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A thermoelastic contact problem with a phase transition

Dietmar Hömberg[1], Alexander Khludnev[2]

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[1] Weierstrass Institute
for Applied Analysis and Stochastics
Mohrenstr. 39
10117 Berlin, Germany
E-Mail: hoemberg@wias-berlin.de

[2] Russian Academy of Sciences
Lavrentev Institute of Hydrodynamics
Lavrentyeva, 15
Novosibirsk, 630090, Russia
E-Mail: khlud@hydro.nsc.ru

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We investigate a thermomechanical contact problem with phase transitions. The system of equations consists of a quasistatic momentum balance and a semilinear energy balance. The phase transition is described by an ordinary differential equation. Different mechanical properties of the respective phases are taken care of by a mixture ansatz.

We prove the existence of a weak solution and a uniqueness result, the latter only being valid in one space dimension.

1 Introduction

Heat treatment of steel, especially hardening, is usually accompanied by the evolution of large residual stresses, i.e. stresses that exist without any external load on the part considered. Reasons for this behaviour are the thermal expansion or contraction in an inhomogeneous temperature field with phase dependent thermal expansion coefficients and density changes due to the phase transformations ([4, p. 13]).

In this paper we investigate a thermomechanical contact problem with phase transitions. The particular application we have in mind is a mathematical model for the Jominy end-quench test. In this test a steel specimen is heated until the high temperature phase *austenite* is reached. Then it is put in a fixation and cooled from below by water quenching. Upon cooling, two phase transitions occur (depending on the steel species, even some more, see, e.g., [10]). Close to the cooling boundary where the highest cooling rates appear *martensite* is formed, a hard and brittle steel phase. With increasing distance from the quenched end, mainly *pearlite* is formed, a soft and ductile steel phase.

Figure 1 depicts the setting of the Jominy end-quench test. The steel specimen rests on the support stabilized by its own weight. Hence the appropriate mathematical description of the Jominy test is in terms of a thermomechanical contact problem with phase transitions. The complete model will be described in Section 2. Thermoelastic contact problems related to resistance welding have been studied by the authors in [11]. Further results on thermoelastic contact can be found in [1], [2], [12] and [15]. The new contribution of this paper is in the inclusion of phase transitions.

We conclude the introduction with a short presentation of a typical phase transition model. To this end let z_1 and z_2 denote the volume fractions of pearlite and martensite, respectively. Then the evolution of the phases z_1 and z_2 can be described by the following system:

$$\dot{z}_1(t) = (1 - z_1 - z_2)g_1(\theta)\,, \tag{1.1}$$

$$\dot{z}_2(t) = [\bar{m}(\theta) - z_1 - z_2]_+ g_2(\theta)\,, \tag{1.2}$$

$$z_1(0) = z_2(0) = 0\,. \tag{1.3}$$

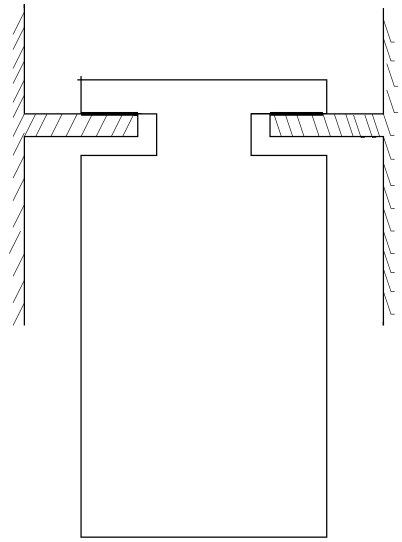

Figure 1: The Jominy end-quench test

Here g_1 and g_2 are positive data functions that can be identified from *isothermal transformation diagrams* (cf. [8]). $\bar{m}(\theta)$ is a monotonic equilibrium volume fraction of martensite satisfying $\bar{m}(\theta_{M_s}) = 0$ and $\bar{m}(\theta_{M_f}) = 1$, where $\theta_{M_s} > \theta_{M_f}$ are critical temperatures between which martensite is formed, and the dot means a derivative with respect to t. We omit the details of proving the existence of a solution to (1.1)–(1.3) for a given temperature evolution θ (cf. [9]) and confine ourselves to showing that the sum of the volume fractions of the new phases is less then one.

Adding (1.1) to (1.2) and substituting $\varphi := z_1 + z_2$ we obtain

$$\dot{\varphi}(t) \le (1-\varphi)(g_1(\theta) + g_2(\theta)) \,.$$

¿From this one can conclude

$$0 \le \varphi < 1 \quad \forall \;\; t \in [0, T] \,. \tag{1.4}$$

Since the main goal of this paper is to investigate the coupling between phase transition and volume fraction, in the sequel we will restrict ourselves to the case of one phase transition with volume fraction φ produced during cooling.

The paper is organized as follows: In Section 2 we describe the complete thermomechanical model. We prove the existence of a weak solution in Section 3. In the last section we present a uniqueness result for our problem in one space dimension.

2 Problem formulation

Let $\Omega \subset R^3$ be a bounded domain with a smooth boundary Γ, $\Gamma = \Gamma_0 \cup \Gamma_c$, $\Gamma_0 \cap \Gamma_c = \emptyset$, meas $\Gamma_0 > 0$, $Q = \Omega \times (0, T)$, $T > 0$. Denote by ν a unit exterior vector to Γ, $\nu = (\nu_1, \nu_2, \nu_3)$. We assume that Γ_c is a regular part of the boundary Γ. In the domain Q, we want to find a displacement vector $u = (u_1, u_2, u_3)$, stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2, 3$, a temperature θ, and a volume fraction φ of the product phase, e.g., *pearlite*, such that

$$-\operatorname{div}\sigma = f \quad \text{in } Q\,, \tag{2.1}$$
$$\varepsilon(u) = C\sigma + q(\varphi)B\theta \quad \text{in } Q\,, \tag{2.2}$$
$$\theta_t - \Delta\theta + q(\varphi)\operatorname{div} u_t = \varphi_t \quad \text{in } Q\,, \tag{2.3}$$
$$\varphi_t = h(\theta, \varphi) \quad \text{in } Q\,, \tag{2.4}$$
$$\theta = \theta_0\,,\ \varphi = 0 \quad \text{for } t = 0\,, \tag{2.5}$$
$$\theta = 0\,,\ u = 0 \quad \text{on } \Gamma_0 \times (0,T)\,, \tag{2.6}$$
$$\frac{\partial\theta}{\partial\nu} = g \quad \text{on } \Gamma_c \times (0,T)\,, \tag{2.7}$$
$$u\cdot\nu \le 0\,,\ \sigma_\nu \le 0\,,\ \sigma_\mu = 0\,,\ u\cdot\nu\sigma_\nu = 0 \quad \text{on } \Gamma_c \times (0,T)\,. \tag{2.8}$$

The thermal expansion coefficient is defined by the mixture ansatz

$$q(\varphi) = \delta_1\varphi + \delta_2(1-\varphi)\,, \tag{2.9}$$

where δ_i are positive constants, $i = 1, 2$, and B is a constant matrix $B = \{b_{ij}\}$, $i, j = 1, 2, 3$,

$$\varepsilon(u) = \{\varepsilon_{ij}(u)\}_{i,j=1}^{3}\,,\ \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\,,\ (C\sigma)_{ij} = c_{ijke}\sigma_{ke}\,.$$

Tensor c_{ijke} satisfies a usual property of a symmetry and positive definitness, i.e.

$$c_{ijke} = c_{keij} = c_{jike}\,,\ c_{ijke}\xi_{ke}\xi_{ij} \ge c|\xi|^2$$

for all $\xi_{ij} = \xi_{ji}$, $c = \mathit{const} > 0$, $c_{ijke} \in L^\infty(\Omega)$,

$$\sigma_\nu = \sigma_{ij}\nu_j\nu_i\,,\ \sigma_\mu = \sigma\nu - \sigma_\nu\nu\,,\ \sigma\nu = \{\sigma_{ij}\nu_j\}_{i=1}^{3}\,.$$

Let us remark that conditions (2.8) describe a contact between Γ_c and a rigid body with zero friction. Functions $f = (f_1, f_2, f_3)$ and g are given,

$$f_i \in H^1(0,T;L^2(\Omega))\,, i = 1,2,3, \quad g \in H^1(0,T;L^2(\Gamma_c))\,.$$

The term φ_t in (2.3) models the latent heat of the phase transition. We assume that the given function h describing a phase transition provides the following properties. For any $\theta \in L^1(Q)$ there exists a solution φ of the problem

$$\varphi_t = h(\theta,\varphi) \quad \text{in } Q\,, \tag{2.10}$$
$$\varphi(0) = 0\,, \tag{2.11}$$

such that

$$0 \le \varphi(t,x) \le 1 \quad \text{a.e. in } Q\,,$$

$$\|\varphi_t\|_{L^\infty(Q)} \le c^* \,, \tag{2.12}$$

with a constant c^* independent of θ. Moreover, let $\theta^n \to \theta$ in $L^1(Q)$. Then

$$\varphi^n \to \varphi \quad \text{in } W^{1,p}(0,T;L^p(\Omega))\,,\ p \in [1,\infty)\,, \tag{2.13}$$

where φ^n, φ are the solutions of (2.10), (2.11) corresponding to θ^n, θ, respectively. Specific functions h providing the above properties can be found in [13].

Summation convention is used over repeated indices. All functions with two below indices are assumed to be symmetric in those indices, i.e. $\sigma_{ij} = \sigma_{ji}$ etc. Note that (2.1) is the quasistatic momentum balance, equation (2.2) provides the constitutive law; where the thermal expansion coefficient $q(\varphi)$ is defined by the mixture ansatz (2.9) with δ_1 being the expansion coefficient in the new phase and δ_2 the one of the old phase $(1-\varphi)$. Equation (2.3) describes an energy balance law; and (2.5) is the equation for the phase transition. We introduce some notations of functional spaces useful in further considerations:

$$\begin{aligned}
H^1_{\Gamma_0}(\Omega) &= \{v \in H^1(\Omega) \,|\, v = 0 \quad \text{on } \Gamma_0\}\,,\\
H &= H^1(0,T;[H^1_{\Gamma_0}(\Omega)]^3)\,,\\
N &= \{\{\sigma_{ij}\} \,|\, \sigma_{ij} \in L^2(\Omega)\,,\ 1 \le i,j \le 3\}\,,\\
\Sigma &= H^1(0,T;N)\,,\\
\Xi &= \{\theta \in L^2(0,T;H^1_{\Gamma_0}(\Omega)) \,|\, \theta_t \in L^2(Q)\}\,,\\
K &= \{u \in [H^1_{\Gamma_0}(\Omega)]^3 \,|\, u\cdot\nu \le 0 \quad \text{almost everywhere on } \Gamma_c\}\,,\\
\mathcal{K} &= \{v \in L^2(0,T;[H^1_{\Gamma_0}(\Omega)]^3) \,|\, v\cdot\nu \le 0 \quad \text{a.e. on } \Gamma_c \times (0,T)\}\,,\\
U &= \{\varphi \,|\, \varphi, \varphi_t \in L^\infty(Q)\,,\ 0 \le \varphi \le 1\,,\ |\varphi_t| \le c^*\}\,.
\end{aligned}$$

Here c^* is taken from the inequality (2.12).

The scalar product in $\mathbb{R}^n$ is denoted by '$\cdot$', its counterpart in $\mathbb{R}^{(n,n)}$ will be denoted by '$:$'.

3 Theorems of existence

Denote $\delta = \max\{\delta_1, \delta_2\}$. The main result of this section is a proof of solution existence of the problem (2.1)–(2.8).

Theorem 3.1 *Let $\theta_0 \in H^1_{\Gamma_0}(\Omega)$ and all above assumptions be fulfilled. Then for small δ there exists a solution u, σ, θ, φ of the problem* (2.1)–(2.8) *such that*

$$u \in \mathcal{K} \cap H\,,\ \sigma \in \Sigma\,,\ \theta \in \Xi\,,\ \varphi \in U\,, \tag{3.1}$$

$$\int_Q \sigma : \varepsilon(v-u) \ge \int_Q f\cdot(v-u) \quad \forall v \in \mathcal{K}\,, \tag{3.2}$$

$$\varepsilon(u) = C\sigma + q(\varphi)B\theta \quad \text{in } Q\,, \tag{3.3}$$

$$\int_Q (\theta_t + q(\varphi)\mathrm{div}\, u_t - \varphi_t)\eta - \int_{\Gamma_c\times(0,T)} g\eta + \int_Q \nabla\theta\nabla\eta = 0 \qquad \forall \eta \in L^2(0,T;H^1_{\Gamma_0}(\Omega)), \tag{3.4}$$

$$\varphi_t = h(\theta,\varphi), \tag{3.5}$$

$$\theta = \theta_0, \ \varphi = 0 \quad for\ t = 0. \tag{3.6}$$

Proof. General idea of the proof is to use the Schauder fixed point theorem. We shall construct a compact operator

$$A : L^2(Q) \to L^2(Q) \tag{3.7}$$

which has a fixed point. In order to construct the operator A we need some preliminary estimates for solutions of auxiliary problems. The first step is to find a solution $u = (u_1, u_2, u_3)$, $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2, 3$, of the problem

$$-\mathrm{div}\,\sigma = f \quad \text{in } Q, \tag{3.8}$$

$$\varepsilon(u) = C\sigma + g(\varphi)B\theta \quad \text{in } Q, \tag{3.9}$$

$$u = 0 \quad \text{on } \Gamma_0 \times (0,T), \tag{3.10}$$

$$u\cdot\nu \le 0,\ \sigma_\nu \le 0,\ \sigma_\mu = 0,\ u\cdot\nu\sigma_\nu = 0 \quad \text{on } \Gamma_c \times (0,T) \tag{3.11}$$

for given $\theta \in \Xi$, $\varphi \in U$, and to derive a priori estimates for this solution.

Solution to the problem (3.8)–(3.11) can be defined as follows:

$$(u,\sigma) \in \mathcal{K} \times L^2(Q), \tag{3.12}$$

and for almost all $t \in (0,T)$

$$\int_\Omega \sigma(t) : \varepsilon(v - u(t)) \ge \int_\Omega f(t)\cdot(v - u(t)) \quad \forall v \in K, \tag{3.13}$$

$$\varepsilon(u(t)) = C\sigma(t) + q(\varphi(t))B\theta(t). \tag{3.14}$$

We omit the details of the proof since it can be done similar to [11]. Just remark that it can be performed be regularizing the equilibrium equation (3.8) with the term $-\alpha\Delta u$, α being a positive parameter. For any $\alpha > 0$ there exists a solution of the regularized problem, and it is possible to pass to the limit as $\alpha \to 0$.

Moreover, the solution (u,σ) of the problem (3.8)–(3.11) satisfies the estimate

$$\|u\|_{L^2(0,T;[H^1_{\Gamma_0}(\Omega)]^3)} + \|\sigma\|_{L^2(0,T;N)} \le c_1\delta^{1/2}\|\theta\|_{L^2(Q)} + c_2 \tag{3.15}$$

with constants c_1, c_2 uniform in δ, $\delta \le \delta_0$. In deriving (3.15) we take into account the inequality $0 \le q(\varphi) \le \delta$. Now we verify that the solution (u,σ) to (3.8)–(3.11) has an additional regularity in t. Denote

$$v^t = v(t),\ d_\tau v^t = \frac{v^{t+\tau} - v^t}{\tau},\ q^t = q(\varphi^t).$$

Choose $\tau > 0$ and take $v = u^{t+\tau}$ in (3.13). Next we consider (3.13) at the point $t+\tau$ and choose $v = u^t$. Summing the relations obtained we derive

$$\int_\Omega d_\tau \sigma^t : \varepsilon(d_\tau u^t) \leq \int_\Omega d_\tau f^t \cdot d_\tau u^t \,. \tag{3.16}$$

For further considerations we need to introduce an auxiliary function $\psi = \{\psi_{ij}\}$, $i, j = 1, 2, 3$, such that ψ_{ij}, $\dot{\psi}_{ij} \in L^2(Q)$, and the equation

$$-\mathrm{div}\, \psi = f \quad \text{in } Q \tag{3.17}$$

is satisfied in the following sense

$$\int_\Omega \psi : \varepsilon(v) = \int_\Omega f \cdot v \quad \forall\ v \in [H^1_{\Gamma_0}(\Omega)]^3$$

for almost all $t \in (0, T)$. It is clear that a function ψ with the above properties exists.

Now multiply (3.14) by $\bar{\sigma} = \sigma^{t+\tau} - \sigma^t - \psi^{t+\tau} + \psi^t$ considering (3.14) at times t and $t+\tau$. Substracting the relations obtained and dividing by τ^2 we obtain the equality

$$\int_\Omega \left\{ C d_\tau \sigma^t - \varepsilon(d_\tau u^t) + \frac{1}{\tau}(q^{t+\tau} B\theta^{t+\theta} - q^t B\theta^t) \right\} : (d_\tau \sigma^t - d_\tau \psi^t) = 0 \,. \tag{3.18}$$

A substitution of $\frac{1}{\tau}(q^{t+\tau} B\theta^{t+\tau} - q^t B\theta^t)$ with $q^t B d_\tau \theta^t + d_\tau q^t \cdot B\theta^{t+\tau}$ can be done in (3.18). Hence, summing (3.16), (3.18) and integrating in t from 0 to $T - \tau$, and taking into account (3.15), (3.17), it follows

$$\int_0^{T-\tau} \|d_\tau \sigma^t\|^2_M dt \leq c_3 \delta \int_0^{T-\tau} \|d_\tau \theta^t\|^2_0 dt + c_4 \delta \int_0^T \|\theta^t\|^2_0 dt + c_5 \,. \tag{3.19}$$

Here $\|\cdot\|_0$ stands for the norm in $L^2(\Omega)$, and the constants c_3,c_4,c_5 are independent of τ, δ, $\delta \leq \delta_0$. In deriving (3.19) we use the upper bound $c\delta$ for $d_\tau q^t$, $c = const > 0$, which follows from the problem

$$\frac{d\varphi^t}{dt} = h(\theta^t, \varphi^t)\,, \ \varphi^0 = 0 \,.$$

Since $\theta \in \Xi$, we have the estimate

$$\int_0^{T-\tau} \|d_\tau \theta^t\|^2_0 dt \leq \int_0^T \|\dot{\theta}\|^2_0 dt \,.$$

Consequently, from (3.19) it follows

$$\|\dot{\sigma}\|^2_{L^2(0,T;N)} \leq c_3 \delta \|\dot{\theta}\|^2_{L^2(Q)} + c_4 \delta \|\theta\|^2_{L^2(Q)} + c_5 \,. \tag{3.20}$$

Relations (3.15), (3.20) imply

$$\|\sigma\|_\Sigma \le c_6 \delta^{1/2} \|\theta\|_\Xi + c_7 \,. \tag{3.21}$$

Hence, equation (3.9) yields the estimate for u,

$$\|u\|_H \le c_8 \delta^{1/2} \|\theta\|_\Xi + c_9 \tag{3.22}$$

with constants c_8, c_9 being uniform in δ, $\delta \le \delta_0$. We complete the analysis of problem (3.8)–(3.11) and look at the following auxiliary problem. For given $u \in H$, $\varphi \in U$ we want to find θ such that

$$\theta_t - \Delta\theta + q(\varphi)\operatorname{div} u_t = \varphi_t \quad \text{in } Q \,, \tag{3.23}$$

$$\theta = \theta_0 \quad \text{for } t = 0 \,, \tag{3.24}$$

$$\theta = 0 \quad \text{on } \Gamma_0 \times (0,T); \; \frac{\partial\theta}{\partial\nu} = g \quad \text{on } \Gamma_c \times (0,T) \,. \tag{3.25}$$

Using standard theory for parabolic problems we can find a unique function θ, such that

$$\theta \in \Xi \,, \tag{3.26}$$

$$\int\limits_Q (\theta_t + q(\varphi)\operatorname{div} u_t - \varphi_t)\eta - \int\limits_{\Gamma_c \times (0,T)} g\eta + \int\limits_Q \nabla\theta\nabla\eta = 0 \qquad \forall \eta \in L^2(0,T; H^1_{\Gamma_0}(\Omega)) \,, \tag{3.27}$$

$$\theta(0) = \theta_0 \,. \tag{3.28}$$

Moreover, the following estimate is valid

$$\|\theta\|_\Xi \le c_{10} \delta^{1/2} \|u\|_H + c_{11} \|g\|_{H^1(0,T;L^2(\Gamma_c))} + c_{12} \,, \tag{3.29}$$

where the constants c_{10}, c_{11}, c_{12} are independent of δ, $\delta \le \delta_0$. In deriving (3.29) we take into account the boundedness of φ_t since $\varphi \in U$.

Now we combine the solvability of the problems (3.8)–(3.11) and (3.23)–(3.25) to prove the solvability of the following problem. Given $\varphi \in U$, we want to find the functions $u = (u_1, u_2, u_3)$, θ, $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2, 3$, such that

$$-\operatorname{div} \sigma = f \quad \text{in } Q \,, \tag{3.30}$$

$$\varepsilon(u) = C\sigma + q(\varphi)B\theta \quad \text{in } Q \,, \tag{3.31}$$

$$\theta_t - \Delta\theta + q(\varphi)\operatorname{div} u_t = \varphi_t \quad \text{in } Q \,, \tag{3.32}$$

$$\theta = \theta_0 \quad \text{for } t = 0 \,, \tag{3.33}$$

$$\theta = 0 \,, \; u = 0 \quad \text{on } \Gamma_c \times (0,T) \,, \tag{3.34}$$

$$\frac{\partial\theta}{\partial\nu} = g \quad \text{on } \Gamma_c \times (0,T) \,, \tag{3.35}$$

$$u \cdot \nu \le 0 \,, \; \sigma_\nu \le 0 \,, \; \sigma_\mu = 0 \,, \; u \cdot \nu\sigma_\nu = 0 \quad \text{on } \Gamma_c \times (0,T) \,. \tag{3.36}$$

To prove existence of a solution to (3.30)–(3.36), we denote $W = \Xi \times H$ and consider a linear and bounded operator

$$M : W \to W'$$

defined by the formula

$$\langle M(\theta, u), (\bar{\theta}, \bar{u}) \rangle = \int_Q (\theta_t + q(\varphi) \operatorname{div} u_t) \bar{\theta} - \int_{\Gamma_c \times (0,T)} g \bar{\theta} + \int_Q \nabla \theta \nabla \bar{\theta} + \int_Q \sigma : \varepsilon(\bar{u}) \,,$$

where $\sigma = \sigma(\theta, u, \varphi)$ is determined from (3.31). Next, we introduce a convex closed set in the space W,

$$S = \{(\theta, u) \in W \mid u \in \mathcal{K} \,,\ \theta(0) = \theta_0\} \,.$$

The solution to the problem (3.30)–(3.36) is defined as follows

$$(\theta, u, \sigma) \in W \times \Sigma \,,$$

$$\langle M(\theta, u), (\bar{\theta}, \bar{u}) - (\theta, u) \rangle \geq \int_Q \{ f \cdot (\bar{u} - u) + \varphi_t (\bar{\theta} - \theta) \} \qquad \forall (\bar{\theta}, \bar{u}) \in S \,; \quad (\theta, u) \in S \,. \tag{3.37}$$

We can prove that a solution of the problem (3.37) exists. Indeed, consider two closed sets,

$$S_1 = \{\theta \in \Xi \mid \theta(0) = \theta_0\} \,,\ S_2 = \{u \in H \mid u \in \mathcal{K}\} \,.$$

In this case inequality (3.37) is equivalent to the following relations

$$\int_Q (\theta_t + q(\varphi) \operatorname{div} u_t - \varphi_t)(\bar{\theta} - \theta) + \int_{\Gamma_c \times (0,T)} g(\bar{\theta} - \theta) + \int_Q \nabla \theta (\nabla \bar{\theta} - \nabla \theta) \geq 0 \quad \forall \bar{\theta} \in S_1 \,;\ \theta \in S_1 \,, \tag{3.38}$$

$$\int_Q \sigma : \varepsilon(\bar{u} - u) \geq \int_Q f \cdot (\bar{u} - u) \quad \forall \bar{u} \in S_2 \,;\ u \in S_2 \,, \tag{3.39}$$

$$\varepsilon(u) = C\sigma + q(\varphi) B \theta \quad \text{in } Q \,. \tag{3.40}$$

We should notice at this point that (3.38) is equivalent to (3.26)–(3.28), and (3.39)–(3.40) can be written in the equivalent form (3.12)–(3.14).

Now we are able to solve a variational inequality for a pseudomonotonous operator whose solution coincides with the solution of (3.37). Indeed, denote $c_0 = \max_{8 \le i \le 12}\{c_i\}$, where c_i are taken from the inequalities (3.22), (3.29). Denote also

$$c_g = ||g||_{H^1(0,T;L^2(\Gamma_c))} \tag{3.41}$$

and introduce two numbers κ_1, κ_2 such that

$$\kappa_1 = \frac{c_0(1 + c_g + c_0\delta^{1/2})}{1 - c_0^2\delta}\,,\ \kappa_2 = \frac{c_0(1 + c_0 c_g \delta^{1/2} + c_0\delta^{1/2})}{1 - c_0^2\delta}\,. \tag{3.42}$$

We choose

$$\delta < \frac{1}{c_0^2}\,. \tag{3.43}$$

In this case κ_1, κ_2 are positive. Note that if $||\theta||_\Xi \le \kappa_1$ from (3.22) it follows that $||u||_H \le \kappa_2$ and conversely, if $||u||_H \le \kappa_2$, the inequality (3.29) implies $||\theta||_\Xi \le \kappa_1$.

Introduce next a set in the space W,

$$S^0 = \{(\theta, u) \in S \,|\, ||\theta||_\Xi \le \kappa_1\,,\ ||u||_H \le \kappa_2\}\,.$$

Since the operator $M : W \to W'$ is pseudomonotonous (see definition in [16]) and the set $S^0 \subset W$ is bounded, there exists a solution (θ, u) of the problem

$$\begin{gathered}(\theta, u) \in S^0\,,\ \langle M(\theta, u), (\bar\theta, \bar u) - (\theta, u)\rangle \ge \int\limits_Q \{f \cdot (\bar u - u) + \varphi_t(\bar\theta - \theta)\} \\ \forall (\bar\theta, \bar u) \in S^0\,.\end{gathered} \tag{3.44}$$

Introducing the sets

$$\begin{aligned} S_1^0 &= \{\theta \in S_1 \,|\, ||\theta||_\Xi \ \le \kappa_1\}\,, \\ S_2^0 &= \{\theta \in S_2 \,|\, ||u||_H \le \kappa_2\}\,, \end{aligned}$$

we can rewrite equivalently variational inequality (3.44) in the form of the following inequalities:

$$\begin{gathered}\int\limits_Q (\theta_t + q(\varphi)\mathrm{div}\, u_t + \varphi_t)(\bar\theta - \theta) - \int\limits_{\Gamma_c \times (0,T)} g(\bar\theta - \theta) \\ + \int\limits_Q \nabla\theta(\bar\theta - \nabla\theta) \ge 0 \quad \forall\ \bar\theta \in S_1^0\,;\ \theta \in S_1^0\,,\end{gathered} \tag{3.45}$$

$$\begin{gathered}\int\limits_Q \{(C\sigma + q(\varphi)B\theta - \varepsilon(u)) : (\bar\sigma - \sigma) + \sigma : \varepsilon(\bar u - u)\} \\ \ge \int\limits_Q f \cdot (\bar u - u) \quad \forall\ (\bar u, \bar\sigma) \in S_2^0 \times \Sigma\,;\ (u, \sigma) \in S_2^0 \times \Sigma\,.\end{gathered} \tag{3.46}$$

It suffices to verify that the solution of (3.44) coincides with the solution of (3.38)–(3.40) and to take into account that (3.38)–(3.40) is equivalent to (3.37). Let (θ, u) be a solution of (3.44). Then $u \in S_2^0$. Find $\tilde{\theta}$ as a solution of (3.38) for a given $u \in S_2^0$. By the estimate (3.29) and (3.43), we have $\tilde{\theta} \in S_1^0$. But (3.45) has a unique solution for a given $u \in S_2^0$, whence $\tilde{\theta} = \theta$. On the other hand, $\theta \in S_1^0$. Consequently, for a solution $(\tilde{u}, \tilde{\sigma})$ of (3.39), (3.40) with the given $\theta \in S_1^0$ we obtain $\tilde{u} \in S_2^0$ by the estimate (3.22) and (3.43), hence $(\tilde{u}, \tilde{\sigma})$ is a unique solution of (3.46), i.e. $(\tilde{u}, \tilde{\sigma}) = (u, \sigma)$. So we have proved coincidence of solutions to (3.44) and (3.38)–(3.40) what is needed.

Solvability of the problem (3.30)–(3.36) is proved for any fixed $\varphi \in U$. This solvability will be used below to find a fixed point in the Schauder theorem.

Now we construct an operator

$$A : L^2(Q) \to L^2(Q)$$

which has a fixed point due to the Schauder theorem. Take a function $\bar{\theta} \in L^2(Q)$ and find a solution of the problem

$$\varphi_t = h(\bar{\theta}, \varphi)\,,$$
$$\varphi(0) = 0\,,$$

which provides an existence of $\varphi = \varphi(\bar{\theta})$, $\varphi \in U$. Substitute this function φ in (3.31), (3.32) and find a solution (u, σ, θ) of the problem (3.30)–(3.36) for small δ satisfying (3.43). In particular, we obtain

$$||\theta||_{\Xi} \le c\,,$$

where c is independent of $\bar{\theta}$. It is clear that if we take

$$||\bar{\theta}||_{L^2(Q)} \le R$$

and R is large enough then $\theta \in \Xi$,

$$||\theta||_{L^2(Q)} \le R\,,$$

and the imbedding $\Xi \subset L^2(Q)$ is compact. Hence, we have constructed operator $A : \bar{\theta} \to \theta$ which is compact and maps a ball $B_R \subset L^2(Q)$ into intself. To guarantee a fixed point of this operator is suffices to verify its continuity. So, let

$$\bar{\theta}^n \to \bar{\theta} \quad \text{in } L^2(Q)\,.$$

We have to prove

$$A(\bar{\theta}^n) \to A(\bar{\theta}) \quad \text{in } L^2(Q)\,.$$

As we know, strong convergence of $\bar{\theta}^n$ in $L^2(Q)$ provides (see (2.13))

$$\varphi^n \to \varphi \quad \text{in } H^1(0, T; L^2(\Omega))\,,$$

where $\varphi^n = \varphi(\bar{\theta}^n)$, $\varphi = \varphi(\bar{\theta})$. Hence, we can assume

$$\varphi^n, \varphi^n_t \to \varphi, \varphi_t \quad \text{almost everywhere in } \ Q\,.$$

Consider the equations

$$\theta^n_t - \Delta\theta^n + q(\varphi^n)\mathrm{div}\, u^n_t = \varphi^n_t \quad \text{in } \ Q\,, \tag{3.47}$$

$$\theta_t - \Delta\theta + q(\varphi)\mathrm{div}\, u_t = \varphi_t \quad \text{in } \ Q\,, \tag{3.48}$$

where $\theta^n = A(\bar{\theta}^n)$, $\theta = A(\bar{\theta})$, and u^n, u correspond to $\bar{\theta}^n$, $\bar{\theta}$, and can be defined from the boundary value problem (3.30)–(3.36) for $\varphi = \varphi^n$ and $\varphi = \varphi(\bar{\theta})$, respectively.

¿From the variational inequalities

$$\int\limits_\Omega \sigma : \varepsilon(v-u) \geq \int\limits_\Omega f\cdot(v-u) \qquad \forall v \in K\,,$$

$$\int\limits_\Omega \sigma^n : \varepsilon(v-u^n) \geq \int\limits_\Omega f\cdot(v-u^n) \quad \forall v \in K\,,$$

holding for almost all $t \in (0,T)$ it follows

$$\int\limits_\Omega (\sigma - \sigma^n) : \varepsilon(u-u^n) \leq 0\,. \tag{3.49}$$

Hence, taking into account the relation

$$\varepsilon(u) - \varepsilon(u^n) = C(\sigma-\sigma^n) + q(\varphi)B\theta - q(\varphi^n)B\theta^n \tag{3.50}$$

we derive for almost all $t \in (0,T)$

$$\|\sigma - \sigma^n\|^2_N \leq \int\limits_\Omega (q(\varphi^n)B\theta^n - q(\varphi)B\theta \pm q(\varphi^n)B\theta) : (\sigma - \sigma^n)\,.$$

Defining $\alpha_n = \|(q(\varphi^n) - q(\varphi))B\theta\|^2_{L^2(Q)}$ this inequality provides the estimate

$$\|\sigma - \sigma^n\|^2_{L^2(0,T;N)} \leq c\delta\|\theta - \theta^n\|^2_{L^2(Q)} + \alpha_n\,, \tag{3.51}$$

where $\alpha_n \to 0$ by the Lebesgue convergence theorem, and the constant c is independent of n, δ.

By (3.51), equality (3.50) implies

$$\|u - u^n\|^2_{L^2(0,T;[H^1_{\Gamma_0}(\Omega)]^3)} \leq c\delta\|\theta - \theta^n\|^2_{L^2(Q)} + c\delta\|\varphi - \varphi^n\|^2_{L^2(Q)} + \alpha_n\,. \tag{3.52}$$

Note that equations (3.47), (3.48) imply after integrating in time from 0 to t,

$$\begin{aligned} &(\theta - \theta^n)(t) - \Delta\int\limits_0^t (\theta-\theta^n) - (\varphi - \varphi^n)(t) \\ &= \int\limits_0^t (q(\varphi^n)\mathrm{div}\, u^n_t - q(\varphi)\mathrm{div}\, u_t \pm q(\varphi^n)\mathrm{div}\, u_t)\,. \end{aligned} \tag{3.53}$$

We use here the equality $(\theta-\theta^n)(0)=0$. Integrate next by parts (in t) in the right-hand side of (3.53) taking into account zero initial conditions $q(\varphi^n)(0)=q(\varphi)(0)=0$, and multiply (3.53) by $(\theta-\theta^n)(t)$. This gives the equality

$$\|\theta-\theta^n\|_0^2+\frac{1}{2}\frac{d}{dt}\left\|\int\limits_0^t \nabla(\theta-\theta^n)\right\|_0^2=\int\limits_\Omega(\varphi-\varphi^n)(\theta-\theta^n)$$

$$-\int\limits_\Omega(\theta-\theta^n)\left(\int\limits_0^t q(\varphi_t^n)\mathrm{div}\,(u^n-u)\right)-\int\limits_\Omega(\theta-\theta^n)\left(\int\limits_0^t (q(\varphi_t^n)-q(\varphi_t))\mathrm{div}\,u\right)$$

$$+\int\limits_\Omega(\theta-\theta^n)q(\varphi^n)\mathrm{div}\,(u^n-u)+\int\limits_\Omega(\theta-\theta^n)(q(\varphi^n)-q(\varphi))\mathrm{div}\,u$$

which implies

$$\|\theta-\theta^n\|_0^2(t)+\frac{1}{2}\frac{d}{dt}\left\|\int\limits_0^t \nabla(\theta-\theta^n)\right\|_0^2\le c\|\varphi-\varphi^n\|_0^2(t)$$

$$+c\|u-u^n\|^2_{H^1_{\Gamma_0}(\Omega)}(t)+c\int\limits_0^t\|u-u^n\|^2_{H^1_{\Gamma_0}(\Omega)}$$

$$+c\|\mathrm{div}\,u(q(\varphi^n)-q(\varphi))\|_0^2(t)+c\int\limits_0^t\|\mathrm{div}\,u(q(\varphi_t^n)-q(\varphi_t))\|_0^2\,.$$

We can integrate this inequality in t from 0 to T then take into account (3.52) and the strong convergence of φ^n to φ in $H^1(0,T;L^2(\Omega))$. For small δ, and by the Lebesgue convergence theorem, this provides as $n\to\infty$

$$\|\theta^n-\theta\|_{L^2(Q)}\to 0$$

what is needed.

So, we have proved the continuity of the operator A. Hence, due to the Schauder fixed point theorem, there exists $\bar\theta\in L^2(Q)$ such that

$$A(\bar\theta)=\bar\theta$$

which completes the proof of Theorem 3.1.

4 Uniqueness of the solution

We can prove uniqueness of the solution to (2.1)–(2.8) only in the one-dimensional case. Let us reformulate the problem (2.1)–(2.8) for this case and provide necessary explanations.

Denote $\Omega = (0,1)$, $Q = \Omega \times (0,T)$, $T > 0$. We have to find displacement $u(x,t)$, stress $\sigma(x,t)$, temperature $\theta(x,t)$, and volume fraction $\varphi(x,t)$ of pearlite such that

$$-\sigma_x = f \quad \text{in } Q\,, \tag{4.1}$$

$$u_x = c_0\sigma + bq(\varphi)\theta \quad \text{in } Q\,, \tag{4.2}$$

$$\theta_t - \theta_{xx} + q(\varphi)u_{tx} = \varphi_t \quad \text{in } Q\,, \tag{4.3}$$

$$\varphi_t = h(\theta,\varphi) \quad \text{in } Q\,, \tag{4.4}$$

$$\theta = \theta_0(x)\,,\ \varphi = 0 \quad \text{for } t = 0\,, \tag{4.5}$$

$$\theta = 0\,,\ u = 0 \quad \text{at } x = 0\,, \tag{4.6}$$

$$\theta_x = g \quad \text{at } x = 1\,, \tag{4.7}$$

$$u \le 0\,,\ \sigma \le 0\,,\ u\sigma = 0 \quad \text{at } x = 1\,. \tag{4.8}$$

Here $(x,t) \in Q$; $c_0 > 0, b$ are constants which are assumed to be equal to 1 for a simplicity; the index x below means a derivative with respect to x, $x \in \Omega$.

To refer to Theorem 3.1, we have to specify the definitions of functional spaces given in Section 2. Let

$$\begin{aligned} H^1_{\Gamma_0}(\Omega) &= \{v \in H^1(\Omega) \,|\, v = 0 \quad \text{at } x = 0\}\,, \\ K &= \{v \in H^1_{\Gamma_0}(\Omega) \,|\, v \le 0 \quad \text{at } x = 1\}\,, \\ \mathcal{K} &= \{v \in L^2(0,T; H^1_{\Gamma_0}(\Omega)) \,|\, v \le 0 \quad \text{a.e. on } \{1\} \times (0,T)\}\,. \end{aligned}$$

According to Theorem 3.1, for small δ there exists a solution $(u,\sigma,\theta,\varphi)$ of the problem (4.1)–(4.8) such that

$$u \in \mathcal{K} \cap H\,,\ \sigma \in \Sigma\,,\ \theta \in \Xi\,,\ \varphi \in U\,, \tag{4.9}$$

$$\int_Q \sigma(v_x - u_x) \ge \int_Q f(v-u) \quad \forall v \in \mathcal{K}\,, \tag{4.10}$$

$$u_x = \sigma + q(\varphi)\theta \quad \text{in } Q\,, \tag{4.11}$$

$$\begin{aligned} \int_Q (\theta_t + q(\varphi)u_{tx} - \varphi_t)\eta - \int_0^T g(1,t)\eta(1,t) \\ + \int_Q \theta_x\eta_x = 0 \quad \forall \eta \in L^2(0,T; H^1_{\Gamma_0}(\Omega))\,, \end{aligned} \tag{4.12}$$

$$\varphi_t = h(\theta, \varphi) \quad \text{in } Q\,, \tag{4.13}$$
$$\theta = \theta_0\,,\ \varphi = 0 \quad \text{for } t = 0\,. \tag{4.14}$$

Since the equation (4.3) implies $\theta \in L^2(0,T;H^2(\Omega))$, $\theta_t \in L^2(0,T;L^2(\Omega))$, we have $\theta \in C(0,T;H^1(\Omega))$. Taking into account the imbedding $H^1(\Omega) \subset C(\bar{\Omega})$ valid for the one-dimensional case, it follows

$$\theta \in L^\infty(Q)\,. \tag{4.15}$$

We have $\sigma, \dot{\sigma} \in L^2(0,T;L^2(\Omega))$, hence $\sigma \in C(0,T;L^2(\Omega))$. On the other hand, from (4.1) it follows

$$\sigma_x \in C(0,T;L^2(\Omega))\,,$$

whence

$$\sigma \in L^\infty(Q)\,. \tag{4.16}$$

By (4.15), (4.16), equation (4.11) yields

$$u_x \in L^\infty(Q)\,.$$

Using the boundedness of θ, $u_x \in L^\infty(Q)$, we can prove the uniqueness of the solution to (4.9)–(4.14). We also make an additional assumption concerning solution properties of (2.10)–(2.11). We assume that the following estimate holds (see [13])

$$\|\varphi^1(t) - \varphi^2(t)\|_0^2 \leq c \int_0^t \|\theta^1 - \theta^2\|_0^2\,, \tag{4.17}$$

where φ^1, φ^2 are solutions of (2.10)–(2.11) corresponding to θ^1, $\theta^2 \in L^2(Q)$ respectively, with a positive constant c independent of θ^1, θ^2. In fact, we can assume that the function h satisfies the Lipschitz conditions in both variables. In this case from the equations

$$\varphi_t^i = h(\theta^i, \varphi^i)\,,\ i = 1,2\,,$$

we easily derive

$$(\varphi^1 - \varphi^2)_t \leq c\,|\,\varphi^1 - \varphi^2\,| + c\,|\,\theta^1 - \theta^2\,|\,,$$

which provides (4.17) and also the following inequality

$$(\varphi^1 - \varphi^2)_t^2 \leq c\,|\,\varphi^1 - \varphi^2\,|^{\,2} + c\,|\,\theta^1 - \theta^2\,|^{\,2} \tag{4.18}$$

with constants c independent of φ^i, θ^i. Now we can formulate a uniqueness result for the problem (4.1)–(4.8).

Theorem 4.1 *Let the assumptions of Section 2 be fulfilled, and the function h satisfy the Lipschitz conditions in both variables. Then the solution to (4.9)–(4.14) is unique for small δ.*

Proof. Assume that there are two solutions $(u^1, \sigma^1, \theta^1, \varphi^1)$, $(u^2, \sigma^2, \theta^2, \varphi^2)$ satisfying (4.9)–(4.14). Denote $u = u^1 - u^2$, $\sigma = \sigma^1 - \sigma^2$, $\theta = \theta^1 - \theta^2$, $\varphi = \varphi^1 - \varphi^2$. From inequalities like (4.10) valid for each σ^i, u^i, $i = 1, 2$, we have (cf. (3.49))

$$\int_\Omega \sigma u_x \leq 0\,. \tag{4.19}$$

Relations (4.2), (4.3) imply

$$u_x = \sigma + q(\varphi^1)\theta^1 - q(\varphi^2)\theta^2 \quad \text{in } Q\,, \tag{4.20}$$

$$\begin{aligned} \theta - \frac{\partial^2}{\partial x^2}\int_0^t \theta + \int_0^t (q(\dot\varphi^2)u_x^2 - q(\dot\varphi^1)u_x^1) \\ + q(\varphi^1)u_x^1 - q(\varphi^2)u_x^2 = \varphi \quad \text{in } Q\,. \end{aligned} \tag{4.21}$$

Multiply next (4.20) by σ and integrate over Ω. Note that

$$|\,q(\varphi^1) - q(\varphi^2)\,| \leq \delta\,|\,\varphi^1 - \varphi^2\,|\,.$$

By (4.19), we obtain

$$\|\sigma(t)\|_0^2 \leq c\delta\|\theta(t)\|_0^2 + c\delta\|\varphi(t)\|_0^2\,. \tag{4.22}$$

Hence, relation (4.20) implies

$$\|u(t)\|^2_{H^1_{\Gamma_0}(\Omega)} \leq c\delta\|\theta(t)\|_0^2 + c\delta\|\varphi(t)\|_0^2 \tag{4.23}$$

with positive constants c. In deriving (4.22), (4.23) we take into account the boundedness of φ^i, θ^i in $L^\infty(Q)$, $i = 1, 2$.

Now multiply (4.21) by θ and integrate over Ω. This gives

$$\begin{aligned} \|\theta(t)\|_0^2 + \frac{1}{2}\frac{d}{dt}\left\|\int_0^t \theta_x\right\|_0^2 = \int_\Omega \varphi\theta + \int_\Omega \theta\left(\int_0^t q(\dot\varphi^2)u_x\right) \\ - \int_\Omega \theta\left(\int_0^t (q(\dot\varphi^2) - q(\dot\varphi^1))u_x^1\right) - \int_\Omega \theta q(\varphi^2)u_x + \int_\Omega \theta(q(\varphi^1) - q(\varphi^2))u_x^1\,, \end{aligned}$$

and, by the boundedness of u_x^i in $L^\infty(Q)$, $i=1,2$, we arrive at the inequality

$$\|\theta(t)\|_0^2 + \frac{1}{2}\frac{d}{dt}\left\|\int_0^t \theta_x\right\|_0^2 \le c\|\varphi(t)\|_0^2 + c\|u(t)\|^2_{H^1_{\Gamma_0}(\Omega)} + c\int_0^t \|u\|^2_{H^1_{\Gamma_0}(\Omega)} + c\delta\int_0^t \|\dot{\varphi}\|_0^2 . \tag{4.24}$$

Since the function h satisfies the Lipschitz condition in both variables we obtain for almost all $\xi \in (0,T)$ (see (4.18)

$$\|\dot{\varphi}(\xi)\|_0^2 \le c(\|\theta(\xi)\|_0^2 + \|\varphi(\xi)\|_0^2) . \tag{4.25}$$

Integrating (4.24), by the Gronwall lemma, (4.17), (4.23), (4.25), we derive for small δ

$$\int_0^t \|\theta(\xi)\|_0^2 d\xi \le 0\,,\ t\in(0,T)\,.$$

Hence $\theta = \theta^1 - \theta^2 = 0$. Since the temperature θ is unique, relations (4.10), (4.11), (4.13) provide uniqueness for u, σ, φ. Theorem 4.1 is proved. □

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