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Thermodynamics of simple two-component thermo-poroelastic media

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Abstract

   The paper is devoted to the thermodynamic construction of a two-component model of poroelastic media undergoing, in contrast to earlier works on this subject, nonisothermal processes. Under the constitutive dependence on partial mass densities, deformation gradient of skeleton, relative velocity, temperature, temperature gradient and porosity (simple poroelastic material) as well as the assumption of small deviations from the thermodynamic equilibrium we construct explicit relations for fluxes, prove the splitting of the free energy into partial contributions without mechanical couplings, construct a chemical potential for the fluid component important for the formulation of boundary conditions on permeable boundaries. We discuss as well a modification of the porosity balance equation in which we account for time changes of equilibrium porosity. This modification yields the behavior of the model characteristic for granular materials.

1   Introduction

Thermodynamic modeling of saturated poroelastic materials by means of a two-component continuum has been limited to isothermal processes. R. M. Bowen who initiated the work in this field [1],[2] constructed a model for a multicomponent system with large deformations of the skeleton and internal variables (volume fractions) for which he postulated evolution equations. Such an approach may be appropriate for some biomaterials but it fails for granular media. For the latter, relaxation processes for internal variables are almost immaterial and the main mechanism driving changes of porosity are volume changes of components. Such phenomena are described within the model with the porosity balance equation. For a two-component system it has been proposed in the work [4] and developed in [5]. Some fundamental properties, its microscopic motivation and a transition to a linear model is presented in [6]. As a model belonging to the so-called extended thermodynamics it has been extended to multicomponent systems in [7] and [9]. All these papers concern so-called simple poroelastic materials in which there is no constitutive dependence on higher gradients of fields. Linearization of such models does not lead to the Biot's model which is successfully applied in various fields of geotechnics. It has been shown that additional couplings appearing in Biot's model require an extension of constitutive variables and thermodynamics of such a model in which the gradient of porosity is the constitutive variable is the subject of the work [8].

In the present work we present a thermodynamic analysis of a model of simple poroelastic materials which differs from these mentioned above in two essential points

- processes are not isothermal; the system is characterized by a single temperature field which may vary in space and time,

- the porosity balance equation is extended by changes of equilibrium porosity.

Changes in the porosity balance equation result from the analysis of earlier models for granular materials. Micro-macro transition as well as solutions of simple boundary problems show that the original model yields a very stiff behavior of the skeleton which is appropriate for rocks but not for granular materials (compare [10]). In addition, if we neglect the relaxation of porosity the modified equation of porosity yields in the linear case changes of porosity indicated by Gassmann relations.

We base our considerations on the assumption that deviations from thermodynamic equilibrium are small. As these deviations are described by three variables: temperature gradient $\mathbf{G} := \text{Grad}\,T$, Lagrangian (relative) velocity $\acute{\mathbf{X}}^F$ and a deviation of porosity from its equilibrium value $\Delta_n := n - n_E$, this assumption means that we assume the dissipation to be a quadratic function of these variables.

Under this assumption we prove the following properties: 1) the Helmholtz free energy splits into two partial potentials which are not coupled by mechanical variables (*simple mixture*), 2) thermal parts of energy and entropy fluxes are connected by the classical Fourier relation and 3) the flux of porosity contains solely a linear contribution of the Lagrangian velocity with a coefficient proportional solely to volume changes of the skeleton.

We complete the work with a presentation of few simplified models. We show that the fully linear model does not coincide with the classical Biot's model due to the lack of coupling between partial stresses. This property has been proven earlier for isothermal processes in simple poroelastic materials.

2   Balance equations

We use the Lagrangian description referring to the reference configuration $\mathcal{B}$ of skeleton [3] in which its deformation gradient $\mathbf{F}^S = \mathbf{1}$. In the two-component medium considered in this work the partial balance equations for the skeleton are defined on a family of volume measurable sets $\left\{ \mathcal{P}^S \left| \mathcal{P}^S \subset \mathcal{B} \right. \right\}$ material with respect to the skeleton, i.e. independent of time. Simultaneously partial balance equations for the fluid are defined on a time dependent family of volume measurable sets $\left\{ \mathcal{P}^F \left| \mathcal{P}^F \subset \mathcal{B} \right. \right\}$ with the kinematics defined by the Lagrangian field of the relative velocity

$$\acute{\mathbf{X}}^F \left(\mathbf{X}, t \right) = \mathbf{F}^{S-1} \left(\acute{\mathbf{x}}^F - \acute{\mathbf{x}}^S \right), \quad \mathbf{X} \in \mathcal{B} \subset \Re^3, \quad t \in \mathcal{T} \subset \Re, \tag{1}$$

in which $\acute{\mathbf{x}}^F, \acute{\mathbf{x}}^S$ are velocities of the fluid component and of the skeleton, respectively. Clearly, for the existence of a function of motion of skeleton, we require the following conditions to be satisfied

$$\mathbf{F} := \frac{\partial \mathbf{F}^S}{\partial t} - \text{Grad}\,\acute{\mathbf{x}}^S = 0, \quad \text{Grad}\,\mathbf{F}^S = \left(\text{Grad}\,\mathbf{F}^S \right)^{\overset{23}{T}}. \tag{2}$$

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We say that a field φ^S, describing a property of the skeleton, whose flux is $\mathbf{\Psi}^S$, and supply is $\hat{\varphi}^S$, satisfies a balance equation if for any set $\mathcal{P}^S$

$$\frac{d}{dt}\int_{\mathcal{P}^S}\varphi^S dV = \oint_{\partial\mathcal{P}^S}\mathbf{\Psi}^S \cdot \mathbf{N}dS + \int_{\mathcal{P}^S}\hat{\varphi}^S dV, \tag{3}$$

where $\partial\mathcal{P}^S$ is the oriented boundary of $\mathcal{P}^S$, and $\mathbf{N}$ – the field of its unit outward normal vectors. A similar equation is assumed to hold for a field φ^F describing a property of the fluid.

Quantities appearing in the above equations are assumed to have at most finite singularities on a set of volume measure zero. For the purpose of this work it is sufficient to assume that this set forms an oriented surface $\mathcal{S}$ given by the equation

$$S\left(\mathbf{X},t\right) = 0 \quad \implies \quad \mathbf{N} = \frac{\operatorname{Grad}S}{|\operatorname{Grad}S|}, \quad U = -\frac{\frac{\partial S}{\partial t}}{|\operatorname{Grad}S|}, \quad \mathbf{X} \in \mathcal{B}, \tag{4}$$

where $U\left(\mathbf{X},t\right)$ is its normal speed of propagation through the reference configuration $\mathcal{B}$.

Under this assumption we can write the above equations in the following local form (e.g. [6])

- in regular points a.e. in $\mathcal{B}$

$$\frac{\partial\varphi^S}{\partial t} = \operatorname{Div}\mathbf{\Psi}^S + \hat{\varphi}^S,$$

$$\frac{\partial\varphi^F}{\partial t} + \operatorname{Div}\left(\varphi^F\acute{\mathbf{X}}^F\right) = \operatorname{Div}\mathbf{\Psi}^F + \hat{\varphi}^F, \tag{5}$$

- in singular points on $\mathcal{S}$

$$-U\left[\left[\varphi^S\right]\right] = \left[\left[\mathbf{\Psi}^S\right]\right]\cdot\mathbf{N}, \quad \left[\left[\varphi^F\left(\acute{\mathbf{X}}^F\cdot\mathbf{N} - U\right)\right]\right] = \left[\left[\mathbf{\Psi}^F\right]\right]\cdot\mathbf{N}, \tag{6}$$

where $[[\ldots]] = (\ldots)^+ - (\ldots)^-$ denotes the difference of limits on both sides of the surface $\mathcal{S}$.

Further on, the fields appearing in the balance equations are partial mass densities in the reference configuration ρ^S, ρ^F, partial momentum densities $\rho^S\acute{\mathbf{x}}^S, \rho^F\acute{\mathbf{x}}^F$, partial energies $\rho^S\left(\varepsilon^S + \frac{1}{2}\acute{x}^{S2}\right), \rho^F\left(\varepsilon^F + \frac{1}{2}\acute{x}^{F2}\right)$, porosity n, and partial entropies $\rho^S\eta^S, \rho^F\eta^F$. If we neglect a mass exchange between components then they have the form

- partial mass conservation laws in regular points a.e. in $\mathcal{B}$ and singular points on $\mathcal{S}$

$$R^S := \frac{\partial\rho^S}{\partial t} = 0, \quad R^F := \frac{\partial\rho^F}{\partial t} + \operatorname{Div}\left(\rho^F\acute{\mathbf{X}}^F\right) = 0, \tag{7}$$

$$U\left[\left[\rho^S\right]\right] = 0, \quad \left[\left[\rho^F\left(\acute{\mathbf{X}}\cdot\mathbf{N} - U\right)\right]\right] = 0,$$

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- partial momentum balance equations in regular points a.e. on $\mathcal{B}$

$$\mathbf{M}^S := \rho^S \frac{\partial \acute{\mathbf{x}}^S}{\partial t} - \operatorname{Div} \mathbf{P}^S - \hat{\mathbf{p}} = 0, \tag{8}$$

$$\mathbf{M}^F := \rho^F \left(\frac{\partial \acute{\mathbf{x}}^F}{\partial t} + \acute{\mathbf{X}}^F \cdot \operatorname{Grad} \acute{\mathbf{x}}^F \right) - \operatorname{Div} \mathbf{P}^F + \hat{\mathbf{p}} = 0,$$

where $\mathbf{P}^S, \mathbf{P}^F$ denote Piola-Kirchhoff partial stress tensors, and $\hat{\mathbf{p}}$ is the source of momentum, and in singular points on $\mathcal{S}$

$$-U \left[\left[\rho^S \acute{\mathbf{x}}^S\right]\right] = \left[\left[\mathbf{P}^S\right]\right] \mathbf{N}, \quad \left[\left[\rho^S \acute{\mathbf{x}}^S \left(\acute{\mathbf{X}}^F \cdot \mathbf{N} - U\right)\right]\right] = \left[\left[\mathbf{P}^S\right]\right] \mathbf{N}, \tag{9}$$

- partial energy balance equations in regular points a.e. on $\mathcal{B}$

$$\frac{\partial \left(\rho^S \varepsilon^S + \frac{1}{2}\rho^S \acute{x}^{S2}\right)}{\partial t} + \operatorname{Div} \left(\mathbf{Q}^S - \mathbf{P}^{ST} \acute{\mathbf{x}}^S\right) = 0, \tag{10}$$

$$\frac{\partial \left(\rho^F \varepsilon^F + \frac{1}{2}\rho^F \acute{x}^{F2}\right)}{\partial t} + \operatorname{Div} \left(\left(\rho^F \varepsilon^F + \frac{1}{2}\rho^F \acute{x}^{F2}\right) \acute{\mathbf{X}}^F + \mathbf{Q}^F - \mathbf{P}^{FT} \acute{\mathbf{x}}^F\right) = 0,$$

and in singular points on $\mathcal{S}$

$$-U \left[\left[\rho^S \left(\varepsilon^S + \frac{1}{2}\acute{x}^{S2}\right)\right]\right] + \left[\left[\mathbf{Q}^S - \mathbf{P}^{ST} \acute{\mathbf{x}}^S\right]\right] \cdot \mathbf{N} = 0,$$

$$\left[\left[\rho^F \left(\acute{\mathbf{X}}^F \cdot \mathbf{N} - U\right) \left(\varepsilon^F + \frac{1}{2}\acute{x}^{F2}\right)\right]\right] + \left[\left[\mathbf{Q}^F - \mathbf{P}^{FT} \acute{\mathbf{x}}^F\right]\right] \cdot \mathbf{N} = 0, \tag{11}$$

- porosity balance equation in regular points a.e. on $\mathcal{B}$

$$N := \frac{\partial \Delta_n}{\partial t} + \operatorname{Div} \mathbf{J} - \hat{n} = 0, \quad \Delta_n := n - n_E, \tag{12}$$

where $\mathbf{J}$ denotes the flux of porosity, $\hat{n}$ – its source, and n_E is the porosity in thermodynamic equilibrium, and in singular points on $\mathcal{S}$

$$-U \left[\left[\Delta_n\right]\right] + \left[\left[\mathbf{J}\right]\right] \cdot \mathbf{N} = 0, \tag{13}$$

- partial entropy balance equations in regular points a.e. on $\mathcal{B}$

$$\frac{\partial \left(\rho^S \eta^S\right)}{\partial t} + \operatorname{Div} \mathbf{H}^S = \hat{\eta}^S, \quad \frac{\partial \left(\rho^F \eta^F\right)}{\partial t} + \operatorname{Div} \left(\rho^F \eta^F \acute{\mathbf{X}}^F + \mathbf{H}^F\right) = \hat{\eta}^F, \tag{14}$$

and in singular points on $\mathcal{S}$

$$-U \left[\left[\rho^S \eta^S\right]\right] + \left[\left[\mathbf{H}^S\right]\right] \cdot \mathbf{N} = 0, \tag{15}$$

$$\left[\left[\rho^F \left(\acute{\mathbf{X}}^F \cdot \mathbf{N} - U\right) \eta^F\right]\right] + \left[\left[\mathbf{H}^F\right]\right] \cdot \mathbf{N} = 0.$$

The partial energy balance equations and partial entropy balance equations are used solely in the bulk form which we explain further in this work. It means that we add corresponding partial equations to each other After easy calculations, the following balance equation for the internal energy follows

$$E := \frac{\partial (\rho \varepsilon)}{\partial t} + \operatorname{Div} \mathbf{Q} - \mathbf{P}^S \cdot \operatorname{Grad} \acute{\mathbf{x}}^S - \mathbf{P}^F \cdot \operatorname{Grad} \acute{\mathbf{x}}^F - \left(\mathbf{F}^{ST} \hat{\mathbf{p}} \right) \cdot \acute{\mathbf{X}}^F = 0, \quad (16)$$

where

$$\rho := \rho^S + \rho^F, \quad \rho \varepsilon := \rho^S \varepsilon^S + \rho^F \varepsilon^F, \quad (17)$$
$$\mathbf{Q} := \mathbf{Q}^S + \mathbf{Q}^F + \rho^F \varepsilon^F \acute{\mathbf{X}}^F,$$

i.e. $\varepsilon, \mathbf{Q}$ are so-called intrinsic parts of the bulk internal energy and energy flux, respectively.

Simultaneously for the entropy we obtain

$$\hat{\eta}^S + \hat{\eta}^F = \frac{\partial (\rho \eta)}{\partial t} + \operatorname{Div} \mathbf{H}, \quad (18)$$

where

$$\rho \eta := \rho^S \eta^S + \rho^F \eta^F, \quad \mathbf{H} := \mathbf{H}^S + \mathbf{H}^F + \rho^F \eta^F \acute{\mathbf{X}}^F. \quad (19)$$

For the purpose of this work we use conditions on the singular surface solely for the boundary of the skeleton on which $U \equiv 0$. We have then

$$\left[\left[\rho^F \acute{\mathbf{X}}^F \cdot \mathbf{N} \right]\right] = 0, \quad \left[\left[\mathbf{P}^S \right]\right] \mathbf{N} = 0, \quad \left[\left[\rho^F \acute{\mathbf{X}}^F \cdot \mathbf{N} \acute{\mathbf{x}}^F \right]\right] = \left[\left[\mathbf{P}^F \right]\right] \mathbf{N},$$

$$\left[\left[\rho^F \acute{\mathbf{X}}^F \cdot \mathbf{N} \left(\frac{1}{2} \acute{x}^{F2} \right) \right]\right] + \left[\left[\mathbf{Q} - \mathbf{P}^{FT} \acute{\mathbf{x}}^F \right]\right] \cdot \mathbf{N} = 0, \quad (20)$$

$$\left[\left[\mathbf{H} \right]\right] \cdot \mathbf{N} = 0.$$

In addition, according to the compatibility condition $(2)_1$, we have on $\mathcal{S}$

$$-U \left[\left[\mathbf{F}^S \right]\right] = \left[\left[\acute{\mathbf{x}}^S \right]\right] \otimes \mathbf{N} \quad \Longrightarrow \quad \left[\left[\acute{\mathbf{x}}^S \right]\right] = 0 \quad \text{for } U \equiv 0. \quad (21)$$

3 Fields and field equations

For two-component poroelastic materials there are the following (macroscopic) fields

$$\mathcal{F} := \left\{ \rho^S, \rho^F, \acute{\mathbf{x}}^S, \acute{\mathbf{x}}^F, \mathbf{F}^S, T, n \right\} \quad (22)$$

where the first two fields are partial mass densities of the skeleton, and the fluid in the reference configuration, respectively, $\acute{\mathbf{x}}^S, \acute{\mathbf{x}}^F$ are macroscopic velocities of these two components, $\mathbf{F}^S$ is the deformation gradient of skeleton, T denotes the common temperature of components, and n is the porosity.

The balance equations of the previous section form field equations for 7 fields $\mathcal{F}$ (22) provided the following constitutive quantities

$$\mathcal{C} := \left\{ \mathbf{P}^S, \mathbf{P}^F, \hat{\mathbf{p}}, \varepsilon, \mathbf{Q}, n_E, \mathbf{J}, \hat{n} \right\}, \tag{23}$$

are given as sufficiently smooth functions of the following constitutive variables

$$\mathcal{V} := \left\{ \rho^S, \rho^F, \mathbf{F}^S, \acute{\mathbf{X}}^F, \Delta_n, T, \mathbf{G} \right\}, \quad \mathbf{G} := \mathrm{Grad}\, T. \tag{24}$$

This set of constitutive variables defines the *simple two-component thermo-poroelastic medium.*

Substitution of the functions $\mathcal{C}\left(\mathcal{V}\right)$ in the balance equations yields field equations whose solutions are called *thermodynamic processes.* These processes are thermodynamically *admissible* if the entropy production $\hat{\eta}^S + \hat{\eta}^F$ is nonnegative, i.e. the following entropy inequality

$$\frac{\partial\left(\rho\eta\right)}{\partial t} + \mathrm{Div}\,\mathbf{H} \geq 0, \quad \eta = \eta\left(\mathcal{V}\right), \quad \mathbf{H} = \mathbf{H}\left(\mathcal{V}\right), \tag{25}$$

where η is the entropy density, and $\mathbf{H}$ – its flux, is identically satisfied. This is the *second law of thermodynamics* for thermo-poroelastic media.

All constitutive quantities depend as well on an initial constant porosity n_0. This dependence is not limited by the second law because the initial porosity does not evolve in time. It shall not be indicated in further relations of this work.

In the next section we exploit the second law of thermodynamics.

4   Conditions following from the second law of thermodynamics

In the exploitation of the second law we use the standard procedure of Lagrange multipliers. According to Liu's theorem the following inequality should hold for arbitrary fields

$$\frac{\partial\left(\rho\eta\right)}{\partial t} + \mathrm{Div}\,\mathbf{H} - \Lambda^{\rho^S} R^S - \Lambda^{\rho^F} R^F -$$

$$-\mathbf{\Lambda}^S \cdot \mathbf{M}^S - \mathbf{\Lambda}^F \cdot \mathbf{M}^F - \Lambda^\varepsilon E - \mathbf{\Lambda}^F \cdot \mathbf{F} - \Lambda^n N \geq 0, \tag{26}$$

where the multipliers $\Lambda^{\rho^S}, \Lambda^{\rho^F}, \mathbf{\Lambda}^S, \mathbf{\Lambda}^F, \Lambda^\varepsilon, \mathbf{\Lambda}^F, \Lambda^n$ are functions of variables $\mathcal{V}$.

After application of the chain rule of differentiation we see that the above inequality is linear with respect to the following time derivatives

$$\left\{ \frac{\partial\rho^S}{\partial t}, \frac{\partial\rho^F}{\partial t}, \frac{\partial\acute{\mathbf{x}}^S}{\partial t}, \frac{\partial\acute{\mathbf{x}}^F}{\partial t}, \frac{\partial T}{\partial t}, \frac{\partial\Delta_n}{\partial t}, \frac{\partial\mathbf{G}}{\partial t}, \frac{\partial\mathbf{F}^S}{\partial t} \right\}. \tag{27}$$

This yields the following relations

$$\Lambda^{\rho^S} = \frac{\partial \rho\eta}{\partial \rho^S} - \Lambda^\varepsilon \frac{\partial \rho\varepsilon}{\partial \rho^S}, \quad \Lambda^{\rho^F} = \frac{\partial \rho\eta}{\partial \rho^F} - \Lambda^\varepsilon \frac{\partial \rho\varepsilon}{\partial \rho^F}, \tag{28}$$

$$\mathbf{\Lambda}^S = \mathbf{\Lambda}^F = 0, \tag{29}$$

$$\Lambda^n = \frac{\partial \rho\eta}{\partial \Delta_n} - \Lambda^\varepsilon \frac{\partial \rho\varepsilon}{\partial \Delta_n}, \quad \mathbf{\Lambda}^F = \frac{\partial \rho\eta}{\partial \mathbf{F}^S} - \Lambda^\varepsilon \frac{\partial \rho\varepsilon}{\partial \mathbf{F}^S}, \tag{30}$$

$$\frac{\partial \rho\eta}{\partial T} - \Lambda^\varepsilon \frac{\partial \rho\varepsilon}{\partial T} = 0, \quad \frac{\partial \rho\eta}{\partial \mathbf{G}} - \Lambda^\varepsilon \frac{\partial \rho\varepsilon}{\partial \mathbf{G}} = 0. \tag{31}$$

The linearity with respect to the following spatial derivatives

$$\left\{ \operatorname{Grad} \rho^S, \; \operatorname{Grad} \rho^F, \; \operatorname{Grad} \acute{\mathbf{x}}^S, \; \operatorname{Grad} \acute{\mathbf{x}}^F, \; \operatorname{Grad} \mathbf{F}^S, \; \operatorname{Grad} \mathbf{G}, \; \operatorname{Grad} \Delta_n \right\}, \tag{32}$$

shall be investigated under two simplifying assumptions.

First of all we assume that a dependence on vectorial variables $\acute{\mathbf{X}}^F, \mathbf{G}$ is linear. This shall be justified later. Consequently, for *isotropic* materials, constitutive vector functions must have the following representation

$$\mathbf{Q} = Q_V \acute{\mathbf{X}}^F - K\mathbf{G}, \quad \mathbf{H} = H_V \acute{\mathbf{X}}^F + H_T \mathbf{G}, \tag{33}$$
$$\mathbf{J} = \Phi \acute{\mathbf{X}}^F + J_T \mathbf{G}, \quad \mathbf{F}^{ST}\hat{\mathbf{p}} = \Pi_V \acute{\mathbf{X}}^F + \Pi_T \mathbf{G},$$

where all scalar coefficients are independent of $\acute{\mathbf{X}}^F$ and $\mathbf{G}$.

Secondly we assume that the dissipation $\mathcal{D}$ is quadratic in variables describing a deviation from the thermodynamic equilibrium. The dissipation $\mathcal{D}$ is determined by the *residual inequality* which follows after the elimination of the linear part containing the derivatives (27) and (32). Under the assumption (33) it has the form

$$\mathcal{D} := \frac{\partial H_V}{\partial T}\acute{\mathbf{X}}^F \cdot \mathbf{G} + \frac{\partial H_T}{\partial T}G^2 - \Lambda^\varepsilon \left(\frac{\partial Q_V}{\partial T}\acute{\mathbf{X}}^F \cdot \mathbf{G} - \frac{\partial K}{\partial T}G^2 \right) -$$

$$- \Lambda^n \left(\frac{\partial \Phi}{\partial T}\acute{\mathbf{X}}^F \cdot \mathbf{G} + \frac{\partial J_T}{\partial T}G^2 \right) + \Lambda^\varepsilon \left(\Pi_V \acute{\mathbf{X}}^F \cdot \acute{\mathbf{X}}^F + \Pi_T \acute{\mathbf{X}}^F \cdot \mathbf{G} \right) + \Lambda^n \hat{n} \geq 0, \tag{34}$$

$$G^2 := \mathbf{G} \cdot \mathbf{G}.$$

As the quantity Δ_n describes the deviation of porosity from its equilibrium value the above assumption yields

$$\hat{n} = -\frac{\Delta_n}{\tau}, \tag{35}$$

where τ is independent of vector variables and of Δ_n, and, simultaneously, the multiplier Λ^n must be a homogeneous linear function of Δ_n. Consequently

$$\frac{\partial \Phi}{\partial T} = 0, \quad \frac{\partial J_T}{\partial T} = 0. \tag{36}$$

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In addition Λ^ε must be independent of Δ_n. Then, according to $(30)_1$, both ε and η are quadratic even functions of Δ_n.

Let us return to the conditions following from the linearity with respect to the derivatives (32). We have

$$\frac{\partial H_V}{\partial \rho^S}\acute{\mathbf{X}}^F + \frac{\partial H_T}{\partial \rho^S}\mathbf{G} - \Lambda^\varepsilon\left(\frac{\partial Q_V}{\partial \rho^S}\acute{\mathbf{X}}^F - \frac{\partial K}{\partial \rho^S}\mathbf{G}\right) -$$

$$-\Lambda^n\left(\frac{\partial \Phi}{\partial \rho^S}\acute{\mathbf{X}}^F + \frac{\partial J_T}{\partial \rho^S}\mathbf{G}\right) = 0, \tag{37}$$

$$\frac{\partial H_V}{\partial \rho^F}\acute{\mathbf{X}}^F + \frac{\partial H_T}{\partial \rho^F}\mathbf{G} - \mathbf{\Lambda}^{\rho^F} - \Lambda^\varepsilon\left(\frac{\partial Q_V}{\partial \rho^S}\acute{\mathbf{X}}^F - \frac{\partial K}{\partial \rho^S}\mathbf{G}\right)$$

$$-\Lambda^n\left(\frac{\partial \Phi}{\partial \rho^S}\acute{\mathbf{X}}^F + \frac{\partial J_T}{\partial \rho^S}\mathbf{G}\right) = 0, \tag{38}$$

$$-\left[H_V - \rho^F\Lambda^{\rho^F} - \Lambda^\varepsilon Q_V - \Lambda^n\Phi\right]\mathbf{F}^{S-T} + \Lambda^\varepsilon\mathbf{P}^S + \mathbf{\Lambda}^F = 0, \tag{39}$$

$$\left[H_V - \rho^F\Lambda^{\rho^F} - \Lambda^\varepsilon Q_V - \Lambda^n\Phi\right]\mathbf{F}^{S-T} + \Lambda^\varepsilon\mathbf{P}^F = 0, \tag{40}$$

$$\mathrm{sym}_{23}\left\{-\left[H_V - \rho^F\Lambda^{\rho^F} - \Lambda^\varepsilon Q_V - \Lambda^n\Phi\right]\left(\mathbf{F}^{S-T}\otimes\acute{\mathbf{X}}^F\right) + \right.$$

$$+ \left(\frac{\partial H_V}{\partial \mathbf{F}^S}\otimes\acute{\mathbf{X}}^F + \frac{\partial H_T}{\partial \mathbf{F}^S}\otimes\mathbf{G}\right) -$$

$$\left.-\Lambda^\varepsilon\left(\frac{\partial Q_V}{\partial \mathbf{F}^S}\otimes\acute{\mathbf{X}}^F - \frac{\partial K}{\partial \mathbf{F}^S}\otimes\mathbf{G}\right) - \Lambda^n\left(\frac{\partial \Phi}{\partial \mathbf{F}^S}\otimes\acute{\mathbf{X}}^F + \frac{\partial J_T}{\partial \mathbf{F}^S}\otimes\mathbf{G}\right)\right\} = 0. \tag{41}$$

$$H_T + \Lambda^\varepsilon K - \Lambda^n J_T = 0, \tag{42}$$

$$\frac{\partial H_V}{\partial \Delta_n}\acute{\mathbf{X}}^F + \frac{\partial H_T}{\partial \Delta_n}\mathbf{G} - \Lambda^\varepsilon\left(\frac{\partial Q_V}{\partial \Delta_n}\acute{\mathbf{X}}^F - \frac{\partial K}{\partial \Delta_n}\mathbf{G}\right) - \Lambda^n\left(\frac{\partial \Phi}{\partial \Delta_n}\acute{\mathbf{X}}^F + \frac{\partial J_T}{\partial \Delta_n}\mathbf{G}\right) = 0. \tag{43}$$

These conditions must hold for arbitrary $\acute{\mathbf{X}}^F, \mathbf{G}, \Delta_n$. Hence we obtain a series of identities which we proceed to investigate.

The condition (42) yields immediately

$$J_T = 0, \quad H_T + \Lambda^\varepsilon K = 0, \tag{44}$$

According to the conditions following from (37), (41) for coefficients of $\mathbf{G}$, we obtain

$$\frac{\partial H_T}{\partial \rho^S} + \Lambda^\varepsilon\frac{\partial K}{\partial \rho^S} = 0, \quad \frac{\partial H_T}{\partial \rho^F} + \Lambda^\varepsilon\frac{\partial K}{\partial \rho^F} = 0, \quad \frac{\partial H_T}{\partial \mathbf{F}^S} + \Lambda^\varepsilon\frac{\partial K}{\partial \mathbf{F}^S} = 0.$$

Consequently, bearing (44) in mind,

$$\Lambda^\varepsilon = \Lambda^\varepsilon(T).$$

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Let us turn our attention to coefficients of $\acute{\mathbf{X}}^F$. We have

$$\frac{\partial H_V}{\partial \rho^S} - \Lambda^\varepsilon \frac{\partial Q_V}{\partial \rho^S} - \Lambda^n \frac{\partial \Phi}{\partial \rho^S} = 0, \quad \frac{\partial H_V}{\partial \rho^F} - \Lambda^\varepsilon \frac{\partial Q_V}{\partial \rho^F} - \Lambda^n \frac{\partial \Phi}{\partial \rho^F} = \Lambda^{\rho^F},$$

$$-\left[H_V - \rho^F \Lambda^{\rho^F} - \Lambda^\varepsilon Q_V - \Lambda^n \Phi \right] \mathbf{F}^{S-T} + \left(\frac{\partial H_V}{\partial \mathbf{F}^S} - \Lambda^\varepsilon \frac{\partial Q_V}{\partial \mathbf{F}^S} - \Lambda^n \frac{\partial \Phi}{\partial \mathbf{F}^S} \right) = 0.$$

Consequently

$$\frac{\partial \Phi}{\partial \rho^S} = 0, \quad \frac{\partial \Phi}{\partial \rho^F} = 0, \quad \Phi \mathbf{F}^{S-T} = \frac{\partial \Phi}{\partial \mathbf{F}^S} \quad \Longrightarrow \quad \Phi = \Phi_0 J^S, \quad \Phi_0 = const. \quad (45)$$

where (36) was used.

There remain the identities

$$\frac{\partial}{\partial \rho^S} \left(H_V - \Lambda^\varepsilon Q_V \right) = 0, \quad \frac{\partial}{\partial \rho^F} \left(H_V - \Lambda^\varepsilon Q_V \right) = \Lambda^{\rho^F},$$

$$\left[H_V - \rho^F \Lambda^{\rho^F} - \Lambda^\varepsilon Q_V \right] \mathbf{F}^{S-T} = \frac{\partial}{\partial \mathbf{F}^S} \left(H_V - \Lambda^\varepsilon Q_V \right). \quad (46)$$

The integrability condition of the first two conditions yields immediately

$$\frac{\partial \Lambda^{\rho^F}}{\partial \rho^S} = 0. \quad (47)$$

On the other hand, substitution of $(46)_2$ in $(46)_3$ leads to the equation

$$\rho^F \frac{\partial \Lambda^{\rho^F}}{\partial \rho^F} \mathbf{F}^{S-T} + \frac{\partial \Lambda^{\rho^F}}{\partial \mathbf{F}^S} = 0.$$

This equation can be easily integrated[1] and we obtain

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[1] For isotropic materials considered further in this section the dependence of Λ^{ρ^F} on $\mathbf{F}^S$ reduces to a dependence on the three invariants I, II, III of the tensor $\mathbf{C}^S$. Then

$$\frac{\partial \Lambda^{\rho^F}}{\partial \mathbf{F}^S} = \frac{\partial \Lambda^{\rho^F}}{\partial I} \frac{\partial I}{\partial \mathbf{F}^S} + \frac{\partial \Lambda^{\rho^F}}{\partial II} \frac{\partial II}{\partial \mathbf{F}^S} + \frac{\partial \Lambda^{\rho^F}}{\partial III} \frac{\partial III}{\partial \mathbf{F}^S} =$$
$$= 2 \frac{\partial \Lambda^{\rho^F}}{\partial I} \mathbf{F}^S + 2 \frac{\partial \Lambda^{\rho^F}}{\partial II} \mathbf{F}^S (I\mathbf{1} - \mathbf{C}^S) + \frac{\partial \Lambda^{\rho^F}}{\partial J^S} J^S \mathbf{F}^{S-T}.$$

Hence we obtain the equation

$$\left(\rho^F \frac{\partial \Lambda^{\rho^F}}{\partial \rho^F} + J^S \frac{\partial \Lambda^{\rho^F}}{\partial J^S} \right) \mathbf{1} + 2 \left(\frac{\partial \Lambda^{\rho^F}}{\partial I} + I \frac{\partial \Lambda^{\rho^F}}{\partial II} \right) \mathbf{C}^S - 2 \frac{\partial \Lambda^{\rho^F}}{\partial II} \mathbf{C}^{S2} = 0.$$

According to the Cayley-Hamilton theorem tensors $\left\{ \mathbf{1}, \mathbf{C}^S, \mathbf{C}^{S2} \right\}$ span the space of tensor functions of $\mathbf{C}^S$. Consequently, the coefficients in this equation should vanish separately, i.e.

$$\frac{\partial \Lambda^{\rho^F}}{\partial I} = 0, \quad \frac{\partial \Lambda^{\rho^F}}{\partial II} = 0,$$

$$\rho^F \frac{\partial \Lambda^{\rho^F}}{\partial \rho^F} + J^S \frac{\partial \Lambda^{\rho^F}}{\partial J^S} = 0.$$

Integration of this equation yields the relation (48). It can be easily shown that this is also a solution in the general case without the assumption of isotropy.

$$\Lambda^{\rho^F} = \Lambda^{\rho^F}\left(T, \rho_t^F\right), \quad \rho_t^F := \rho^F J^{S-1}. \tag{48}$$

It is convenient to introduce the following notation

$$\psi := \varepsilon - \Lambda^{\varepsilon-1}\eta. \tag{49}$$

Obviously ψ corresponds to the classical Helmholtz free energy function.

Before we proceed with the exploitation of the above results let us summarize the results for multipliers which follow from the above considerations. We have

$$\Lambda^{\rho^S} = -\Lambda^\varepsilon \frac{\partial \rho\psi}{\partial \rho^S}, \quad \Lambda^{\rho^F} = -\Lambda^\varepsilon \frac{\partial \rho\psi}{\partial \rho^F}, \tag{50}$$

$$\Lambda^n = -\Lambda^\varepsilon \frac{\partial \rho\psi}{\partial \Delta_n}, \quad \mathbf{\Lambda}^F = -\Lambda^\varepsilon \frac{\partial \rho\psi}{\partial \mathbf{F}^S}, \quad \frac{\partial \rho\psi}{\partial \mathbf{G}} = 0.$$

Consequently, the integration of (48) yields the splitting of the free energy into two constitutive parts

$$\rho\psi = \rho^S \psi^S + \rho^F \psi^F, \quad \psi^S = \psi^S\left(T, \rho^S, \mathbf{F}^S, \Delta_n\right), \quad \psi^F = \psi^F\left(T, \rho_t^F, \Delta_n\right). \tag{51}$$

This separation justifies the name *simple porous materials*. Similarly to simple mixtures of fluids, partial free energies of components depend solely on their own measures of deformation: skeleton on the deformation gradient $\mathbf{F}^S$, and fluid – on the current mass density of the fluid ρ_t^F. There is no energy of interaction between components.

We are now in the position to integrate the relations between H_V and Q_V. After intergration of $(46)_2$ we obtain

$$H_V - \Lambda^\varepsilon Q_V = -\Lambda^\varepsilon \rho^F \psi^F. \tag{52}$$

Hence the fluxes can be written in the following final form

$$\mathbf{H} = \Lambda^\varepsilon \left(\mathbf{Q} - \rho^F \psi^F \acute{\mathbf{X}}^F\right), \quad \text{i.e.} \quad \mathbf{H}^S + \mathbf{H}^F = \Lambda^\varepsilon \left(\mathbf{Q}^S + \mathbf{Q}^F\right),$$

$$\mathbf{J} = \Phi_0 J^S \acute{\mathbf{X}}^F, \tag{53}$$

$$H_T = -\Lambda^\varepsilon K, \quad H_V = \Lambda^\varepsilon Q_V - \Lambda^\varepsilon \rho^F \psi^F \acute{\mathbf{X}}^F.$$

where the relations (17) and (19) have been used.

Let us consider an impermeable boundary between a saturated porous material and a fluid which is physically identical with the fluid filling pores of the skeleton. If the temperature is continuous on this boundary and simultaneously jump conditions (20) are fulfilled we call it an *ideal wall* for the fluid component. This surface is material simultaneously with respect to the skeleton and the fluid, i.e.

$$\acute{\mathbf{X}}^F = 0 \quad \Longrightarrow \quad \left[\left[\acute{\mathbf{x}}^F\right]\right] = 0 \quad \Longrightarrow$$
$$\Longrightarrow \quad [[\mathbf{Q}]] \cdot \mathbf{N} = 0 \;\&\; [[\mathbf{H}]] \cdot \mathbf{N} = 0 \quad \Longrightarrow$$
$$\Longrightarrow \quad [[\Lambda^\varepsilon]] = 0.$$

As the multiplier in the fluid outside porous material is the reciprocal of the temperature the above result yields

$$\Lambda^\varepsilon = \frac{1}{T}, \tag{54}$$

in the porous material as well.

Finally inspection of relations (39), (40) for partial stresses shows that they satisfy the following relations

$$\mathbf{P}^F = -\rho_t^{F2} \frac{\partial \psi^F}{\partial \rho_t^F} J^S \mathbf{F}^{S-T} - \Phi_0 \frac{\partial \rho \psi}{\partial \Delta_n} J^S \mathbf{F}^{S-T},$$

$$\mathbf{P}^S = \frac{\partial \rho^S \psi^S}{\partial \mathbf{F}^S} + \Phi_0 \frac{\partial \rho \psi}{\partial \Delta_n} J^S \mathbf{F}^{S-T}. \tag{55}$$

Transformation to Cauchy stresses yields

$$\mathbf{T}^F := J^{S-1} \mathbf{P}^F \mathbf{F}^{ST} = -p^F \mathbf{1}, \quad p^F := \rho_t^{F2} \frac{\partial \psi^F}{\partial \rho_t^F} + \beta \Delta_n, \tag{56}$$

$$\mathbf{T}^S := J^{S-1} \mathbf{P}^S \mathbf{F}^{ST} = \rho_t^S \frac{\partial \psi^S}{\partial \mathbf{F}^S} \mathbf{F}^{ST} + \beta \Delta_n \mathbf{1}, \quad \rho_t^S := \rho^S J^{S-1},$$

$$\beta := \Phi_0 \frac{\partial^2 \rho \psi}{\partial \Delta_n^2}, \tag{57}$$

where we use the property that the free energy is a quadratic even function of Δ_n.

Bearing the above results in mind we analyze a jump condition on the permeable boundary of the skeleton. According to relations (20) and (21) we have

$$\left[\left[\rho^F \acute{\mathbf{X}}^F \cdot \mathbf{N} \frac{1}{2} \acute{x}^{F2} + \mathbf{Q} \cdot \mathbf{N}\right]\right] = \left[\left[\acute{\mathbf{x}}^F \cdot \mathbf{P}^F \mathbf{N}\right]\right] =$$

$$= \left[\left[\rho^F \acute{\mathbf{X}}^F \cdot \mathbf{N} \left(\frac{p^F}{\rho^F} J^S - \acute{\mathbf{x}}^F \cdot \acute{\mathbf{x}}^S\right)\right]\right].$$

Hence the relations $(20)_5$ and $(53)_1$ yield immediately

$$\left[\left[\mu^F\right]\right] = 0, \quad \mu^F := \psi^F + \frac{p^F}{\rho_t^F} + \frac{1}{2} \left(\acute{\mathbf{x}}^F - \acute{\mathbf{x}}^S\right) \cdot \left(\acute{\mathbf{x}}^F - \acute{\mathbf{x}}^S\right). \tag{58}$$

The quantity μ^F is the *chemical potential* of the fluid component. Its continuity on the permeable boundary replaces the mechanical condition on continuity of partial pressures on impermeable boundaries.

It remains to write the residual inequality (34) in which we account for the above results. We obtain

$$\mathcal{D} = \frac{1}{T} K G^2 - \left[\frac{Q_V}{T} + T \frac{\partial}{\partial T} \left(\frac{\rho^F \psi^F}{T}\right) - \Pi_T\right] \acute{\mathbf{X}}^F \cdot \mathbf{G} + \Pi_V \acute{\mathbf{X}}^F \cdot \acute{\mathbf{X}}^F + \frac{\beta}{\Phi_0 \tau} \Delta_n^2 \geq 0. \tag{59}$$

It is now obvious that the simplifying assumptions which we have made in this section amount indeed to a quadratic form of the dissipation $\mathcal{D}$, i.e. to small deviations of processes from the thermodynamic equilibrium in which $\acute{\mathbf{X}}^F\big|_E = 0$, $\mathbf{G}\big|_E = 0$, $\Delta_n\big|_E = 0$. Consequently, material parameters should satisfy the following conditions

$$K > 0, \quad \Pi_V > 0, \quad K\Pi_V + \left[\frac{Q_V}{T} + T\frac{\partial}{\partial T}\left(\frac{\rho^F \psi^F}{T}\right) - \Pi_T\right]^2 > 0, \quad (60)$$

$$\frac{\beta}{\Phi_0 \tau} > 0.$$

Further restrictions on material parameters follow from the stability analysis of thermodynamic equilibrium. We do not consider this problem in the present work.

5 Particular cases

In this section we demonstrate a few examples of simplified models of a thermoporoelastic material which have an important bearing in applications. We begin with the exploitation of the isotropy assumption with respect to deformations of the skeleton. Obviously, contributions of the fluid are already isotropic.

We use the right and left Cauchy-Green deformation tensors

$$\mathbf{C}^S := \mathbf{F}^{ST}\mathbf{F}^S, \quad \mathbf{B}^S := \mathbf{F}^S\mathbf{F}^{ST},$$

$$I = \operatorname{tr}\mathbf{C}^S \equiv \operatorname{tr}\mathbf{B}^S, \quad II = \frac{1}{2}\left(I^2 - \operatorname{tr}\mathbf{C}^{S2}\right) \equiv \frac{1}{2}\left(I^2 - \operatorname{tr}\mathbf{B}^{S2}\right), \quad (61)$$

$$III \equiv J^{S2} = \det\mathbf{C}^S \equiv \det\mathbf{B}^S,$$

where I, II, III are the main invariants common for both deformation tensors.

According to the polar decomposition theorem we have

$$\mathbf{F}^S = \mathbf{R}^S\sqrt{\mathbf{C}^S}, \quad \mathbf{R}^{ST} = \mathbf{R}^{S-1}. \quad (62)$$

Under the assumption of material objectivity the free energy function ψ^S is independent of rotations $\mathbf{R}^S$. Consequently, if we drop a trivial dependence on ρ^S, we have

$$\psi^S = \psi^S\left(T, \mathbf{C}^S, \Delta_n\right) = \psi^S\left(T, I, II, III, \Delta_n\right), \quad (63)$$

where the second part of the relation follows from the assumption on isotropy.

Bearing this relation in mind we obtain from (56) the following relation for the partial Cauchy stress in skeleton

$$\mathbf{T}^S = \beth_{-1}\mathbf{B}^{S-1} + \beth_0\mathbf{1} + \beth_1\mathbf{B}^S + \beta\Delta_n\mathbf{1}, \quad (64)$$

where

$$\beth_{-1} := -2\rho_t^S III\left.\frac{\partial\psi^S}{\partial II}\right|_{\Delta_n=0}\mathbf{B}^{S-1}, \quad \beth_0 := 2\rho_t^S\left(II\frac{\partial\psi^S}{\partial II} + III\frac{\partial\psi^S}{\partial III}\right)\Bigg|_{\Delta_n=0},$$

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$$\beth_1 := 2\rho_t^S \left. \frac{\partial \psi^S}{\partial I} \right|_{\Delta_n = 0}, \tag{65}$$

and the Cayley-Hamilton theorem

$$\mathbf{B}^{S3} - I\mathbf{B}^{S2} + II\mathbf{B}^S - III\mathbf{1} = 0, \tag{66}$$

has been used.

Problems with the practical determination of the elasticities $\beth_{-1}, \beth_0, \beth_1$ for poroelastic materials yields the necessity of a further simplification. In the classical elasticity theory a quadratic isotropic model has been proposed by Signorini. The constitutive relation for this model follows from the above model by the truncation on the second term in expansion around the point $(T = T_0, \mathbf{F}^S = \mathbf{1})$ (i.e. $I = 3, II = 3, III = 1$) and it has the form

$$\mathbf{T}^S = \mathbf{T}_0^S + \left[\lambda^S I_e + c^S II_e + \frac{1}{2} \left(\lambda^S + \mu^S - \frac{1}{2} c^S \right) I_e^2 \right] \mathbf{1} +$$

$$+ 2 \left[\mu^S - \left(\lambda^S + \mu^S + \frac{1}{2} c^S \right) I_e \right] \mathbf{e}^S + 2 c^S \mathbf{e}^{S2} - \alpha_T^S \frac{T - T_0}{T_0} \mathbf{1} + \beta \Delta_n \mathbf{1}, \tag{67}$$

where

$$\mathbf{e}^S := \frac{1}{2} \left(\mathbf{1} - \mathbf{B}^{S-1} \right), \quad I_e := \operatorname{tr} \mathbf{e}^S, \quad II_e := \frac{1}{2} \left(I_e^2 - \operatorname{tr} \mathbf{e}^{S2} \right), \tag{68}$$

$\mathbf{e}^S$ is the Almansi-Hamel deformation tensor, α_T^S is the thermal expansion coefficient of skeleton and it may be linearly dependent on I_e, while material parameters λ^S, μ^S, c^S depend solely on the reference temperature T_0.

Finally the classical linear model for small deformations

$$\left\| \mathbf{e}^S \right\| \ll 1, \quad \left\| \mathbf{e}^S \right\| := \min \left(\{ \lambda_1, \lambda_2, \lambda_3 \} \right),$$

$$|\epsilon| \ll 1, \quad \epsilon := \frac{\rho_0^F - \rho_t^F}{\rho_0^F},$$

$$\det \left(\mathbf{e}^S - \lambda_i \mathbf{1} \right) = 0, \quad i = 1, 2, 3,$$

follows from (67) in the following form

$$\mathbf{T}^S = \mathbf{T}_0^S + \lambda^S I \mathbf{1} + 2\mu^S \mathbf{e}^S - \alpha_T^S \frac{T - T_0}{T_0} \mathbf{1} + \beta \Delta_n \mathbf{1}, \tag{69}$$

$$\mathbf{T}^F = \mathbf{T}_0^F - \kappa \rho_0^F \epsilon \mathbf{1} - \alpha_T^F \frac{T - T_0}{T_0} \mathbf{1} - \beta \Delta_n \mathbf{1},$$

where κ is the compressibility coefficient of the fluid, and α_T^S, α_T^F denote constant thermal expansion coefficients of skeleton and fluid, respectively.

The last model corresponds to the classical Biot's model but it does not contain the Biot's coupling term.

6   Concluding remarks

We have shown that the assumption on a quadratic form of the dissipation yields a quite explicit form of constitutive relations for thermo-poroelastic materials. Their mechanical part does not differ from this derived earlier for isothermal processes. Relations for energy and entropy fluxes justify the assumption made in the earlier papers on the proportionality of their intrinsic parts (comp. (53)) which was basic for the formulation of the second law of thermodynamics for isothermal processes. In addition we have shown that, similarly to classical miscible mixtures, a chemical potential for the fluid component is continuous on the permeable boundary of the porous body. This property is fundamental for the formulation of boundary conditions on such a boundary.

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