Global solutions to a Penrose–Fife phase–field model under flux boundary conditions for the inverse temperature

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Global Solutions to a Penrose–Fife Phase–Field Model Under Flux Boundary Conditions for the Inverse Temperature[1]

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Abstract. In this paper, we study an initial–boundary value problem for a system of phase–field equations arising from the Penrose–Fife approach to model the kinetics of phase transitions. In contrast to other recent works in this field, the correct form of the boundary condition for the temperature field is assumed which leads to additional difficulties in the mathematical treatment. It is demonstrated that global existence and, in the case of only one or two space dimensions, also uniqueness of strong solutions can be shown under essentially the same assumptions on the data as in the previous papers where a simplified boundary condition for the heat exchange with the surrounding medium has been used.

1 Introduction

In this paper, we study the initial–boundary value problem

$$\phi_t - \Delta\phi \in -F_1'(\phi) - \frac{F_2'(\phi)}{\theta}\,, \quad \text{in } Q := \Omega \times (0, +\infty)\,, \tag{1.1}$$

$$\Big(\rho(\theta)\Big)_t + F_2'(\phi)\,\phi_t = -\Delta\left(\frac{1}{\theta}\right) + g\,, \quad \text{in } Q\,, \tag{1.2}$$

$$\frac{\partial\phi}{\partial n} = 0\,, \quad \frac{\partial}{\partial n}\left(\frac{1}{\theta}\right) = \theta - \theta_\Gamma\,, \quad \text{on } \Sigma := \Gamma \times (0, +\infty)\,, \tag{1.3}$$

$$\phi(\cdot, 0) = \phi_0\,, \quad \theta(\cdot, 0) = \theta_0\,, \quad \text{in } \Omega\,, \tag{1.4}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $1 \le N \le 3$, with smooth boundary Γ.

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The system (1.1)–(1.4) constitutes a (generalized) Penrose–Fife model of phase–field type for diffusive phase transitions with non–conserved order parameter. In this connection, the fields ϕ and θ, respectively, represent the order parameter of the phase transition and the absolute temperature, respectively, and all physical constants are normalized to unity; moreover, the local free energy density $F = F(\phi, \theta)$ is assumed in the form

$$F(\phi, \theta) = F_0(\theta) \, + \, \theta \, F_1(\phi) \, + \, F_2(\phi) \, , \tag{1.5}$$

where the function

$$\rho'(\theta) = - \, \theta \, F_0''(\theta) \tag{1.6}$$

represents the specific heat. A typical form for the nonlinearities F_1 under consideration is given by

$$F_1'(\phi) = \beta(\phi) \, - \, s'(\phi) \, , \tag{1.7}$$

where s is smooth and where β denotes a maximal monotone graph. For example, if the order parameter ϕ represents a phase fraction having only values in the interval $[0,1]$, the graph β denotes the subdifferential of the indicator function of $[0,1]$. For details of the derivation of the field equations (1.1), (1.2) we refer to [12].

Initial–boundary value problems for the field equations (1.1), (1.2) have been studied in a number of recent papers. Indeed, the techniques developed in the fundamental papers [14] and [15] have been extended into various directions; in this connection, we refer to [2], [4–7], and [9–10]. The presently most general results have been established in [11]. However, up to the paper [15], where only the spatially one–dimensional case has been considered, all of these works share one serious physical drawback: they do not assume the correct boundary condition for the temperature field θ. Typically, the linear boundary condition

$$- \frac{\partial \theta}{\partial n} = \theta \, - \, \theta_\Gamma \, , \tag{1.8}$$

or a modification thereof, has been considered. However, in the Penrose–Fife model leading to the internal energy balance (1.2) the heat flux $\mathbf{q}$ is (up to a constant) given by

$$\mathbf{q} = \nabla \left(\frac{1}{\theta} \right) \, . \tag{1.9}$$

We should point out at this place that the classical Fourier law, in which the heat flux points in the direction of the negative temperature gradient, does not seem to be appropriate in this model; apparently, in this case there is no maximum principle hidden in the energy balance (1.2) which could guarantee that the (absolute !) temperature θ stays positive throughout the evolution of the phase transition. On the other hand, if the heat flux is given by (1.9), then the law describing the heat exchange with the surrounding medium should have the form assumed in (1.3).

It is the aim of this note to close the gap between mathematical model and physical interpretation: we are going to show that, except for the additional restriction that the source g must be non–negative, essentially the same conditions as in the above-mentioned papers have to be assumed for the functions arising in the model equations in order to guarantee the global existence of strong solutions which, at least for space dimensions one and two, will also turn out to be unique.

2 Statement of the Main Results

Consider the system (1.1)–(1.4). We make the following general assumptions on the data of the system.

(A1) There exist a maximal monotone graph β on $\mathbb{R}$ with domain $D(\beta)$ satisfying $Int(D(\beta)) \neq \emptyset$ and $0 \in \beta(0)$, and a function s satisfying $s(0) = 0$ which is twice continuously differentiable on the closure J of $D(\beta)$, such that

$$F_1' = \beta - s' . \tag{2.1}$$

We also assume that there exist $c_1 \geq 0$, $c_2 \geq 0$ and $a > 0$ such that

$$-c_1 + a\,\xi^2 \leq -s(\xi)\,, \quad s''(\xi) \leq c_2\,, \quad \forall\, \xi \in J\,. \tag{2.2}$$

(A2) F_2 belongs to $C^2(J)$ and satisfies

$$0 \leq F_2''(\xi) \leq M_2, \quad \forall\, \xi \in J\,, \tag{2.3}$$

where $M_2 \leq 2\,a$, if $F_2'(0) = 0$, and $M_2 < 2\,a$, otherwise.

(A3) $\rho \in C^1([0, +\infty))$ is an increasing Lipschitz continuous function satisfying $\rho(0) = 0$, and there exist constants $c_3 > 0$ and $\xi_\infty \geq 0$ such that

$$\rho'(\xi) \geq c_3 \quad \text{if } \xi \geq \xi_\infty\,. \tag{2.4}$$

We denote by L_ρ the Lipschitz constant for ρ, and by ρ^{-1} its inverse on $[0, +\infty)$.

(A4) For each $T \in [0, +\infty)$, the functions θ_Γ and $(\theta_\Gamma)_t$ belong to $L^\infty(\Sigma_T)$ and it holds

$$\theta_\Gamma > 0 \quad \text{a.e. on } \Sigma_T\,, \quad \frac{1}{\theta_\Gamma} \in L^\infty(\Sigma_T)\,, \tag{2.5}$$

where $\Sigma_T := \Gamma \times (0, T)$.

(A5) For each $T \in [0, +\infty)$, it holds $g \in L^\infty(Q_T)$ and $g_t \in L^\infty(0, T, L^2(\Omega))$, where $Q_T := \Omega \times (0, T)$. We also assume that g is non–negative in Q.

(A6) $\phi_0 \in H^2(\Omega)$ and $\theta_0 \in H^1(\Omega) \cap L^\infty(\Omega)$, respectively, are such that $\phi_0 \in D(\beta)$ almost everywhere in Ω, $\beta^\circ(\phi_0) \in L^2(\Omega)$ and

$$\frac{\partial \phi_0}{\partial n} = 0\,, \quad \text{a. e. on } \Gamma\,, \quad \theta_0 > 0 \quad \text{a.e. in } \Omega\,, \quad \frac{1}{\theta_0} \in L^\infty(\Omega)\,. \tag{2.6}$$

Remark 1: If $D(\beta)$ is bounded, then it suffices to postulate that $s \in C^2(J)$ and that $F_2 \in C^2(J)$ is convex in order to ensure that (2.2) and (2.3) are satisfied.

For some results, we will also need the additional assumption

(A3') There exist constants $c_3' > 0$, $\nu \geq 2$ and $\xi_0 > 0$ such that

$$c_3'\,\xi^{\nu-2} \leq \rho'(\xi)\,, \quad 0 \leq \xi \leq \xi_0\,. \tag{2.7}$$

We now state the main results of this paper.

Theorem 1 *Suppose that the assumptions* **(A1)** *to* **(A6)** *are satisfied, and suppose that in addition the following conditions hold:*

(i) $D(\beta)$ *is open, and* $\beta \in C^2(D(\beta))$, $s \in C^3(J)$, $F_2 \in C^4(J)$,

(ii) $\rho \in C^3([0,+\infty))$, *and* **(A3')** *is satisfied with* $\nu = 2$,

(iii) $\theta_\Gamma \in C^2(\Gamma \times [0,+\infty))$, $g \in C^2(\bar{\Omega} \times [0,+\infty))$,

(iv) $\beta(\phi_0) \in L^\infty(\Omega)$, $\theta_0 \in H^2(\Omega)$.

Then the system (1.1)–(1.4) *has a unique global classical solution*

$$(\phi,\theta) \in C(\bar{\Omega} \times [0,+\infty), D(\beta) \times (0,+\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,+\infty), I\!R^2).$$

The next result is a generalization of the result of Theorem 1.

Theorem 2 *Suppose that the assumptions* **(A1)** *to* **(A6)** *are satisfied. Then there exists a pair* (ϕ,θ) *such that, for each* $T \in (0,+\infty)$,

(i) $\phi \in W^{1,2}(0,T,H^1(\Omega)) \cap L^\infty(0,T,H^2(\Omega)) \left(\subset C(\overline{Q_T})\right)$, $\phi \in D(\beta)$ *a.e. in* Q_T, $\phi(\cdot,0) = \phi_0$,

(*ii*) $\theta \in L^\infty(0,T,H^1(\Omega)) \cap L^\infty(Q_T)$, $\theta > 0$ *a.e. in* Q_T, $\dfrac{1}{\theta} \in L^2(0,T,H^2(\Omega))$,

$\rho(\theta) \in H^1(Q_T)$, $\rho(\theta)(\cdot,0) = \rho(\theta_0)$,

(*iii*) (ϕ,θ) *satisfies* (1.1)–(1.2) *almost everywhere in* Q_T, *and*

$$\frac{\partial \phi}{\partial n} = 0, \quad \frac{\partial}{\partial n}\left(\frac{1}{\theta}\right) = \rho^{-1}(\gamma_\partial(\rho(\theta))) - \theta_\Gamma, \quad a.e. \ on \ \Sigma_T .$$

(Here, γ_∂ *denotes the trace operator on* Γ *).*
Moreover, if also **(A3')** *holds, then, for any* $T \in (0,+\infty)$, (1.3) *is satisfied almost everywhere on* Σ_T, $\dfrac{1}{\theta}$ *belongs to* $L^\infty(Q_T)$, θ *to* $L^2(0,T,H^2(\Omega))$, *and both* θ *and* $\dfrac{1}{\theta}$ *belong to* $W^{1,2}(0,T,L^2(\Omega))$.

We supplement Theorem 2 with a partial uniqueness result.

Proposition 3 *Suppose that the assumptions* **(A1)** *to* **(A6)** *and* **(A3')** *are satisfied. If*

(i) $D(\beta)$ *is open,* $\beta \in C^1(D(\beta))$, $\rho \in C^2((0,+\infty))$,

(ii) $\beta(\phi_0) \in L^\infty(\Omega)$,

(iii) $N \in \{1,2\}$,

then the solution to (1.1)–(1.4) *given by Theorem 2 is unique.*

3 A Regularized System

In this section, we study the well–posedness of the system (1.1)–(1.4) under the following stronger assumptions.

(B1) There exist a maximal monotone graph β on $\mathbb{R}$ with open and nonempty domain $D(\beta)$ which is twice continuously differentiable on $D(\beta)$ and satisfies $\beta(0) = 0$, and a function $s \in C^3(J)$ satisfying $s(0) = 0$, where J denotes the closure of $D(\beta)$, such that

$$F_1' = \beta - s'. \tag{3.1}$$

We also assume that there exist $\gamma_1 \geq 0$, $\gamma_2 \geq 0$ and $\alpha > 0$ such that

$$-\gamma_1 + \alpha\xi^2 \leq -s(\xi), \quad s''(\xi) \leq \gamma_2, \quad \forall \xi \in J, \tag{3.2}$$

and we denote by $\hat\beta$ the primitive of β vanishing at $\xi = 0$.

(B2) $F_2 \in C^4(J)$ satisfies

$$0 \le F_2''(\xi) \le \mu_2\,, \quad \forall\, \xi \in J\,, \tag{3.3}$$

where $\mu_2 \le 2\,\alpha$, if $F_2'(0) = 0$, and $\mu_2 < 2\alpha$, otherwise.

(B3) $\rho \in C^3([0,+\infty))$ is an increasing Lipschitz continuous function satisfying $\rho(0) = 0$, and there exist $\gamma_3 > 0$, $\gamma_4 > 0$ and $\zeta_\infty \ge 0$ such that

$$\begin{cases} \rho'(\xi) \ge \gamma_3 & \text{if } \xi \ge \zeta_\infty, \\ \\ \rho'(\xi) \ge \gamma_4 & \text{if } \xi \le \zeta_\infty. \end{cases} \tag{3.4}$$

We denote by Λ_ρ the Lipschitz constant for ρ and by ρ^{-1} its inverse on $[0,+\infty)$, and we put $\gamma_5 = \min(\gamma_3, \gamma_4) > 0$.

(B4) $\theta_\Gamma \in C^2(\Gamma \times [0,+\infty))$, and $\theta_\Gamma > 0$ on $\Gamma \times [0,+\infty)$.

(B5) $g \in C^2(\bar\Omega \times [0,+\infty))$ is non–negative.

(B6) $(\phi_0, \theta_0) \in H^2(\Omega, \mathbb{R}^2)$ are such that

$$\beta(\phi_0) \in L^\infty(\Omega)\,, \quad \min_{x\in\bar\Omega} \theta_0(x) > 0\,. \tag{3.5}$$

We have the following result.

Proposition 4 *Under the assumptions* **(B1)** *to* **(B6)**, *the problem* (1.1)–(1.4) *has a unique global classical solution*

$$(\phi,\theta) \in C(\bar\Omega \times [0,+\infty), D(\beta) \times (0,+\infty)) \cap C^{2,1}(\bar\Omega \times (0,+\infty), \mathbb{R}^2)\,.$$

Proof:
The proof of Proposition 4 consists of two steps. First, employing abstract results of H. Amann [2], we prove local existence and uniqueness of a classical solution to (1.1)–(1.4), as well as a criterion for its global existence in time which requires uniform estimates for the local solution to (1.1)–(1.4); in the second step, we derive these uniform estimates, using essentially the same techniques as in [14] and [9].

Step 1: Local existence.

The local existence part of the proof is very close to that of Proposition 2.1 in [10], as well as to those of Theorems 17.3 and 17.4 in [2]. We are going to specify how problem

(1.1)–(1.4) fits into the abstract framework developed in [2]. To this end, let $\mathcal{D} := D(\beta) \times (0,+\infty)$, and let $w_1 := \phi$, $w_2 := \rho(\theta)$. For any $x \in \bar{\Omega}$ and $w = (w_1, w_2) \in \mathcal{D}$, we define

$$\begin{aligned}
a(x,w) &= \begin{pmatrix} 1 & 0 \\ & \\ -F_2'(w_1) & \dfrac{(\rho^{-1})'(w_2)}{(\rho^{-1}(w_2))^2} \end{pmatrix}, \\
a_{jk}(x,w) &= \begin{cases} a(x,w) & \text{if } 1 \le j = k \le N, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

and, for any $(y,t) \in \Gamma \times [0,+\infty)$, $w = (w_1, w_2) \in \mathcal{D}$,

$$c(y,t,w) = \begin{pmatrix} 0 & 0 \\ 0 & -\dfrac{\rho^{-1}(w_2) - \theta_\Gamma(y,t)}{w_2} \end{pmatrix}.$$

Then, $a_{jk} \in C^2(\bar{\Omega} \times \mathcal{D}, \mathcal{L}(\mathbb{R}^2))$, and $c \in C^2(\Gamma \times [0,+\infty) \times \mathcal{D}, \mathcal{L}(\mathbb{R}^2))$, where $\mathcal{L}(\mathbb{R}^2)$ denotes the vector space of (2×2)–real–valued matrices. Note that, for each $w \in \mathcal{D}$, the family $\{a_{jk}(\cdot, w), 1 \le j,k \le N\}$ is lower triangular in the sense of [2, Sect. 15]. Furthermore, for each $(t,w) \in [0,+\infty) \times \mathcal{D}$, the boundary value problem $(\mathcal{A}(t,w), \mathcal{B}(t,w))$ defined by

$$\begin{aligned}
\mathcal{A}(t,w)v &= -\sum_{j,k=1}^{N} \frac{\partial}{\partial x_j} \left(a_{jk}(\cdot, w) \frac{\partial v}{\partial x_k} \right), \\
\mathcal{B}(t,w)v &= -\sum_{j,k=1}^{N} \nu^j . \gamma_\partial \left(a_{jk}(\cdot, w) \frac{\partial v}{\partial x_k} \right) + c(\cdot, t, w)\gamma_\partial(v),
\end{aligned}$$

is normally elliptic in the sense of [2, Sect. 4]. Here, γ_∂ and $\nu = (\nu^1, \ldots, \nu^N)$ denote the trace operator and the outer unit normal vector field to Γ, respectively.

Finally, we define $f \in C^2(\bar{\Omega} \times [0,+\infty) \times \mathcal{D} \times \mathbb{R}^{2N}, \mathbb{R}^2)$ by

$$f(x,t,w,z,z') = \begin{pmatrix} -F_1'(w_1) \; - \; \dfrac{F_2'(w_1)}{\rho^{-1}(w_2)} \\ \\ g(x,t) \; + \; F_1'(w_1) \; F_2'(w_1) \; + \; \dfrac{F_2'(w_1)^2}{\rho^{-1}(w_2)} \; + \; F_2''(w_1) \displaystyle\sum_{j=1}^{N} (z_j)^2 \end{pmatrix},$$

where $w = (w_1, w_2)$, $z = (z_1, \ldots, z_N)$, and $z' = (z_1', \ldots, z_N')$.

The derivative of f with respect to the pair (z,z') clearly satisfies assumption (14.4) of [2], that is, for each compact subset K of $\mathcal{D}$ and any $T > 0$, there exists a positive constant $c_{K,T}$ such that

$$|\partial_{(z,z')} f(x,t,w,z,z')| \le c_{K,T} \; (1 + |(z,z')|), \quad (x,t,w,z,z') \in \bar{\Omega} \times [0,T] \times K \times \mathbb{R}^{2N}.$$

In addition, also assumption (15.12) of [2] holds.

We set $w_0 := (\phi_0, \rho(\theta_0))$. By the continuity of the imbeddings $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and $H^1(\Omega) \hookrightarrow L^4(\Omega)$, we have $w_0 \in H^2(\Omega, \mathbb{R}^2)$. It follows from (3.5) and from the continuity of the imbedding $H^2(\Omega) \hookrightarrow W^{9/8,4}(\Omega)$ that

$$w_0 \in W^{9/8,4}(\Omega, \mathcal{D}) = \left\{ v \in W^{9/8,4}(\Omega, \mathbb{R}^2),\ v(\bar{\Omega}) \subset \mathcal{D} \right\} . \tag{3.6}$$

We may then apply the Theorems 14.4, 14.6 and 15.5 of [2] on $[0, T]$ for each positive T to obtain

Lemma 5 *Suppose that the assumptions* **(B1)** *to* **(B6)** *are satisfied. Then the initial boundary value problem*

$$\begin{aligned} w_t + \mathcal{A}(t, w)w &= f(t, w, \nabla w)\,, && (3.7)\\ \mathcal{B}(t, w)w &= 0\,, && (3.8)\\ w(\cdot, 0) &= w_0\,, && (3.9) \end{aligned}$$

has a unique maximal classical solution $w(\cdot, w_0) = (w_1, w_2)$ *on* $[0, t^+(w_0))$ *, that is,*

$$w \in C(\bar{\Omega} \times [0, t^+(w_0)), \mathcal{D}) \cap C^{2,1}(\bar{\Omega} \times (0, t^+(w_0)), \mathbb{R}^2)\,,$$

and w *satisfies* (3.7)–(3.9) *pointwise.*
Moreover, $t^+(w_0) = +\infty$ *, provided that there exist functions*

$$a_1, b_1 : \mathbb{R} \to D(\beta), \quad a_2, b_2 : \mathbb{R} \to (0, +\infty),$$

such that

$$\begin{aligned} a_1(T) \le w_1(t) &\le b_1(T)\,, \quad 0 \le t \le T < +\infty,\ t < t^+(w_0), && (3.10)\\ a_2(T) \le w_2(t) &\le b_2(T)\,, \quad 0 \le t \le T < +\infty,\ t < t^+(w_0)\,. && (3.11) \end{aligned}$$

Step 2: Uniform estimates.

In order to complete the proof of Proposition 4, it remains to check the validity of (3.10) and (3.11), respectively. For this purpose, we have to establish L^∞–estimates for w_1 and for $\beta(w_1)$ (implying (3.10)), as well as positive upper and lower bounds for w_2. To derive the latter estimates, we follow the idea of [14] and try to obtain some L^∞–estimate for the function

$$u := \frac{1}{\rho^{-1}(w_2)}\,.$$

Let $T > 0$. We are going to prove estimates for w_1 in $W^{1,2}(0, T, H^1(\Omega))$ and in $L^\infty(0, T, H^2(\Omega))$, for $\beta(w_1)$ in $L^\infty(Q_T)$, for w_2 in $H^1(Q_T)$, in $L^2(0, T, H^2(\Omega))$ and in

$L^\infty(Q_T)$, and for u in $L^\infty(Q_T)$, which depend only on the global data Ω, T, α, μ_2, γ_1, γ_2, γ_3, γ_4, ζ_∞ and on constants $R_0(T)$ and $R(T)$ satisfying

$$\begin{aligned} & |g|_{L^\infty(Q_T)} + |g_t|_{L^\infty(0,T,L^2(\Omega))} + |\theta_\Gamma|_{L^\infty(\Sigma_T)} + |(\theta_\Gamma)_t|_{L^\infty(\Sigma_T)} \\ + & \left|\frac{1}{\theta_\Gamma}\right|_{L^\infty(\Sigma_T)} + |\phi_0|_{H^2(\Omega)} + |\beta(\phi_0)|_{L^2(\Omega)} + |s'(\phi_0)|_{L^2(\Omega)} + |F_2'(0)| \\ + & |F_1(0)| + |\theta_0|_{H^1(\Omega)} + |\theta_0|_{L^\infty(\Omega)} + \left|\frac{1}{\theta_0}\right|_{L^\infty(\Omega)} + \rho(1) + \zeta_\infty \le R_0(T), \end{aligned} \quad (3.12)$$

$$R_0(T) + |\beta(\phi_0)|_{L^\infty(\Omega)} \le R(T). \quad (3.13)$$

For later use, we will pay much attention to how these estimates depend on γ_4 and $R(T)$. In the sequel, we denote by C, $\bar{C}$ any positive constant depending only on Ω, T, the constants α, γ_1, γ_2 in assumption **(B1)**, μ_2 in assumption **(B2)**, γ_3, ζ_∞, Λ_ρ in assumption **(B3)**, and $R_0(T)$ satisfying (3.12); moreover, any positive constant depending not only on the above mentioned constants but also on the constant γ_4 in assumption **(B3)** and on $R(T)$ satisfying (3.13), will be denoted by D or $\bar{D}$, respectively.

The required uniform estimates will now be proved in several steps, each stated in the form of a separate lemma. To begin with, we recall that the pair $w = (w_1, w_2)$ given by Lemma 5 is a solution to the initial–boundary value problem

$$\begin{aligned} w_{1t} - \Delta w_1 &= -F_1'(w_1) - F_2'(w_1)\, u, && (3.14)\\ w_{2t} + \Delta u &= g - F_2'(w_1)\, w_{1t}, && (3.15)\\ \frac{\partial w_1}{\partial n} &= 0, \quad \frac{\partial u}{\partial n} = \frac{1}{u} - \theta_\Gamma, && (3.16)\\ w_1(\cdot,0) &= \phi_0, \quad w_2(\cdot,0) = \rho(\theta_0), && (3.17) \end{aligned}$$

where $u = \dfrac{1}{\rho^{-1}(w_2)}$. We have

Lemma 6 *There exists a constant $C > 0$ satisfying*

$$|w_1|_{L^\infty(0,T,H^1(\Omega))} + |w_{1t}|_{L^2(Q_T)} + |\nabla u|_{L^2(Q_T)} + |u|_{L^1(0,T,H^1(\Omega))} \le C. \quad (3.18)$$

Proof:
From **(B1)** it follows that

$$F_1(\xi) = F_1(0) + \hat{\beta}(\xi) - s(\xi) \le F_1(0) + \xi\,\beta(\xi) - \xi\, s'(\xi) + \frac{\gamma_2}{2}\xi^2,$$

whence, using (3.12),

$$\int_\Omega F_1(\phi_0)\, dx \le \bar{C} + |\phi_0|_{L^2(\Omega)} \left(|\beta(\phi_0)|_{L^2(\Omega)} + |s'(\phi_0)|_{L^2(\Omega)} + \frac{\gamma_2}{2}|\phi_0|_{L^2(\Omega)}\right) \le C. \quad (3.19)$$

Next, let $\tilde{\rho} : (0,+\infty) \to \mathbb{R}$ denote the function defined by

$$\tilde{\rho}(\rho(1)) = \rho(1)\,, \quad \tilde{\rho}'(\xi) = \frac{1}{\rho^{-1}(\xi)}\,, \quad \xi > 0\,,$$

that is,

$$\tilde{\rho}(\xi) = \rho(1) \,+\, \int_{\rho(1)}^{\xi} \frac{1}{\rho^{-1}(\sigma)}\, d\sigma\,.$$

Then

$$\tilde{\rho}(\xi) \le \xi\,, \ \text{for } \xi \in (0,+\infty)\,. \tag{3.20}$$

Next, note that **(B1)** implies that $\hat{\beta}$ is non–negative. Therefore, by virtue of (3.1)–(3.3),

$$F_1(\xi) \,+\, F_2(\xi) \ge F_1(0) \,+\, \alpha\,\xi^2 \,-\, \gamma_1 \,-\, \frac{\mu_2}{2}\,\xi^2 \,-\, |F_2'(0)|\,|\xi| \ge -\,\bar{C}\,,$$

whence

$$\int_\Omega (F_1(w_1) + F_2(w_1))\; dx \ge -\,C\,. \tag{3.21}$$

We now take the scalar product in $L^2(\Omega)$ of (3.14) with w_{1t}, of (3.15) with $1-u$, and add the resulting equations to obtain

$$\frac{d}{dt}\int_\Omega \left(\frac{|\nabla w_1|^2}{2} + F_1(w_1) + F_2(w_1) + w_2 - \tilde{\rho}(w_2)\right)\, dx \;\;+\;\; \int_\Omega \left(|w_{1t}|^2 + |\nabla u|^2\right)\, dx$$
$$+ \int_\Gamma \left(\frac{1}{u} + \theta_\Gamma\; u\right)\; d\sigma = \int_\Gamma (1+\theta_\Gamma)\; d\sigma \;\;+\;\; \int_\Omega g\;(1-u)\; dx\,.$$

Owing to Poincaré's inequality, we find upon integrating over $(0,t)$, $t \in [0,T]$,

$$\int_\Omega \left(\frac{|\nabla w_1|^2}{2} + F_1(w_1) + F_2(w_1) + w_2 - \tilde{\rho}(w_2)\right)\; dx \,+\, \int_0^t \int_\Omega \left(|w_{1t}|^2 + \frac{|\nabla u|^2}{2}\right)\; dxds$$
$$+\,\bar{C}\int_0^t |u|_{H^1(\Omega)}\; ds \le C \,+\, \int_\Omega (F_1(\phi_0) \,+\, F_2(\phi_0) \,+\, \rho(\theta_0) \,-\, \tilde{\rho}(\rho(\theta_0)))\; dx\,. \tag{3.22}$$

It follows from (3.19) and from **(B2)** that

$$\int_\Omega (F_1(\phi_0) \,+\, F_2(\phi_0))\; dx \le \bar{C} \,+\, \int_\Omega \left(|F_2'(0)|\;|\phi_0| + \mu_2\;|\phi_0|^2\right)\; dx \le C\,. \tag{3.23}$$

We also infer from **(B3)** and from **(B6)** that

$$\int_\Omega (\rho(\theta_0) \,-\, \tilde{\rho}(\rho(\theta_0)))\; dx \le \int_\Omega \left(\Lambda_\rho\;\theta_0 \,-\, \rho(1) \,+\, \frac{|\rho(1) \,-\, \rho(\theta_0)|}{1 \,+\, \sup_{\xi\in\Omega}\,|1/\theta_0(\xi)|}\right) dx \le C\,. \tag{3.24}$$

Combining (3.20)–(3.24), we conclude that

$$\frac{1}{2}\int_\Omega |\nabla w_1|^2\; dx \,+\, \frac{1}{2}\int_0^t\int_\Omega \left(2\,|w_{1t}|^2 \,+\, |\nabla u|^2\right)\; dxds \,+\, \bar{C}\;\int_0^t |u|_{H^1(\Omega)}\; ds \le C\,,$$

which implies (3.18), since $|w_1|_{L^\infty(0,T,L^2(\Omega))} \le C \,+\, \sqrt{T}\;|w_{1t}|_{L^2(Q_T)}\,.$ □

Next, we are going to improve the estimates obtained in Lemma 6.

Lemma 7 *There exists a constant* $C > 0$ *satisfying*

$$|w_{1t}|_{L^\infty(0,T,L^2(\Omega))} + |\nabla w_{1t}|_{L^2(Q_T)} + |u|_{L^\infty(0,T,H^1(\Omega))} + \int_0^T \int_\Omega \rho'\left(\frac{1}{u}\right) \left|\frac{u_t}{u}\right|^2 \, dxds \le C\,, \quad (3.25)$$

$$|w_1|_{L^\infty(0,T,H^2(\Omega))} + |F_1'(w_1)|_{L^\infty(0,T,L^2(\Omega))} \le C\,. \quad (3.26)$$

Proof:
We follow the lines of the proof of Lemma 3.2 in [14]. To this end, we differentiate (3.14) with respect to t, take the scalar product in $L^2(\Omega)$ of the resulting equation with w_{1t}, and add the result to the $L^2(\Omega)$–scalar product of (3.15) with $-u_t$. Using Lemma 6, the positivity of u, the continuity of the trace operator from $H^1(\Omega)$ into $L^1(\Gamma)$, as well as the assumptions **(B1)**, **(B2)**, **(B4)**, **(B5)**, we find that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \left(\frac{|w_{1t}|^2}{2} + \frac{|\nabla u|^2}{2} + g\,u\right) dx + \frac{d}{dt}\int_\Gamma (\theta_\Gamma\, u - \ln u)\, d\sigma \\
&+ \int_\Omega \left(|\nabla w_{1t}|^2 + \rho'\left(\frac{1}{u}\right)\left|\frac{u_t}{u}\right|^2\right) dx \\
= \ &\int_\Omega g_t\, u\, dx + \int_\Gamma (\theta_\Gamma)_t\, u\, d\sigma + \int_\Omega (-F_1''(w_1) - u\, F_2''(w_1))\, |w_{1t}|^2\, dx \le C\,.
\end{aligned}$$

Since

$$w_{1t}(0) = \Delta\phi_0 - F_1'(\phi_0) - \frac{F_2'(\phi_0)}{\theta_0}\,,$$

the above inequality yields, after integration over $(0,t)$, $t \in [0,T]$,

$$\begin{aligned}
&\int_\Omega \left(\frac{|w_{1t}|^2}{2} + \frac{|\nabla u|^2}{2} + g\,u\right) dx + \int_\Gamma (\theta_\Gamma\, u - \ln u)\, d\sigma \\
&+ \int_0^t \int_\Omega \left(|\nabla w_{1t}|^2 + \rho'\left(\frac{1}{u}\right)\left|\frac{u_t}{u}\right|^2\right) dxds \le C\,,
\end{aligned}$$

which implies (3.25), since the function $\xi \mapsto \dfrac{\min\theta_\Gamma}{2}\,\xi - \ln\xi$ is bounded from below on $(0,+\infty)$. Next, note that w_1 satisfies

$$-\Delta w_1 + \gamma_2\, w_1 + F_1'(w_1) = \gamma_2\, w_1 - w_{1t} - F_2'(w_1)\, u\,, \quad \frac{\partial w_1}{\partial n} = 0\,. \quad (3.27)$$

Thus, we can infer from (3.25) and from the continuity of the imbedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ that

$$|\gamma_2\, w_1 - w_{1t} - F_2'(w_1)\, u|_{L^\infty(0,T,L^2(\Omega))} \le C\,.$$

By assumption **(B1)**, the function $\xi \mapsto \gamma_2\,\xi + F_1'(\xi)$ is non–decreasing. Therefore, (3.27) and the last estimate yield

$$|\Delta w_1|_{L^\infty(0,T,L^2(\Omega))} + |F_1'(w_1)|_{L^\infty(0,T,L^2(\Omega))} \le C\,,$$

whence (3.26) follows. $\square$

Next, we note that (3.26) and the continuity of the imbedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ imply that

$$|w_1|_{L^\infty(Q_T)} \le C\,. \tag{3.28}$$

Therefore, putting $h := g - F_2'(w_1)\; w_{1t}$, we can conclude that

$$|h|_{L^2(0,T,L^6(\Omega))} \le C\,. \tag{3.29}$$

Now, recall that w_2 is a solution to the problem

$$w_{2t} \,+\, \Delta u = h\,, \quad \frac{\partial u}{\partial n} = \frac{1}{u} - \theta_\Gamma\,. \tag{3.30}$$

We are going to prove the following result.

Lemma 8 *There exists a constant* $C > 0$ *satisfying*

$$|\theta|_{L^\infty(Q_T)} + |w_2|_{L^\infty(Q_T)} \le C\,, \tag{3.31}$$

where $\theta = \rho^{-1}(w_2)$.

Proof:
The proof makes use of Moser's technique (cf. [1]). To this end, consider the functions $\rho_{p+1} : [0,+\infty) \to [0,+\infty)$ defined by

$$\rho_{p+1}(0) = 0\,, \quad \rho_{p+1}'(\xi) = (p+1)\; \rho'(\xi)\; \xi^p\,, \qquad \text{for } \xi \ge 0\,,$$

for any $p \in [0,+\infty)$. We then infer from **(B3)** that

$$\rho_{p+1}(\xi) \;\le\; \Lambda_\rho\; \xi^{p+1}\,, \qquad \text{for } \xi \ge 0\,, \tag{3.32}$$

$$\gamma_3\; \xi^{p+1} - (\gamma_3 + \Lambda_\rho)\; \zeta_\infty^{p+1} \;\le\; \rho_{p+1}(\xi)\,, \qquad \text{for } \xi \ge \zeta_\infty\,. \tag{3.33}$$

Let $p \in (2,+\infty)$. We take the scalar product in $L^2(\Omega)$ of (3.30) with θ^p . It follows

$$\frac{1}{p+1}\,\frac{d}{dt}\int_\Omega \rho_{p+1}(\theta)\; dx \;+\; \frac{4\,p}{(p-1)^2}\int_\Omega \left|\nabla \theta^{\frac{p-1}{2}}\right|^2 dx \;+\; \int_\Gamma \theta^{p+1}\; d\sigma$$
$$\le\; \int_\Gamma \theta_\Gamma\; \theta^p\; d\sigma \;+\; \int_\Omega |h|\,\theta^p\,dx\,.$$

Using Young's and Poincaré's inequalities and the continuity of the imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we obtain

$$\frac{d}{dt}\int_\Omega \rho_{p+1}(\theta)\; dx \;+\; \bar{C}\;\left(\int_\Omega \theta^{3(p-1)}\; dx\right)^{\frac{1}{3}} \le C R_0(T)^{p+1} \;+\; (p+1)\int_\Omega |h|\,\theta^p\,dx\,. \tag{3.34}$$

Next, we infer from Hölder's and Young's inequalities that

$$(p+1)\int_\Omega |h|\theta^p dx \;\leq\; \frac{\bar{C}}{2}\left(\int_\Omega \theta^{3(p-1)}dx\right)^{\frac{1}{3}} + C(p+1)^2|h|^2_{L^6(\Omega)}\left(\int_\Omega \theta^{\frac{3}{4}(p+1)}dx\right)^{\frac{4}{3}}, \quad (3.35)$$

$$(p+1)\int_\Omega |h|\theta^p dx \;\leq\; \frac{\bar{C}}{2}\left(\int_\Omega \theta^{3(p-1)}dx\right)^{\frac{1}{3}} + C(p+1)^2|h|^2_{L^6(\Omega)}\left(\int_\Omega \theta^{p+1}dx\right). \quad (3.36)$$

We first take $p = 3$ in (3.34). It follows from (3.29), (3.33), (3.34), (3.36), and from Gronwall's lemma that

$$|\theta|_{L^\infty(0,T,L^4(\Omega))} \leq C\,. \quad (3.37)$$

We now consider the sequence (p_k) of real numbers defined by

$$p_0 = 4, \;\; p_{k+1} = \frac{4}{3}\,p_k, \;\; k \in \mathbb{N}\,.$$

Let $k \in \mathbb{N}$, and take $p = p_{k+1} - 1$ in (3.34). In view of (3.12), (3.32), (3.33), and (3.35), we can conclude that

$$\sup_{t\in(0,T)}\int_\Omega \theta^{p_{k+1}}(t)\,dx \leq C\,p^2_{k+1}\,\max\left\{R_0(T)^{p_{k+1}}\,,\; \sup_{t\in(0,T)}\left(\int_\Omega \theta^{p_k}(t)\,dx\right)^{\frac{4}{3}}\right\}.$$

Hence, invoking (3.37) and [9, Lemma A.1], we find that

$$\sup_{t\in(0,T)}|\theta(t)|_{L^{p_k}(\Omega)} \leq C\,, \quad \forall\, k \in \mathbb{N}\,.$$

Taking the limit as $k \to +\infty$, we obtain (3.31), since ρ is Lipschitz continuous. □

The following result is a straightforward consequence of (3.15), Lemma 7 and Lemma 8.

Corollary 9 *There exists a constant $C > 0$ satisfying*

$$|\theta|_{L^\infty(0,T,H^1(\Omega))} + |w_2|_{L^\infty(0,T,H^1(\Omega))} + |w_{2t}|_{L^2(Q_T)} + |u|_{L^2(0,T,H^2(\Omega))} \leq C\,. \quad (3.38)$$

Next, we recall that u is a solution to the problem

$$\rho'\left(\frac{1}{u}\right)\; u_t \,-\, u^2\;\Delta u = -h\;u^2\,, \quad \frac{\partial u}{\partial n} = \frac{1}{u} \,-\, \theta_\Gamma\,. \quad (3.39)$$

We are going to prove the following result.

Lemma 10 *There exists a constant $D > 0$ satisfying*

$$|u|_{L^\infty(Q_T)} \leq D. \quad (3.40)$$

Proof:
The proof of Lemma 10 again relies on Moser's technique. For further use, we introduce the function $\sigma_p : (0,+\infty) \to \mathbb{R}$, $p \geq 1$, by

$$\sigma_p(1) = 1\,, \quad \sigma_p'(\xi) = p\,\rho'\left(\frac{1}{\xi}\right)\,\xi^{p-1}\,, \qquad \text{for } \xi \in (0,+\infty)\,.$$

Owing to **(B3)**, σ_p satisfies

$$1 - \gamma_5 + \gamma_5\,\xi^p \leq \sigma_p(\xi) \leq \Lambda_\rho\,\xi^p + 1 - \Lambda_\rho\,, \qquad \text{for } \xi \in [1,+\infty)\,. \tag{3.41}$$

Let $p \in (1,+\infty)$. We take the scalar product in $L^2(\Omega)$ of (3.39) with u^{p-1}. It follows

$$\begin{aligned}\frac{d}{dt}\int_\Omega \sigma_p(u)\,dx \;+\; &\frac{4\,p}{p+1}\int_\Omega u\left|\nabla\left(u^{\frac{p+1}{2}}\right)\right|^2\,dx + p\int_\Gamma \theta_\Gamma\,u^{p+1}\,d\sigma \\ &= \;p\int_\Gamma u^p\,d\sigma + p\int_\Omega |h|\,u^{p+1}\,dx\,.\end{aligned} \tag{3.42}$$

By (3.31),

$$u(x,t) \geq \bar{C} > 0\,, \quad \forall\,(x,t) \in \bar{\Omega}\times[0,T]\,.$$

Therefore, by virtue of Young's and Poincaré's inequalities,

$$\frac{d}{dt}\int_\Omega \sigma_p(u)\,dx + \bar{C}\,\left|u^{\frac{p+1}{2}}\right|^2_{H^1(\Omega)} \leq C\,R_0(T)^p + p\int_\Omega |h|\,u^{p+1}\,dx\,.$$

Since $H^1(\Omega)$ is continuously imbedded in $L^6(\Omega)$, we find, after integration over $(0,t)$, $t\in[0,T]$, that

$$\int_\Omega \sigma_p(u(t))\,dx + \bar{C}\int_0^t\left(\int_\Omega u^{3(p+1)}\,dx\right)^{\frac{1}{3}}\,ds \leq C R_0(T)^p + p\int_0^t\int_\Omega |h|\,u^{p+1}\,dxds\,. \tag{3.43}$$

Next, Hölder's and Young's inequalities yield

$$\begin{aligned}p\int_0^t\int_\Omega |h|\,u^{p+1}\,dxds &\leq p\int_0^t |h|_{L^6(\Omega)}\left(\int_\Omega u^{3(p+1)}\,dx\right)^{\frac{1}{6}}\left(\int_\Omega u^{\frac{3}{4}(p+1)}\,dx\right)^{\frac{2}{3}}\,ds \\ \leq\; \frac{\bar{C}}{2}\left(\int_0^t\int_\Omega u^{3(p+1)}\,dxds\right)^{\frac{1}{3}} &+ C\;p^2\int_0^t |h|^2_{L^6(\Omega)}\left(\int_\Omega u^{\frac{3}{4}(p+1)}\,dx\right)^{\frac{4}{3}}\,ds\,.\end{aligned}$$

The latter estimate, in combination with (3.29) and (3.43), implies

$$\int_\Omega \sigma_p(u(t))\,dx \leq C\left(R_0(T)^p + p^2\sup_{t\in(0,T)}\left(\int_\Omega u^{\frac{3}{4}(p+1)}(t)\,dx\right)^{\frac{4}{3}}\right)\,. \tag{3.44}$$

Consider now the sequence (q_k) of real numbers defined by

$$q_0 = 6,\quad q_{k+1} = \frac{4}{3}q_k - 1,\quad k\in\mathbb{N}.$$

Let $k \in \mathbb{N}$, and take $p = q_{k+1}$ in (3.44). Using (3.41), we obtain

$$\int_\Omega u^{q_{k+1}}(t)\, dx \le D\left(R_0(T)^{q_{k+1}} + q_{k+1}^2 \sup_{t\in(0,T)} \left(\int_\Omega u^{q_k}(t)\, dx\right)^{\frac{4}{3}}\right).$$

Moreover, (3.25) and the continuity of the imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ imply that $|u|_{L^\infty(0,T,L^6(\Omega))} \le C$. We then infer from [9, Lemma A.1] that

$$\sup_{t\in(0,T)} |u(t)|_{L^{q_k}(\Omega)} \le D, \quad \forall\, k \in \mathbb{N},$$

and (3.40) follows from passing to the limit as $k \to +\infty$. □

We are now in the position to complete the proof of Proposition 4: indeed, it follows from **(B1)**, **(B2)**, (3.28) and (3.40), that

$$|s'(w_1) - F_2'(w_1)\, u|_{L^\infty(Q_T)} \le D, \tag{3.45}$$

and w_1 is a solution to

$$w_{1t} + \beta(w_1) - \Delta w_1 = s'(w_1) - F_2'(w_1)\, u, \quad \frac{\partial w_1}{\partial n} = 0.$$

A monotonicity argument ensures that

$$|\beta(w_1)|_{L^\infty(Q_T)} \le |s'(w_1) - F_2'(w_1)\, u|_{L^\infty(Q_T)} + |\beta(\phi_0)|_{L^\infty(\Omega)},$$

whence, using (3.5) and (3.45),

$$|\beta(w_1)|_{L^\infty(Q_T)} \le D. \tag{3.46}$$

Then, (3.10) follows from **(B1)**, (3.28) and (3.46). Finally, (3.11) is a consequence of (3.31), (3.40) and the monotonicity of ρ. The application of Lemma 5, with $\theta = \rho^{-1}(w_2)$, concludes the proof of the assertion of Proposition 4. □

4 Proof of Theorems 1 and 2

Since Theorem 1 is an immediate consequence of Proposition 4, we concentrate on the proof of Theorem 2.

In the sequel, β is a maximal monotone graph, and s, F_2, ρ, g, θ_Γ, ϕ_0 and θ_0 denote functions satisfying the general assumptions **(A1)** to **(A6)**. We recall that J denotes the closure in $\mathbb{R}$ of the domain $D(\beta)$ of β. In order to prove Theorem 2, we are first going to construct suitable approximations of F_1, F_2, ρ, g, θ_Γ, ϕ_0 and θ_0, so that Proposition 4 can be applied; this will be done in the next lemma. Then we will derive uniform estimates for the solutions to the approximate problems, and the assertion of Theorem 2 will follow from passage to the limit.

The following result follows from standard arguments in approximation theory which need not to be repeated here.

Lemma 11 *There exist constants* $\mu_2 > 0$, $\alpha > 0$, $\gamma_1 \geq 0$, $\gamma_2 \geq 0$, $\gamma_3 > 0$, $\zeta_\infty \geq 0$, $\Lambda_\rho > 0$, *depending only on* M_2 , c_1 , c_2 , a , c_3 , ξ_∞ *and* L_ρ , *and sequences of functions* s_n , β_n , $F_{1,n}$, $F_{2,n}$, ρ_n *such that the following conditions hold:*

(i) $\beta_n \in C^1(I\!R)$ *is an increasing function satisfying*

$$\beta \subset \liminf_{n\to+\infty} \beta_n \,, \tag{4.1}$$

$$|\beta_n(x)| \leq |\beta^\circ(x)| \,. \tag{4.2}$$

(ii) $F_{1,n}$, $F_{2,n}$ *and* ρ_n *satisfy the assumptions* **(B1)**–**(B3)** *with the above constants and with* $\gamma_{4,n} = \dfrac{1}{n}$, *and*

$$F_{1,n}(0) = F_1(0), \quad F'_{1,n} = s'_n - \beta_n \,.$$

(iii) (λ_n) *converges to* λ *in* $C^2(J)$, (ρ_n) *converges to* ρ *in* $C([0,+\infty))$, *and* (s_n) *converges to* s *in* $C^2(J)$.

Moreover, if also **(A3')** *holds, then there exist constants* $\gamma'_3 > 0$ *and* $\zeta_0 > 0$, *depending only on* c'_3 *and* ξ_0 , *such that*

$$\rho'_n(\xi) \geq \gamma'_3 \, \xi^{\nu-2} \,, \qquad 0 \leq \xi \leq \zeta_0 \,. \tag{4.3}$$

(We recall that, if Y is a Banach space and if $(A_\epsilon)_{\epsilon\geq 0}$ denotes a sequence of operators acting on Y , then the notation

$$A_0 \subset \liminf_{\epsilon\to 0} A_\epsilon$$

means that to any $(x_0, y_0) \in A_0$ there exists some sequence $(x_\epsilon, y_\epsilon) \in A_\epsilon$ such that (x_ϵ, y_ϵ) converges to (x_0, y_0) in $Y \times Y$ as ϵ tends to zero.)

A straightforward consequence of Lemma 11, (*iii*), is that

$$-\frac{1}{\rho^{-1}} \subset \liminf_{n\to+\infty} \left(-\frac{1}{\rho_n^{-1}}\right) ; \tag{4.4}$$

in addition to that, to any compact subset K of J there exists a constant $C_{K,s} > 0$, depending only on K and s, such that

$$|s_n|_{C^2(K)} \leq C_{K,s} \,. \tag{4.5}$$

Moreover, we can pick two sequences $g^n \in C^\infty(\bar\Omega \times [0,+\infty))$, $\theta^n_\Gamma \in C^\infty(\Gamma \times [0,+\infty))$, such that the assumptions **(B4)** and **(B5)** hold and, in addition, for every $T > 0$ the following conditions are satisfied:

(i) (g^n) is bounded in $L^\infty(Q_T)$,

(ii) $((g^n)_t)$ is bounded in $L^\infty(0,T,L^2(\Omega))$,

(iii) (g^n) and $((g^n)_t)$, respectively, converge in $L^2(Q_T)$ to g and g_t, respectively,

(iv) (θ_Γ^n), $\left(\dfrac{1}{\theta_\Gamma^n}\right)$ and $\left(\dfrac{\partial\theta_\Gamma^n}{\partial t}\right)$ are bounded in $L^\infty(\Sigma_T)$,

(v) (θ_Γ^n) converges to θ_Γ in $L^2(\Sigma_T)$.

Finally, we put $\phi_0^n = \phi_0$, and there exists some sequence $(\theta_0^n) \subset H^2(\Omega)$ satisfying **(B6)**, which converges to θ_0 in $H^1(\Omega)$, while the sequences (θ_0^n) and $\left(\dfrac{1}{\theta_0^n}\right)$, respectively, remain bounded in $L^\infty(\Omega)$.

Since both β_n and the imbedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ are continuous, it follows that $\beta_n(\phi_0^n) = \beta_n(\phi_0)$ belongs to $L^\infty(\Omega)$ and that (3.5) holds. It also follows from (4.2), (4.5) and **(A6)** that

$$|\beta_n(\phi_0^n)|_{L^2(\Omega)} + |s_n'(\phi_0^n)|_{L^2(\Omega)} \le C_{K_{\phi_0},s} + |\beta^\circ(\phi_0)|_{L^2(\Omega)}, \tag{4.6}$$

where, thanks to the continuity of the imbedding $H^2(\Omega) \hookrightarrow C(\bar\Omega)$, the subset $K_{\phi_0} = \phi_0(\bar\Omega)$ of J is indeed compact.

By virtue of Proposition 4, for each $n \ge 1$ the problem

$$\phi_t^n = \Delta\phi^n - F_{1,n}'(\phi^n) - \frac{F_{2,n}'(\phi^n)}{\theta^n}, \quad \text{in } Q = \Omega \times (0,+\infty), \tag{4.7}$$

$$\rho_n(\theta^n)_t + F_{2,n}'(\phi^n)\phi_t^n = -\Delta\left(\frac{1}{\theta^n}\right) + g^n, \quad \text{in } Q, \tag{4.8}$$

$$\frac{\partial\phi^n}{\partial n} = 0, \quad \frac{\partial}{\partial n}\left(\frac{1}{\theta^n}\right) = \theta^n - \theta_\Gamma^n, \quad \text{on } \Sigma = \Gamma \times (0,+\infty), \tag{4.9}$$

$$\phi^n(\cdot,0) = \phi_0^n, \quad \theta^n(\cdot,0) = \theta_0^n, \quad \text{on } \Omega, \tag{4.10}$$

has a unique classical solution

$$(\phi^n,\theta^n) \in C(\bar\Omega \times [0,+\infty), \mathbb{R} \times (0,+\infty)) \cap C^{2,1}(\bar\Omega \times (0,+\infty), \mathbb{R}^2).$$

Now, let $T > 0$. It follows from (4.6) and from Lemma 11 that we may find some positive constant $R_0(T) > 0$ such that (3.12) holds for $(F_{1,n}, F_{2,n}, \rho_n, g^n, \theta_\Gamma^n, \phi_0^n, \theta_0^n)$, uniformly in $n \in \mathbb{N}$. Invoking the Lemmas 6 to 8, as well as Corollary 9, we conclude that there exists a constant $C_0 > 0$, depending only on Ω, T, α, γ_1, γ_2, μ_2, γ_3, Λ_ρ and $R_0(T)$, such that, for any $n \ge 1$, the pair (ϕ^n,θ^n) satisfies the estimate

$$
\begin{aligned}
&|\phi^n|_{L^\infty(0,T,H^2(\Omega))} + |\phi_t^n|_{L^\infty(0,T,L^2(\Omega))} + |\nabla\phi_t^n|_{L^2(Q_T)} + |F'_{1,n}(\phi^n)|_{L^\infty(0,T,L^2(\Omega))} \\
&\quad + |\theta^n|_{L^\infty(0,T,H^1(\Omega))} + |\theta^n|_{L^\infty(Q_T)} + |\rho_n(\theta^n)|_{L^\infty(0,T,H^1(\Omega))} + |\rho_n(\theta^n)_t|_{L^2(Q_T)} \\
&\qquad + |u^n|_{L^\infty(0,T,H^1(\Omega))} + |u^n|_{L^2(0,T,H^2(\Omega))} \le C_0 \,, \qquad (4.11)
\end{aligned}
$$

where

$$
u^n = \frac{1}{\theta^n} .
$$

In fact, it follows from Lemma 11 that C_0 depends only on Ω, T, α, c_1, c_2, c_3, L_ρ and $R_0(T)$. Thus, in the sequel we denote by C any positive constant depending only on Ω, T, a, c_1, c_2, M_2, c_3, L_ρ and $R_0(T)$.

From (4.11) we can infer that (ϕ^n) is equicontinuous in $C([0,T],H^1(\Omega))$ and that $(\phi^n(t))$ is relatively compact in $H^1(\Omega)$, for any $t \in [0,T]$. Therefore, by the Ascoli theorem, (ϕ^n) is a relatively compact set in $C([0,T],H^1(\Omega))$, and we may assume that there exists some $\phi \in C([0,T],H^1(\Omega))$ such that

$$
\phi^n \to \phi \quad \text{in } C([0,T],H^1(\Omega)) \text{ and a.e. in } Q_T . \qquad (4.12)
$$

Similarly, we infer from (4.11) that the sequence $(\rho_n(\theta^n))$ is relatively compact in $C([0,T],L^2(\Omega))$. It also follows from (4.11) that $(\rho_n(\theta^n))$ forms a bounded subset of the space

$$
\mathcal{W} = \left\{ w \in L^2(0,T,H^1(\Omega)),\ w_t \in L^2(0,T,L^2(\Omega)) \right\} .
$$

A classical compactness result ensures that the imbedding of $\mathcal{W}$ in $L^2(0,T,H^{1/2}(\Omega))$ is compact. Since the trace operator $\gamma_\partial : H^{1/2}(\Omega) \to L^2(\Gamma)$ is continuous, we may assume that there exists some $\zeta \in L^\infty(0,T,H^1(\Omega)) \cap W^{1,2}(0,T,L^2(\Omega))$ such that

$$
\begin{aligned}
\rho_n(\theta^n) &\to \zeta \quad \text{in } L^2(Q_T) \text{ and a.e. in } Q_T , &(4.13)\\
\gamma_\partial(\rho_n(\theta^n)) &\to \gamma_\partial(\zeta) \quad \text{in } L^2(\Sigma_T) \text{ and a.e. on } \Sigma_T . &(4.14)
\end{aligned}
$$

Next, we infer from (4.11), (4.12), Lemma 11 and the continuity of the imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, that we may assume that

$$
\begin{aligned}
F'_{2,n}(\phi^n) &\to F'_2(\phi) \quad \text{in } L^2(0,T,L^6(\Omega)) ,\\
s'_n(\phi^n) &\rightharpoonup s'(\phi) \quad \text{in } L^2(Q_T) , \qquad (4.15)
\end{aligned}
$$

and that there exists some $\psi \in L^2(Q_T)$ such that

$$
\beta_n(\phi^n) \rightharpoonup \psi \quad \text{in } L^2(Q_T) . \qquad (4.16)
$$

Invoking (4.1), (4.12), (4.16), and using a monotonicity argument, we find that

$$
\phi \in D(\beta) , \quad \text{and } \psi \in \beta(\phi) \text{ a.e. in } Q_T . \qquad (4.17)
$$

Next, we turn our interest to the sequences (θ^n) and (u^n). By (4.11), we may assume that there exist functions $\theta \in L^\infty(0,T,H^1(\Omega))$, $u \in L^2(0,T,H^2(\Omega))$ and $b \in L^2(\Sigma_T)$ such that

$$\begin{aligned} \theta^n &\rightharpoonup \theta \ \text{ in } L^2(0,T,H^1(\Omega))\,, \\ \gamma_\partial(u^n) &\rightharpoonup b \ \text{ in } L^2(\Sigma_T)\,, \\ u^n &\rightharpoonup u \ \text{ in } L^2(0,T,H^2(\Omega))\,. \end{aligned} \tag{4.18}$$

To complete the proof of Theorem 2, we need to investigate the relationship between the functions ζ in (4.13), (4.14) and u, θ, b in (4.18), respectively. To this end, note that $\left(-\dfrac{1}{\rho^{-1}}\right)$ is a maximal monotone graph on $\mathbb{R}$ with domain $(0,+\infty)$. Moreover,

$$-\,u^n = -\,\frac{1}{\rho_n^{-1}(\rho_n(\theta^n))}\,.$$

Therefore, it follows from (4.4), (4.13) and (4.18) that

$$\zeta > 0 \ \text{ and } \ u = \frac{1}{\rho^{-1}(\zeta)} \qquad \text{a.e. in } Q_T\,. \tag{4.19}$$

Similarly, using Lemma 11, the convergences (4.13), (4.14) and (4.18), and invoking the positivity of ζ and monotonicity arguments, we can conclude that

$$\begin{aligned} \theta &= \rho^{-1}(\zeta) \qquad \text{a.e. in } Q_T\,, \\ b &= \rho^{-1}(\gamma_\partial(\zeta)) \quad \text{a.e. on } \Sigma_T\,. \end{aligned} \tag{4.20}$$

We are now ready to complete the proof of the first part of Theorem 2. Indeed, it is easily checked that $(F'_{2,n}(\phi^n)\ u^n)$ converges weakly to $F'_2(\phi)/\theta$ in $L^2(Q_T)$ and that $(F'_{2,n}(\phi^n)\ \phi^n_t)$ converges to $F'_2(\phi)\ \phi_t$. Then, (1.1) and (1.2) follow from (4.11) and from the convergence results established in (4.12), (4.13), (4.15), (4.16), (4.18), (4.19) and (4.20); moreover, the properties *(i), (ii), (iii)* are a consequence of (4.20), (4.11) and of the continuity properties (4.12) and (4.13).

To confirm the second part of the assertion of Theorem 2, we now assume that ρ satisfies assumption **(A3')**. Then, for each $n \geq 1$, the function ρ_n satisfies (4.3). We need the following lemma.

Lemma 12 *Suppose that $T > 0$ is a positive real number, and suppose that w, w_Γ, $\mathcal{R}$ and f denote functions having the following properties:*

(i) $\mathcal{R} \in C^3([0,+\infty))$ is an increasing Lipschitz continuous function with $\mathcal{R}(0) = 0$ such that there exist $\zeta_0 > 0$, $\gamma'_3 > 0$ and $\nu \geq 2$ satisfying

$$\mathcal{R}'(\xi) \geq \gamma'_3\ \xi^{\nu-2}\,, \quad 0 \leq \xi \leq \zeta_0\,. \tag{4.21}$$

We denote by $\Lambda_{\mathcal{R}}$ *a Lipschitz constant for* $\mathcal{R}$ *.*

(ii) $w_\Gamma \in C^2(\Gamma \times [0,T])$ *, and* $\min_{\Gamma\times[0,T]} w_\Gamma = m_T > 0$ *.*

(iii) $f \in L^\infty(0,T,L^2(\Omega))$ *.*

(iv) $w \in C(\bar\Omega \times [0,T],(0,+\infty)) \cap C^{2,1}(\bar\Omega \times (0,T),I\!R)$ *satisfies*

$$\mathcal{R}'\left(\frac{1}{w}\right)\ \frac{w_t}{w^2} = \Delta w + f \quad \text{in } Q_T\,, \tag{4.22}$$

$$\frac{\partial w}{\partial n} = \frac{1}{w} - w_\Gamma \quad \text{on } \Sigma_T\,. \tag{4.23}$$

We put

$$w_m = \min_{\bar\Omega\times[0,T]} w(x,t)\,.$$

Then there exists a positive constant C_∞ *, depending only on* Ω *,* γ_3' *,* ν *,* ζ_0 *,* $\Lambda_{\mathcal{R}}$ *,* T *,* m_T *,* $|f|_{L^\infty(0,T,L^2(\Omega))}$ *,* w_m *,* $|w|_{L^\infty(0,T,L^6(\Omega))}$ *and* $|w(0)|_{L^\infty(\Omega)}$ *, such that*

$$|w|_{L^\infty(Q_T)} \le C_\infty\,. \tag{4.24}$$

Proof:

The proof of Lemma 12 involves similar arguments as those used in the proofs of Lemma 2.3 in [6] and of Lemma 6.6 in [11]. The proof includes two steps: first, we are going to show that w is bounded in $L^\infty(0,T,L^p(\Omega))$ by some constant $C_{\infty,p}$ which depends on the same data as C_∞ and also on p; this result will then enable us to employ Moser's technique and thus to establish the validity of (4.24).

In the sequel, we denote by C_∞ any positive constant depending only on Ω, T, γ_3', ν, ζ_0, $\Lambda_{\mathcal{R}}$, m_T, $|f|_{L^\infty(0,T,L^2(\Omega))}$, w_m, $|w|_{L^\infty(0,T,L^6(\Omega))}$ and $|w(0)|_{L^\infty(\Omega)}$.

Let $k \in (\nu,+\infty)$. We denote by $\mathcal{R}_k : [0,+\infty) \to [0,+\infty)$ the function defined by

$$\mathcal{R}_k(0) = 0\,, \quad \mathcal{R}_k'(\xi) = (k+1-\nu)\ \mathcal{R}'\left(\frac{1}{\xi}\right)\ \xi^{k-2}\,, \quad \text{for } \xi \in (0,+\infty)\,.$$

We then infer from (4.21) that

$$0 \le \mathcal{R}_k(\xi) \le \Lambda_{\mathcal{R}}\ \xi^{k-1}\,, \quad \forall\, \xi \ge 0\,, \tag{4.25}$$

$$\gamma_3'\ \left(\xi^{k+1-\nu} - \zeta_0^{\nu-k-1}\right) \le \mathcal{R}_k(\xi)\,, \quad \forall\, \xi \ge \frac{1}{\zeta_0}\,. \tag{4.26}$$

We take the scalar product in $L^2(\Omega)$ of (4.22) with w^k to obtain

$$\frac{1}{k+1-\nu}\frac{d}{dt}\int_\Omega \mathcal{R}_k(w)\ dx + \frac{4}{k}\int_\Omega w\left|\nabla\left(w^{\frac{k}{2}}\right)\right|^2\ dx + m_T\int_\Gamma w^k\ d\sigma$$

$$\leq \int_\Gamma w^{k-1}\, d\sigma + \int_\Omega |f|\, w^k\, dx\,.$$

Using Young's and Poincaré's inequalities, we find that there exists some $\delta > 0$, depending only on Ω, w_m, ν and m_T, such that

$$\frac{d}{dt}\int_\Omega \mathcal{R}_k(w)\, dx + \delta\, \left|w^{\frac{k}{2}}\right|^2_{H^1(\Omega)} \leq C_\infty\, m_T^{-k} + k \int_\Omega |f|\, w^k\, dx\,. \tag{4.27}$$

Next, we estimate the last term on the right-hand side of (4.27). By Hölder's inequality,

$$k \int_\Omega |f|\, w^k\, dx \leq k\, |f|_{L^2(\Omega)}\, |w|_{L^6(\Omega)}\, \left|w^{k-1}\right|_{L^3(\Omega)} \leq C_\infty\, k\, \left|w^{\frac{k}{2}}\right|^{\frac{2(k-1)}{k}}_{L^6(\Omega)}\,.$$

Since $H^1(\Omega)$ is continuously imbedded in $L^6(\Omega)$, it follows from Young's inequality that

$$k \int_\Omega |f|\, w^k\, dx \leq \frac{\delta}{2}\, \left|w^{\frac{k}{2}}\right|^2_{H^1(\Omega)} + C_\infty\, k^k\, \delta^{-k}\,. \tag{4.28}$$

Combining (4.27) and (4.28), we see that

$$\frac{d}{dt}\int_\Omega \mathcal{R}_k(w)\, dx + \frac{\delta}{2}\, \left|w^{\frac{k}{2}}\right|^2_{H^1(\Omega)} \leq C_\infty\, (m_T^{-k} + k^k\, \delta^{-k})\,.$$

We conclude from (4.25), (4.26) and from the above inequality that to each $k \in (\nu, +\infty)$ there exists a positive constant $C_{\infty,k}$ depending on Ω, T, γ_3', ν, ζ_0, $\Lambda_{\mathcal{R}}$, m_T, $|f|_{L^\infty(0,T,L^2(\Omega))}$, w_m, $|w|_{L^\infty(0,T,L^6(\Omega))}$, $|w(0)|_{L^\infty(\Omega)}$ and k, such that

$$|w|_{L^\infty(0,T,L^{k+1-\nu}(\Omega))} \leq C_{\infty,k}\,. \tag{4.29}$$

Next, for any $k > \nu$, it follows from Hölder's and Young's inequalities and from the continuity of the imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ that

$$\begin{aligned} k \int_\Omega |f|\, w^k\, dx &\leq k\, |f|_{L^2(\Omega)} \left(\int_\Omega w^{3k}\, dx\right)^{\frac{5}{18}} \left(\int_\Omega w^{\frac{3k}{4}}\, dx\right)^{\frac{2}{9}} \\ &\leq \frac{\delta}{2}\, \left|w^{\frac{k}{2}}\right|^2_{H^1(\Omega)} + C_\infty\, k^6 \left(\int_\Omega w^{\frac{3k}{4}}\, dx\right)^{\frac{4}{3}}, \end{aligned} \tag{4.30}$$

where $\delta > 0$ is defined in (4.27). Combining (4.27) and (4.29), invoking (4.25) and (4.26) and integrating over $(0,t)$, $t \in [0,T]$, we obtain the estimate

$$\int_\Omega w^{k+1-\nu}(t)\, dx \leq C_\infty\, (m_T^{-k} + \zeta_0^{-k} + |w(0)|^k_{L^\infty(\Omega)}) + C_\infty\, k^6\, |w|^k_{L^\infty(0,T,L^{3k/4}(\Omega))}\,. \tag{4.31}$$

We now consider the sequence (ν_n) of real numbers defined by

$$\nu_0 = 3\nu\,, \quad \nu_{n+1} = \frac{4}{3}\, \nu_n + 1 - \nu\,.$$

Then $\nu_n \geq 1$, for every $n \geq 0$. We may take $k = \nu_{n+1} + \nu - 1 > \nu$ in (4.31) and get

$$\int_\Omega w^{\nu_{n+1}}(t)\ dx \leq C_\infty\ M^{\nu_n}\ +\ C_\infty\ (\nu_n)^6\ |w|^{\frac{4\nu_n}{3}}_{L^\infty(0,T,L^{\nu_n}(\Omega))}\,, \quad t \in [0,T]\,, \tag{4.32}$$

with

$$M = \max\left\{1\,,\ \frac{1}{m_T}\,,\ \frac{1}{\zeta_0}\,,\ |w(0)|_{L^\infty(\Omega)}\right\}.$$

Finally, we infer from (4.29) that

$$|w|_{L^\infty(0,T,L^{\nu_0}(\Omega))} \leq C_{\infty,\nu_0}\ . \tag{4.33}$$

It then follows from (4.32), (4.33) and from [9, Lemma A.1] that

$$|w|_{L^\infty(0,T,L^{\nu_n}(\Omega))} \leq C_\infty\,, \quad \forall\, n \in \mathbb{N}\,.$$

Since $\nu_n \to \infty$ as $n \to \infty$, the assertion follows from passage to the limit as $n \to \infty$ in the above inequality. □

After this preparation, the proof of Theorem 2 can be concluded as follows. In view of (4.11) and of Lemma 11, for each $n \geq 1$ the functions $(u^n\,,\,\rho_n\,,\,\theta^n_\Gamma\,,\,f^n)$ satisfy the assumptions of Lemma 12, uniformly in $n \in \mathbb{N}$, where

$$f^n = F'_{2,n}(\phi^n)\ \phi^n_t\ -\ g^n.$$

Hence, there exists a constant $C_\infty > 0$ such that

$$|u^n|_{L^\infty(Q_T)}\ \leq\ C_\infty\,. \tag{4.34}$$

We then infer from (4.11) and (4.34) that (θ^n) is bounded in $L^2(0,T,H^1(\Omega))$ and that (θ^n_t) is bounded in $L^2(Q_T)$. A classical compactness result and the continuity of the trace operator $\gamma_\partial : H^{1/2}(\Omega) \to L^2(\Gamma)$ ensure that (θ^n) converges to θ almost everywhere in Q_T and that $(\gamma_\partial(\theta^n))$ converges to $\gamma_\partial(\theta)$ almost everywhere on Σ_T. Consequently, (1.3) holds. Finally, the regularity properties are straightforward consequences of (4.11), (4.34) and the Gagliardo–Nirenberg inequality. With this, the proof of the assertion of Theorem 2 is complete. □

5 Uniqueness in One and Two Space Dimensions

In this section, we are going to prove the assertion of Proposition 3. To this end, suppose that (ϕ_1, θ_1) and (ϕ_2, θ_2) are two solutions to (1.1)–(1.4) having the properties stated in Theorem 2, and let $T > 0$ be given. Since

$$\beta(\phi_0) \in L^\infty(\Omega)\,, \qquad -F'_1(\phi_i) - \frac{F'_2(\phi_i)}{\theta_i} \in L^\infty(Q_T)\,,$$

a monotonicity argument and (1.1) imply that $\beta(\phi_i) \in L^\infty(Q_T)$, $i = 1, 2$. We also know that ϕ_i is continuous on $\overline{Q_T}$, $i = 1, 2$. Then, since $D(\beta)$ is open, there exists a compact subset $[a_T, b_T]$ of $D(\beta)$ such that

$$a_T \leq \phi_i(x,t) \leq b_T \quad \text{in } Q_T\,, \quad i = 1,2\,.$$

Next, since both θ_i and $\dfrac{1}{\theta_i}$ belong to $L^\infty(Q_T)$, $i = 1, 2$, there exists a positive constant κ satisfying

$$0 < \frac{1}{\kappa} \leq \theta_i(x,t) \leq \kappa \ \text{ a.e. in } \ Q_T\,, \ \ i = 1,2\,.$$

We put

$$K = \max\left\{|\beta'|_{C([a_T,b_T])}, |F_2'|_{C([a_T,b_T])}, |\rho''|_{C([1/\kappa,\kappa])}\right\}, \ \ \rho_i' = \min_{\xi\in[1/\kappa,\kappa]} \rho'(\xi)\,, \ \ \rho_s' = \max_{\xi\in[1/\kappa,\kappa]} \rho'(\xi)\,,$$

and we define

$$\phi := \phi_1 - \phi_2\,, \ \ \theta := \theta_1 - \theta_2\,, \ \ \zeta := \rho(\theta_1) - \rho(\theta_2)\,, \ \ u := \frac{1}{\theta_1} - \frac{1}{\theta_2}\,.$$

Then

$$\rho_i'\,|\theta| \leq |\zeta| \leq \rho_s'\,|\theta|\,, \tag{5.1}$$

and it holds

$$\phi_t - \Delta\phi + c_2\phi_1 + F_1'(\phi_1) - c_2\phi_2 - F_1'(\phi_2) = c_2\phi - \frac{F_2'(\phi_1)}{\theta_1} + \frac{F_2'(\phi_2)}{\theta_2}\,, \tag{5.2}$$

$$\frac{\partial\phi}{\partial n} = 0\,, \quad \text{on } \Sigma_T\,, \tag{5.3}$$

$$\zeta_t\, +\, \Delta u = -\,F_2'(\phi_1)\,\phi_{1t}\, +\, F_2'(\phi_2)\,\phi_{2t}\,, \quad \text{in } Q_T\,, \tag{5.4}$$

$$\frac{\partial u}{\partial n} = \theta\,, \quad \text{on } \Sigma_T\,. \tag{5.5}$$

In the sequel, we denote by C or $\bar{C}$ any constant depending only on Ω, T, c_2, M_2, a_T, b_T, κ, K, ρ_i' and ρ_s'.

We take the scalar product in $L^2(\Omega)$ of (5.2) with ϕ. Using the monotonicity of the mapping $\xi \mapsto c_2\,\xi\, +\, F_1'(\xi)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \phi^2\,dx + \int_\Omega |\nabla\phi|^2\,dx \ &\leq\ c_2\int_\Omega |\phi|^2\,dx + \int_\Omega |\phi| \left|\frac{F_2'(\phi_1)}{\theta_1} - \frac{F_2'(\phi_2)}{\theta_2}\right| dx \\ &\leq\ (c_2\, +\, M_2\,\kappa)\int_\Omega |\phi|^2\,dx\, +\, K\,\kappa^2\int_\Omega |\phi|\ |\theta|\,dx\,,\end{aligned}$$

whence

$$\frac{d}{dt}\int_\Omega \phi^2\,dx + \int_\Omega |\nabla\phi|^2\,dx \leq C\left(\int_\Omega \phi^2\,dx + \int_\Omega \zeta^2\,dx\right)\,. \tag{5.6}$$

Next, we take the scalar product in $L^2(\Omega)$ of (5.4) with ζ. It follows

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \zeta^2\,dx + \int_\Gamma \theta\,\zeta\,d\sigma + \int_\Omega \nabla\zeta\cdot\nabla(-u)\,dx \\
\le\ & \int_\Omega \Big(|\zeta|\,|F_2'(\phi_1)-F_2'(\phi_2)|\,|\phi_{1t}| + |\zeta|\,|F_2'(\phi_2)|\,|\phi_t|\Big)\,dx \\
\le\ & M_2\,|\phi_{1t}|_{L^4(\Omega)}\,|\zeta|_{L^2(\Omega)}\,|\phi|_{L^4(\Omega)} + K\,|\zeta|_{L^2(\Omega)}\,|\phi_t|_{L^2(\Omega)}\,.
\end{aligned}\tag{5.7}$$

We have

$$\begin{aligned}
&\int_\Omega \nabla\zeta\cdot\nabla(-u)\,dx = \int_\Omega \nabla\zeta\cdot\left(\frac{\nabla\rho(\theta_1)}{\rho'(\theta_1)\theta_1^2} - \frac{\nabla\rho(\theta_2)}{\rho'(\theta_2)\theta_2^2}\right)\,dx \\
\ge\ & \int_\Omega \frac{|\nabla\zeta|^2}{\rho_s'\,\kappa^2}\,dx - \int_\Omega \frac{|\nabla\zeta|\,|\nabla\theta_2|}{\rho'(\theta_1)(\theta_1\theta_2)^2}\,|\rho'(\theta_2)\theta_2^2-\rho'(\theta_1)\theta_1^2|\,dx \\
\ge\ & \frac{1}{\rho_s'\,\kappa^2}\int_\Omega |\nabla\zeta|^2\,dx - \frac{\kappa^4}{(\rho_i')^2}\,(2\,\kappa\,\rho_s' + K\,\kappa^2)\int_\Omega |\nabla\theta_2|\,|\nabla\zeta|\,|\zeta|\,dx\,.
\end{aligned}\tag{5.8}$$

Also, since ρ is non–decreasing, the function $\theta\,\zeta$ is non–negative. Consequently, it follows from (5.7) and (5.8) that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega \zeta^2\,dx + \bar{C}\int_\Omega |\nabla\zeta|^2\,dx \\
\le\ & C\int_\Omega |\nabla\zeta|\,|\nabla\theta_2|\,|\zeta|\,dx + C\,|\phi_t|_{L^2(\Omega)}\,|\zeta|_{L^2(\Omega)} + C\,|\phi_{1t}|_{L^4(\Omega)}\,|\zeta|_{L^2(\Omega)}\,|\phi|_{L^4(\Omega)}\,.
\end{aligned}\tag{5.9}$$

From Theorem 2, we already know that $\theta_2 \in L^\infty(Q_T)\cap L^2(0,T,H^2(\Omega))$. The Gagliardo–Nirenberg inequality implies that for $N\in\{1,2\}$ it holds

$$|\nabla\theta_2|_{L^4(\Omega)} \le C\,|\theta_2|_{H^2(\Omega)}^{\frac12}\,|\theta_2|_{L^\infty(\Omega)}^{\frac12}\,.$$

Hence, $\nabla\theta_2$ belongs to $L^4(Q_T)$. In addition, the Gagliardo–Nirenberg inequality yields for $N\in\{1,2\}$ that

$$|\zeta|_{L^4(\Omega)} \le C\,|\zeta|_{H^1(\Omega)}^{\frac12}\,|\zeta|_{L^2(\Omega)}^{\frac12}\,,$$

so that

$$\int_\Omega |\nabla\theta_2|\,|\nabla\zeta|\,|\zeta|\,dx \le C\,|\nabla\theta_2|_{L^4(\Omega)}\,|\zeta|_{L^2(\Omega)}^{\frac12}\,|\zeta|_{H^1(\Omega)}^{\frac32}\,.\tag{5.10}$$

Therefore, using Young's inequality, we conclude from (5.9) and (5.10) that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \zeta^2\,dx + \bar{C}\,|\zeta|_{H^1(\Omega)}^2 \ \le\ & \frac{\bar{C}}{2}\,|\zeta|_{H^1(\Omega)}^2 + \frac12\,|\phi|_{H^1(\Omega)}^2 + C\,|\phi_t|_{L^2(\Omega)}^2 \\
& + C\left(1+|\nabla\theta_2|_{L^4(\Omega)}^4 + |\phi_{1t}|_{H^1(\Omega)}^2\right)\,|\zeta|_{L^2(\Omega)}^2\,.
\end{aligned}$$

Integrating over $(0,t)$, $t\in[0,T]$, we find:

$$\begin{aligned}
|\zeta(t)|_{L^2(\Omega)}^2 \ \le\ & \frac12\int_0^t |\phi|_{H^1(\Omega)}^2\,ds + C\int_0^t |\phi_t|_{L^2(\Omega)}^2\,ds \\
& + C\int_0^t \left(1+|\nabla\theta_2|_{L^4(\Omega)}^4 + |\phi_{1t}|_{H^1(\Omega)}^2\right)\,|\zeta|_{L^2(\Omega)}^2\,ds\,.
\end{aligned}\tag{5.11}$$

Next, we infer from [8] that

$$
\begin{aligned}
|\phi_t|_{L^2(0,t,L^2(\Omega))} &\le C \left| F_1'(\phi_1) + \frac{F_2'(\phi_1)}{\theta_1} - F_1'(\phi_2) - \frac{F_2'(\phi_2)}{\theta_2} \right|_{L^2(0,t,L^2(\Omega))} \\
&\le C \left(|\phi|_{L^2(0,t,L^2(\Omega))} + |\zeta|_{L^2(0,t,L^2(\Omega))} \right) .
\end{aligned}
$$

Thus, (5.11) becomes

$$
\begin{aligned}
|\zeta(t)|^2_{L^2(\Omega)} &\le \frac{1}{2} \int_0^t |\phi|^2_{H^1(\Omega)} \, ds + C \int_0^t |\phi|^2_{L^2(\Omega)} \, ds \\
&\quad + C \int_0^t \left(1 + |\nabla \theta_2|^4_{L^4(\Omega)} + |\phi_{1t}|^2_{H^1(\Omega)}\right) \, |\zeta|^2_{L^2(\Omega)} \, ds . \qquad (5.12)
\end{aligned}
$$

Finally, we integrate (5.6) over $(0,t)$ and add the result to (5.12) to obtain the estimate

$$
|\phi(t)|^2_{L^2(\Omega)} + |\zeta(t)|^2_{L^2(\Omega)} \le C \int_0^t \left(1 + |\nabla \theta_2|^4_{L^4(\Omega)} + |\phi_{1t}|^2_{H^1(\Omega)}\right) \, \left(|\phi|^2_{L^2(\Omega)} + |\zeta|^2_{L^2(\Omega)}\right) \, ds .
$$

Since both $|\nabla \theta_2|^4_{L^4(\Omega)}$ and $|\phi_{1t}|^2_{H^1(\Omega)}$ belong to $L^1(0,T)$, we can employ Gronwall's lemma to obtain the asserted result. □

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