Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Galerkin Approximation for Rayleigh-Bénard Convection

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submitted: Feb. 10, 2004

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No. 907

Berlin 2004

1991 *Mathematics Subject Classification.* 74D10,74F05,74F10,77N25.

Key words and phrases. Galerkin approximation, stability, pattern formation.

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Abstract

We employ a Galerkin approximation for the system of equations governing Rayleigh-Bénard convection. This approximation reduces the dimension of the problem by one, while it captures the nonlinear behavior even when only a few basis functions are used. We prove convergence of the method and finally demonstrate the effectiveness of this method for the problems of feedback controlled Rayleigh-Bénard convection in three dimensions and the complex dynamics of spiral-defect chaos.

1 Introduction

Rayleigh-Bénard convection [13], [1] (RBC) plays a fundamental role in the theory of pattern forming systems. In the most basic experimental setting it arises when a quiescent fluid layer, inside a closed container is heated from below. Above a certain horizontal temperature difference, measured by the Rayleigh number (Ra) buoyancy forces destabilize the fluid in form of rolls and more complicated patterns when boundary conditions are varied or the temperature difference is increased, or when a feedback control mechanism is used. Also, due to experimental progress, the ability to investigate RBC in containers with larger horizontal dimensions, lead to the discovery of new instabilities, involving for example spiral-defect chaos (SDC) [10].

Understanding of the underlying physical processes is greatly increased if it would be possible to have meaningful comparisons between experiment and theory, beyond the visual aspects. For the large scale three dimensional structures there is therefore a need to develop theoretical tools leading to reduced description and effective numerical computations. In case of SDC for example where one is in a regime close to onset, model equations such as the Swift-Hohenberg equations have been developed, [3], for which however there exist no derivation from the underlying Boussinesq equations.

In this paper we present an alternative method addressing above class of problems. It rests on the property that the instability in such pattern forming systems has fine scale structure in some dimensions and large scale coherent structures in the other dimensions, e.g. the vertical direction in the above problems. The idea of the Galerkin approximation method is to represent the flow variables by a linear combination of basis functions, using only a small number (one or two) of low degree polynomials for the vertical direction, such that the boundary conditions are explicitly satisfied. In this way one is able to reduce the dimension of the boundary value problem by one. This approximation can be systematically derived form the underlying governing equations, here, the Boussinesq equations, and its accuracy can be determined and is therfore much less phenomenological. It is also easily extendable to include higher order structures by allowing for more basis functions.

The idea for such a reduction has been suggested by [9], taken up again by [5] and extended to include boundary effects by [16]. The Galerkin approximation method is similar in spirit to the lubrication approximation. There, one integrates out the laminar flow in one direction to reduce the Navier-Staokes equations to the corresponding lubrication equations, while here one has to incorparate one or two basis functions to account for the simple flow structure in one dimension and upon integrating that out to obtain a dimension reduced problem.

In this paper we detail the construction of the Galerkin approximation and prove convergence of the method. We illustrate the method starting with the problem of feedback controlled RBC in two dimensions [8],[15], [12], including a discussion of linear and weakly nonlinear stability. After presenting the proof, we apply the method to three-dimensional controlled and uncontrolled RBC. Finally we show how we can use this method to efficiently compute SDC.

2 Formulation

We present first the problem of feedback control of Rayleigh-Bénard convection. This has a wide range of technological application, where it is often desired to optimize material processes by suppressing or enhancing the onset of instability. For example in Czochralski crystal growth [11] suppression of the instability would be desirable in order to prevent defect and dopant heterogeneity caused by convection, while in other applications one seeks to advance the onset for example to enhance mixing in biochemical reactors.

The governing equations for the convection layer are the Boussinesq approximation together with continuity and energy equation. In dimensionless form they are:

$$\mathrm{Pr}^{-1}\left[\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\,\mathbf{u}\right] = -\nabla p + R\,T^+\;(0,1)^t + \nabla^2\mathbf{u} \tag{2.1}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{2.2}$$

$$\partial_t T + \mathbf{u}\cdot\nabla T = \nabla^2 T + w \tag{2.3}$$

where the scalings

$$\left.\begin{array}{rclrclrcl} (x,z) &=& (\frac{x^*}{d},\frac{z^*}{d}), & t &=& \frac{\kappa}{d^2}t^*, & (v,w) &=& \frac{d}{\kappa}(v^*,w^*),\\ T^+ &=& \frac{k}{\bar{q}d}T^*, & p &=& \frac{d^2}{\rho\kappa^2}p^*. \end{array}\right\} \tag{2.4}$$

have been used. We denote by d, κ, ρ and $\bar{q}$ the height of the fluid layer, thermal diffusivity, fluid density and spatially averaged heat flux, respectively. We further write

$$\mathbf{u} = (v,w), \qquad T^+ = T_c + T = \theta_u - (z - \frac{1}{2}) + T \tag{2.5}$$

with the conductive state T_c and the temperature θ_u on the upper boundary $z = 1/2$.

$$R = \frac{g\alpha\bar{q}d^4}{\kappa\nu k_{th}} \quad \text{and} \quad Pr = \frac{\nu}{\kappa} \tag{2.6}$$

denote the Rayleigh and the Prandtl number, with k_{th}, α, g and ν the thermal conductivity, thermal expansion coefficient, gravity and viscosity, respectively. Except for the treatment of the spiral-defect chaos we assume the Prandtl number to be large and therefore neglect the left hand side of (2.1). This is in accordance with the experimental situation of [7].

For the boundary conditions we assume no-slip and impermeability for the velocity at the upper and lower boundaries

$$v = w = 0 \qquad \text{at} \quad z = \pm\frac{1}{2} \tag{2.7}$$

In experiments by [8] the temperature is kept fixed at the upper boundary. Hence, we have

$$T = 0 \qquad \text{at} \quad z = +\frac{1}{2}\,. \tag{2.8}$$

The feedback control boundary condition at the lower boundary is

$$\partial_z T = -\omega\ \partial_x^2 \int_{-1/2}^{1/2} T\,dz \qquad \text{at} \quad z = -\frac{1}{2}. \tag{2.9}$$

where

$$\omega = \frac{2gH}{d}\left(-\frac{d\eta}{dT}\right)$$

is the control parameter. Note that for most fluids the refractive index η decreases with temperature, and so $\omega > 0$, see [7], [8], [16] for details on the derivation of this boundary condition. We also refer to [16] where we found that the problem may become ill-posed for $\omega < -1$ and explained the importance of details of the boundary condition, such as heater thickness and boundary thickness there, while in the range above that value their influence becomes negligible.

2.1 Galerkin approximation method for controlled Rayleigh-Bénard convection

We now represent the flow variables by a linear combination of basis function, using only a small number of low degree polynomials for the z-direction. By testing with the basis functions (i.e. multiplying by the basis functions and integrating over the domain) each equation of the governing system is replaced by a small number of lower dimension equations.

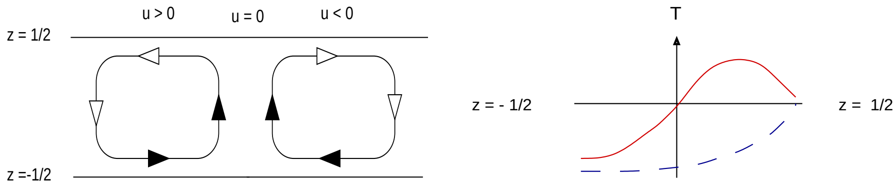

Figure 1: Streamlines for (2.10)–(2.12) (left). Polynomials for the temperature $H_0(z)$ (dashed) and $H_1(z)$ (solid).

One modelling aspect of this method is to determine the minimal number of polynomials necessary to capture the dominant nonlinearity. The usefulness of this method is due to the fact that in many cases the patterns that arise in many hydrodynamic instabilities can be approximated by one or two polynomials.

The flow pattern we need to capture for our problem are 2D rolls. The minimal polynomial representation for the velocity components (v, w) that satisfiy the no-slip boundary and non-permeability conditions at $z = \pm 1/2$ and the continuity equation is

$$v(x,z,t) = u(x,t)\,\mu_z(z)\,, \tag{2.10}$$

$$w(x,z,t) = -u_x(x,t)\,\mu(z)\,, \tag{2.11}$$

where

$$\mu(z) = \frac{1}{4}\left(z^2 - \frac{1}{4}\right)^2 \tag{2.12}$$

Figure 1 shows streamlines of a roll pattern produced by (2.10)–(2.12) for periodic $u(x,t)$.

The temperature satisfies a nonhomogeneous boundary condition with feedback control. We take this into account by making the following ansatz for the Galerkin approximation of the temperature field:

$$T(x,z,t) = h(x,t)\,H_0(z) + s(x,t)\,\ell(z) \tag{2.13}$$

where we have split the temperature into a contribution for the problem with homogenous boundary conditions plus a term that models the control boundary conditions. This means that

$$H_0(1/2) = 0 \qquad \text{and} \quad H_0'(-1/2) = 0\,, \tag{2.14}$$

$$\text{while} \qquad \ell(1/2) = 0 \qquad \text{and} \quad \ell'(-1/2) = 1 \tag{2.15}$$

The lowest order polynomial $H_0(z)$ that satisfies the conditions (2.14) is

$$H_0(z) = \left(z - \frac{1}{2}\right)\left(z + \frac{3}{2}\right)\,, \tag{2.16}$$

see the dashed curve in figure 1. This representation of the temperature is capable of producing temperature fields which are not symmetric with respect to zero.

Rayleigh-Bénard convection rolls diminish the temperature difference between $z = \pm 1/2$ in that they carry hot fluid from the lower side to the top (filled arrows in figure 1), while cold fluid will be transported from the upper side to the bottom (empty arrows). This is necessary to achieve nonlinear saturation of the rolls.

This is not so, if for example we had chosen Neuman boundary conditions on both sides, see [5]. In this case we would have needed a third order polynomial as well in order to break the symmetry of the temperature profile.

In the next section we show that the polynomial $\ell(z)$ not only needs to satisfy conditions (2.15) but in order to prevent artificial singularities that arise through this approximation, for positive feedback control, we need to require

$$\rho_1 = \int_{-1/2}^{1/2} \ell(z)\, dz \;=\; \langle \ell, H_0 \rangle \int_{-1/2}^{1/2} H_0(z)\, dz \tag{2.17}$$

$$\text{with} \quad \langle \ell, H_0 \rangle \;=\; \int_{-1/2}^{1/2} \ell(z) H_0(z)\, dz \tag{2.18}$$

In order to simplify calculations we choose $\ell(z)$ to be orthogonal to $H_0(z)$, i.e. the scalar product $\langle \ell, H_0 \rangle = 0$, and therefore also $\rho_1 = 0$. This leads us to the following polynomial:

$$\ell(z) = \left(z - \frac{1}{2}\right)\left(z^2 + \frac{1}{8} z - \frac{1}{16}\right) \tag{2.19}$$

Finally, we obtain the Galerkin approximation by testing the full problem with the test functions

$$\theta_0 = \delta(x)\, \mu_z(z)\,, \quad \theta_1 = -\delta'(x)\, \mu_z(z)\,, \quad \phi_0 = \delta(x) H_0(z)\,, \tag{2.20}$$

to obtain

$$\partial_x^4 u - 24\, \partial_x^2 u + 504\, u = -R\; \partial_x \left(60\, h - \frac{3}{2}\, s\right) \tag{2.21}$$

$$\partial_t h - \partial_x^2 h + \frac{5}{2} h + \frac{9}{448}\left(u \partial_x h + \frac{1}{2} h \partial_x u\right) = \tag{2.22}$$

$$\frac{15}{8} s + \frac{5}{448} \partial_x u + \frac{u \partial_x s - 3/2\, s\, \partial_x u}{448 \cdot 24} \tag{2.23}$$

with $s = \omega\, \frac{2}{3}\, \partial_x^2 h$ representing the control boundary condition.

The system of equations are not only much easier to treat numerically because of the reduced dimension, but, as we will see in the following section, its analytical treatment is much simpler.

Table 1: Comparison of critical parameters

Critical Parameter	Full Problem	Galerkin Approx. 2 polynomials	Galerkin Approx. 1 polynomial
R_c	1296	1350 (4%)	1446 (12%)
k_c	2.55	2.52 (1%)	2.39 (6%)

3 Stability

3.1 Linear stability

We first observe, that for this Galerkin approximation, linearization about the conductive state $h(x,t)=0$, $u(x,t)=0$, reduces the linear stability problem to solving

$$\partial_t \hat{h} = \sigma(k,\omega)\hat{h} \tag{3.1}$$

with growth rate

$$\sigma(k,\omega) = -\left(k^2+\frac{5}{2}\right)+\frac{75}{112}MR+\omega k^2\left(\frac{5}{448}MR-\frac{5}{4}\right). \tag{3.2}$$

$\hat{h}(k,t)$ denotes the Fourier transform of $h(x,t)$ and

$$M = \frac{k^2}{k^4+24\,k^2+504}. \tag{3.3}$$

The simplicity of the formula for the growth rate enables us to write down the expression for the critical Rayleigh number in as a function of the feedback control parameter,

$$R_c(\omega) = \frac{28}{15}\frac{(4+5\omega)\left(k_c^4+24k_c^2+504\right)^2}{(2\omega-5)\,k_c^4+84\omega k_c^2+2520}. \tag{3.4}$$

where $k_c(\omega)$ is the solution of the polynomial

$$(4+5\omega)\left(\omega k^8+120k^6\right)+\left(6360+4944\omega-2520\omega^2\right)k^4 \tag{3.5}$$
$$-10080\omega k^2-302400=0 \tag{3.6}$$

Surprisingly, the approximation yields rather good results, even though only the smallest number of basis functions have been used. For example when $\omega=0$ (uncontrolled Rayleigh-Bénard convection) we have $R_c=1446$ and $k_c=2.39$ compared to $R_c=1296$ and $k_c=2.55$ for the full problem, which a difference of about 12% and 6%, respectively. If we add just one more polynomial in the temperature approximation, we obtain $R_c=1350$ and $k_c=2.52$, which is just a difference of 4% and 1%, respectively, see table I. The model for two basis functions is included in the appendix.

3.2 Weakly nonlinear stability

The nonlinear behavior near R_c is described by the Landau equations for the amplitude. Their derivation from the original governing equations is often not feasable if the boundary conditions are other than Neumann conditions. The Galerkin approximation removes boundaries in z-direction We show how the Landau equation for controlled RBC can be derived on the basis of multiple-scale method.

Suppose the system experiences a small initial perturbation

$$\mathbf{w}(x,0) = \delta\left(\bar{u}(x), \bar{h}(x)\right), \quad \mathbf{w} = (u,h) \tag{3.7}$$

where $\delta \ll 1$. When we rescale the problem by δ as

$$u = \delta u^*, \quad h = \delta h^* \tag{3.8}$$

Drop the $*$ and denote by

$$\mathcal{L}_u(\mathbf{w}) := \partial_x^4 u - 24\,\partial_x^2 u + 504\,u + R\,\partial_x\left(60\,h - \frac{3}{2}\,s\right) \tag{3.9}$$

$$\mathcal{L}_h(\mathbf{w}) := \partial_t h - \partial_x^2 h + \frac{5}{2}h - \frac{15}{8}s - \frac{5}{448}\partial_x u \tag{3.10}$$

then in the scaled problem the nonlinear terms appear as a small correction.

$$\mathcal{L}_u(\mathbf{w}) = 0 \tag{3.11}$$

$$\mathcal{L}_h(\mathbf{w}) = \frac{\delta}{448}\left[-9\left(u\partial_x h + \frac{1}{2}h\partial_x u\right) + \frac{\omega}{24}\left(\frac{2}{3}u\partial_{xxx}h - \partial_{xx}h\partial_x u\right)\right] \tag{3.12}$$

In x-direction we assume periodic boundary conditions. For this perturbation problem we make the ansatz

$$\mathbf{w}(x,t;\delta) := \mathbf{w}_0(x,t,\tau) + \delta\mathbf{w}_1(x,t,\tau) + \delta^2\mathbf{w}_2(x,t,\tau) + O(\delta^3) \tag{3.13}$$

$$R = R_c + \delta^2\alpha, \quad \tau = \delta^2 t \tag{3.14}$$

To leading order we basically get the linear stability problem

$$\mathcal{L}_u(\mathbf{w}_0) = 0, \quad \mathcal{L}_h(\mathbf{w}_0) = 0 \tag{3.15}$$

$$\text{with initial conditions: } \mathbf{w}_0(x,0) = (\bar{u}(x),\, \bar{h}(x)) \tag{3.16}$$

To O(δ) we get

$$\mathcal{L}_u(\mathbf{w}_1) = 0 \tag{3.17}$$

$$\mathcal{L}_h(\mathbf{w}_1) = \frac{1}{448}\left(-9\left[u_0\partial_x h_0 + \frac{1}{2}h_0\partial_x u_0\right] + \frac{\omega}{24}\left[\frac{2}{3}u_0\partial_{xxx}h_0 - \partial_{xx}h_0\partial_x u_0\right]\right) \tag{3.18}$$

To $O(\delta^2)$ we find

$$\mathcal{L}_u(\mathbf{w}_2) = -\alpha\,(60\,\partial_x h_0 - \omega\partial_{xxx}h_0) \tag{3.19}$$

$$\begin{aligned}\mathcal{L}_h(\mathbf{w}_2) &= -\partial_\tau h_0 - \frac{9}{448}\left(u_1\partial_x h_0 + u_0\partial_x h_1 + \frac{1}{2}\left(h_0\partial_x u_1 + h_1\partial_x u_0\right)\right) \\ &+\frac{\omega}{448\cdot 24}\left(\frac{2}{3}u_0\partial_{xxx}h_1 + \frac{2}{3}u_1\partial_{xxx}h_0 - \partial_{xx}h_0\partial_x u_1 - \partial_{xx}h_1\partial_x u_0\right)\end{aligned} \tag{3.20}$$

Since we have periodic boundary conditions the solution can be written in form

$$h_0(x,t,\tau) = \sum_{n=1}^{\infty} A_n(t,\tau)\sin(nk_c x) + B_n(t,\tau)\cos(nk_c x) \tag{3.21}$$

$$u_0(x,t,\tau) = \sum_{n=1}^{\infty} E_n(t,\tau)\sin(nk_c x) + F_n(t,\tau)\cos(nk_c x) \tag{3.22}$$

with

$$\begin{aligned}&A_n(t,\tau) = K(\tau)e^{\sigma_n t} \qquad B_n(t,\tau) = L(\tau)e^{\sigma_n t} \\ &E_n(t,\tau) = V_n A_n(t,\tau)\,, \quad F_n(t,\tau) = -V_n B_n(t,\tau)\,, \\ &V_n = \frac{R_c M_n}{nk_c}\left(60 + \omega(k_c n)^2\right)\,, \quad M_n = \frac{(nk_c)^2}{(nk_c)^4 + 24(nk_c)^2 + 504}\,, \\ &\sigma_n = -\left((nk_c)^2 + \frac{5}{2}\right) + \frac{75}{112}M_n R_c + \omega(k_c n)^2\left(\frac{5}{448}M_n R_c - \frac{5}{4}\right)\end{aligned}$$

The leading order solution corresponds to the solution to the linear stability problem at criticality. This means that there $\sigma_1 = 0$, while for all other σ_{in} we have $\mathrm{Re}(\sigma_{in}) < 0$. Hence, the dominant terms in the expansions are

$$h_0 = K(\tau)\sin(k_c x) + L(\tau)\cos(k_c x) \tag{3.23}$$

$$u_0 = V_1\left[L(\tau)\sin(k_c x) - K(\tau)\cos(k_c x)\right] \tag{3.24}$$

while all other terms decay.

The unknown functions $K(\tau)$ and $L(\tau)$ have to be determined by solving the higher order problems To $O(\delta)$ we obtain the solution

$$h_1 = \rho_1(K^2 + L^2) + \rho_2\left[(L^2 - K^2)\cos(2k_c x) - 2KL\sin(2k_c x)\right] \tag{3.25}$$

$$u_1 = q_1\left[(L^2 - K^2)\sin(2k_c x) - 2KL\cos(2k_c x)\right] \tag{3.26}$$

where

$$\begin{aligned}\rho_1 &= \frac{V_1 k_c}{448}\left(\frac{9}{10} + \omega\frac{k_c^2}{72}\right)\,, \ \rho_2 = \frac{V_1 k_c}{448\sigma_2}\left(\frac{27}{4} + \omega\frac{k_c^2}{144}\right) \\ q_1 &= \frac{1}{448}\frac{V_1 V_2 k_c}{\sigma_2}\left(\frac{27}{4} - \omega\frac{k_c^2}{144}\right)\,.\end{aligned}$$

Note, that the right hand side of (3.19)–(3.20) contains linear combinations of $\sin(k_c x)$, $\cos(k_c x)$ etc. Hence, we make the following ansatz for the solution to u_2 and h_2 :

$$u_2 = \mu_1(\tau)\sin(k_c x) + \nu_1(\tau)\cos(k_c x) \tag{3.27}$$

$$h_2 = \mu_2(\tau)\sin(k_c x) + \nu_2(\tau)\cos(k_c x) \tag{3.28}$$

If we now sort both sides of the $O(\delta^2)$ equation with respect to $\sin(k_c x)$ and $\cos(k_c x)$ we obtain four equations for the unknowns μ_1, ν_1, μ_2, ν_2. In vector notation this reads :

$$\begin{pmatrix} 1 & 0 & 0 & -R_c M_1 \zeta \\ 0 & 1 & R_c M_1 \zeta & 0 \\ 0 & 0 & \sigma_1 & 0 \\ 0 & 0 & 0 & \sigma_1 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \nu_1 \\ \mu_2 \\ \nu_2 \end{pmatrix} = \begin{pmatrix} \alpha L \zeta \\ -\alpha K \zeta \\ \text{Eq. for } K \\ \text{Eq. for } L \end{pmatrix} \tag{3.29}$$

where

$$\zeta = \frac{M_1\,(60 + \omega k_c^2)}{k_c(\omega)} \quad \text{and} \quad \sigma_1 = \sigma_1\,(k_c(\omega), R_c(\omega))$$

Note, that $\sigma_1\,(k_c(\omega), R_c(\omega)) = 0$. Therefore, the solvability condition requires that the equation for K and the equation for L on the right hand side are zero:

$$\frac{dK}{d\tau} - a(\alpha,\omega)\,K - b(\omega)\,K\,(K^2 + L^2) = 0 \tag{3.30}$$

$$\frac{dL}{d\tau} - a(\alpha,\omega)\,L - b(\omega)\,L\,(K^2 + L^2) = 0 \tag{3.31}$$

where

$$a(\alpha,\omega) = \frac{5}{448}\alpha M_1(60 + \omega k_c^2)$$

$$b(\omega) = \frac{k_c}{448}\left[9V_1\left(\rho_1 - \frac{1}{2}\rho_2\right) - \frac{\omega k_c^2}{24}\left(\frac{4}{3}q_1 + \frac{14}{3}\rho_2 V_1\right)\right]$$

These are often also called the Landau equations. From linear theory we note that

$$\text{sgn}(a) = \text{sgn}(\alpha) = \text{sgn}(R - R_c)$$

and that $b(\omega) > 0$, so that we always have a supercritical bifurcation for any ω.

4 Galerkin approximation for problems with feedback control

Before we prove the convergence of the Galerkin approximation method, we like to dicuss the problem of artificial singularities for problems with boundary conditions different than homogeneous Neumann conditions. We discuss this using the

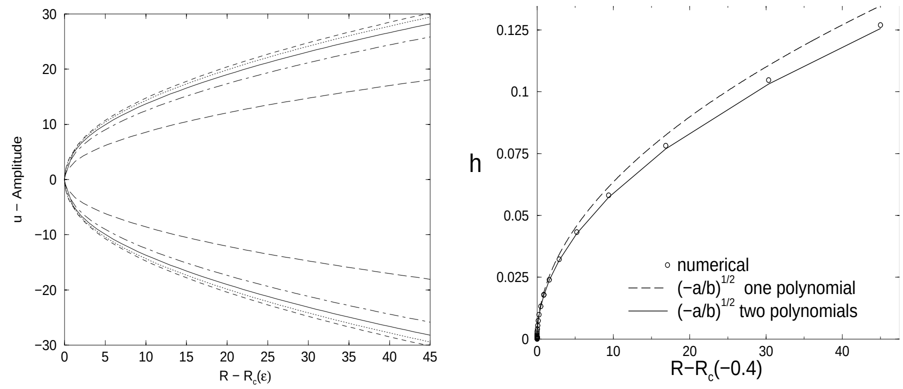

Figure 2: Amplitude for u using the Galerkin approximation with two temperature function (left). Comparison of the amplitude for the temperature using weakly nonlinear results from the Galerkin approximation with one and two temperature function (right)

boundary conditions above but for simplification use the heat equation in 2D as the governing equation instead. This problem can be further simplified if we Fourier transform it w.r.t. x.

$$T_t = T_{zz} - k^2 T \tag{4.1}$$

$$T_z(1/2,t) = 0 \tag{4.2}$$

$$T_z(-1/2,t) = \omega\ k^2 \int_{-1/2}^{1/2} T\, dz \tag{4.3}$$

$$T(z,0) = g(z). \tag{4.4}$$

Let us assume, that this problem has a solution. We can always find a function $\ell(z)$ such that $\ell'(1/2) = 0$ and $\ell'(-1/2) = 0$. Next we can define the following functions:

$$s(t) \;=\; \omega\ k^2 \int_{-1/2}^{1/2} T\, dz \tag{4.5}$$

$$v(z,t) \;=\; T(z,t) - \ell(z)\, s(t)\,, \tag{4.6}$$

so that $v(z,t)$ satisfies the equation

$$v_t + \ell\, s_t \;=\; v_{zz} - k^2\, v + (\ell'' - k^2\, \ell)\, s \tag{4.7}$$

with homogeneous boundary conditions

$$v_z(1/2,t) \;=\; 0 \tag{4.8}$$

$$v_z(-1/2,t) \;=\; 0. \tag{4.9}$$

For the initial conditions for v we note that from (4.5)–(4.6)

$$T(z,0) = g(z) \;=\; v(z,0) + \ell(z)\, s(0) \tag{4.10}$$

$$\;=\; v(z,0) + \ell(z) \frac{\omega k^2}{1 - \rho\omega k^2} \int_{-1/2}^{1/2} v(z,0)\, dz \tag{4.11}$$

where

$$\rho = \int_{-1/2}^{1/2} \ell(z)\, dz$$

Integration of (4.11) yields

$$\int_{-1/2}^{1/2} v(z,0)\, dz = (1 - \rho\omega\, k^2) \int_{-1/2}^{1/2} g(z)\, dz$$

and hence

$$v(z,0) = g(z) - \omega\, k^2 \ell(z) \int_{-1/2}^{1/2} g(z)\, dz \tag{4.12}$$

Conversely, it is clear that for given v that satisfies (4.7)–(4.9), with s and ℓ having above properties, we can define T that satisfies (4.1)–(4.4).

Clearly, if we integrate (4.7) and denote

$$V(t) = \int_{-1/2}^{1/2} v(z,t)\, dz \qquad \text{and} \qquad \gamma = \int_{-1/2}^{1/2} g(z)\, dz \tag{4.13}$$

we see immediately (as before in section 2), that the resulting initial value problem

$$V_t = -(\omega + 1)\, k^2\, V \qquad \text{with} \qquad V(0) = \gamma(1 - \rho\omega k^2) \tag{4.14}$$

will be ill-posed for $\omega < -1$

For our Galerkin approximation we prodeed in a similar fashion. We approximate $v(z,t)$ by

$$v_N(z,t) = \sum_i^N H_i(z)\, h_i(t) \tag{4.15}$$

where the H_i form a sequence of orthonormal polynomials w.r.t. the standard inner product,

$$\langle H_i, H_j \rangle = \int_{-1/2}^{1/2} H_i(z) H_j(z)\, dz = \delta_{ij} \tag{4.16}$$

which satisfy

$$H_i'(1/2) = 0, \qquad H_i'(-1/2) = 0 \tag{4.17}$$

and set

$$T_N(z,t) \;=\; v_N(z,t) + \ell_N(z)\, s(t) \tag{4.18}$$

where $\ell_N(z)$ is a polynomial with

$$\ell_N'(1/2) \;=\; 0, \qquad \ell_N'(-1/2) = 1 \tag{4.19}$$

Substitution of (4.18) into (4.1)–(4.3) and taking the inner product with H_j yields for each j the equation

$$h_{j_t} + \langle \ell_{\scriptscriptstyle N}, H_j\rangle\, s_t = \sum_i^N \langle H_i'', H_j\rangle h_i - k^2\, h_j + \langle \ell_{\scriptscriptstyle N}'' - k^2 \ell_{\scriptscriptstyle N}, H_j\rangle s \tag{4.20}$$

$$s(t) = \frac{\omega k^2}{1-\rho_{\scriptscriptstyle N}\omega k^2}\int_{-1/2}^{1/2} v_{\scriptscriptstyle N}\, dz \quad \text{with} \quad \rho_{\scriptscriptstyle N} = \int_{-1/2}^{1/2} \ell_{\scriptscriptstyle N}\, dz \tag{4.21}$$

The problem for $v_{\scriptscriptstyle N}(z,t)$ is now obtained by summation of the product of H_j and (4.20). If we integrate the resulting equation by using (4.21) as well as the properties of H_i and $\ell_{\scriptscriptstyle N}$ and denote

$$V_{\scriptscriptstyle N}(t) \;=\; \int_{-1/2}^{1/2} \sum_j^N H_j\, h_j\, dz = \int_{-1/2}^{1/2} v_{\scriptscriptstyle N}\, dz \tag{4.22}$$

$$\text{and the projection } P(Q) \;=\; \sum_i^N \langle Q, H_j\rangle H_j \;, \text{ for some} \tag{4.23}$$

polynomial Q, we obtain the following equation

$$\begin{aligned} &\left[1+\omega\, k^2 \left(\int_{-1/2}^{1/2} P(\ell_{\scriptscriptstyle N}) - \rho_{\scriptscriptstyle N}\, dz\right)\right] V_{\scriptscriptstyle N t} \\ &\quad= -k^2 \left[1+\omega+\omega\, k^2 \left(\int_{-1/2}^{1/2} P(\ell_{\scriptscriptstyle N}) - \rho_{\scriptscriptstyle N}\, dz\right)\right] V_{\scriptscriptstyle N} \\ &\quad\quad +\omega\, k^2 \left[\int_{-1/2}^{1/2} P(\ell_{\scriptscriptstyle N}'') - \ell_{\scriptscriptstyle N}''\, dz\right] V_{\scriptscriptstyle N} \\ &\quad\quad + \left(1-\rho_{\scriptscriptstyle N}\omega k^2\right) \int_{-1/2}^{1/2} \sum_i^N \left[P(H_i'') - H_i''\right] h_i\, dz \end{aligned} \tag{4.24}$$

In this form, we observe that, for $\omega < 0$, the approximate problem will produce artificial singularities, which are not present in the exact problem, if

$$1+\omega\, k^2 \left(\int_{-1/2}^{1/2} P(\ell_{\scriptscriptstyle N}) - \rho_{\scriptscriptstyle N}\, dz\right) = 0\,.$$

However, the sequence of orthonormal polynomials H_i that produce the approximation $v_{\scriptscriptstyle N}$, all have property (4.17). Hence the constant polynomial $H_0(z) = 1$ is always a member. But this means that

$$\int_{-1/2}^{1/2} P(Q)\, dz = \langle Q, H_0\rangle = \int_{-1/2}^{1/2} Q\, dz \tag{4.25}$$

Therefore, (4.25) reduces to

$$V_{\scriptscriptstyle N t} = -(1+\omega)\, k^2\, V_{\scriptscriptstyle N} \qquad \text{with} \qquad V_{\scriptscriptstyle N}(0) = \gamma(1-\rho_{\scriptscriptstyle N}\omega k^2) \tag{4.26}$$

The important property (4.25) though is not necessarily satisfied for general boundary condition. If we change for example the top ($z = 1/2$) boundary condition to be of Dirichlet type, then $H_i(1/2) = 0$ and (4.25) can not be derived anymore. Therefore, if we want to approximate the problem (4.1)–(4.4) with $T(1/2, t) = 0$, by (4.15)–(4.19) with $H_i(1/2) = 0$ and $\ell_N(1/2) = 0$, we find again the same coefficient of V_{Nt}. In this case however we have to explicitely require

$$\int_{-1/2}^{1/2} \ell_N \, dz = \int_{-1/2}^{1/2} P(\ell_N) \, dz \tag{4.27}$$

in order to avoid artificial singularities for negative *varepsilon*. This in turn gives an additional constraint on ℓ_N.

4.1 Convergence

For the problem

$$\begin{aligned} T_t &= T_{xx} + T_{zz} \qquad \text{on} \quad \Omega && (4.28)\\ T_z(x, 1/2, t) &= 0 && (4.29)\\ T_z(x, -1/2, t) &= -\omega \int_{-1/2}^{1/2} T_{xx} \, dz && (4.30)\\ T(x, z, 0) &= g(x, z), && (4.31) \end{aligned}$$

where Ω denotes the domain $]0, L[\times] - 1/2, 1/2[$, and where T satisfies periodic boundary conditions in x, and $t \in [0, t_f]$. we analyse the convergence properties of a Galerkin-scheme designed to approximate the solution of (4.28)–(4.31). For later use, we set $I :=]-1/2, 1/2[$. For this purpose, we first make some assumptions regarding the solution of the continuous problem. We will assume that for $\omega > -1$ the problem has, for sufficiently smooth data g, a unique solution with $T \in L^2(H^2(\Omega))$, and that this solution has additional regularity properties, T and $T_t \in L^2(H^{7/2}(\Omega))$.

In paragraph 1, we reformulate the continuous problem by splitting T into two variables, θ, that satisfies homogeneous boundary conditions at $z = \pm 1/2$, and a second term $s(x,t)l(z)$ which accounts for the (unusual) boundary conditions (4.30). We also pass to the Fourier-transform with respect to x. In paragraph 2, we set up the weak formulation and the Galerkin-scheme. In paragraph 3, we derive estimates for the difference of the solution T and the discrete solution T^N in terms of the norm of the continuous solution for θ. The bound for the difference of T and T^N provided by this estimate tends to zero as N tends to ∞, where N is the dimension of the sub-space used for the discretization.

4.1.1 Reformulation

Now fix a polynomial $l(z)$ so that

$$l'(1/2) = 0, \qquad l'(-1/2) = 1, \qquad \text{and} \qquad \int_{-1/2}^{1/2} l \, dz = 0,$$

and let

$$s(x,t) = T_z(x,-1/2,t) \tag{4.32}$$
$$\theta(x,z,t) = T(x,z,t) - s(x,t)l(z) \tag{4.33}$$

for $t \geq 0$.

Then, s and θ, θ_t are in $L^2(H^2(]0,L[))$ and $L^2(H^2(\Omega))$, respectively, and satisfy

$$\theta_t + s_t l = \theta_{xx} + \theta_{zz} + s_{xx} l + s l'' \tag{4.34}$$
$$s = -\omega \int_{-1/2}^{1/2} \theta_{xx}\, dz \tag{4.35}$$
$$\theta_z(x,\pm 1/2,t) = 0 \tag{4.36}$$
$$s(x,0) = g_z(x,-1/2) \tag{4.37}$$
$$\theta(x,z,0) = g(x,z) - s(x,0)l(z). \tag{4.38}$$

Conversely, any solution s and θ of (4.34)-(4.38) of this regularity class generates, via

$$T(x,z,t) = \theta(x,z,t) + s(x,t)l(z) \tag{4.39}$$

a solution of (4.28)–(4.31) within the class $L^2(H^2(\Omega))$ (or better). Since we assumed that the solution T of (4.28)–(4.31) is unique, the solution s and θ of (4.34)–(4.38) must be unique too. For, assume we have two solutions, s_1, θ_1 and s_2, θ_2, then from uniqueness of T, it follows

$$\theta_1(x,z,t) + s_1(x,t)l(z) = \theta_2(x,z,t) + s_2(x,t)l(z). \tag{4.40}$$

Evaluating this at $z = -1/2$ yields $s_1 = s_2$ and plugging this into (4.40) yields $\theta_1 = \theta_2$.

In the following, we will assume that the solution (4.34)–(4.38) has additional regularity properties,

$$s(t) \in H^2(]0,L[) \quad \text{and} \quad \theta(t) \in H^2(\Omega) \quad \text{both for all } t \in [0,t_f]. \tag{4.41}$$

We now Fourier-transform (4.34)–(4.38), via

$$s(x,t) = \sum_{j=0}^{\infty} \hat{s}(j,t)e^{ik_j x}, \quad \theta(x,z,t) = \sum_{j=0}^{\infty} \hat{\theta}(j,z,t)e^{ik_j x} \text{ with } k_j = \frac{2\pi}{L} j.$$

In the following, we will typically surpress the dependence on j, e.g. by writing k instead of k_j. The transformed equations then read

$$\hat{\theta}_t + \hat{s}_t l = -k^2\hat{\theta} + \hat{\theta}_{zz} - k^2 \hat{s} l + \hat{s} l'', \qquad \text{on } I, \text{ and for } t > 0 \tag{4.42}$$
$$\hat{s} = \omega k^2 \int_{-1/2}^{1/2} \hat{\theta}\, dz, \qquad \text{on } I, \text{ and for } t > 0, \tag{4.43}$$
$$\hat{\theta}_z(j,\pm 1/2) = 0, \tag{4.44}$$
$$\hat{s}(j,0) = \hat{g}_z(j,-1/2), \tag{4.45}$$
$$\hat{\theta}(j,z,0) = \hat{g}(j,z) - \hat{s}(j,0)l(z). \tag{4.46}$$

These equations have to be solved for all $j = 0, 1, \ldots$. From our above considerations, we conclude that (4.42)–(4.46) can be assumed to have, for each j, a unique solution $\hat{s}$, $\hat{\theta}$ within the class of functions that satisfy

$$\hat{\theta}(t) \in H^2(I), \quad \text{for all } t \in [0, t_f]. \tag{4.47}$$

and

$$\int_0^{t_f} |\hat{s}|^2 \, dt < \infty \,, \quad \text{and} \quad \hat{\theta} \in L^2\left(H^2(I)\right) .$$

4.1.2 Weak formulation and Discretization

Let

$$\mathcal{M}_c := \{\psi \in H^2(I);\ \psi_z(\pm 1/2) = 0\}.$$

Then, for the above solution we have $\hat{\theta}(t) \in \mathcal{M}_c$ and $\hat{\theta}$, $\hat{s}$ satisfy (where $(\cdot,\cdot)$ denotes the inner product of $L^2(I)$),

$$\begin{aligned}(\hat{\theta}_t, \psi) + \hat{s}_t(l, \psi) &= -(\hat{\theta}_z, \psi_z) - \hat{s}(l', \psi_z) - \hat{s}\psi(-1/2) \\ &\quad -k^2(\hat{\theta}, \psi) - k^2 \hat{s}(l, \psi), \quad \text{for all } \psi \in \mathcal{M}_c. \end{aligned} \tag{4.48}$$

The remaining conditions, (4.43)–(4.46), carry over from before.

For the discrete subspaces of $\mathcal{M}_c$, we take

$$\mathcal{M}_N := \operatorname{span}\{\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_N\},$$

where $\mathcal{H}_i$ are polynomials in z, ordered by their degree, that satisfy

$$\mathcal{H}_i'(\pm 1/2) = 0, \qquad (\mathcal{H}_i, \mathcal{H}_j) = \delta_{ij}. \tag{4.49}$$

Note that, in particular, $\mathcal{H}_0 \equiv 1$. We then formulate the following problem (discretized with respect to z):

Find $\hat{s}^N$, $\hat{\theta}^N$, with $\hat{\theta}^N(t) \in \mathcal{M}_N$, so that

$$\begin{aligned}(\hat{\theta}^N_t, \psi) + \hat{s}^N_t(l, \psi) &= -(\hat{\theta}^N_z, \psi_z) - \hat{s}^N(l', \psi_z) - \hat{s}^N\psi(-1/2) \\ &\quad -k^2(\hat{\theta}^N, \psi) - k^2 \hat{s}^N(l, \psi), \quad \text{for all } \psi \in \mathcal{M}_N, \end{aligned} \tag{4.50}$$

$$\hat{s}^N = \omega k^2 \int_{-1/2}^{1/2} \hat{\theta}^N \, dz \tag{4.51}$$

$$\hat{s}^N(j, 0) = \hat{s}(j, 0) \tag{4.52}$$

$$\hat{\theta}^N(j, z, 0) = \sum_{i=0}^{N} (\hat{\theta}(j, \cdot, 0), \mathcal{H}_i)\, \mathcal{H}_i(z). \tag{4.53}$$

By setting $\psi = 1$ in (4.48) and in (4.50), respectively, we find that $\hat{s}$ and $\hat{s}^N$ satisfy the same equation,

$$\hat{s}_t = -(1+\omega)k^2\hat{s} \qquad \text{and} \qquad \hat{s}^N_t = -(1+\omega)k^2\hat{s}^N,$$

so that in view of (4.52), we get $\hat{s}(j,t) = \hat{s}^N(j,t)$ for all $t \in [0, t_f]$. Therefore, when we subtract (4.48) and (4.50), all $\hat{s}$ and $\hat{s}^N$ terms cancel

$$(\hat{\theta}_t - \hat{\theta}^N_t, \psi) = -(\hat{\theta}_z - \hat{\theta}^N_z, \psi_z) - k^2(\hat{\theta} - \hat{\theta}^N, \psi). \tag{4.54}$$

4.1.3 Error Analysis

Let $\pi_N : \mathcal{M}_c \to \mathcal{M}_N$ denote a projection onto $\mathcal{M}_N$, and let

$$\hat{\zeta} := \hat{\theta} - \pi_N(\hat{\theta}), \qquad \hat{\zeta}^N := \hat{\theta}^N - \pi_N(\hat{\theta}).$$

Using this, (4.54) becomes

$$(\hat{\zeta}_t - \hat{\zeta}_t^N, \psi) = -(\hat{\zeta}_z - \hat{\zeta}_z^N, \psi_z) - k^2(\hat{\zeta} - \hat{\zeta}^N, \psi).$$

For the special choice $\psi = \hat{\zeta}^N$, this becomes

$$(\hat{\zeta}_t, \hat{\zeta}^N) - \frac{1}{2}\frac{d}{dt}||\hat{\zeta}^N||^2 = -(\hat{\zeta}_z, \hat{\zeta}_z^N) + ||\hat{\zeta}_z^N||^2 - k^2(\hat{\zeta}, \hat{\zeta}^N) + k^2||\hat{\zeta}^N||^2.$$

By an application of the Cauchy-Schwarz inequality, and Young's inequality, we get

$$\frac{d}{dt}||\hat{\zeta}^N||^2 + ||\hat{\zeta}_z^N||^2 + k^2||\hat{\zeta}^N||^2 \leq ||\hat{\zeta}_t||^2 + ||\hat{\zeta}_z||^2 + k^2||\hat{\zeta}||^2 + ||\hat{\zeta}^N||^2. \tag{4.55}$$

Here as further below, the unspecified norm denotes the $L^2(I)$-norm.

If we forget in (4.55), for the moment, the second and third term on the left hand side, we can use Gronwall's lemma to get an estimate for $||\hat{\zeta}_N||^2$,

$$\begin{aligned} ||\hat{\zeta}^N||^2 &\leq ||\hat{\zeta}^N(0)||^2 e^t + \int_0^t (||\hat{\zeta}_t||^2 + ||\hat{\zeta}_z||^2 + k^2||\hat{\zeta}||^2) e^{t-s}\, ds \\ &\leq (t+1)e^t \left[||\hat{\zeta}^N(0)||^2 + \int_0^t \left(||\hat{\zeta}_t||^2 + ||\hat{\zeta}_z||^2 + k^2||\hat{\zeta}||^2 \right) dt \right]. \end{aligned}$$

Recall that $t = 0$, $\hat{\theta}^N$ was chosen to be the $L^2(I)$ projection of $\hat{\theta}$, see (4.53). From this, we conclude

$$\begin{aligned} ||\hat{\zeta}^N(0)||^2 &= ||\hat{\theta}^N(0) - \pi_N(\hat{\theta}(0))||^2 \\ &= \left(\hat{\theta}^N(0) - \pi_N(\hat{\theta}(0)), 2\hat{\theta}(0) - 2\hat{\theta}^N(0) + \hat{\theta}^N(0) - \pi_N(\hat{\theta}(0)) \right), \end{aligned}$$

Note, the inserted terms do not contribute to the right hand side, because of the choice of $\hat{\theta}^N$ as $L^2(I)$-projection of $\hat{\theta}(0)$ onto $\mathcal{M}_N$, i.e. $(\,\hat{\theta}^N(0) - \pi_N(\hat{\theta}(0)), 2\hat{\theta}(0) - 2\hat{\theta}^N(0)\,) = 0$.

$$\begin{aligned} &= \left(\hat{\theta}^N(0) - \pi_N(\hat{\theta}(0)), 2\hat{\theta}(0) - \pi_N(\hat{\theta}(0)) - \hat{\theta}^N(0) \right) \\ &= ||\hat{\theta}(0) - \pi_N(\hat{\theta}(0)||^2 - ||\hat{\theta}(0) - \hat{\theta}^N(0)||^2 \\ &\leq ||\hat{\theta}(0) - \pi_N(\hat{\theta}(0)||^2 = ||\hat{\zeta}(0)||^2. \end{aligned}$$

so that

$$||\hat{\zeta}^N||^2 \leq (t+1)e^t \left[||\hat{\zeta}(0)||^2 + \int_0^t \left(||\hat{\zeta}_t||^2 + ||\hat{\zeta}_z||^2 + k^2||\hat{\zeta}||^2 \right) dt \right]. \tag{4.56}$$

Evaluating the right hand side at $t = t_f$ and plugging the result into (4.55), we get

$$\begin{aligned}\frac{d}{dt}||\hat{\zeta}^N||^2 + ||\hat{\zeta}_z^N||^2 + k^2||\hat{\zeta}^N||^2 &\leq ||\hat{\zeta}_t||^2 + ||\hat{\zeta}_z||^2 + k^2||\hat{\zeta}||^2 \\ &+(t_f+1)e^{t_f}\left[||\hat{\zeta}(0)||^2 + \int_0^{t_f}\left(||\hat{\zeta}_t||^2 + ||\hat{\zeta}_z||^2 + k^2||\hat{\zeta}||^2\right)\,dt\right].\end{aligned}$$

Integrating over $[0,t]$, we get after a little algebra

$$\begin{aligned}&\sup_{0\leq t\leq t_f}||\hat{\zeta}^N||^2 + \int_0^{t_f}\left(||\hat{\zeta}_z^N||^2 + k^2||\hat{\zeta}^N||^2\right)\,dt \\ &\leq 2(1+t_f^2e^{t_f})\left[||\hat{\zeta}(0)||^2 + \int_0^{t_f}\left(||\hat{\zeta}_t||^2 + ||\hat{\zeta}_z||^2 + k^2||\hat{\zeta}||^2\right)\,dt\right],\end{aligned}$$

in short,

$$\begin{aligned}&||\hat{\zeta}^N||^2_{L^\infty(L^2(I))} + ||\hat{\zeta}_z^N||_{L^2(L^2(I))} + k^2||\hat{\zeta}^N||_{L^2(L^2(I))} \\ &\leq C(t_f)\,\left(||\hat{\zeta}(0)||^2 + ||\hat{\zeta}_t||^2_{L^2(L^2(I))} + ||\hat{\zeta}_z||^2_{L^2(L^2(I))} + k^2||\hat{\zeta}||^2_{L^2(L^2(I))}\right).\end{aligned}$$

We are now in a position to estimate $T - T^N$, where T^N can be reconstructed from the discrete solutions θ^N and s^N via

$$\begin{aligned}T^N(x,\,t) &= \theta^N(x,z,t) + s^N(x,t)l(z), \\ \theta^N(x,z,t) &= \sum_{j=0}^{\infty}\hat{\theta}^N(j,z,t)e^{ik_jx}, \\ s^N(x,t) &= \sum_{j=0}^{\infty}\hat{s}^N(j,t)e^{ik_jx}.\end{aligned}$$

We find, using our finding that $\hat{s} = \hat{s}^N$,

$$\begin{aligned}&||T-T^N||^2_{L^\infty(L^2(\Omega))} + ||T-T^N||^2_{L^2(H_1(\Omega))} \\ &\quad= ||\theta-\theta^N||^2_{L^\infty(L^2(\Omega))} + ||\theta-\theta^N||^2_{L^2(H_1(\Omega))} \\ &\quad\leq ||\zeta||^2_{L^\infty(L^2(\Omega))} + ||\zeta^N||^2_{L^\infty(L^2(\Omega))} + ||\zeta||^2_{L^2(H_1(\Omega))} + ||\zeta^N||^2_{L^2(H_1(\Omega))} \\ &\quad\leq \sum_{j=0}^{\infty}\Big[||\hat{\zeta}||^2_{L^\infty(L^2(I))} + ||\hat{\zeta}^N||^2_{L^\infty(L^2(I))} + ||\hat{\zeta}||^2_{H^1(L^2(I))} + ||\hat{\zeta}_z^N||^2_{L^2(L^2(I))} \\ &\qquad\qquad + (1+k^2)||\hat{\zeta}^N||^2_{L^2(L^2(I))}\Big]\end{aligned}$$

The terms on the right hand side containing $\hat{\zeta}^N$ can be estimated using (4.56) and (4.57); this introduces $||\hat{\zeta}(0)||^2$. We wish to replace this term (and $||\hat{\zeta}||^2_{L^\infty(L^2(I))}$) by L^2-estimates of $\hat{\zeta}_t$, in the following manner:

Let $\tilde{t} \in [0,t_f]$ be chosen so that

$$\left|\left|\hat{\zeta}(\tilde{t})\right|\right|^2_{L^2(I)} = \min_{t\in[0,t_f]}\left|\left|\hat{\zeta}(t)\right|\right|^2_{L^2(I)} \leq \frac{1}{t_f}\int_0^{t_f}\left|\left|\hat{\zeta}(t)\right|\right|^2_{L^2(I)}$$

Then, we get

$$\begin{aligned}
\left|\left|\hat{\zeta}(t)\right|\right|^2_{L^2(I)} &= \left|\left|\hat{\zeta}(\tilde{t})\right|\right|^2_{L^2(I)} + \int_{\tilde{t}}^t 2(\hat{\zeta}_t(s), \hat{\zeta}(s))\, ds \\
&\leq \frac{1}{t_f}\int_0^{t_f} \left|\left|\hat{\zeta}(t)\right|\right|^2_{L^2(I)} + \int_t^{t_f} \left(\left|\left|\hat{\zeta}_t(s)\right|\right|^2_{L^2(I)} + \left|\left|\hat{\zeta}(s)\right|\right|^2_{L^2(I)}\right) ds \\
&\leq (1+1/t_f)\int_t^{t_f} \left(\left|\left|\hat{\zeta}_t(s)\right|\right|^2_{L^2(I)} + \left|\left|\hat{\zeta}(s)\right|\right|^2_{L^2(I)}\right) ds.
\end{aligned}$$

Setting $t = 0$ on the left hand side yields the estimate for $||\hat{\zeta}(0)||^2$. Furthermore, taking the supremum on the right hand side, we get

$$||\hat{\zeta}||^2_{L^\infty(L^2(I))} \leq (1+1/t_f)\left(||\hat{\zeta}_t||_{L^2(L^2(I))} + ||\hat{\zeta}||_{L^2(L^2(I))}\right) \tag{4.57}$$

Now using (4.56), (4.57) and (4.57), we get

$$\begin{aligned}
&||T - T^N||^2_{L^\infty(L^2(\Omega))} + ||T - T^N||^2_{L^2(H_1(\Omega))} \\
&\qquad \leq C(t_f)\sum_{j=0}^{\infty}\left(||\hat{\zeta}_t||^2_{L^2(L^2(I))} + ||\hat{\zeta}_z||^2_{L^2(L^2(I))} + k^2||\hat{\zeta}||^2_{L^2(L^2(I))}\right)
\end{aligned} \tag{4.58}$$

We will now make a special choice for π_N. Let, for $N > 0$,

$$\mathrm{proj}_N : \{\psi_z;\ \psi \in \mathcal{M}_c\} \to \{\psi_z;\ \psi \in \mathcal{M}_N\}$$

be the interpolation operator which assigns, to every function h from the left set, the polynomial which interpolates this function at the $N+1$ Gauss-Lobatto nodes. Note that this polyomial has degree $N+1$, and since the left and right end points of I are included in the Gauss-Lobatto nodes, it is zero at $\pm 1/2$. In other words, it arises as the derivative of a polynomial of degree $N+2$ with vanishing derivatives at $z = \pm 1/2$, i.e. as the derivative of a polynomial of $\mathcal{M}_N$. So, proj_N is well defined.

We know from [2] that

$$||h - \mathrm{proj}_N(h)||_{L^2(I)} \leq CN^{-1}||h||_{H^1(I)}. \tag{4.59}$$

We now define π_N to be, for $f \in \mathcal{M}_c$

$$\begin{aligned}
\pi_0(f) &= f(-1/2), \\
\pi_N(f) &= f(-1/2) + \int_{-1/2}^z \mathrm{proj}_N(f_z)\, dz, \qquad \text{for } N > 0.
\end{aligned}$$

From the construction of proj_N it is easy to see that $\pi_N(f) \in \mathcal{M}_N$. Since

$$f - \pi_N(f) = \int_{-1/2}^z f_z - \mathrm{proj}_N(f_z)\, dz$$

we get from (4.59)

$$||(f - \pi_N(f))_z||_{L^2(I)} = ||f_z - \mathrm{proj}_N(f_z)||_{L^2(I)} \leq CN^{-1}||f_z||_{H^1(I)}$$

and, with a little algebra using Cauchy-Schwarz

$$\begin{aligned} ||f - \pi_N(f)||_{L^2(I)} &\leq ||f_z - \text{proj}_N(f_z)||_{L^2(I)} \\ &\leq CN^{-1}||f_z||_{H^1(I)} \\ &\leq CN^{-1}||f||_{H^2(I)} \end{aligned}$$

Furthermore, if in addition to $f(t) \in \mathcal{M}_c$ we also have $f_t(t) \in H^2(I)$, we have the following estimate

$$||(f - \pi_N(f))_t||_{L^2(I)} = ||f_t - \pi_N(f_t)||_{L^2(I)} \leq CN^{-1}||f_t||_{H^2(I)}.$$

We use this to get

$$\begin{aligned} &||T - T^N||^2_{L^\infty(L^2(\Omega))} + ||T - T^N||^2_{L^2(H_1(\Omega))} \\ &\quad \leq \; C(t_f)\, N^{-1} \sum_{j=0}^{\infty} \left(||\hat{\theta}_t||^2_{L^2(H^2(I))} + ||\hat{\theta}_z||^2_{L^2(H^1(I))} + k^2 ||\hat{\theta}||^2_{L^2(H^2(I))} \right) \\ &\quad \leq \; C(t_f)\, N^{-1} \left(||\theta_t||^2_{L^2(H^2(\Omega))} + ||\theta||^2_{L^2(H^2(\Omega))} \right) \end{aligned} \tag{4.60}$$

5 Pattern selection for 3-D Rayleigh-Bénard convection

5.1 Controlled Rayleigh-Bénard convection

In the three-dimensional version of (2.1)–(2.3), (2.7), (2.8), (2.9), is

$$\text{Pr}^{-1} \left[\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \mathbf{u} \right] = -\nabla p + R\, T^+ \; (0,0,1)^t + \nabla^2 \mathbf{u} \tag{5.1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{5.2}$$

$$\partial_t T + \mathbf{u} \cdot \nabla T = \nabla^2 T + w \tag{5.3}$$

With boundary conditions

$$u = v = w = 0 \qquad \text{at} \quad z = \pm\frac{1}{2} \tag{5.4}$$

$$T = 0 \qquad \text{at} \quad z = +\frac{1}{2}\,. \tag{5.5}$$

and

$$\partial_z T = -\omega\ \Delta_2 \int_{-1/2}^{1/2} T\, dz \qquad \text{at} \quad z = -\frac{1}{2}. \tag{5.6}$$

where $\Delta_2 = \partial_x^2 + \partial_y^2$. We let

$$\mathbf{u} = \langle u, v, w \rangle = \nabla \times \mathbf{B} \quad \text{with the} \quad \mathbf{B} = \langle \phi, \varphi, \psi \rangle \tag{5.7}$$

Taking the *curl* of (5.1) and noting that $Pr^{-1} \ll 1$ we obtain for (5.1)

$$-\Delta^2\phi + \Delta\left(\partial_x^2\phi + \partial_y\partial_x\varphi + \partial_z\partial_x\psi\right) + \mathrm{Ra}\,\partial_y T = 0 \tag{5.8}$$

$$-\Delta^2\varphi + \Delta\left(\partial_x\partial_y\phi + \partial_y^2\varphi + \partial_z\partial_y\psi\right) - \mathrm{Ra}\,\partial_x T = 0 \tag{5.9}$$

$$-\Delta^2\psi + \Delta\left(\partial_x\partial_z\phi + \partial_y\partial_z\varphi + \partial_z^2\psi\right) = 0 \tag{5.10}$$

The minimal polynomial representation for the velocity components that enables us to capture the three-dimensional convection cell pattern and that is such that $\mathbf{u}$ satisfies the continuity equation and the boundary conditions at $z = \pm 1/2$ is again $\mu(z) = 1/4(z^2 - 1/4)^2$. Consequently, we make the ansatz

$$\phi = U(x,y,t)\,\mu(z)\,, \quad \varphi = V(x,y,t)\,\mu(z)\,, \quad \psi = W(x,y,t)\,\mu(z) \tag{5.11}$$

so that

$$u = \mu\,\partial_y W - V\,\partial_z\mu,\; v = -\mu\,\partial_x W + U\,\partial_z\mu,\; w = \mu\,\partial_x V - \mu\,\partial_y U \tag{5.12}$$

For the temperature we make analogously to the two-dimensional case the ansatz

$$T = h(x,y,t)\,H(z) + s(x,y,t)\,\ell(z) \tag{5.13}$$

where $H(z)$ and $\ell(z)$ are as before and

$$s(x,y,t) = \omega\frac{2}{3}\Delta_2 h \tag{5.14}$$

If we substitute the ansatz (5.12)–(5.13) into (5.8)–(5.10) and (5.3) and testing the result with

$$\delta(\tilde{x}-x,\tilde{y}-y)\,\mu(z) \quad \text{and} \quad \delta(\tilde{x}-x,\tilde{y}-y)\,H(z) \tag{5.15}$$

respectively, we obtain the problem

$$\begin{aligned}
&\partial_y^4 U - 24\,\partial_y^2 U + 504\,U + \partial_x^2\left(\partial_y^2 U - 12\,U\right)\\
&\qquad = -\mathrm{Ra}\,\partial_y\left(60\,h - \frac{3}{2}s\right) + \partial_x\partial_y\left(\Delta_2 V - 12\,V\right) &(5.16)\\
&\partial_x^4 V - 24\,\partial_x^2 V + 504\,V + \partial_y^2\left(\partial_x^2 V - 12\,V\right)\\
&\qquad = \mathrm{Ra}\,\partial_x\left(60\,h - \frac{3}{2}s\right) + \partial_x\partial_y\left(\Delta_2 U - 12\,U\right) &(5.17)\\
&\partial_t h - \Delta_2 h + \frac{5}{2}h + \frac{9}{448}\left[\left(U\,\partial_y h - V\,\partial_x h\right) + \frac{1}{2}\left(\partial_y U - \partial_x V\right)\,h\right]\\
&\qquad = \frac{15}{8}s + \frac{5}{448}\left(\partial_y U - \partial_x V\right) &(5.18)\\
&\qquad\quad + \frac{1}{448\cdot 24}\left[\left(U\partial_y s - V\partial_x s\right) - \frac{3}{2}\left(\partial_y U - \partial_x V\right)\,s\right] &(5.19)
\end{aligned}$$

This system we solve numerically using a finite difference code and an implicit Euler scheme for the time discretisation and a Newton scheme combined with an iterative solver BICSTAB [14]) for the linear subproblems. We solve the problem with homogeneous Dirichlet boundary conditions on an (L_x, L_y) square. For the initial condition we use

$$h(x, y, 0) = xy\left(\frac{x}{L_x} - 1\right)\left(\frac{y}{L_y} - 1\right)10^{-n} \tag{5.20}$$

where we let $n = 4$. The other variables we set to zero. For all runs we let the Rayleigh number $\mathrm{Ra} = 1.1 * Ra_c$ For the uncontrolled problem we expect a quadratic pattern of convection cells, for Dirichlet and Neumann boundary conditions on $z = 1/2$ and $z = -1/2$ respectively. Figure 3 shows the streamlines of the vertical velocity. We see in the first column, that after going through a transient phase of a lozenges pattern, squares starts to appear and fill the whole horizontal domain.

We now ask if feedback control not only supresses (or enhances) the instability as well as changes its wavelength, but if it can also have an effect on the three-dimensional pattern. Starting with the same initial condition, we observe in the right column of figure 3 that for feedback control of $\omega = -0.9$ also here a lozenges pattern appears, followowed by a pattern of square cells, all having smaller wavelength. At some point, when the amplitude has become large enough the control effects a change in the up-down symmetry and a new hexagonal pattern eventually establishes itself.

5.2 Spiral-defect chaos

In the middle of the 90's [10] discovered in a Rayleigh-Bénard experiment using large containers, that for a set of parameters and boundary conditions, for which only parallel rolls should occur, a new instability appeared. At certain locations these rolls started to become unstable and form a spiral pattern over the whole horizontal region being chaotic in time. In order to avoid full 3D computations for a long time in order to capture these patterns, researches have developed model equations, such as the Swift-Hohenberg equation [3]. However not all terms in this equations can be derived from the underlying Boussinesq equations. So here we have an alternative method to reduce this problem to a 2D situation, where numerical computations can capture SDC in a reasonable time.

For this problem we can not neglect the left hand side of (5.1). In the flow regime considered here, the Prandtl number is of O(1). Also here, the boundary conditions at $z = \pm 1/2$ are both homogeneous Neumann conditions, so that in our Galerkin approximation the minimal set are two temperature functions to capture the vertical structure of the flow. Otherwise, we proceed similarly as in the previous paragraph We make the ansatz:

$$\phi = U(x, y, t)\,\mu(z)\,, \quad \varphi = V(x, y, t)\,\mu(z)\,, \quad \psi = W(x, y, t)\,\nu(z) \tag{5.21}$$

$$T = hH_0 + fH_1\,, \qquad H_0 = \nu\,,\ H_1 = \mu_z\,, \quad \int_{-1/2}^{1/2} H_0 H_1\, dz \tag{5.22}$$

$$\text{where} \quad \nu = \frac{1}{2}\left(z^2 - \frac{1}{4}\right)\,, \quad \mu = \nu^2 \tag{5.23}$$

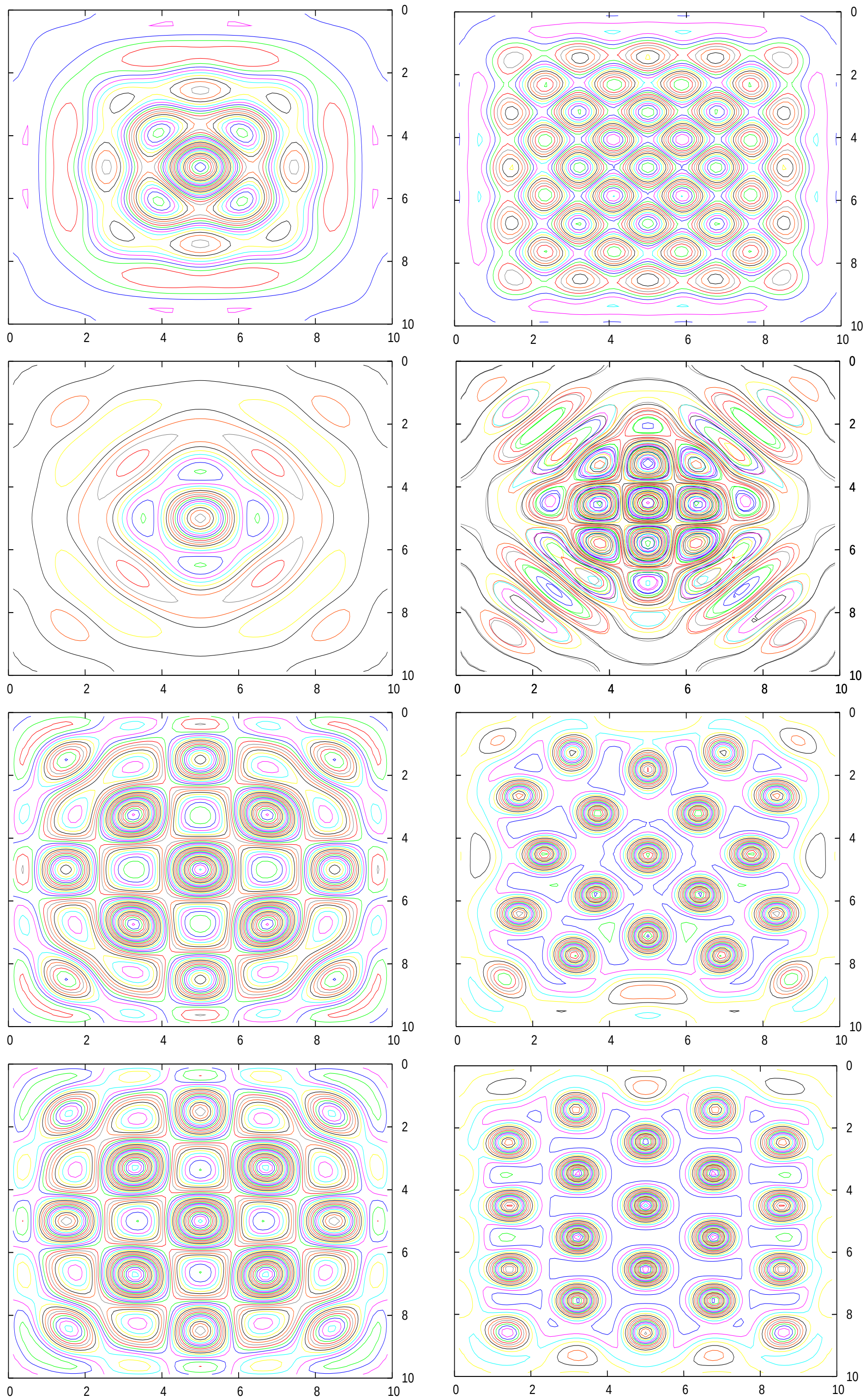

Figure 3: Uncontrolled (left) and controlled ($\omega = -0.9$ right) Rayleigh-Bénard convection for $\mathrm{Ra} = 1.1\mathrm{Ra}_c$

and arrive at the Galerkin approximation

$$
\begin{aligned}
&12\,\partial_t U - \partial_t(\partial_y^2 U - \partial_x\partial_y V) - \frac{5}{44}\partial_y\left(\partial_y W(\partial_x^2 V - \partial_x\partial_y U) + \partial_x W(\partial_y^2 U - \partial_x\partial_y V)\right)\\
&-\frac{1}{2}\partial_y(V\partial_y W + U\partial_x W) + \frac{1}{2}\partial_x W(\partial_x V - \partial_y U) - \partial_y W(\partial_x U + \partial_y V) - V\Delta_2 W\\
&= -\mathrm{Pr}\left[\partial_y^4 U - 24\,\partial_y^2 U + 504\,U + \partial_x^2\left(\partial_y^2 U - 12\,U\right)\right.\\
&\qquad\left. +\mathrm{Ra}\,9\,\partial_y h - \partial_x\partial_y\left(\Delta_2 V - 12\,V\right)\right]
\end{aligned}
\tag{5.24}
$$

$$
\begin{aligned}
&12\,\partial_t V - \partial_t(\partial_x^2 V - \partial_x\partial_y U) + \frac{5}{44}\partial_x\left(\partial_y W(\partial_x^2 V - \partial_x\partial_y U) + \partial_x W(\partial_y^2 U - \partial_x\partial_y V)\right)\\
&+\frac{1}{2}\partial_x(V\partial_y W + U\partial_x W) + \frac{1}{2}\partial_y W(\partial_x V - \partial_y U) + \partial_x W(\partial_x U + \partial_y V) + U\Delta_2 W\\
&= -\mathrm{Pr}\left[\partial_x^4 V - 24\,\partial_x^2 V + 504\,V + \partial_y^2\left(\partial_x^2 V - 12\,V\right)\right.\\
&\qquad\left. -\mathrm{Ra}\,9\,\partial_x h - \partial_x\partial_y\left(\Delta_2 U - 12\,U\right)\right]
\end{aligned}
\tag{5.25}
$$

$$
\begin{aligned}
&\partial_t\Delta W + \frac{3}{28}\left(\partial_x W\partial_y\Delta W - \partial_y W\partial_x\Delta W\right) + \frac{1}{56}\partial_x\left(U(\partial_y U - \partial_x V)\right)\\
&+\frac{1}{56}\partial_y\left(V(\partial_y U - \partial_x V)\right) - \frac{1}{84}\partial_x\left(V(\partial_x U + \partial_y V)\right) + \frac{1}{84}\partial_y\left(U(\partial_x U - \partial_y V)\right)\\
&= \mathrm{Pr}\left(\Delta^2 W - 10\,\Delta W\right)
\end{aligned}
\tag{5.26}
$$

$$
\begin{aligned}
&\partial_t h + \frac{3}{28}\left(\partial_x W\partial_y h - \partial_y W\partial_x h\right) + \frac{1}{84}\left(V\partial_x f - U\partial_y f\right) + \frac{1}{56}(\partial_x V - \partial_y U)f\\
&= \Delta h - 10\,h - \frac{3}{28}(\partial_x V - \partial_y U)
\end{aligned}
\tag{5.27}
$$

$$
\begin{aligned}
&\partial_t f + \frac{1}{12}\left(\partial_x W\partial_y f - \partial_y W\partial_x f\right) + \frac{1}{12}\left(V\partial_x h - U\partial_y h\right) - \frac{1}{24}(\partial_x V - \partial_y U)h\\
&= \Delta h - 42\,h
\end{aligned}
\tag{5.28}
$$

We solve this system using a pseudo-spectral method and an implicit Euler for the time discretisation. We choose periodic boundary conditions for the horizontal boundaries. We set $\mathrm{Pr} = 1$ and $\mathrm{Ra} = 1776 \cdot 1.70 = \mathrm{Ra}_c * 1.7$. In figure 4 we see a snapshot of the streamlines for the temperature.

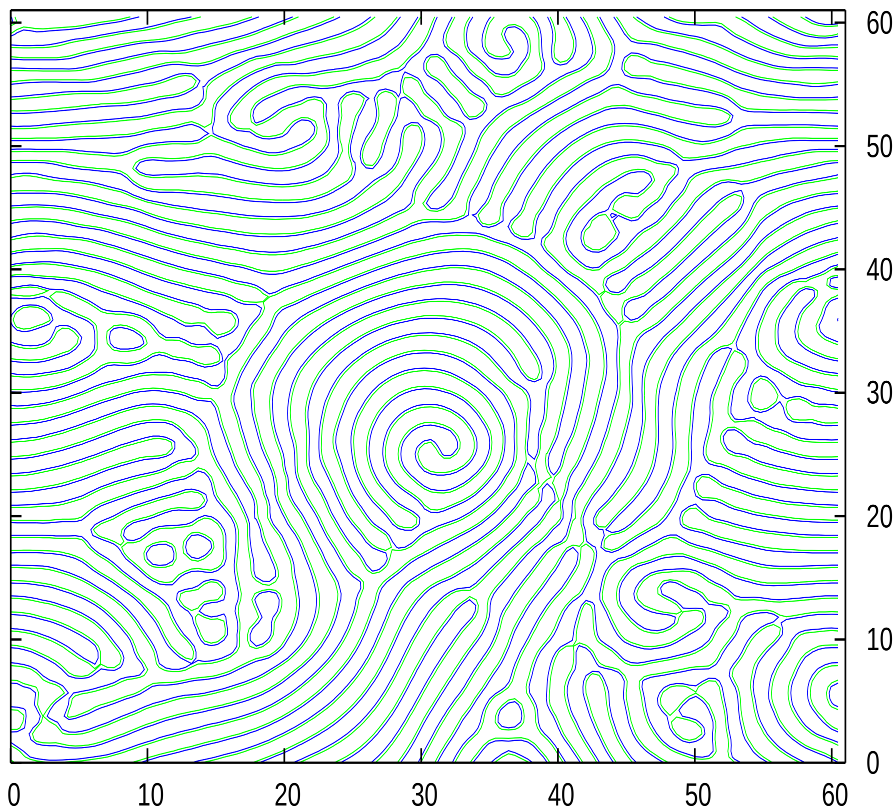

Figure 4: Streamlines of the temperature at $z = -1/2$ and $z = 1/2$ for $\mathrm{Pr} = 1$ and $\mathrm{Ra} = 1.7 \cdot 1776 = 1.7 \cdot \mathrm{Ra}_c$

A Galerkin approximation with two temperature functions fot controlled Rayleigh-Bénard convection

A.1 Governing equations

Above discussion will now be exploited for our controlled RB problem. Here, we derive the Galerkin approximation with two temperature functions. We let

$$T(x,z,t) = h(x,t)\, H_0(z) + f(x,t)\, H_1(z) + s(x,t)\, \ell_2(z) \tag{1.29}$$

where $H_0(z)$ and $H_1(z)$ are chosen such that $H_0(1/2) = 0$ and $H_0'(-1/2) = 0$ which yields a second order polynomial, while the same conditions for $H_1(z)$ together with

$$\langle H_0, H_1 \rangle = 0 \tag{1.30}$$

yields a third order polynomial. We arrive at

$$H_0(z) = \left(z - \frac{1}{2}\right)\left(z + \frac{3}{2}\right), \tag{1.31}$$

$$H_1(z) = \left(z - \frac{1}{2}\right)\left(z^2 + \frac{29}{32}z + \frac{7}{64}\right). \tag{1.32}$$

The polynomial $\ell_2(z)$ naturally must satify $\ell_2(1/2) = 0$. The order will be further increased by requiring the boundary condition at $z = -1/2$ to be satisfied. However, when considering the possibility of negative gain $\omega < 0$ we obtain artificial singularities, not present in the full problem, unless

$$\rho_2 = \int_{-1/2}^{1/2} \ell_2(z)\, dz = \langle \ell_2, H_0 \rangle \int_{-1/2}^{1/2} H_0(z)\, dz + \langle \ell_2, H_1 \rangle \int_{-1/2}^{1/2} H_1(z)\, dz$$

is satisfied. Calculations can be further simplified, if we choose $\ell_2(z)$ to also be orthogonal to H_0 and H_1. As a consequence, we obtain a polynomial of fourth order such that $\rho_2 = 0$ and normalize it such that $\ell_2'(-1/2) = 1$. This yields

$$\ell_2(z) = -\frac{7}{4}\left(z - \frac{1}{2}\right)\left(z^3 + \frac{1}{10}z^2 - \frac{17}{140}z - \frac{1}{280}\right) \tag{1.33}$$

For the velocity function we follow Hosoi and Dupont and require $\mathbf{u}$ to be divergence free. Additionally, we require no-slip boundary conditions at $z = 1/2$ and $z = -1/2$. This yields

$$v(x,z,t) = u(x,t)\, \mu_z(z), \tag{1.34}$$

$$w(x,z,t) = -u_x(x,t)\, \mu(z), \tag{1.35}$$

where

$$\mu(z) = \frac{1}{4}\left(z^2 - \frac{1}{4}\right)^2 \tag{1.36}$$

We can now derive the Galerkin approximation by testing the full problem with the test functions

$$\theta_0 = \delta(x)\,\mu_z(z) \qquad \theta_1 = -\delta'(x)\,\mu_z(z) \tag{1.37}$$

$$\phi_0 = \delta(x) H_0(z) \qquad \phi_1 = \delta(x) H_1(z) \tag{1.38}$$

to obtain

$$u_{xxxx} - 24\,u_{xx} + 504\,u = -R\left(60\,h + \frac{27}{8}\,f - \frac{1}{5}\,s\right)_x \tag{1.39}$$

$$\begin{aligned} h_t - h_{xx} + \frac{5}{2}\,h + \frac{9}{448}\left(h_x u + \frac{1}{2}\,h u_x\right) &= -\frac{5}{64}\,f + \frac{15}{8}\,s + \frac{5}{448}\,u_x \\ &+\frac{1}{448}\left(\frac{97}{96} f_x u + \frac{91}{64}\,f u_x\right) + \frac{1}{448\cdot 20}\left(3\,s_x u + \frac{37}{12}\,s u_x\right) \end{aligned} \tag{1.40}$$

$$\begin{aligned} f_t - f_{xx} + \frac{3059}{130}\,f - \frac{173}{390\cdot 32}\left(f_x u + \frac{1}{2}\,f u_x\right) &= -\frac{112}{13}\,h - \frac{6132}{325}\,s + \frac{9}{130}\,u_x \\ +\frac{1}{390}\left(97\,h_x u - \frac{79}{2}\,h u_x\right) + \frac{1}{429\cdot 200}\left(31\,s_x u + \frac{3\cdot 329}{4}\,s u_x\right) \end{aligned} \tag{1.41}$$

$$\text{with} \quad s = \omega\left(\frac{2}{3}\,h_{xx} + \frac{1}{48}\,f_{xx}\right) \tag{1.42}$$

A.2 Linear stability for two temperature functions

We first like to determine the critical Raleigh number (R_c) of above problem. We linearize about the conductive state, hence about $u(x,t) = 0$, $h(x,t) = 0$, $f(x,t) = 0$ and $s(x,t) = 0$.

$$u_{xxxx} - 24\,u_{xx} + 504\,u = -R\left(60\,h + \frac{27}{8}\,f - \frac{1}{5}\,s\right)_x \tag{1.43}$$

$$h_t - h_{xx} + \frac{5}{2}\,h = -\frac{5}{64}\,f + \frac{15}{8}\,s + \frac{5}{448}\,u_x \tag{1.44}$$

$$f_t - f_{xx} + \frac{3059}{130}\,f = -\frac{112}{13}\,h - \frac{6132}{325}\,s + \frac{9}{130}\,u_x \tag{1.45}$$

$$\text{with} \quad s = \omega\left(\frac{2}{3}\,h_{xx} + \frac{1}{48}\,f_{xx}\right) \tag{1.46}$$

Fourier transform of above equations yields

$$\hat{h}_t = -\left(k^2 + \frac{5}{2}\right)\hat{h} - \frac{5}{64}\,\hat{f} + \frac{15}{8}\,\hat{s} + \frac{5}{448}\,i\,k\,\hat{u} \tag{1.47}$$

$$\hat{f}_t = -\left(k^2 + \frac{3059}{130}\right)\hat{f} - \frac{112}{13}\,\hat{h} - \frac{6132}{325}\,\hat{s} + \frac{9}{130}\,i\,k\,\hat{u} \tag{1.48}$$

$$\text{where} \quad \hat{s} = -\omega\,k^2\left(\frac{2}{3}\,\hat{h} + \frac{1}{48}\,\hat{f}\right) \quad \text{and}$$

$$i\,k\,\hat{u} = M\,R\left(60\,\hat{h} + \frac{27}{8}\,\hat{f} - \frac{1}{5}\,\hat{s}\right) \text{ with } M = \frac{k^2}{k^4 + 24\,k^2 + 504}$$

with the solution of (1.47)–(1.48)

$$\begin{aligned}
\hat{h}(k,t) &= K_1\, a_1\, \exp(\sigma_1\, t) + K_2\, a_2\, \exp(\sigma_2\, t) \qquad (1.49)\\
\hat{f}(k,t) &= K_1\, \exp(\sigma_1\, t) + K_2\, \exp(\sigma_2\, t) \qquad (1.50)
\end{aligned}$$

where K_1 and K_2 are constants and

$$\begin{aligned}
a_1 &= \frac{1}{2D}\left(A - C + \sqrt{(A-C)^2 + 4DB}\right), \qquad (1.51)\\
a_2 &= \frac{1}{2D}\left(A - C - \sqrt{(A-C)^2 + 4DB}\right), \qquad (1.52)\\
\sigma_1 &= \frac{1}{2}\left(A + C + \sqrt{(A-C)^2 + 4DB}\right), \qquad (1.53)\\
\sigma_2 &= \frac{1}{2}\left(A + C - \sqrt{(A-C)^2 + 4DB}\right) \qquad (1.54)
\end{aligned}$$

with

$$\begin{aligned}
A(k^2,\omega) &= -k^2 - \frac{5}{2} + \mathrm{R}\frac{5}{448}\left(60 + \frac{2}{15}\omega k^2\right) M - \frac{5}{4}\omega k^2 \qquad (1.55)\\
B(k^2,\omega) &= \mathrm{R}\frac{5}{448}\left(\frac{27}{8} + \frac{1}{240}\omega k^2\right) M - \frac{5}{64} - \frac{5}{128}\omega k^2 \qquad (1.56)\\
C(k^2,\omega) &= -k^2 - \frac{3059}{130} + \mathrm{R}\frac{9}{130}\left(\frac{27}{8} + \frac{1}{240}\omega k^2\right) M \\
&\quad + \frac{511}{1300}\omega k^2 \qquad (1.57)\\
D(k^2,\omega) &= \mathrm{R}\frac{9}{130}\left(60 + \frac{2}{15}\omega k^2\right) M - \frac{112}{13} + \frac{4088}{325}\omega k^2 \qquad (1.58)
\end{aligned}$$

From this we calculate from the dominant growth rate σ_1, by solving

$$\sigma_1 = 0 \quad \text{and} \quad \frac{\partial \sigma_1}{\partial R} = 0 \qquad (1.59)$$

the critical Rayleigh number together with the critical wavenumber.

Acknowledgments

The authors thank Andrea Bertozzi for many helpful discussions and for suggesting the problem of SDC to us. AM is supported by the DFG Heisenberg fellowship.

References

[1] Bénard, H. Les tourbillons cellulaires dans une nappe liquide transportant de la chaleur par convection en régime permanent *Ann. Chim. Phys.*, 23, 62–144,1901.

[2] Canuto C., Quateroni A. Approximation results for orthogonal polynomials in Sobolev spaces *Math. Comp.*, 38 (157), 1982

[3] Cross M.C., Tu Y. Defect dynamics for spiral chaos in Rayleigh-Bénard convection *Phys. Rev. Lett.*, 75 (5), 834–837, 1996.

[4] Cross M.C., Meiron D., Tu Y. Chaotic Dynamics: Anumerical investigation *Chaos*, 4, 607–619, 1994.

[5] Dupont, T. F. and Hosoi, A. E. A reduced-dimension model for Rayleigh-Bénard convection *preprint*, 1999.

[6] Getling, A. V. Rayleigh-Bénard Convection Structures and Dynamics *Word Scientific. Singapore.* , 1998.

[7] Howle, L. E. Control of Rayleigh-Bénard convection in a small aspect ratio container *Int. J. Heat Mass Trans.* , 40, 817–822, 1997.

[8] Howle, L. E. The effect of boundary properties on controlled Rayleigh-Bénard convection *J. Fluid Mech.*, 411, 39–58, 2000.

[9] Manneville, P. Dissipative Structures and Weak Turbulence *Academic Press, Inc.*, New York, 1990.

[10] Morris S.W., Bodenschatz E., Cannell D.S., Ahlers G. Spiral defect chaos in large aspect ratio Rayleigh-Bénard convection *Phys. Rev. Lett.*, 71 (13), 2026–2029, 1993.

[11] Müller, G. Crystals: Growth, Properties, and Applications; 12: Convection and Inhomogeneities in Crystal Growth from the Melt. *H.C. Freyhardt et al. (Eds.)*, Springer-Verlag, 1988.

[12] Or, A.C., Cortelezzi, L, Speyer, J.L. Robust feedback control of Rayleigh-Bénard convection *J.Fluid Mech.*, 437, 175–202, 2001.

[13] Lord Rayleigh On convection currents in a horizontal layer of fluid, when the higher temperature is on the under side. *Phil. Mag.*, 32, 529–546, 1916.

[14] Skalický T. LASPack Reference Manual *Version 1.12.3*, Dresden University of Technology, Institute for Fluid Mechanics, 1996.

[15] Tang J., Bau H.H. Stabilization of the no-motion state in Rayleigh-Bénard convection through the use of feedback control *Phys. Rev. Lett.*, 70, 1795–1798, 1993.

[16] Wagner, B. A., Bertozzi A.L., Howle L.E. Positive feedback control of Rayleigh-Bénard convection *Discrete Cont. Dyn.-B*, 3 (4), 619-642, 2003.