tilde{f}$ allows us to rewrite the first order necessary conditions. Hence (24) can be rewritten as $c^* = k(x\tilde{f}')^{-1}$ and for (25) we have

$$u^* = \left(\frac{A(1-\alpha)\tilde{f}'(x,t;u_a)}{B\big(\frac{1}{\rho} - x\tilde{f}'(x,t;u_a)\big)}\right)^{\frac{1}{\alpha}} x = \left(\frac{A(1-\alpha)}{B\big(\frac{1}{\rho}q^* - x\big)}\right)^{\frac{1}{\alpha}} x. \qquad (28)$$

The condition on the left hand side shows us that the average allocation of human capital u_a influences the individual's decision of allocating human capital. Furthermore, using $q^* = 1/\tilde{f}'$ the condition on the right corresponds to (9) in the planer's case. Again, we reduce the HJB-equation and arrive at

$$\rho\tilde{f} + 1 + \ln\tilde{f}' = \frac{A^{\frac{1}{\alpha}}\alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}}\tilde{f}'^{\frac{1}{\alpha}}x}{B^{\frac{1-\alpha}{\alpha}}(\frac{1}{\rho} - \tilde{f}'x)^{\frac{1-\alpha}{\alpha}}} + B\big(\tfrac{1}{\rho} - \tilde{f}'x\big) + B\tfrac{\gamma-\gamma u_a}{1-\alpha}\big(\tfrac{1}{\rho} - \tilde{f}'x\big) + \dot{\tilde{f}}. \qquad (29)$$

This is still a partial differential equation in the variables t and x. In a first step we only seek the balanced growth path solution, where the state-like variable x and the control variables q and u remain constant over time. In this case, we take the exogenous value $u_{a,t}$ to equal a constant $u_a \in [0,1]$ whence the coefficients of the reduced HJB-equation are time independent and we look for time autonomous solutions $\tilde{f}$. Using (29) and treating u_a as a given constant we obtain

$$\rho\tilde{f} + 1 - \tfrac{B(1-\alpha+\gamma-\gamma u_a)}{\rho(1-\alpha)} + \ln\tilde{f}' = \tfrac{B(1-\alpha+\gamma-\gamma u_a)}{1-\alpha}x\left(\frac{\varphi_a^{\frac{1-\alpha}{\alpha}}\tilde{f}'^{\frac{1}{\alpha}}}{(\frac{1}{\rho} - \tilde{f}'x)^{\frac{1-\alpha}{\alpha}}} - \tilde{f}'\right), \qquad (30)$$

where φ_a depends on the value of u_a:

$$\varphi_a := \left(\frac{A(1-\alpha)\alpha^\alpha}{B(1-\alpha+\gamma-\gamma u_a)^\alpha}\right)^{\frac{1}{1-\alpha}} > 0.$$

A slightly modified version of the function in (11) namely

$$\tilde{f}(x;u_a) = \frac{B\frac{1-\alpha+\gamma-\gamma u_a}{1-\alpha} + \rho\ln\rho - \rho}{\rho^2} + \tfrac{1}{\rho}\ln(x+\varphi_a) \qquad (31)$$

is a solution of this reduced HJB equation, which only yields admissible controls for the steady state (x^{ss}, u_a^{ss}) of equation (27), which is given by

$$x^{ss} = \frac{\rho\alpha\varphi_a^{ss}}{B(1-\alpha+\gamma-\gamma u_a^{ss})} \quad \text{and} \quad u_a^{ss} = \tfrac{\rho}{B}, \qquad (32)$$

where φ_a^{ss} denotes φ_a for $u_a = u_a^{ss}$. The corresponding optimal controls are found to be

$$u^{ss} = \tfrac{\rho}{B} \quad \text{and} \quad q^{ss} = \rho(x^{ss} + \varphi_a^{ss}) = \rho\varphi_a^{ss}\tfrac{\rho(\alpha-\gamma)+B(1-\alpha+\gamma)}{B(1-\alpha+\gamma)-\rho\gamma}. \qquad (33)$$

We stress that $\tilde{f}(x^{ss}; u_a^{ss})$ determines u^{ss} independently of u_a such that the decentralized steady state is unique. A posteriori we conclude that the only possible parameter choice in order to have a symmetric Nash equilibrium in steady state is $u_a = u_a^{ss} = \rho/B$. Conversely, this steady state solution is indeed a Nash equilibrium for the DOP at hand. It equals the corresponding values derived by the Pontryagin maximum principle along the lines of Benhabib and Perli (1994). Compared to the centralized case, the steady-state allocation of human capital between the two sectors is not influenced by the external effect of human capital in goods production captured by γ. Since the realized marginal productivity in the schooling sector remains constant, human capital just grows at a constant rate and the average stock of human capital affects total factor productivity in the goods sector in a constant manner. On the other hand human capital has diminishing private marginal returns in goods production such that physical capital investment provides the mechanism to reconcile the (private) marginal returns of human capital in both sectors. Hence, for the allocation of human capital along the balanced growth path, the degree of the external effect in goods production plays no role in the decentralized economy. On a transition path to the steady state, however, the value of γ is important, since the goods sector's productivity path influences the path of the opportunity costs of schooling and therefore the convergence of the state-like variable towards its steady state.

4.3 Determining the decentralized solution

Off the steady state we can only assert that in a Nash equilibrium with time homogeneous controls $u_{a,t} = u^*(x_t)$ holds along the optimal controls. The first intuition to replace $u_{a,t}$ by $u^*(x)$ from equation (27) in the reduced HJB-equation (30) is wrong because the dynamic programming principle needs to consider all possible values (x,t) in order to derive the HJB-equation, which is not only defined on the one-dimensional solution submanifold (x_t, t) corresponding to the optimal control. Hence, this idea would introduce a dependence of u_a on x and f in the dynamic programming principle from which the agent can benefit, i.e. he would be in the position to (partially) exploit the externality. For instance, starting in $x > x^{ss}$ the agent would have an incentive to decrease the state variable x by shifting human capital to the schooling sector. Although the choice of a smaller u has a lowering effect on goods sector productivity, for positive γ this effect is more than offset by the higher values of h and h_a. Hence the increased productivity would afford the opportunity to increase consumption. Numerical implementations of this wrong approach yield significantly deviating policy rules. Yet, if we transform the reduced HJB-equation into a differential equation for $\tilde{f}_x(x,t)$, like in the centralized case of Section 3.2, we shall find that the differential equation along the solution path simplifies to an ordinary differential equation.

From there on, the same techniques as before yield an initial value problem which is easily solved numerically. We restrict our attention of the HJB-equation $f(x,t)$ to the homogeneous form $G^{(t)}(x, f_x(x,t), f_t(x,t))$ with

$$G^{(t)}(x,p,d) := \frac{B(1-\alpha+\gamma-\gamma u_{a,t})}{\rho^2(1-\alpha)} - \frac{\ln p}{\rho} + \frac{B(1-\alpha+\gamma-\gamma u_{a,t})}{\rho(1-\alpha)} xp(\psi_{a,t}(\tfrac{1}{\rho p} - x)^{\frac{\alpha-1}{\alpha}} - 1) + \frac{d}{\rho}$$

and

$$\psi_{a,t} := \left(\frac{A(1-\alpha)\alpha^\alpha}{B(1-\alpha+\gamma-\gamma u_{a,t})^\alpha}\right)^{\frac{1}{\alpha}}.$$

Consequently, $p(x,t) := f_x(x,t)$ solves the partial differential equation

$$p = G_x^{(t)} + G_p^{(t)} p_x + G_t^{(t)} p_t.$$

Solving for $p_x(x,t)$ would already yield an explicit differential equation in p. Note again that $\dot{x}_t = \rho G_p^{(t)}$ holds along the optimal control path. From economic theory we immediately infer that x_t converges monotonically to the steady state x^{ss} such that $\dot{x}_t = \rho G_p^{(t)} \neq 0$ holds off the balanced growth path. Denoting the inverse function of $t \mapsto x_t$ by $x \mapsto t(x)$ we put $\tilde{p}(x) = p(x, t(x))$ for $x > 0$ and $x \neq x^{ss}$. From $G_d^{(t)} = \rho^{-1}$ we thus infer

$$\tilde{p}(x) = G_x^{(t)} + G_p^{(t)}\left(\tilde{p}'(x) - p_t t'(x)\right) + \frac{p_t}{\rho} = G_x^{(t)} + G_p^{(t)} \tilde{p}'(x).$$

where $G^{(t)}$ means of course $G^{(t(x))}$ along the solution path. There we know from the Nash condition that $u_{a,t} = u^*(x_t)$ holds which we can finally insert to obtain the ordinary differential equation $\tilde{p} = G_x + G_p \tilde{p}'$ with

$$\begin{aligned} G_x(x,p) &= \frac{B(1-\alpha+\gamma-\gamma u(p,x))}{\rho(1-\alpha)} p\left(\psi(p,x)(\tfrac{1}{\rho p} - x)^{\frac{-1}{\alpha}}\left(\tfrac{1}{\rho p} + \tfrac{1-2\alpha}{\alpha}x\right) - 1\right), \\ G_p(x,p) &= -\frac{1}{\rho p} + \frac{B(1-\alpha+\gamma-\gamma u(p,x))}{\rho(1-\alpha)} x\left(\psi(p,x)(\tfrac{1}{\rho p} - x)^{\frac{-1}{\alpha}}\left(\tfrac{1}{\alpha\rho p} - x\right) - 1\right) \end{aligned}$$

Here $u(p,x)$ is derived from formula (28), that is

$$u(p,x) = \left(\frac{A(1-\alpha)}{B\left(\frac{1}{\rho p} - x\right)}\right)^{\frac{1}{\alpha}} x. \tag{34}$$

and $\psi(p,x)$ is obtained from $\psi_{a,t}$, replacing $u_{a,t}$ by $u(p,x)$. The explicit ordinary differential equation $\tilde{p}' = (\tilde{p} - G_x)/G_p$ can be rewritten in terms of the optimal control-like variable $q = \tilde{p}^{-1}$ such that the following explicit differential equation in q has to be solved:

$$q' = q\frac{(1-\alpha)\rho - B(1-\alpha+\gamma-\gamma u(q,x))\left(\psi(q,x)(\frac{1}{\rho}q - x)^{\frac{-1}{\alpha}}(\frac{1}{\rho}q + \frac{1-2\alpha}{\alpha}x) - 1\right)}{(1-\alpha)q - B(1-\alpha+\gamma-\gamma u(q,x))x\left(\psi(q,x)(\frac{1}{\rho}q - x)^{\frac{-1}{\alpha}}(\frac{1}{\alpha\rho}q - x) - 1\right)}. \tag{35}$$

As in the centralized case, the right hand side will be indeterminate at the steady state since numerator and denominator vanish. Again L'Hôpital's rule gives an explicit expression for the derivative q' at x^{ss}

$$q'(x^{ss}) = \frac{[u^{ss}(\alpha-\gamma)+1-\alpha+\gamma]\left[(2-2\alpha+\gamma)(1-\alpha+\gamma)(u^{ss})^{-1} - (2-2\alpha+\gamma)\gamma + 1 - \alpha + \gamma\right]}{(1-\alpha^2)(1-\alpha+\gamma-\gamma u^{ss}) + \alpha(1-\alpha+\gamma)u^{ss}}. \tag{36}$$

The Appendix gives some intermediate results that we obtained when determining this expression. Finally, a simple initial value problem remains to be solved. Note, that the differential equation (35) together with the restated first order necessary condition (34) gives an explicit formula for the optimal control-like variable q. After solving for the optimal q^* given a certain state x, one can easily determine the optimal human capital allocation u^*. In the more general model considered by Mulligan and Sala-i-Martin (1993) the corresponding optimal controls are determined by two coupled differential equations.

Easy numerical schemes now allow to determine the solution, which does not yield indeterminacy in the differential equation except at the steady state x^{ss} and at $x = 0$ such that the value function is twice continuously differentiable. The optimal controls found are indeed admissible except for small values of x where u may be larger than one. As before we set $u = 1$ in this region and solve the control problem for the single variable q. Altogether, this verifies that we have obtained the symmetric Nash equilibrium we had looked for. Moreover, it is seen to be unique.

The solution method presented here is also applicable for the parameter space studied by Xie (1994). Xie analyzes the model with isoelastic preferences where the parameter σ denotes the inverse of the intertemporal elasticity of substitution. Since he restricts σ to be equal to the output elasticity of physical capital, the number of parameters studied there is equal to ours and his choice of the more general utility function is bought by imposing this restriction[2].

5 The solution method revisited

The solution method of the Uzawa-Lucas model of endogenous growth presented here is of global character and works in the centralized case as well as in the decentralized case. We reduce the problem of determining the value function to solving an explicit ordinary differential equation in q with the initial value $q(x^{ss})$ prescribed by the balanced growth path solution. We show analytically that x^{ss} is a saddle point of the dynamics of q. By exhibiting a closed form solution we find the unstable branch, which in turn easily yields the value $q'(x^{ss})$ for the stable solution branch by L'Hôpital's rule. Using this knowledge, the initial value problem for the stable solution is determinate and becomes easily tractable by standard numerical solvers for explicit ordinary differential equations. While the differential equation for q is highly nonlinear, it is still possible to study analytically the dependence of the solution on the different parameters. The standard method is to linearize the right hand side with respect to this parameter and to investigate the dynamics locally around one particular parameter value. The knowledge of $q(x)$ then allows to determine the optimal human capital allocation

[2]The functional form of the unstable solution branch is then given by:
$\Xi(k, h, h_a; u_a) := \frac{\rho^{1-\sigma}\left(k+\varphi_a h h_a^{\gamma/(1-\sigma)}\right)^{1-\sigma}}{\rho(1-\sigma)\left(1-\frac{\rho}{B}(1-\sigma+\gamma-u_a\gamma)\right)^{\sigma}} - \frac{2-\sigma}{\rho(1-\sigma)}$, with $\varphi_a := \left(\frac{\sigma^{\sigma}(1-\sigma)A}{(1-\sigma+\gamma-u_a\gamma)^{\sigma}B}\right)^{\frac{1}{1-\sigma}}$

$u(x)$ via formula (34). Of course, the value function V itself is calculated on the basis of the formula $q(x) = f'(x)^{-1}$ together with its value at x^{ss} and the relationship between V and f. The transition from an initial state off the balanced growth path is simulated by inserting the optimal controls into the two dynamic state equations for k and h.

The main steps in simplifying the original nonlinear implicit HJB equation in the variables k and h were (a) to use the inherent homogeneity by introducing the state-like variable x, the control-like variables q and u and the value function-like function f and (b) to transform the implicit one-dimensional differential equation for f into an explicit differential equation for q. To what extent does this approach hinge on the special model considered? The first step (a) is a standard simplification trick and applies to almost all two-sector models considered in the economical literature. So, we focus on the second step (b). For the transformation of the implicit HJB equation for f it was essential that this equation had the form

$$f(x) = G(x, f'(x)) \text{ for some function } G. \tag{37}$$

Given this representation, our method readily yields an explicit differential equation for $f'(x)$, which together with some specific value of f, e.g. at the steady state, permits a simple analytical and numerical analysis of the value function. The additional knowledge of a solution facilitates the calculations in the saddle point case, but is not strictly necessary since a local analysis around the steady state would already provide enough information. If we now consider an autonomous two-sector growth model in general form

$$\begin{aligned} \dot{k}_t &= F_k(k_t, h_t, u_t) - c, \\ \dot{h}_t &= F_h(k_t, h_t, u_t), \end{aligned}$$

with the optimization problem for (u_t) and (c_t)

$$\int_0^\infty e^{-\rho t} g(c_t)\, dt \longrightarrow \max!,$$

we end up with the autonomous HJB equation

$$\rho V = \max_{c,u}\Big(g(c) + V_k(F_k(k,h,u) - c) + V_h F_h(k,h,u)\Big).$$

At first glance it seems that we always obtain the desired representation (37). However, the reduction to state-like and control-like variables may introduce a non-derivative term on the right. For instance, the use of an isoelastic utility function $g(c) = c^\beta$, $\beta \in (0,1)$, in the centralized Uzawa-Lucas model lets us set by homogeneity

$$V(k,h) = f(kh^{-\frac{1-\alpha+\gamma}{1-\alpha}})h^{\frac{(1-\alpha+\gamma)\beta}{1-\alpha}} = f(x)h^{\frac{(1-\alpha+\gamma)\beta}{1-\alpha}}.$$

Hence, the derivative V_h is expressed in terms of both, f and f'. This shows that the choice of a logarithmic utility function is crucial for our approach to work so easily. In the general case, but with the specific choice $g(u) = \ln(u)$ we will not face this problem. Assuming homogeneity of the production functions F_k and F_h with respect to k and h, we can introduce a suitable state-like variable $x = x(k,h) = k^{\alpha_k} h^{-\alpha_h}$ and the value function can be written as

$$V(k,h) = f(x) + \rho^{-1} \ln(h^{\alpha_h}).$$

This additive structure thus allows to write V_k and V_h in terms of f' only. In summary, our approach allows to simplify drastically the HJB-equations for all two-sector growth models with logarithmic utility. Naturally, a wider scope beyond two-sector growth models is feasible, but the general structure of the mathematical optimization problem should be preserved.

Let us discuss briefly the similarities and differences of our and alternative solution strategies. The reason for choosing the value function approach rather than the more widely used Pontryagin maximum principle is twofold. First, the value function approach is able to cope with non-concave DOPs which can occur in the centralized version of the model if γ is positive. Second, the value function has the advantage that it is deterministic even if we allow for stochastic disturbances in the model, such that this approach is the natural one to choose if one wants to extend the model in this direction. The HJB-equations are single implicit partial differential equations. In the centralized case $V(k,h)$ is two dimensional and in the decentralized case $\tilde{V}(k,h,h_a,t;u_a)$ is four dimensional. At this stage the Pontryagin maximum principle leads to a system of four explicit coupled differential equations for $\dot{k}$, $\dot{h}$, $\dot{c}$, and $\dot{u}$. Then we exploit the inherent homogeneity of the DOP by introducing the state-like variable x and the control-like variable q. Using these transformed variables we arrive at an implicit ordinary differential equation in x in the centralized case and at an implicit partial differential equation in x and t in the decentralized case. The application of the maximum principle, however, would have lead to a three dimensional system of explicit differential equations. In the centralized case we use a standard technique in order to transform the implicit into an explicit rational differential equation (21) for the control-like variable q. Since we know the steady state value x^{ss} and the unstable solution branch, we are able to determine the initial condition $q(x^{ss})$ and the local linearization corresponding to the stable branch $q'(x^{ss})$.

The time elimination method proposed by Mulligan and Sala-i-Martin (1991) uses similar arguments, although it is based on the maximum principle. It uses the fact that the derivative of the control-like variable with respect to the state-like variable is equal to the time derivative of the control-like variable divided by the time derivative of the state-like variable. To obtain initial conditions in this method, basically the same procedure as ours is used in the steady state. Focusing on two-sector growth models in a second paper, Mulligan and Sala-i-Martin (1993) derive a system of two explicit ordinary differential equations, which in our terminology involve q and u as functions of x. Even in the Uzawa-Lucas model they do not simplify further to one equation, which is due to the non-logarithmic utility assumed in their model, compare the discussion above. We want to stress here that numerically both methods can be implemented with high speed (our examples were computed in less than one second) and high accuracy by standard mathematical packages. Yet, the reduction to merely one explicit differential equation has the great advantage that the behavior of the solution can be studied analytically much more easily than in coupled systems, especially with respect to the influence of certain parameters.

The preceding arguments are even more stringent when compared with the adjoint equations obtained by the pure maximum principle. This approach is adopted e.g. by Benhabib and Perli (1994) and Bond, Wang, and Yip (1996) and yields, even after reduction, three nonlinear coupled dynamic equations where only the values for $t \to \infty$ are known from the balanced growth path. The more demanding techniques of shooting or backward solving for this kind of problems, as advocated by Brunner and Strulik (2002), are thus avoided.

Instead of following the maximum principle for our concrete model, we deliberately favored the dynamic programming or value function approach because it allows an easier generalization to the stochastic setting. An interesting natural example would be random shocks in the production of physical capital. Very generally, if we conserve homogeneity in the noise modelling, the HJB equation in terms of the state-like variable x will be of the form

$$f = G(x, f', f''),$$

involving a second derivative due to the Itô term of the corresponding Markov generator. Consequently, the same principle as before yields one second order differential equation for $p = f'$:

$$p = G_x(x,p,p') + G_p(x,p,p')p' + G_P(x,p,p')p'' \Rightarrow p'' = \frac{p - G_x + G_p p'}{G_P}.$$

Note that the equation is explicit in the second order term and even elliptic. For special model setups this may be simplified further, but already the general model allows the use of standard analytical and numerical tools for elliptic equations. The only possible difficulty might be a singularity in the first order term. A detailed analysis is left for future work. Here, we only want to mention two advantages we anticipate for our approach: First, in the maximum principle approach the adjoint equations are backward stochastic differential equations and an extension of the time elimination method is not obvious. Second, the reduction to only one equation, even though of second order, should simplify the analysis, the numerical implementation of Monte Carlo simulations and the fitting to empirical data considerably.

6 Discussion of the numerical results

In this section we make use of our method and show that on the transition path the output growth rate towards the balanced growth rate obeys a U-shaped

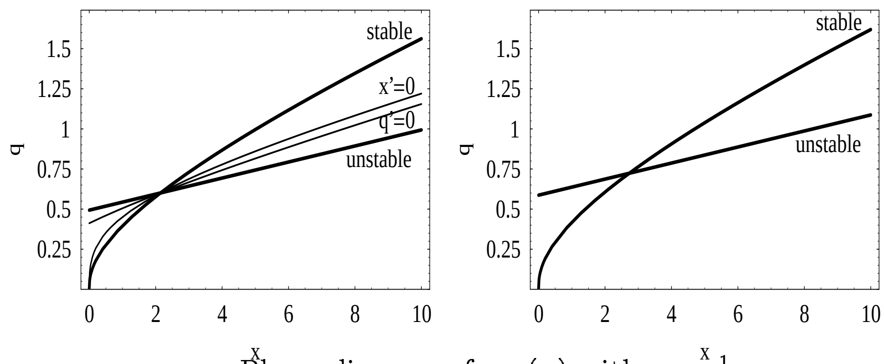

Figure 1: Phase diagrams for $q(x)$ with $\gamma = \frac{1}{10}$
left: centralized solution right: decentralized solution

course. In order to study the economy's performance, we consider the following typical calibration of the parameter values

$$A = 1, \quad B = \tfrac{1}{10}, \quad \rho = \tfrac{1}{20}, \quad \text{and} \quad \alpha = \tfrac{1}{3}\,. \tag{38}$$

Figure 1 shows the phase diagrams for $q(x)$ where we have set γ equal to $\frac{1}{10}$. The left part displays the solution of the centralized economy and the right part refers to the decentralized economy. The linear thick lines are derived from the solutions (11) and (31), whereas the concave thick lines that start in the respective origin are the optimal controls q derived from the true value function V, that is the numerical solution of (21) and (35). Both functions meet in the saddle points (x^{ss}, q^{ss}). Note, that the central planer's high valuation of human capital leads to a lower steady state value of x in the centralized economy. In the centralized case, the thin lines indicate the curves where the functions $L(x,q)$ and $K(x,q)$ vanish. The steeper thin line corresponds to values (x,q) where $\dot{x} = 0$. Above this line $\dot{x}$ is negative and below it is positive. Similarly, the flatter thin line corresponds to values (x,q) where $\dot{q} = 0$, above this line q decreases and below it increases. This shows that the controls derived from (11) correspond to the unstable solution branch, whereas the numerically determined optimal controls are indeed globally stable, i.e. they induce adjustment towards the balanced growth path solution. Since u_a is not well-defined off the two solution branches the corresponding lines are missing in the decentralized case.

Figure 2 shows the optimal human capital allocation u in the (x,γ) space as a surface. The left part represents the social planner's solution and the right part the decentralized case. The black lines correspond to the respective steady-state values u^{ss}. Since the social returns are taken into account by the planner, his valuation of human capital is higher than that of the representative agent. This explains why for positive γ the planner's allocation of human capital to the goods sector, respectively the cooperative allocation, is always smaller than the corresponding value of u in the decentralized economy. For small values of x and γ, they both are about to set u larger than one.

Let us now turn to the social planner's solution. Keeping γ fixed, the fraction of

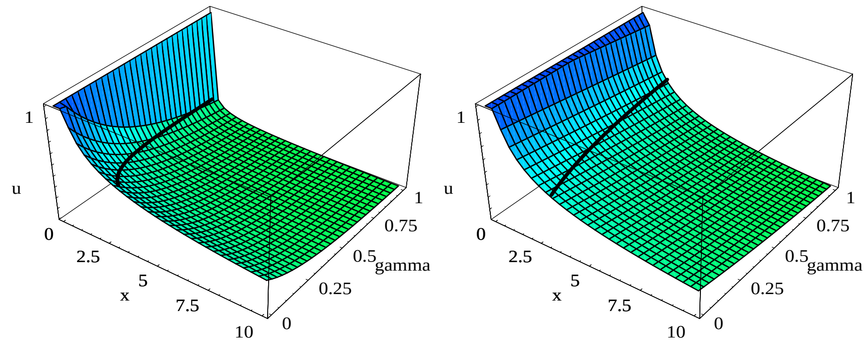

Figure 2: optimal time share u with respect to (x, γ)
left: centralized solution right: decentralized solution

time allocated to goods production decreases when x increases. This observation can be rationalized as follows. A high value of x indicates that the economy's endowment of human capital is relatively low. This leads to high marginal social returns of human capital in the goods sector. Arbitrage reasoning implies that the realized marginal productivity of human capital in the schooling sector must also be relatively high. Hence a comparatively high fraction of human capital is attracted by the schooling sector. This explains the relatively low value of u. Since the marginal social returns of human capital in goods production and γ are positively related, this reasoning can also explain the negative slope of the surface with respect to γ. Hence, we observe for the steady state x^{ss} that the value of u^{ss} decreases in the value of γ.

Next, we consider the representative agent's solution. For fixed γ, the behavior of u is qualitatively similar to the centralized case. However, the agent considers private returns instead of social returns. The fact that γ has no influence on the private marginal returns of human capital in goods production explains why the surface of u is not simply declining in γ. Instead it is increasing to the left and decreasing to the right of x^{ss}. Given a certain level of $x > x^{ss}$, u is decreasing in γ. Since the agent anticipates aggregate schooling activities and hence the evolution of total factor productivity, he defers human capital investment for lower values of γ in order to exploit the relatively low opportunity costs of schooling in the future. Otherwise, if $x < x^{ss}$ holds, he prefers schooling today for large values of γ because today's opportunity costs in this case are relatively low. The benefits from anticipation are the larger the higher the effect of h_a on total factor productivity, i.e. the bigger γ. On the other hand, the costs of non-coordination are increasing in γ. For the steady state u^{ss}, human capital returns in both sectors move parallel and the benefits from anticipation vanish, such that $u^{ss} = \rho/B$ must be independent of γ.

Figure 3 plots the numerical results for the growth rate of output in (x, γ) space. The left part shows the social planner's case and the right part refers to the decentralized economy. The black lines denote the respective balanced growth rates. Note that we confirm Barro and Sala-i-Martin's (1995) finding that output

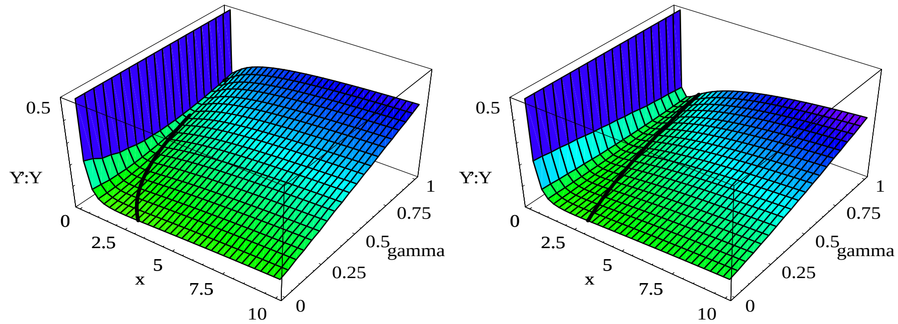

Figure 3: growth rate of output with respect to the state x
left: centralized solution right: decentralized solution

growth rates are U-shaped for fixed γ. The minima of the U-shaped growth rates are always to the left of the balanced growth rate. Analytically the growth rate of output can be decomposed as follows:

$$\frac{\dot{y}_t}{y_t} = \alpha\frac{\dot{x}_t}{x_t} + (1-\alpha)\frac{\dot{u}_t}{u_t} + \frac{1-\alpha+\gamma}{1-\alpha}\frac{\dot{h}_t}{h_t}, \quad \text{where } \frac{\dot{x}_t}{x_t} = \frac{\dot{k}_t}{k_t} - \frac{1-\alpha+\gamma}{1-\alpha}\frac{\dot{h}_t}{h_t}. \tag{39}$$

In the following we consider the individual terms in the sum as separate functions of x. The first term $\dot{x}/x$ is positive for $x < x_{ss}$ and negative for $x > x_{ss}$, tending to infinity for $x \to 0$. From the decay of $u(x)$ we infer that the second term $\dot{u}/u$ has the reverse property: it is negative for $x < x_{ss}$ and positive for $x > x_{ss}$. The last term $\dot{h}/h$ is never negative and monotonically increasing with $\lim_{x\to\infty} \dot{h}/h = B$. Furthermore $\dot{h}(x_{ss})/h(x_{ss})$ determines the growth rate of output along the balanced growth path and is equal to $B - \frac{1-\alpha}{1-\alpha+\gamma}\rho$ in the centralized case and to $B - \rho$ in the decentralized case. Hence, the three terms behave very differently and yield in sum the U-shape of $\dot{y}/y$ that we observe. For positive γ the growth rate of output implied by the planner's solution is always higher than the corresponding rate in the decentralized economy. This can be interpreted as the welfare loss due to the lack of a coordination mechanism. Qualitatively, however, the implied growth rates of both solutions behave very similar. For $x < x^{ss}$ with fixed γ, the growth rate first declines in x before it starts to rise again. For $x > x^{ss}$, the growth rate is increasing in x.

Again we study first the centralized case. The case $x < x^{ss}$ corresponds to a relatively high endowment of human capital. Let us assume that the social opportunity costs of schooling are higher than the potential benefits of educational activities. Obviously the planner wishes to set u bigger than one, i.e. he wants to disinvest in human capital. Since he must set u equal to one, he can only adjust physical capital. Therefore, we know that the output stream is larger than the smoothed consumption stream. But with u equal to one, goods technology has diminishing returns in physical capital and any additional unit of k lowers productivity in goods production. As a consequence, the output growth rate is high but declining. When the restriction $u \leq 1$ is no longer binding the planner starts to shift human capital to the schooling sector. Two opposite effects start to set in. The lower value of u directly causes a smaller productivity in goods production while the growing stock of h pushes productivity in the opposite direction. Initially the decline of productivity due to the physical capital effect and to the allocation effect dominates. After a while when u adjusts slower the human capital effect is dominating such that goods sector productivity starts to rise again and the output growth rate follows. Obviously, the dominance of the human capital effect arises the faster the larger γ. The case $x > x^{ss}$ means that human capital is relatively scarce such that the productivity of h in goods production is high. Here the effect due to human capital dominates during the whole transition process, although its dominance is becoming weaker when the state-like variable x moves towards its steady state.

Finally we focus on the decentralized case. Concerning the influence of x, the representative agent's reasoning is similar to the planner's except that it refers to private instead of social opportunity costs of schooling. The larger the external effect γ, the higher the agent's benefits of anticipating the evolution of the average stock of human capital h_a. These benefits are reflected in high growth rates of either human or physical capital which are then transmitted to high output growth rates. However, due to the growing costs of non-coordination, the increase in γ is not as steep as in the centralized case.

7 Conclusion

In this paper, we have introduced a simple method of analyzing global transitional dynamics of the Uzawa-Lucas endogenous growth model with logarithmic utility. As a result, we merely have to solve one ordinary differential equation in the parameters of the model. Since we know the functional form of the unstable solution branch, we are able to determine analytically the initial condition for the differential equation. Finally, we solve the initial value problem. This is a clear advantage compared to the three-dimensional system of coupled differential equations encountered when applying standard approaches based on the maximum principle. Furthermore, our method yields global results, such as the U-shaped course of output growth rates, which are not captured by a local linearization.

We present numerical results to the social planner's optimization problem for wide ranging degrees of the external effect γ. In the decentralized case, the representative agent's HJB equation is restricted to the economy's average decision u_a. Therefore, the agent has to determine his optimal choice u given the economy-wide average decision u_a. Our method formalizes this reasoning by treating the aggregate human capital allocation rule u_a as an exogenously given parameter. This formalization illustrates the reasoning of agents in a clear manner and reveals the economic intuition of symmetric Nash equilibria. Anticipating the evolution of the average human capital stock means finding a fixed point in the space of admissible time dependent policy functions. In a symmetric equilibrium, the agent's optimal policy rule u should coincide with the economy's average decision u_a. Finally, we argue that the model's inherent asymmetry is responsible for the U-shaped course of output growth rates.

We show that our method is generally applicable for two sector growth models with logarithmic utility. Due to this restriction on the preferences, our approach is even easier to solve than the time-elimination method proposed by Mulligan and Sala-i-Martin (1991). The homogeneity in the production technologies allows us to rewrite the value function as the sum of the value function-like function f and a logarithmic expression in the human capital stock. As a result the shadow values of physical and human capital only depend on f'. Furthermore the HJB-equation can be stated in two terms only, in the state-like variable x and in f'. When choosing more general preferences, e.g. iso-elastic utility, the additive structure disappears and our approach does not work so easily.

By and large, we have limited our discussion mainly to the deterministic setup. However, theorists are often interested in the impact of cyclical volatility on the economy's performance. For instance, Canton (2002) asks whether long-term economic growth increases or decreases with increased cyclical volatility. Since the value function is deterministic even in a model with uncertainty, the value function approach is the natural one to choose if one wants to introduce stochastic disturbances. We argue that if the modelling of the noise preserves the model's homogeneity, the HJB-equation can be stated in three terms, in x, in f', and in f''. The second derivative of the value function-like function f leads us to one second order differential equation. Since this equation is explicit and even elliptic it allows the use of standard mathematical tools. Finally, we think that a rigorous treatment of the stochastic case would be a worthwhile project.

Appendix

W is an upper bound for the true value function V

For all $k_0, h_0, t > 0$ and all controls $(c_t, u_t) \in \mathcal{X}$ with $\tau > t$ we have

$$\begin{aligned} e^{-\rho t} W(k_t, h_t) &= W(k_0, h_0) + \int_0^t (-\rho e^{-\rho s} W(k_s, h_s) \\ &\qquad + e^{-\rho s} W_k(k_s, h_s) \dot{k}_s + e^{-\rho s} W_h(k_s, h_s) \dot{h}_s) \, ds \\ &\leq W(k_0, h_0) + \int_0^t (-\rho e^{-\rho s} W(k_s, h_s) \\ &\qquad + e^{-\rho s} (\rho W(k_s, h_s) - \ln(c_s)) \, ds \\ &= W(k_0, h_0) + \int_0^t e^{-\rho s} (-\ln(c_s)) \, ds, \end{aligned}$$

where we have used the fact that W solves the HJB equation or is at least an upper bound, i.e. a supersolution, if $(c_t^*, u_t^*) \notin \mathcal{X}$ since we have not excluded the values $u_t^* > 1$. Hence, by rearranging the terms and taking the limit $t \to \infty$, we obtain

$$V(k_0, h_0) \leq W(k_0, h_0) - \lim_{t \to \infty} e^{-\rho t} W(k_t, h_t),$$

if the latter limit exists. Note that this inequality is always trivially valid for $\tau < \infty$. Since h_t grows exponentially and $c_t \geq 0$ holds, also k_t cannot grow faster than exponentially and thus $W(k_t, h_t)$ grows at most linearly, which implies that the limit cannot be larger than zero. On the other hand, $h_t \geq h_0$ holds for all $t \geq 0$ so that $W(k_t, h_t)$ is uniformly bounded from below for all $t \geq 0$. Hence, this last limit exists, equals zero, i.e. the transversality condition is fulfilled, and

$$V(k_0, h_0) \leq W(k_0, h_0)$$

holds, as asserted.

Proposition 1

Obviously, the controls c^{ss} and u^{ss} are admissible because $u^{ss} \leq 1$ by assumption. The value of x^{ss} ensures that we are on the balanced growth path and $\dot{x}_t = 0$ holds so that the values of the controls are easily derived from (13). Hence by the preceding remark on W, the controls are indeed optimal.

$\tilde{W}$ is an upper bound for the true value function $\tilde{V}$

For all $k_v, h_v, t > v$ and any controls $(c_s, u_s) \in \mathcal{X}_v$ with $\tau_v = \infty$ we find

$$\begin{aligned} e^{-\rho t} W(k_t, h_t, t) &= e^{-\rho v} W(k_v, h_v, v) + \int_v^t e^{-\rho s} \big(-\rho W(k_s, h_s) + W_k(k_s, h_s, s) \dot{k}_s \\ &\quad + W_h(k_s, h_s, s) \dot{h}_s + W_t(k_s, h_s, s) \big) \, ds \\ &\leq e^{-\rho v} W(k_v, h_v, v) + \int_v^t (-\rho e^{-\rho s} \ln(c_s)) \, ds. \end{aligned}$$

The exponential growth bounds for $h_{a,t}$ and h_t imply exponential bounds for k_t and c_t so that $\lim_{t \to \infty} e^{-\rho t} W(k_t, h_t, t) = 0$ is guaranteed. We infer

$$W(k_v, h_v, v) \geq e^{\rho v} \int_v^\infty e^{-\rho s} \ln(c_s) \, ds.$$

Since the controls were arbitrary, we have shown $W(k, h, v) \geq V(k, h, v)$ under the only hypothesis $\tau_v = \infty$. If $\tau_v < \infty$ were true, the value function would equal $-\infty$ and the asserted inequality is trivially true.

The initial value $q(x^{ss})$ in the decentralized case

Note that the derivatives of $\psi(q, x)$ at $q = q(x^{ss})$ and $x = x^{ss}$ are found to be

$$\psi_x = \frac{(\varphi_a^{ss})^{\frac{1-\alpha}{\alpha}} \gamma u_x}{B(1-\alpha+\gamma) - \rho\gamma} \quad \text{and} \quad \psi_q = \frac{(\varphi_a^{ss})^{\frac{1-\alpha}{\alpha}} \gamma u_q}{B(1-\alpha+\gamma) - \rho\gamma}.$$

The respective derivatives of (34) are given by

$$u_x = \frac{B(1-\alpha+\gamma)-\rho\gamma+\rho}{B\alpha\varphi_a^{ss}} \quad \text{and} \quad u_q = \frac{-1}{B\alpha\varphi_a^{ss}}.$$

Let $K(q(x^{ss}), x^{ss})$ and $L(q(x^{ss}), x^{ss})$ denote the numerator and denominator of the differential equation (35). Again $q'(x^{ss}) = \frac{-K_x(q(x^{ss}),x^{ss})}{\rho L_q(q(x^{ss}),x^{ss})}$ holds and the two derivatives are as follows:

$$K_x\left(q(x^{ss}), x^{ss}\right) = -\frac{\rho q^{ss}\left[(2-2\alpha+\gamma)\left((u^{ss})^{-1}(1-\alpha+\gamma)-\gamma\right)+1-\alpha+\gamma\right]}{\alpha\varphi_a^{ss}}$$

$$L_q\left(q(x^{ss}), x^{ss}\right) = \frac{1-\alpha^2}{\alpha} + \frac{u^{ss}(1-\alpha+\gamma)}{1-\alpha+\gamma-\gamma u^{ss}}$$

This implies the expression given in equation (36).

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