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Conditional Log-Laplace Functionals of Immigration Superprocesses with Dependent Spatial Motion

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Abstract

A non-critical branching immigration superprocess with dependent spatial motion is constructed and characterized as the solution of a stochastic equation driven by a time-space white noise and an orthogonal martingale measure. A representation of its conditional log-Laplace functionals is established, which gives the uniqueness of the solution and hence its Markov property. Some properties of the superprocess including an ergodic theorem are also obtained.

1 Introduction

A class of superprocesses with dependent spatial motion (SDSM) over the real line $\mathbb{R}$ were introduced and constructed in Wang (1997, 1998). A generalization of the model was then given in Dawson *et al* (2001). Let $c \in C_b^2(\mathbb{R})$ and $h \in C_b^2(\mathbb{R})$ and assume both h and h' are square-integrable. Let

$$\rho(x) = \int_{\mathbb{R}} h(y-x)h(y)dy, \quad x \in \mathbb{R},$$

and $a(x) = c(x)^2 + \rho(0)$. Let $\sigma \in C_b^2(\mathbb{R})^+$. We denote by $M(\mathbb{R})$ the space of finite Borel measures on $\mathbb{R}$ endowed with a metric compatible with its topology of weak convergence. For $f \in C_b(\mathbb{R})$ and $\mu \in M(\mathbb{R})$ set $\langle f, \mu \rangle = \int f d\mu$. Then an SDSM $\{X_t : t \geq 0\}$ is characterized by the following martingale problem: For each $\phi \in C_b^2(\mathbb{R})$,

$$M_t(\phi) = \langle \phi, X_t \rangle - \langle \phi, X_0 \rangle - \frac{1}{2}\int_0^t \langle a\phi'', X_s \rangle ds, \quad t \geq 0, \tag{1.1}$$

is a continuous martingale with quadratic variation process

$$\langle M(\phi) \rangle_t = \int_0^t \langle \sigma\phi^2, X_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z-\cdot)\phi', X_s \rangle^2 dz. \tag{1.2}$$

Clearly, the SDSM reduces to a usual critical branching Dawson-Watanabe superprocess if $h(\cdot) \equiv 0$; see e.g. Dawson (1993). A general SDSM arises as the weak limit of critical branching particle systems with dependent spatial motion. Consider a family of independent Brownian motions $\{B_i(t) : t \geq 0, i = 1, 2, \cdots\}$, the individual noises, and a time-space white noise $\{W_t(B) : t \geq 0, B \in \mathcal{B}(\mathbb{R})\}$, the common noise. The migration of a particle in the approximating system with label i is defined by the stochastic equation

$$dx_i(t) = c(x_i(t))dB_i(t) + \int_{\mathbb{R}} h(y - x_i(t))W(dt, dy), \tag{1.3}$$

where $W(ds, dy)$ denotes the time-space stochastic integral relative to $\{W_t(B)\}$. When $c(\cdot) \equiv 0$, the SDSM lives in the space of purely atomic measures; see Dawson and Li (2003), Li *et al* (2004) and Wang (1997, 2002).

In this paper, we consider a further extension of the model of Dawson *et al* (2001) and Wang (1997, 1998). Let $b \in C_b^2(\mathbb{R})$ and let $m \in M(\mathbb{R})$. A modification of the above martingale problem is to replace (1.1) by

$$M_t(\phi) = \langle \phi, X_t \rangle - \langle \phi, X_0 \rangle - t\langle \phi, m \rangle - \frac{1}{2}\int_0^t \langle a\phi'', X_s \rangle ds + \int_0^t \langle b\phi, X_s \rangle ds. \tag{1.4}$$

We shall prove that the martingale problem given by (1.2) and (1.4) really has a solution $\{X_t : t \geq 0\}$. The process may be regarded as a non-critical branching *SDSM with immigration* (SDSMI), where $b(\cdot)$ is the linear growth rate and $m(dx)$ gives the immigration rate. This modification is closely related to the recent work of Dawson and Li (2003), where an interactive immigration given by

$$\int_0^t \langle q(\cdot, X_s)\phi, m \rangle ds$$

was considered, where $q(\cdot,\cdot)$ is a function on $\mathbb{R} \times M(\mathbb{R})$ giving a state dependent immigration density. However, Dawson and Li (2003) assumed $b(\cdot) \equiv c(\cdot) \equiv 0$ and used essentially the purely atomic property of the process, which is not available for the present model.

The main purpose of this paper is to give a representation of the conditional log-Laplace functionals of solution of (1.2) and (1.4) and to illustrate some applications of the representation. This work was stimulated by Xiong (2003), who established a similar characterization for the model of Skoulakis and Adler (2001). The key idea of the representation is to decompose the martingale (1.4) into two orthogonal components, which arise respectively from the migration and the branching. Since the decomposition uses additional information which is not provided by (1.2) and (1.4), we shall start with the corresponding particle system and consider the high density limit following Dawson *et al* (2000). In this way, we can easily separate the two kinds of noises. It turns out that the common migration noise $\{W(ds,dy)\}$ remains after the limit procedure and the limit process satisfies the following martingale problem: For each $\phi \in C_b^2(\mathbb{R})$,

$$\begin{aligned} Z_t(\phi) &= \langle \phi, X_t \rangle - \langle \phi, X_0 \rangle - t\langle \phi, m \rangle - \frac{1}{2}\int_0^t \langle a\phi'', X_s \rangle ds \\ &\quad + \int_0^t \langle b\phi, X_s \rangle ds - \int_0^t \int_{\mathbb{R}} \langle h(y-\cdot)\phi', X_s \rangle W(ds,dy) \end{aligned} \tag{1.5}$$

is a continuous martingale orthogonal to $\{W_t(\phi)\}$ with quadratic variation process

$$\langle Z(\phi) \rangle_t = \int_0^t \langle \sigma\phi^2, X_s \rangle ds. \tag{1.6}$$

This formulation suggests that we may regard $\{X_t : t \geq 0\}$ as a generalized inhomogeneous Dawson-Watanabe superprocess with immigration, where

$$\int_{\mathbb{R}} h(y-\cdot) W(dt,dy)$$

gives a generalized drift in the underlying migration. Based on the techniques developed in Kurtz and Xiong (1999) and Xiong (2003), we prove that for each $\phi \in W_2^1(\mathbb{R}) \cap C_b(\mathbb{R})$ there is a unique strong solution of the non-linear SPDE

$$\begin{aligned} \psi_{r,t}(x) &= \phi(x) + \int_r^t \left[\frac{1}{2}a(x)\psi''_{s,t}(x) - \frac{1}{2}\sigma(x)\psi_{s,t}(x)^2\right] ds \\ &\quad - \int_r^t b(x)\psi_{s,t}(x) ds + \int_r^t \int_{\mathbb{R}} h(y-x)\psi'_{s,t}(x) \cdot W(ds,dy), \end{aligned} \tag{1.7}$$

where the last term on the right hand side denotes the backward stochastic integral relative to the white noise. Then we show that the conditional log-Laplace functionals of $\{X_t : t \geq 0\}$ given $\{W(ds, dy)\}$ can be represented by the solution of (1.7). Since the parameters in (1.7) do not meet the requirements of the results of Kurtz and Xiong (1999) and Xiong (2003), the investigation of (1.7) itself is of interest from the point of view of non-linear SPDE's. The representation of the conditional log-Laplace functionals is proved by direct analysis based on (1.5), (1.6) and (1.7). This approach is different from that of Xiong (2003), where a Wong-Zakai type approximation was used. The idea of conditional log-Laplace approach has also used by Crisan (2002) for another different model. In fact, the approach in Section 5 is adapted from Crisan (2002) which simplifies our original arguments. It is well-known that non-conditional log-Laplace functionals play very important roles in the study of classical Dawson-Watanabe superprocesses. We shall see that conditional Laplace functionals are almost as efficient as the non-conditional Laplace functionals in studying some properties of the SDSMI. In particular, the characterization of the conditional Laplace functionals gives immediately the uniqueness of solution of (1.5) and (1.6), which in turn implies the Markov property of $\{X_t : t \geq 0\}$. We also prove some properties of the SDSMI including an ergodic theorem, which show the potential of other applications of the conditional log-Laplace functionals.

The remainder of the paper is organized as follows. In Section 2 we give a formulation of the system of branching particle with dependent spatial motion and immigration. Some useful estimates of the moments of the system are also given. In Section 3 we obtain a solution of the martingale problem (1.5) and (1.6) as the high density limit of a sequence of particle systems. The existence and uniqueness of the strong solution of (1.7) is established in Section 4. In Section 5 we give the representation of the conditional log-Laplace functionals of the solution of (1.5) and (1.6). Some properties of the SDSMI are discussed in Section 6.

2 Branching particle systems

The main purpose of this section is to give an explicit construction for the immigration branching particle system with dependent spatial motion by modifying the constructions of Dawson *et al* (2000) and Walsh (1986). This construction set up the process in a useful form.

We start with a simple interacting particle system. Let $\theta > 0$ be a constant and (c, h) be given as in the introduction. Let $N(\mathbb{R}) \subset M(\mathbb{R})$ be the set of integer-valued measures on $\mathbb{R}$ and let $M_\theta(\mathbb{R}) := \{\theta^{-1}\sigma : \sigma \in N(\mathbb{R})\}$. Given $\{a_i : i = 1, \cdots, n\}$, let $\{x_i(t) : t \geq 0, i = 1, \cdots, n\}$ be given by

$$x_i(t) = a_i + \int_0^t c(x_i(s))dB_i(s) + \int_0^t \int_{\mathbb{R}} h(y - x_i(s))W(dy, ds). \tag{2.1}$$

We may define a measure-valued process $\{X_t : t \geq 0\}$ by

$$\langle \phi, X_t \rangle = \sum_{i=1}^n \theta^{-1}\phi(x_i(t)), \qquad t \geq 0. \tag{2.2}$$

By the discussions in Dawson *et al* (2001) and Wang (1997, 1998), $\{X_t : t \geq 0\}$ is a diffusion process in $M_\theta(\mathbb{R})$ with generator $\mathcal{A}_\theta$ given by

$$\mathcal{A}_\theta F(\mu) \quad = \quad \frac{1}{2}\int_{\mathbb{R}^2} \rho(x - y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy)$$

$$
+\frac{1}{2}\int_{\mathbb{R}} a(x)\frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx)
$$

$$
+\frac{1}{2\theta}\int_{\mathbb{R}^2} c(x)c(y)\frac{d^2}{dxdy}\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\delta_x(dy)\mu(dx), \tag{2.3}
$$

where

$$
\frac{\delta F(\mu)}{\delta\mu(x)} = \lim_{r\to 0} r^{-1}[F(\mu + r\delta_x) - F(\mu)]
$$

and $\delta^2 F(\mu)/\delta\mu(x)\delta\mu(y)$ is defined by the above limit with $F(\mu)$ replaced by $\delta F(\mu)/\delta\mu(y)$. In particular, if $F_{f,\{\phi_i\}}(\mu) := f(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)$ with $f \in C_b^2(\mathbb{R}^n)$ and $\{\phi_i\} \subset C_b^2(\mathbb{R})$, then

$$
\begin{aligned}
\mathcal{A}_\theta F_{f,\{\phi_i\}}(\mu) \;=\; & \frac{1}{2}\sum_{i,j=1}^{n} f_{ij}''(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\int_{\mathbb{R}^2}\rho(x-y)\phi_i'(x)\phi_j'(y)\mu(dx)\mu(dy) \\
& +\frac{1}{2}\sum_{i=1}^{n} f_i'(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\langle a\phi_i'',\mu\rangle \\
& +\frac{1}{2\theta}\sum_{i,j=1}^{n} f_{ij}''(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\langle c^2\phi_i'\phi_j',\mu\rangle,
\end{aligned} \tag{2.4}
$$

is well-defined.

A more interesting particle system involves branching and immigration. Let $\gamma > 0$ be a constant and let $m \in M(\mathbb{R})$. Let $p(x,\cdot) = \{p_0(x), p_1(x), p_2(x), \cdots\}$ be a family of discrete probability distributions which measurably depends on the index $x \in \mathbb{R}$ and satisfies $p_1(\cdot) \equiv 0$. Suppose that

$$
q(x) := \sum_{i=1}^{\infty} i p_i(x), \quad x \in \mathbb{R}, \tag{2.5}
$$

is a bounded function. We shall construct a immigration branching particle system with parameters $(a,\rho,\gamma,p,\theta m,1/\theta)$. Let $\mathcal{A}$ be the set of all strings of the form $\alpha = n_0 n_1 \cdots n_{l(\alpha)}$, where $l(\alpha)$ is the length of α and the n_j are non-negative integers with $0 \le n_0 \le 1$ and $n_j \ge 1$ for $j \ge 1$. We provide $\mathcal{A}$ with the arboreal ordering, that is, $m_0\cdots m_p \prec n_0\cdots n_q$ if and only if $p \le q$ and $m_0 = n_0, \cdots, m_p = n_p$. Then α has exactly $l(\alpha)$ predecessors, which we denote respectively by $\alpha - 1$, $\alpha - 2$, $\cdots$, $\alpha - l(\alpha)$. For example, if $\alpha = 12431$, then $\alpha - 2 = 124$ and $\alpha - 4 = 1$.

Consider a sequence of random variables $\{a_{01},\cdots,a_{0n}\} \subset \mathbb{R}$. Let $\{W(ds,dx) : s \ge 0, x \in \mathbb{R}\}$ be a time-space white noise and $\{N(ds,dx) : s \ge 0, x \in \mathbb{R}\}$ a Poisson random measure with intensity $\theta ds m(dx)$. We shall assume $\langle 1, m\rangle > 0$, otherwise the construction of the immigration part is trivial. In this case, we can enumerate the atoms of $N(ds,dx)$ as

$$
\{(s_i, a_{1i}) : 0 < s_1 < s_2 < \cdots, a_{1i} \in \mathbb{R}\}. \tag{2.6}
$$

We also define the families

$$
\{B_\alpha(t) : t \ge 0, \alpha \in \mathcal{A}\}, \quad \{S_\alpha : \alpha \in \mathcal{A}\}, \quad \{\eta_{a,\alpha} : a \in \mathbb{R}, \alpha \in \mathcal{A}\}, \tag{2.7}
$$

where $\{B_\alpha\}$ are independent standard Brownian motions, $\{S_\alpha\}$ are i.i.d. exponential random variables with parameter γ, and $\{\eta_{a,\alpha}\}$ are independent random variables with distribution $p(a,\cdot)$. We assume that the families $\{W(ds,dx)\}$, $\{N(ds,dx)\}$, $\{a_{0i}\}$, $\{B_\alpha\}$, $\{S_\alpha\}$ and $\{\eta_{a,\alpha}\}$ are independent.

We define $\beta_{0n_1} = 0$ if $1 \leq n_1 \leq n$ and $\beta_{0n_1} = \infty$ if $n_1 > n$, and define $\beta_{1n_1} = s_{n_1}$ for all $n_1 \geq 1$. For $\alpha \in \mathcal{A}$ with $l(\alpha) = 1$ we let $\zeta_\alpha = \beta_\alpha + S_\alpha$. Heuristically, S_α is the life length of the particle with label α, β_α is its birth time and ζ_α is its death time. The random variables a_α defined above can be interpreted as the birth place of the particle with label α. The trajectory $\{x_\alpha(t) : t \geq \beta_\alpha\}$ of the particle is the solution of the equation

$$x(\beta_\alpha + t) = a_\alpha + \int_{\beta_\alpha}^{\beta_\alpha + t} c(x(s)) dB_\alpha(s) + \int_{\beta_\alpha}^{\beta_\alpha + t} \int_{\mathbb{R}} h(y - x(s)) W(ds, dy). \tag{2.8}$$

For $\alpha \in \mathcal{A}$ with $l(\alpha) > 1$ the trajectory $\{x_\alpha(t) : t \geq \beta_\alpha\}$ is defined by the above equation with $a_\alpha = x_{\alpha-1}(\zeta_{\alpha-1}^-)$, $\zeta_\alpha = \beta_\alpha + S_\alpha$ and

$$\beta_\alpha = \begin{cases} \zeta_{\alpha-1} & \text{if } n_{l(\alpha)} \leq \eta_{x_{\alpha-1}(\zeta_{\alpha-1}-),\alpha-1} \\ \infty & \text{if } n_{l(\alpha)} > \eta_{x_{\alpha-1}(\zeta_{\alpha-1}-),\alpha-1}, \end{cases} \tag{2.9}$$

where $x_{\alpha-1}(\zeta_{\alpha-1}-)$ denotes the left limit of $x_{\alpha-1}(t)$ at $t = \zeta_{\alpha-1}$. Then

$$\langle \phi, Y_t \rangle = \sum_{\alpha \in \mathcal{A}} \theta^{-1} \phi(x_\alpha(t)) 1_{[\beta_\alpha, \zeta_\alpha)}(t), \qquad t \geq 0, \tag{2.10}$$

defines a process $\{Y_t : t \geq 0\}$ in $M_\theta(\mathbb{R})$. This process has countably many jumps, and between those jumps it behaves just as the diffusion process $\{X_t : t \geq 0\}$ constructed by (2.2). The jumps of $\{Y_t : t \geq 0\}$ corresponds to the generator

$$\begin{aligned} \mathcal{B}_\theta F(\mu) &= \sum_{j=0}^{\infty} \int_{\mathbb{R}} \theta \gamma p_j(x) [F(\mu + (j-1)\theta^{-1}\delta_x) - F(\mu)] \mu(dx) \\ &\quad + \int_{\mathbb{R}} \theta [F(\mu + \theta^{-1}\delta_x) - F(\mu)] m(dx). \end{aligned} \tag{2.11}$$

Note that

$$\begin{aligned} \mathcal{B}_\theta F_{f,\{\phi_i\}}(\mu) &= \sum_{j=0}^{\infty} \int_{\mathbb{R}} \theta \gamma p_j(x) [f(\langle \phi_1, \mu \rangle + \theta^{-1}\phi_1(x), \cdots, \langle \phi_n, \mu \rangle + \theta^{-1}\phi_n(x)) \\ &\qquad - f(\langle \phi_1, \mu \rangle, \cdots, \langle \phi_n, \mu \rangle)] \mu(dx) \\ &\quad + \int_{\mathbb{R}} \theta [f(\langle \phi_1, \mu \rangle + \theta^{-1}\phi_1(x), \cdots, \langle \phi_n, \mu \rangle + \theta^{-1}\phi_n(x)) \\ &\qquad - f(\langle \phi_1, \mu \rangle, \cdots, \langle \phi_n, \mu \rangle)] m(dx). \end{aligned} \tag{2.12}$$

Indeed, we may regard $\{Y_t : t \geq 0\}$ as a concatenation of a sequence of independent copies of $\{X_t : t \geq 0\}$. See e.g. Sharpe (1988) for discussions of concatenation of general Markov processes. This analysis shows that $\{Y_t : t \geq 0\}$ is a Markov process with generator $\mathcal{L}_\theta := \mathcal{A}_\theta + \mathcal{B}_\theta$. We call the process an *immigration branching particle system* with parameters $(c, h, \gamma, p, \theta m, 1/\theta)$. Let $\mathcal{D}_1(\mathcal{L}_\theta)$ denote the collection of all functions $F_{f,\{\phi_i\}}$ with $f \in C_0^2(\mathbb{R}^n)$ and $\{\phi_i\} \subset C_b^2(\mathbb{R})$. Then we have

Theorem 2.1 *The process $\{Y_t : t \geq 0\}$ constructed by (2.10) solves the $(\mathcal{L}_\theta, \mathcal{D}_1(\mathcal{L}_\theta))$-martingale problem.*

Let us give another useful formulation of the immigration particle system. From (2.8), (2.10) and Itô's formula we get

$$\langle \phi, Y_t \rangle = \langle \phi, Y_0 \rangle + \sum_{i=1}^{\infty} \theta^{-1} \phi(a_{1i}) 1_{(0,t]}(s_i)$$

$$+\sum_{\alpha\in\mathcal{A}}[\eta_{x_\alpha(\zeta_\alpha-),\alpha}-1]\theta^{-1}\phi(x_\alpha(\zeta_\alpha-))1_{(0,t]}(\zeta_\alpha)$$
$$+\sum_{\alpha\in\mathcal{A}}\int_0^t\theta^{-1}\phi'(x_\alpha(s))1_{[\beta_\alpha,\zeta_\alpha)}(s)c(x_\alpha(s))dB_\alpha(s)$$
$$+\sum_{\alpha\in\mathcal{A}}\int_0^t\int_{\mathbb{R}}\theta^{-1}\phi'(x_\alpha(s))1_{[\beta_\alpha,\zeta_\alpha)}(s)h(y-x_\alpha(s))W(ds,dy)$$
$$+\frac{1}{2}\sum_{\alpha\in\mathcal{A}}\int_0^t\theta^{-1}\phi''(x_\alpha(s))1_{[\beta_\alpha,\zeta_\alpha)}(s)a(x_\alpha(s))ds,$$

which can be rewritten as

$$\begin{aligned}\langle\phi,Y_t\rangle &= \langle\phi,Y_0\rangle+\int_{(0,t]}\int_{\mathbb{R}}\theta^{-1}\phi(x)N(ds,dx)\\ &\quad+\sum_{\alpha\in\mathcal{A}}[\eta_{x_\alpha(\zeta_\alpha-),\alpha}-1]\theta^{-1}\phi(x_\alpha(\zeta_\alpha-))1_{(0,t]}(\zeta_\alpha)\\ &\quad+\sum_{\alpha\in\mathcal{A}}\int_0^t\theta^{-1}\phi'(x_\alpha(s))1_{[\beta_\alpha,\zeta_\alpha)}(s)c(x_\alpha(s))dB_\alpha(s)\\ &\quad+\int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\phi',Y_s\rangle W(ds,dy)+\frac{1}{2}\int_0^t\langle a\phi'',Y_s\rangle ds. \end{aligned}\tag{2.13}$$

It is not hard to see that, for any $\psi\in C_b(\mathbb{R})$,

$$U_t(\psi):=\sum_{\alpha\in\mathcal{A}}\int_0^t\theta^{-1}\psi(x_\alpha(s))1_{[\beta_\alpha,\zeta_\alpha)}(s)c(x_\alpha(s))dB_\alpha(s)\tag{2.14}$$

is a continuous local martingale with quadratic variation process

$$\langle U(\psi)\rangle_t:=\int_0^t\langle\theta^{-1}c^2\psi^2,Y_s\rangle ds.\tag{2.15}$$

In the sequel, we assume

$$\sigma(x)=\sum_{i=0}^\infty p_i(x)(i-1)^2,\qquad x\in\mathbb{R},\tag{2.16}$$

is a bounded function on $\mathbb{R}$.

Proposition 2.1 *For any $\phi\in C_b(\mathbb{R})$,*

$$Z_t(\phi):=\sum_{\alpha\in\mathcal{A}}[\eta_{x_\alpha(\zeta_\alpha-),\alpha}-1]\theta^{-1}\phi(x_\alpha(\zeta_\alpha-))1_{(0,t]}(\zeta_\alpha)-\int_0^t\langle\gamma(q-1)\phi,Y_s\rangle ds\tag{2.17}$$

is a local martingale with predictable quadratic variation process

$$\langle Z(\phi)\rangle_t=\int_0^t\langle\theta^{-1}\gamma\sigma\phi^2,Y_s\rangle ds.\tag{2.18}$$

Proof. Recall that $\{S_\alpha\}$ are i.i.d. exponential random variables with parameter γ. Let

$$J_t(\phi)=\sum_{\alpha\in\mathcal{A}}\theta^{-1}[\eta_{x_\alpha(\zeta_\alpha-),\alpha}-1]\phi(x_\alpha(\zeta_\alpha-))1_{(0,t]}(\zeta_\alpha).\tag{2.19}$$

Observe that the process $\{J_t(\phi) : t \geq 0\}$ jumps only if a particle in the population splits. It is not hard to show that $\{(Y_t, J_t(\phi)) : t \geq 0\}$ is a Markov process with generator $\mathcal{J}_\theta$ such that

$$\begin{aligned}\mathcal{J}_\theta F(\mu, u) &= \mathcal{A}_\theta F(\cdot, u)(\mu) + \int_{\mathbb{R}} \theta[F(\mu + \theta^{-1}\delta_x, u) - F(\mu, u)]m(dx) \\ &\quad + \sum_{j=0}^{\infty} \int_{\mathbb{R}} \theta\gamma p_j(x)[F(\mu + (j-1)\theta^{-1}\delta_x, u + (j-1)\theta^{-1}\phi(x)) - F(\mu, u)]\mu(dx).\end{aligned}$$

In particular, if $F(\mu, u) = u$, then

$$\mathcal{J}_\theta F(\mu, \nu) = \sum_{j=0}^{\infty} \int_{\mathbb{R}} \gamma p_j(x)(j-1)\phi(x)\mu(dx) = \langle \gamma(q-1)\phi, \mu\rangle.$$

This shows that (2.17) is a local martingale. For $l > 0$ let $\tau_l = \inf\{s \geq 0 : \langle 1, Y_s\rangle \geq l\}$. Let $\Delta_n := \{0 = t_{n,0} < t_{n,1} < \cdots < t_{n,n} = t\}$ be a sequence of partitions of $[0, t]$ such that $\|\Delta_n\| := \max_{1 \leq i \leq n} |t_{n,i} - t_{n,i-1}| \to 0$ as $n \to \infty$. Observe that

$$\begin{aligned}&\sum_{i=0}^{n} \left(\int_{t_{n,i-1}\wedge\tau_l}^{t_{n,i}\wedge\tau_l} |\langle \gamma(q-1)\phi, X_s\rangle| ds \right)^2 \\ &\leq \gamma l \|q-1\| \|\Delta_n\| \int_0^{t\wedge\tau_l} |\langle \gamma(q-1)\phi, X_s\rangle| ds.\end{aligned}$$

The right hand side goes to zero a.s. as $n \to \infty$. Let $\mathcal{A}_{t,l} = \{\alpha \in \mathcal{A} : 0 < \zeta_\alpha \leq t \wedge \tau_l\}$ and for $\alpha \in \mathcal{A}_{t,l}$ let $(r_{n,\alpha}, t_{n,\alpha}]$ be the unique interval in $\{(t_{n,i-1} \wedge \tau_l, t_{n,i} \wedge \tau_l] : i = 1, \cdots, n\}$ containing the jump time ζ_α of (2.19). Then we have

$$\begin{aligned}&\sum_{i=0}^{n} |J_{t_{n,i}\wedge\tau_l}(\phi) - J_{t_{n,i-1}\wedge\tau_l}(\phi)| \int_{t_{n,i-1}\wedge\tau_l}^{t_{n,i}\wedge\tau_l} |\langle \gamma(q-1)\phi, Y_s\rangle| ds \\ &\leq \sum_{\alpha \in \mathcal{A}_{t,l}} \theta^{-1} |\eta_{x_\alpha(\zeta_\alpha-),\alpha} - 1| |\phi(x_\alpha(\zeta_\alpha-))| \int_{r_{n,\alpha}}^{t_{n,\alpha}} |\langle \gamma(q-1)\phi, Y_s\rangle| ds \\ &\leq \theta^{-1}\gamma l \|q-1\| \|\Delta_n\| \sum_{\alpha \in \mathcal{A}_{t,l}} |\eta_{x_\alpha(\zeta_\alpha-),\alpha} - 1| |\phi(x_\alpha(\zeta_\alpha-))|.\end{aligned}$$

Under the assumptions, $\mathcal{A}_{t,l}$ is a.s. a finite set so the right hand side goes to zero a.s. as $n \to \infty$. It follows that

$$\begin{aligned}[Z(\phi)]_{t\wedge\tau_l} &:= \lim_{n\to\infty} \sum_{i=0}^{n} |Z_{t_{n,i}\wedge\tau_l}(\phi) - Z_{t_{i-1}\wedge\tau_l}(\phi)|^2 \\ &= \lim_{n\to\infty} \sum_{i=0}^{n} |J_{t_{n,i}\wedge\tau_l}(\phi) - J_{t_{i-1}\wedge\tau_l}(\phi)|^2 \\ &= \sum_{\alpha \in \mathcal{A}} \theta^{-2} [\eta_{x_\alpha(\zeta_\alpha-),\alpha} - 1]^2 \phi(x_\alpha(\zeta_\alpha-))^2 1_{(0,t\wedge\tau_l]}(\zeta_\alpha).\end{aligned}$$

By martingale theory, $Z_{t\wedge\tau_l}(\phi)^2 - [Z(\phi)]_{t\wedge\tau_l}$ is a martingale. Note that $[Z(\phi)]_{t\wedge\tau_l}$ has same jump times as $J_{t\wedge\tau_l}(\phi)$ with squared jump sizes. By an argument similar to the beginning of this proof, we conclude that $[Z(\phi)]_{t\wedge\tau_l} - \langle Z(\phi)\rangle_{t\wedge\tau_l}$ is a martingale. Then $\langle Z(\phi)\rangle_{t\wedge\tau_l}$ is a predictable process such that $Z_{t\wedge\tau_l}(\phi)^2 - \langle Z(\phi)\rangle_{t\wedge\tau_l}$ is a martingale, implying the desired result. ∎

Let $\tilde{N}(ds,dx) = N(ds,dx) - \theta ds m(dx)$. Note that the assumptions on independence imply that the four martingale measures $\{W(ds,dx)\}$, $\{\tilde{N}(ds,dx)\}$, $\{Z(ds,dx)\}$ are $\{U(ds,dx)\}$ are orthogonal to each other. Now we may rewrite (2.13) into

$$\begin{aligned}\langle \phi, Y_t\rangle &= \langle \phi, Y_0\rangle + t\langle \phi, m\rangle + \int_{(0,t]}\int_{\mathbb{R}} \theta^{-1}\phi(x)\tilde{N}(ds,dx) \\ &\quad + \int_0^t \langle \gamma(q-1)\phi, Y_s\rangle ds + Z_t(\phi) + U_t(\phi') \\ &\quad + \int_0^t \int_{\mathbb{R}} \langle h(y-\cdot)\phi', Y_s\rangle W(ds,dy) + \frac{1}{2}\int_0^t \langle a\phi'', Y_s\rangle ds. \end{aligned} \tag{2.20}$$

Clearly, the third term on the right hand side of (2.20) has a càdlàg modification. By Dellacherie and Meyer (1982, p.69, Theorem VI.4), the martingale $\{Z_t(\phi) : t \geq 0\}$ has a càdlàg modification. All other terms on the right hand side have continuous modifications. Therefore, the measure-valued process $\{Y_t : t \geq 0\}$ has a càdlàg modification and (2.20) gives an SPDE formulation of this immigration branching particle system. The following result shows that (2.14) and (2.17) are in fact square-integrable martingales.

Proposition 2.2 *Let $B_1 := \|\gamma(q-1)\|$ and $B_2 := \|\theta\gamma\sigma\|$, where $\|\cdot\|$ denotes the supremum norm. Then there is a locally bounded function C_2 on $\mathbb{R}_+^3$ such that*

$$\boldsymbol{E}\{\sup_{0\leq s\leq t}\langle 1, Y_s\rangle^2\} \leq C_2(B_1, B_2, t)(1 + \langle 1,\mu\rangle^2 + \langle 1, m\rangle^2), \quad t \geq 0. \tag{2.21}$$

Proof. Applying (2.20) to $\phi \equiv 1$ we get

$$\langle 1, Y_t\rangle = \langle 1,\mu\rangle + \theta^{-1}N((0,t]\times\mathbb{R}) + \int_0^t \langle \gamma(q-1), Y_s\rangle ds + Z_t(1), \tag{2.22}$$

where $\{Z_t(1) : t \geq 0\}$ is a local martingale with quadratic variation process

$$\langle Z(1)\rangle_t = \int_0^t \langle \theta^{-1}\gamma\sigma, Y_s\rangle ds.$$

For $l > 0$ let $\tau_l = \inf\{s \geq 0 : \langle 1, Y_s\rangle \geq l\}$. Then we have

$$\begin{aligned}\boldsymbol{E}\{\langle 1, Y_{t\wedge\tau_l}\rangle\} &\leq \langle 1,\mu\rangle + t\langle 1, m\rangle + \boldsymbol{E}\Big\{\int_0^{t\wedge\tau_l} \langle |\gamma(q-1)|, Y_s\rangle ds\Big\} \\ &\leq \langle 1,\mu\rangle + t\langle 1, m\rangle + B_1\int_0^t \boldsymbol{E}\{\langle 1, Y_{s\wedge\tau_l}\rangle\} ds.\end{aligned}$$

By Gronwall's inequality, we have

$$\boldsymbol{E}\{\langle 1, Y_{t\wedge\tau_l}\rangle\} \leq (\langle 1,\mu\rangle + t\langle 1, m\rangle)e^{B_1 t}, \quad t \geq 0.$$

By Fatou's lemma we may let $m \to \infty$ in the above to get

$$\boldsymbol{E}\{\langle 1, Y_t\rangle\} \leq (\langle 1,\mu\rangle + t\langle 1, m\rangle)e^{B_1 t}, \quad t \geq 0. \tag{2.23}$$

By (2.22) there is a universal constant $C \geq 0$ such that

$$\begin{aligned}\boldsymbol{E}\{\sup_{0\leq s\leq t}\langle 1, Y_{s\wedge\tau_l}\rangle^2\} &\leq C\Big[\langle 1,\mu\rangle^2 + \theta^{-2}\boldsymbol{E}\{N((0,t]\times\mathbb{R})^2\} + \boldsymbol{E}\{\sup_{0\leq s\leq t} Z_{s\wedge\tau_l}(1)^2\} \\ &\quad + B_1^2\boldsymbol{E}\Big\{\Big(\int_0^{t\wedge\tau_l}\langle 1, Y_s\rangle ds\Big)^2\Big\}\Big],\end{aligned}$$

where $\boldsymbol{E}\{N((0,t]\times\mathbb{R})^2\} = \theta t\langle 1,m\rangle + \theta^2 t^2\langle 1,m\rangle^2$ by a formula for the Poisson random measure. By a martingale inequality,

$$\boldsymbol{E}\{\sup_{0\leq s\leq t} Z_{s\wedge\tau_l}(1)^2\} \leq 4\boldsymbol{E}\{Z_{t\wedge\tau_l}(1)^2\} \leq 4B_2\int_0^t \boldsymbol{E}\{\langle 1,Y_s\rangle\}ds.$$

Then we use Hölder's inequality to see that

$$\begin{aligned}\boldsymbol{E}\{\sup_{0\leq s\leq t}\langle 1,Y_{s\wedge\tau_l}\rangle^2\} &\leq C\bigg[\langle 1,\mu\rangle^2 + \theta^{-1}t\langle 1,m\rangle + t^2\langle 1,m\rangle^2 + 4B_2\int_0^t \boldsymbol{E}\{\langle 1,Y_s\rangle\}ds \\ &\quad + B_1^2 t\int_0^t \boldsymbol{E}\{\sup_{0\leq s\leq t}\langle 1,Y_{s\wedge\tau_l}\rangle^2\}ds\bigg].\end{aligned}$$

By Gronwall's inequality, we get an estimate for $\boldsymbol{E}\{\sup_{0\leq s\leq t}\langle 1,Y_{t\wedge\tau_l}\rangle^2\}$. Then we obtain (2.21) by Fatou's lemma. ∎

3 Stochastic equation of the SDSMI

Let (c,h,σ,b,m) be given as in the introduction. Suppose that $W(ds,dx)$ is a time-space white noise. For $\mu\in M(\mathbb{R})$ we consider the stochastic equation:

$$\begin{aligned}\langle\phi,X_t\rangle &= \langle\phi,\mu\rangle + t\langle\phi,m\rangle + \frac{1}{2}\int_0^t\langle a\phi'',X_s\rangle ds - \int_0^t\langle b\phi,X_s\rangle ds \\ &\quad + \int_0^t\int_{\mathbb{R}}\phi(y)Z(ds,dy) + \int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\phi',X_s\rangle W(ds,dy), \qquad (3.1)\end{aligned}$$

where $Z(ds,dy)$ is an orthogonal martingale measure which is orthogonal to the white noise $W(ds,dy)$ and has covariation measure $\sigma(y)X_s(dy)ds$. Clearly, this is equivalent with the martingale problem given by (1.5) and (1.6). We shall prove that (3.1) has a weak solution $\{X_t: t\geq 0\}$, which will serve as a candidate of the SDSMI with parameters (c,h,σ,b,m). For a function F on $M(\mathbb{R})$, let

$$\begin{aligned}\mathcal{A}F(\mu) &= \frac{1}{2}\int_{\mathbb{R}^2}\rho(x-y)\frac{d^2}{dxdy}\frac{\delta^2F(\mu)}{\delta\mu(x)\delta\mu(y)}\mu(dx)\mu(dy) \\ &\quad + \frac{1}{2}\int_{\mathbb{R}} a(x)\frac{d^2}{dx^2}\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) \qquad (3.2)\end{aligned}$$

and

$$\begin{aligned}\mathcal{B}F(\mu) &= \frac{1}{2}\int_{\mathbb{R}}\sigma(x)\frac{\delta^2F(\mu)}{\delta\mu(x)^2}\mu(dx) - \int_{\mathbb{R}} b(x)\frac{\delta F(\mu)}{\delta\mu(x)}\mu(dx) \\ &\quad + \int_{\mathbb{R}}\frac{\delta F(\mu)}{\delta\mu(x)}m(dx) \qquad (3.3)\end{aligned}$$

if the right hand sides are meaningful. We shall also prove that $\{X_t: t\geq 0\}$ solves a martingale problem associated with $\mathcal{L} := \mathcal{A}+\mathcal{B}$. In particular, if $F_{f,\{\phi_i\}}(\mu) = f(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)$ for $f\in C_0^2(\mathbb{R}^n)$ and $\{\phi_i\}\subset C_b^2(\mathbb{R})$, then

$$\begin{aligned}\mathcal{A}F_{f,\{\phi_i\}}(\mu) &= \frac{1}{2}\sum_{i,j=1}^n f''_{ij}(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\int_{\mathbb{R}^2}\rho(x-y)\phi'_i(x)\phi'_j(y)\mu(dx)\mu(dy) \\ &\quad + \frac{1}{2}\sum_{i=1}^n f'_i(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\langle a\phi''_i,\mu\rangle \qquad (3.4)\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}F_{f,\{\phi_i\}}(\mu) &= \frac{1}{2}\int_{\mathbb{R}}\sigma(x)\Big[\sum_{i,j=1}^{n} f''_{ij}(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\phi_i(x)\phi_j(x)\Big]\mu(dx) \\
&\quad -\int_{\mathbb{R}} b(x)\Big[\sum_{i=1}^{n} f'_i(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\phi_i(x)\Big]\mu(dx) \\
&\quad +\int_{\mathbb{R}}\Big[\sum_{i=1}^{n} f'_i(\langle\phi_1,\mu\rangle,\cdots,\langle\phi_n,\mu\rangle)\phi_i(x)\Big]m(dx). \qquad (3.5)
\end{aligned}$$

Let $\mathcal{D}_1(\mathcal{L})$ denote the collection of all functions $F_{f,\{\phi_i\}}$ with $f \in C_0^2(\mathbb{R}^n)$ and $\{\phi_i\} \subset C_b^2(\mathbb{R})$.

We shall obtain (3.1) as the limit of a sequence of equations of immigration branching particle systems. Let $(c, h, \gamma_k, p^{(k)}, \theta_k m, \theta_k^{-1})$ be a sequence of parameters such that $\theta_k \to \infty$ as $k \to \infty$. Let q_k and σ_k be defined by (2.5) and (2.16) in terms of $(\gamma_k, p^{(k)}, \theta_k)$. We assume that $\{X_t^{(k)} : t \geq 0\}$ is a immigration particle system which satisfies

$$\begin{aligned}
\langle\phi, X_t^{(k)}\rangle &= \langle\phi, X_0^{(k)}\rangle + t\langle\phi, m\rangle + \int_{(0,t]}\int_{\mathbb{R}}\theta_k^{-1}\phi(x)\tilde{N}^{(k)}(ds,dx) \\
&\quad + \int_0^t \langle\gamma_k(q_k-1)\phi, X_s^{(k)}\rangle ds + Z_t^{(k)}(\phi) + U_t^{(k)}(\phi') \\
&\quad + \int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\phi', X_s^{(k)}\rangle W^{(k)}(ds,dy) + \frac{1}{2}\int_0^t\langle a\phi'', X_s^{(k)}\rangle ds, \qquad (3.6)
\end{aligned}$$

where $(N^{(k)}, Z^{(k)}, M^{(k)}, W^{(k)})$ are as in (2.20) with parameters $(c, h, \gamma_k, p^{(k)}, \theta_k m, \theta_k^{-1})$. We assume that the $X_0^{(k)}$ are deterministic and $X_0^{(k)} \to \mu$ as $k \to \infty$.

Lemma 3.1 *Suppose that $B_1 := \sup_{k\geq 1}\|\gamma_k(q_k-1)\| < \infty$ and $B_2 := \sup_{k\geq 1}\|\theta_k^{-1}\gamma_k\sigma_k\| < \infty$. Then for any $\phi \in C_b^2(\mathbb{R})$, each term in equation (3.6) gives a tight sequence in $D([0,\infty),\mathbb{R})$.*

Proof. The tightness of $\{\langle\phi, X_t^{(k)}\rangle : t \geq 0, k = 1, 2, \cdots\}$ will follow if we can prove each term on the right hand side is tight. The tightness of the first two term is immediate. Let $\{\tau_k\}$ be an arbitrary sequence of stopping times bounded above by some constant $T > 0$. Let

$$V_t^{(k)}(\phi') = \int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\phi', X_s^{(k)}\rangle W^{(k)}(ds,dy).$$

Then we have

$$\begin{aligned}
\boldsymbol{E}\{|V_{\tau_k+t}^{(k)}(\phi') - V_{\tau_k}^{(k)}(\phi')|^2\} &= \boldsymbol{E}\Big\{\int_0^t ds\int_{\mathbb{R}}\langle h(y-\cdot)\phi', X_{\tau_k+s}^{(k)}\rangle^2 dy\Big\} \\
&= \boldsymbol{E}\Big\{\int_0^t ds\int_{\mathbb{R}^2}\rho(x-z)\phi'(x)\phi'(z)X_{\tau_k+s}^{(k)}(dx)X_{\tau_k+s}^{(k)}(dz)\Big\} \\
&\leq \|\rho\|\int_0^t \boldsymbol{E}\{\langle\phi', X_{\tau_k+s}^{(k)}\rangle^2\}ds.
\end{aligned}$$

By Proposition 2.1, the right hand side is bounded by a constant independent of $k \geq 1$. In particular, the estimate holds if $\tau_k = 0$. By Chebyshev's inequality, we have

$$\sup_{k\geq 1}\boldsymbol{P}\{|V_t^{(k)}(\phi')| > \eta\} \to 0 \quad (\eta \to \infty)$$

and

$$\sup_{k\geq 1}\boldsymbol{P}\{|V^{(k)}_{\tau_k+t}(\phi') - V^{(k)}_{\tau_k}(\phi')| > \eta\} \to 0 \quad (t \to 0).$$

Then $\{V_t^{(k)}(\phi') : t \geq 0, k = 1, 2, \cdots\}$ is tight in $D([0,\infty),\mathbb{R})$; see Adlous (1978). Let

$$Y_t^{(k)}(\phi) = \int_0^t \langle \gamma_k(q_k - 1)\phi, X_s^{(k)}\rangle ds. \tag{3.7}$$

By Hölder's inequality,

$$\boldsymbol{E}\{|Y^{(k)}_{\tau_k+t}(\phi) - Y^{(k)}_{\tau_k}(\phi)|^2\} \leq B_1^2 t \int_0^t \boldsymbol{E}\{\langle\phi, X^{(k)}_{\tau_k+s}\rangle^2\}ds.$$

By the same reason as the above, $\{Y_t^{(k)}(\phi) : t \geq 0, k = 1, 2, \cdots\}$ is also a tight sequence in $D([0,\infty),\mathbb{R})$. The tightness of the remaining four terms follows by similar arguments. ∎

Lemma 3.2 *Suppose that* $\gamma_k(1 - q_k(\cdot)) \to b(\cdot)$ *and* $\theta_k^{-1}\gamma_k\sigma_k(\cdot) \to \sigma(\cdot)$ *uniformly for* $b \in C_b(\mathbb{R})$ *and* $\sigma \in C_b(\mathbb{R})^+$. *Then the sequence* $\{X_t^{(k)} : t \geq 0, k = 1, 2, \cdots\}$ *is tight in* $D([0,\infty), M(\mathbb{R}))$. *Moreover, the limit process* $\{X_t : t \geq 0\}$ *of any subsequence of* $\{X_t^{(k)} : t \geq 0, k = 1, 2, \cdots\}$ *is a.s. continuous and solves the* $(\mathcal{L}, \mathcal{D}_1(\mathcal{L}))$*-martingale problem.*

Proof. By Lemma 3.1 and a result of Roelly-Coppoletta (1986), $\{X_t^{(k)} : t \geq 0, k = 1, 2, \cdots\}$ is tight in $D([0,\infty), M(\bar{\mathbb{R}}))$. We write $\phi \in C_b^2(\bar{\mathbb{R}})$ if $\phi \in C_b^2(\mathbb{R})$ and its derivatives up to the second degree can be extended continuously to $\bar{\mathbb{R}}$. If $\{\phi_i\} \subset C^2(\bar{\mathbb{R}})$, we can extend $F_{f,\{\phi_i\}}$, $\mathcal{A}F_{f,\{\phi_i\}}$ and $\mathcal{B}F_{f,\{\phi_i\}}$ continuously to $M(\bar{\mathbb{R}})$. Let $\bar{F}_{f,\{\phi_i\}}$, $\bar{\mathcal{A}}\bar{F}_{f,\{\phi_i\}}$ and $\bar{\mathcal{B}}\bar{F}_{f,\{\phi_i\}}$ denote respectively those extensions. Let $(\mathcal{A}_k, \mathcal{B}_k)$ and $(\bar{\mathcal{A}}_k, \bar{\mathcal{B}}_k)$ denote the corresponding operators associated with $\{X_t^{(k)} : t \geq 0\}$. Clearly, if $\mu_k \in M_k(\bar{\mathbb{R}})$ and $\mu_k \to \mu$, then $\bar{\mathcal{A}}_k\bar{F}_{f,\{\phi_i\}}(\mu_k) \to \bar{\mathcal{A}}\bar{F}_{f,\{\phi_i\}}(\mu)$. By Taylor's expansion,

$$\begin{aligned}
&\bar{\mathcal{B}}_k\bar{F}_{f,\{\phi_i\}}(\mu_k) \\
= \;& \sum_{j=0}^{\infty}\int_{\mathbb{R}} \theta_k\gamma_k p_j(x)[f(\langle\phi_1,\mu_k\rangle + (j-1)\theta_k^{-1}\phi_1(x), \cdots, \langle\phi_n,\mu_k\rangle + (j-1)\theta_k^{-1}\phi_n(x)) \\
&\qquad\qquad - f(\langle\phi_1,\mu_k\rangle, \cdots, \langle\phi_n,\mu_k\rangle)]\mu_k(dx) \\
&+ \int_{\mathbb{R}} \theta_k[f(\langle\phi_1,\mu_k\rangle + \theta_k^{-1}\phi_1(x), \cdots, \langle\phi_n,\mu_k\rangle + \theta_k^{-1}\phi_n(x)) \\
&\qquad\qquad - f(\langle\phi_1,\mu_k\rangle, \cdots, \langle\phi_n,\mu_k\rangle)]m(dx) \\
= \;& \int_{\mathbb{R}} \gamma_k(q_k(x) - 1)\left[\sum_{i=1}^n f_i'(\langle\phi_1,\mu_k\rangle, \cdots, \langle\phi_n,\mu_k\rangle)\phi_i(x)\right]\mu_k(dx) \\
&+ \int_{\mathbb{R}} \frac{\gamma_k\sigma_k(x)}{2\theta_k}\left[\sum_{i,j=1}^n f_{ij}''(\langle\phi_1,\mu_k\rangle + \eta_k\phi_1(x), \cdots, \langle\phi_n,\mu_k\rangle + \eta_k\phi_n(x))\phi_i(x)\phi_j(x)\right]\mu_k(dx) \\
&+ \int_{\mathbb{R}} \sum_{i=1}^n \left[f_i'(\langle\phi_1,\mu_k\rangle + \zeta_k\phi_1(x), \cdots, \langle\phi_n,\mu_k\rangle + \zeta_k\phi_n(x))\phi_i(x)\right]m(dx),
\end{aligned}$$

where $0 < \eta_k, \zeta_k < \theta_k^{-1}$. Then $\bar{\mathcal{B}}_k\bar{F}_{f,\{\phi_i\}}(\mu_k) \to \bar{\mathcal{B}}\bar{F}_{f,\{\phi_i\}}(\mu)$ under the assumption. Let $\{X_t : t \geq 0\}$ be the limit of any subsequence of $\{X_t^{(k)} : t \geq 0, k = 1, 2, \cdots\}$. As in the proof of Lemma 4.2 of Dawson *et al* (2001) one can show that

$$\bar{F}_{f,\{\phi_i\}}(X_t) - \bar{F}_{f,\{\phi_i\}}(X_0) - \int_0^t \bar{\mathcal{L}}\bar{F}_{f,\{\phi_i\}}(X_s)ds \tag{3.8}$$

is a martingale, where $\bar{\mathcal{L}} = \bar{\mathcal{A}} + \bar{\mathcal{B}}$. It is not hard to check that the "gradient squared" operator associated with $\bar{\mathcal{L}}$ satisfies the derivation property of Barkry and Emery (1985). Then $\{X_t : t \geq 0\}$ is actually almost surely continuous as an $M(\bar{\mathbb{R}})$-valued process. By a modification of the proof of Theorem 4.1 of Dawson *et al* (2001) one can show that $\{X_t : t \geq 0\}$ is almost surely supported by $\mathbb{R}$. Thus $\{X_t^{(k)} : t \geq 0, k = 1, 2, \cdots\}$ is tight in $D([0,\infty), M(\mathbb{R}))$ and $\{X_t : t \geq 0\}$ is a.s. continuous as an $M(\mathbb{R})$-valued process. ∎

Lemma 3.3 *If $\{X_t : t \geq 0\}$ is the continuous solution of the $(\mathcal{L}, \mathcal{D}_1(\mathcal{L}))$-martingale problem, then for each integer $n \geq 1$ there is a locally bounded function C_n on $\mathbb{R}_+^3$ such that*

$$\boldsymbol{E}\{\sup\nolimits_{0\leq s\leq t}\langle 1, X_s\rangle^n\} \leq C_n(\|b\|, \|\sigma\|, t)(1 + \langle 1, \mu\rangle^n + \langle 1, m\rangle^n), \quad t \geq 0. \tag{3.9}$$

Proof. If $\{X_t : t \geq 0\}$ is the continuous solution of the $(\mathcal{L}, \mathcal{D}_1(\mathcal{L}))$-martingale problem, then

$$Z_t(1) := \langle 1, X_t\rangle - \langle 1, \mu\rangle - t\langle 1, m\rangle + \int_0^t \langle b, X_s\rangle ds \tag{3.10}$$

is a continuous local martingale with quadratic variation process

$$\langle Z(1)\rangle_t = \int_0^t \langle \sigma, X_s\rangle ds. \tag{3.11}$$

For $l > 0$ let $\tau_l = \inf\{s \geq 0 : \langle 1, X_s\rangle \geq l\}$. The inequalities for $n = 1$ and $n = 2$ can be proved as in the proof of Proposition 2.2. Now the Burkholder-Davis-Gundy inequality implies that

$$\begin{aligned}\boldsymbol{E}\{\sup\nolimits_{0\leq s\leq t}\langle 1, X_{s\wedge\tau_l}\rangle^{2n}\} \;\leq\; & C\bigg[\langle 1, \mu\rangle^{2n} + t^{2n}\langle 1, m\rangle^{2n} + \boldsymbol{E}\bigg\{\bigg(\int_0^{t\wedge\tau_l}\langle |b|, X_s\rangle ds\bigg)^{2n}\bigg\} \\ & + \boldsymbol{E}\bigg\{\bigg(\int_0^{t\wedge\tau_l}\langle \sigma, X_s\rangle ds\bigg)^{n}\bigg\}\bigg].\end{aligned}$$

where $C \geq 0$ is a universal constant. Then we use the Hölder's inequality to see that

$$\begin{aligned}\boldsymbol{E}\{\sup\nolimits_{0\leq s\leq t}\langle 1, X_{s\wedge\tau_l}\rangle^{2n}\} \;\leq\; & C_n\bigg[\langle 1, \mu\rangle^{2n} + t^{2n}\langle 1, m\rangle^{2n} + \theta^{-n}t^n\langle 1, m\rangle^n \\ & + \|b\|^{2n}t^{2n-1}\int_0^t \boldsymbol{E}\{\sup\nolimits_{0\leq r\leq s}\langle 1, X_{r\wedge\tau_l}\rangle^{2n}\}ds\bigg] \\ & + \|\sigma\|^n t^{n-1}\int_0^t \boldsymbol{E}\{\langle 1, X_s\rangle^n\}ds.\end{aligned} \tag{3.12}$$

By using (3.12) and Gronwall's inequality inductively, we get some estimates for $\boldsymbol{E}\{\sup_{0\leq s\leq t}\langle 1, X_{t\wedge\tau_l}\rangle^n\}$. Then we obtain the inequalities for $\boldsymbol{E}\{\sup_{0\leq s\leq t}\langle 1, X_t\rangle^n\}$ by Fatou's lemma. ∎

Lemma 3.4 *Suppose there are constants $d_0 > 0$ and $\delta > 1/2$ such that $h(x) \leq d_0(1 + |x|)^{-\delta}$ for all $x \in \mathbb{R}$. If $\gamma_k(1 - q_k(\cdot)) \to b(\cdot)$ and $\theta_k^{-1}\gamma_k\sigma_k(\cdot) \to \sigma(\cdot)$ uniformly for $b \in C_b(\mathbb{R})$ and $\sigma \in C_b(\mathbb{R})^+$, then the limit process $\{X_t : t \geq 0\}$ of any subsequence of $\{X_t^{(k)} : t \geq 0, k = 1, 2, \cdots\}$ is a weak solution of (3.1).*

Proof. By Lemma 3.1, $\{(X_t^{(k)}, U_t^{(k)}, W_t^{(k)}, Z_t^{(k)}) : t \geq 0, k = 1, 2, \cdots\}$ is tight in $D([0,\infty), M(\bar{\mathbb{R}}) \times \mathcal{S}'(\mathbb{R})^3)$; see Mitoma (1983). By passing to a subsequence, we simply assume that $\{(X_t^{(k)}, U_t^{(k)}, W_t^{(k)}, Z_t^{(k)}) : t \geq 0\}$ converges in distribution to some process $\{(X_t, U_t, W_t, Z_t) : t \geq 0\}$. By Lemma 3.2, $\{X_t : t \geq 0\}$ is a.s. continuous and solves the $(\mathcal{L}, \mathcal{D}_1(\mathcal{L}))$-martingale problem. Considering the Skorokhod representation, we assume $\{(X_t^{(k)}, U_t^{(k)}, W_t^{(k)}, Z_t^{(k)}) : t \geq 0\}$ converges almost surely to the process $\{(X_t, U_t, W_t, Z_t) : t \geq 0\}$ in the topology of $D([0,\infty), M(\bar{\mathbb{R}}) \times \mathcal{S}'(\mathbb{R})^3)$. Since each $\{W_t^{(k)} : t \geq 0\}$ is a time-space white noise, so is $\{W_t : t \geq 0\}$. In view of (2.15), we have a.s. $U_t(\phi) = 0$ for all $t \geq 0$ and $\phi \in \mathcal{S}(\mathbb{R})$. Then the theorem follows once it is proved that $\{(X_t, W_t, Z_t) : t \geq 0\}$ satisfies (3.1). Clearly, it is sufficient to prove this for $\phi \in \mathcal{S}(\mathbb{R})$ with compact support $\mathrm{supp}(\phi)$. Let $Y_t(y) = \langle h(y-\cdot)\phi', X_t\rangle$ and $Y_t^{(k)}(y) = \langle h(y-\cdot)\phi', X_t^{(k)}\rangle$. Note that the weak convergence of measures can be induced by the Vasershtein metric; see Ethier and Kurtz (1986, p.150). For $l > 0$ let $\tau_l = \inf\{s \geq 0 : \langle 1, X_s^{(k)}\rangle \geq l$ for some $k \geq 1\}$. Then it is easy to see that $\{Y_t^{(k)} 1_{\{t<\tau_l\}} : t \geq 0\}$ converges to $\{Y_t 1_{\{t<\tau_l\}} : t \geq 0\}$ in $D([0,\infty), C_0(\mathbb{R}))$, where $C_0(\mathbb{R})$ is furnished with the uniform norm. By Cho (1995, Theorem 2.1), for $\psi \in \mathcal{S}(\mathbb{R})$ we have almost surely

$$\lim_{k\to\infty} \int_0^t \int_{\mathbb{R}} \psi(y) Y_s^{(k)}(y) 1_{\{s<\tau_l\}} W^{(k)}(ds,dy) = \int_0^t \int_{\mathbb{R}} \psi(y) Y_s(y) 1_{\{s<\tau_l\}} W(ds,dy). \quad (3.13)$$

Setting $a = \sup\{|x|, x \in \mathrm{supp}(\phi)\}$ we have

$$\sup_{|z|\leq a} |h(y-z)| \leq d(y) := d_0[1_{\{|y|\leq a\}} + 1_{\{|y|>a\}}(1+|y|-|a|)^{-2\delta}], \quad (3.14)$$

and hence

$$|Y_t(y)| \leq \langle \phi', X_t\rangle d(y) \quad \text{and} \quad |Y_t^{(k)}(y)| \leq \langle \phi', X_t^{(k)}\rangle d(y). \quad (3.15)$$

By the Burkholder-Davis-Gundy inequality,

$$\begin{aligned} &\boldsymbol{E}\Big\{\Big(\int_0^t \int_{\mathbb{R}} \psi(y) Y_s^{(k)}(y) 1_{\{s<\tau_l\}} W^{(k)}(ds,dy)\Big)^4\Big\} \\ &\qquad \leq \mathrm{const} \cdot \boldsymbol{E}\Big\{\Big(\int_0^t \int_{\mathbb{R}} \psi(y)^2 Y_s^{(k)}(y)^2 1_{\{s<\tau_l\}} ds dy\Big)^2\Big\} \\ &\qquad \leq \mathrm{const} \cdot l^4 \|\phi'\|^4 \langle \psi^2 d^2, \lambda\rangle^2 t^2, \end{aligned} \quad (3.16)$$

where λ denotes the Lebesgue measure on $\mathbb{R}$. Since the right hand side of (3.16) is independent of $k \geq 1$, the convergence (3.13) also holds in the L^2-sense. For each $\epsilon > 0$, it is not hard to choose $\psi \in \mathcal{S}(\mathbb{R})$ so that

$$\begin{aligned} &\boldsymbol{E}\Big\{\Big(\int_0^t \int_{\mathbb{R}} (1-\psi(y)) Y_s^{(k)}(y) 1_{\{s<\tau_l\}} W^{(k)}(ds,dy)\Big)^2\Big\} \\ &\qquad \leq \mathrm{const} \cdot l^2 \|\phi'\|^2 \langle |1-\psi|^2 d^2, \lambda\rangle t \leq \epsilon. \end{aligned} \quad (3.17)$$

The same estimate is available with $Y^{(k)}$ and $W^{(k)}$ replaced respectively by Y and W. Clearly, (3.13) and (3.17) imply that

$$\lim_{k\to\infty} \int_0^t \int_{\mathbb{R}} Y_s^{(k)}(y) 1_{\{s<\tau_l\}} W^{(k)}(ds,dy) = \int_0^t \int_{\mathbb{R}} Y_s(y) 1_{\{s<\tau_l\}} W(ds,dy) \quad (3.18)$$

in the L^2-sense. Passing to a suitable subsequence we get the almost sure convergence for (3.18). Now letting $k \to \infty$ in (3.6) we get

$$\begin{aligned}\langle \phi, X_{t\wedge\tau_l}\rangle &= \langle \phi,\mu\rangle + (t\wedge\tau_l)\langle\phi, m\rangle + \frac{1}{2}\int_0^{t\wedge\tau_l}\langle a\phi'', X_s\rangle ds - \int_0^{t\wedge\tau_l}\langle b\phi, X_s\rangle ds \\ &\quad + \int_0^{t\wedge\tau_l}\int_{\mathbb{R}}\phi(y)Z(ds,dy) + \int_0^{t\wedge\tau_l}\int_{\mathbb{R}}\langle h(y-\cdot)\phi', X_s\rangle W(ds,dy),\end{aligned}$$

from which (3.1) follows. The extensions from $\phi \in \mathcal{S}(\mathbb{R})$ to $\phi \in C_b^2(\mathbb{R})$ is immediate. ∎

Theorem 3.1 *Suppose there are constants $d_0 > 0$ and $\delta > 1/2$ such that $h(x) \le d_0(1+|x|)^{-\delta}$ for all $x \in \mathbb{R}$. Then the stochastic equation (3.1) has a continuous weak solution $\{X_t : t \ge 0\}$. Moreover, $\{X_t : t\ge 0\}$ also solves the $(\mathcal{L}, \mathcal{D}_1(\mathcal{L}))$-martingale problem.*

Proof. Given $b \in C_b(\mathbb{R})$ and $\sigma \in C_b(\mathbb{R})^+$, we set $\theta_k = k$, $\gamma_k = \sqrt{k}$ and

$$p_0^{(k)} = 1 - p_2^{(k)} - p_k^{(k)}, \quad p_2^{(k)} = \frac{(k-1)^2(1-b/\sqrt{k}) - k\sigma_k}{2(k-1)^2 - k}, \quad p_k^{(k)} = \frac{2\sigma_k - 1 + b/\sqrt{k}}{2(k-1)^2 - k},$$

where $\sigma_k(\cdot) = \sqrt{k}\sigma(\cdot) + 1$. Then the sequence $(\gamma_k, p^{(k)}, \theta_k)$ satisfies the conditions of Lemma 3.4. By Lemmas 3.2 and 3.4, equation (3.1) has a continuous weak solution $\{X_t : t \ge 0\}$ which solves the $(\mathcal{L}, \mathcal{D}_1(\mathcal{L}))$-martingale problem. ∎

4 Stochastic log-Laplace equations

In this section, we establish the existence and uniqueness of solution of the stochastic log-Laplace equation (1.7). The techniques here are based on Kurtz and Xiong (1999) and have been stimulated by the recent work Xiong (2003), which considers a model of Skoulakis and Adler (2001). Let (c, h, σ, b, m) be given as in the introduction. Suppose that $W(ds, dx)$ is a time-space white noise. The main objective is to discuss the non-linear SPDE:

$$\begin{aligned}\psi_t(x) &= \phi(x) + \int_0^t \left[\frac{1}{2}a(x)\partial_x^2\psi_s(x) - b(x)\psi_s(x) - \frac{1}{2}\sigma(x)\psi_s(x)^2\right]ds \\ &\quad + \int_0^t\int_{\mathbb{R}} h(y-x)\partial_x\psi_s(x)W(ds,dy), \qquad t \ge 0. \end{aligned} \tag{4.1}$$

We write $\phi \in W_2^k(\mathbb{R})$ if ϕ together with its derivatives up to the kth degree are square-integrable. For $\phi \in W_2^k(\mathbb{R})$ let

$$\|\phi\|_k^2 = \sum_{i=0}^k \|\partial_x^{(i)}\phi\|_0^2. \tag{4.2}$$

Following Xiong (2003), we first consider a smoothed version of equation (4.1). Let $(T_t)_{t\ge 0}$ denote the transition semigroup of a standard Brownian motion. Let $\{h_j : j = 1, 2, \cdots\}$ be a complete orthonormal system of $L^2(\mathbb{R})$. Then

$$W_j(t) = \int_0^t\int_{\mathbb{R}} h_j(y)W(ds,dy), \qquad t \ge 0 \tag{4.3}$$

defines a sequence of independent standard Brownian motions $\{W_j : j = 1, 2, \cdots\}$. For $\epsilon > 0$ let

$$W^\epsilon(dt, dx) = \sum_{j=1}^{[1/\epsilon]} h_j(x) W_j(dt) dx, \qquad s \geq 0, y \in \mathbb{R}. \tag{4.4}$$

For any bounded non-negative $\phi \in L^2(\mathbb{R})$ define $d_\epsilon(\phi) = (\|T_\epsilon \phi\| \wedge \epsilon^{-1})\|T_\epsilon \phi\|^{-1}$. By a general result in Kurtz and Xiong (1999, Theorem 3.5), there is a unique strong solution $\psi_t^\epsilon(x)$ of the equation

$$\begin{aligned} \psi_t^\epsilon(x) &= T_\epsilon \phi(x) + \int_0^t \left[\frac{1}{2} a(x) \partial_x^2 \psi_s^\epsilon(x) - b(x) \psi_s^\epsilon(x) - \frac{1}{2} \sigma(x) \psi_s^\epsilon(x) d_\epsilon(\psi_s^\epsilon) T_\epsilon \psi_s^\epsilon(x)\right] ds \\ &\quad + \int_0^t \int_{\mathbb{R}} h(y - x) \partial_x \psi_s^\epsilon(x) W^\epsilon(ds, dy), \qquad t \geq 0; \end{aligned} \tag{4.5}$$

see also Rozovskii (1990).

Lemma 4.1 *For the solution $\{\psi_t^\epsilon : t \geq 0\}$ of (4.5) we have a.s. $\|\psi_t^\epsilon\|_{ess} \leq e^{-b_0 t}\|\phi\|_{ess}$ for all $t \geq 0$, where $b_0 = \inf_x b(x)$.*

Proof. Indeed, for a non-trivial $\phi \in L^2(\mathbb{R})^+$, the solution of (4.5) can be obtained in the following way. Let $\{B_i(t)\}$ be a sequence of independent Brownian motions which are also independent of the white noise $\{W(ds, dy)\}$. Let "$\|\cdot\|_0$" and "$\langle \cdot, \cdot \rangle_0$" denote respectively the norm and the inner product in $L^2(\mathbb{R})$. By Kurtz and Xiong (1999, Theorems 2.1 and 2.2), there is a unique strong solution $\psi_t^\epsilon(x)$ of the stochastic system

$$\begin{aligned} \xi_i(t) - \xi_i(0) &= \int_0^t c(\xi_i(s)) dB_i(s) + 2\int_0^t c(\xi_i(s)) c'(\xi_i(s)) ds \\ &\quad - \int_0^t \int_{\mathbb{R}} h(y - \xi_i(s)) W^\epsilon(ds, dy), \end{aligned} \tag{4.6}$$

$$\begin{aligned} m_i(t) - m_i(0) &= \int_0^t \left[\frac{1}{2} a''(\xi_i(s)) - b(\xi_i(s))\right] m_i(s) ds \\ &\quad - \frac{1}{2} \int_0^t \sigma(\xi_i(s)) d_\epsilon(\psi_s) T_\epsilon \psi_s(\xi_i(s)) m_i(s) ds \\ &\quad - \int_0^t \int_{\mathbb{R}} h'(y - \xi_i(s)) m_i(s) W^\epsilon(ds, dy), \end{aligned} \tag{4.7}$$

and

$$\psi_t(x) dx = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n m_i(t) \delta_{\xi_i(t)}(dx), \qquad t \geq 0, x \in \mathbb{R}, \tag{4.8}$$

where $\{(m_i(0), \xi_i(0)) : i = 1, 2, \cdots\}$ is a sequence of exchangeable random variables on $[0, \infty) \times \mathbb{R}$ which are independent of $\{B_i(t)\}$ and $\{W(ds, dy)\}$ and satisfy $\lim_{n \to \infty} n^{-1} \sum_{i=1}^n m_i(0) \delta_{\xi_i(0)}(dx)$ $= T_\epsilon \phi(x) dx$. By Kurtz and Xiong (1999, Theorems 3.1 – 3.5), $\psi_t^\epsilon(x)$ is also the pathwise unique solution of (4.5). By a duality argument similar to the proof of Xiong (2003, Lemma 2.2) we get $\|\psi_t^\epsilon\|_{\text{ess}} \leq e^{-b_0 t}\|\phi\|_{\text{ess}}$. ∎

Lemma 4.2 *For the solution $\{\psi_t^\epsilon : t \geq 0\}$ of (4.5) we have*

$$\boldsymbol{E}\left\{\sup_{0\leq r\leq t} \|\psi_r^\epsilon\|_0^4\right\} \leq K(t) \tag{4.9}$$

for a locally bounded function $K(\cdot)$ on $[0,\infty)$.

Proof. Although the arguments are similar to those of Xiong (2003), we shall give the detailed proof for the convenience of the reader. For any $f \in C_b^\infty(\mathbb{R})$ with compact support,

$$\begin{aligned}\langle\psi_t^\epsilon, f\rangle_0 &= \langle T_\epsilon\phi, f\rangle_0 + \int_0^t \left[\frac{1}{2}\langle a\partial_x^2\psi_s^\epsilon, f\rangle_0 - \langle b\psi_s^\epsilon, f\rangle_0 - \frac{1}{2}\langle\sigma\psi_s^\epsilon d_\epsilon(\psi_s^\epsilon)T_\epsilon\psi_s^\epsilon, f\rangle_0\right] ds \\ &\quad + \sum_{j=1}^{[1/\epsilon]}\int_0^t\int_{\mathbb{R}} h_j(z)\left[\int_{\mathbb{R}} h_j(y)\langle h(y-\cdot)\partial_x\psi_s^\epsilon, f\rangle_0 dy\right] W(ds,dz).\end{aligned}$$

By Itô's formula,

$$\begin{aligned}\langle\psi_t^\epsilon, f\rangle_0^2 &= \langle T_\epsilon\phi, f\rangle_0^2 + \int_0^t \langle\psi_s^\epsilon, f\rangle_0\langle a\partial_x^2\psi_s^\epsilon - 2b\psi_s^\epsilon - \sigma\psi_s^\epsilon d_\epsilon(\psi_s^\epsilon)T_\epsilon\psi_s^\epsilon, f\rangle_0 ds \\ &\quad + 2\int_0^t\int_{\mathbb{R}}\langle\psi_s^\epsilon, f\rangle_0\langle h(y-\cdot)\partial_x\psi_s^\epsilon, f\rangle_0 W^\epsilon(ds,dy) \\ &\quad + \sum_{j=1}^{[1/\epsilon]}\int_0^t\left[\int_{\mathbb{R}}\langle h_j(y)h(y-\cdot)\partial_x\psi_s^\epsilon, f\rangle_0 dy\right]^2 ds.\end{aligned}$$

Adding f over in a complete orthonormal system of $L^2(\mathbb{R})$ we get

$$\begin{aligned}\|\psi_t^\epsilon\|_0^2 &= \|T_\epsilon\phi\|_0^2 + \int_0^t\langle a\partial_x^2\psi_s^\epsilon - 2b\psi_s^\epsilon - \sigma\psi_s^\epsilon d_\epsilon(\psi_s^\epsilon)T_\epsilon\psi_s^\epsilon, \psi_s^\epsilon\rangle_0 ds \\ &\quad + 2\int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\partial_x\psi_s^\epsilon, \psi_s^\epsilon\rangle_0 W(ds,dy) \\ &\quad + \sum_{j=1}^{[1/\epsilon]}\int_0^t ds\int_{\mathbb{R}}\left[\int_{\mathbb{R}} h_j(y)h(y-z)\partial_x\psi_s^\epsilon(z)dy\right]^2 dz \\ &\leq \|T_\epsilon\phi\|_0^2 + \int_0^t\langle c^2\partial_x^2\psi_s^\epsilon, \psi_s^\epsilon\rangle_0 ds + \int_0^t\langle -2b\psi_s^\epsilon - \sigma\psi_s^\epsilon d_\epsilon(\psi_s^\epsilon)T_\epsilon\psi_s^\epsilon, \psi_s^\epsilon\rangle_0 ds \\ &\quad + 2\int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\partial_x\psi_s^\epsilon, \psi_s^\epsilon\rangle_0 W(ds,dy) \\ &\quad + \int_0^t\langle\rho(0)\partial_x^2\psi_s^\epsilon, \psi_s^\epsilon\rangle_0 ds + \int_0^t ds\int_{\mathbb{R}}\left[\int_{\mathbb{R}} h(y-z)^2(\partial_x\psi_s^\epsilon(z))^2 dy\right] dz.\end{aligned} \tag{4.10}$$

Since $\psi_s^\epsilon \in W_2^2(\mathbb{R})$, there exists a sequence $f_n \in C_0^\infty(\mathbb{R})$ such that $f_n \to \psi_s^\epsilon$ in $W_2^2(\mathbb{R})$. Note that

$$\langle c^2 f_n'', f_n\rangle = \langle (c^2)'', f_n^2\rangle/2 - \langle c^2, (f_n')^2\rangle \leq K\|f_n\|_0^2.$$

Taking $n \to \infty$ we have

$$\langle c^2\partial_x^2\psi_s^\epsilon, \psi_s^\epsilon\rangle_0 \leq K\|\psi_s^\epsilon\|_0^2. \tag{4.11}$$

It is easy to see that

$$\langle -2b\psi_s^\epsilon - \sigma\psi_s^\epsilon d_\epsilon(\psi_s^\epsilon)T_\epsilon\psi_s^\epsilon, \psi_s^\epsilon\rangle_0 \leq K\|\psi_s^\epsilon\|_0^2.$$

Therefore, we can continue (4.10) with

$$\begin{aligned}\|\psi_t^\epsilon\|_0^2 &\leq \|T_\epsilon\phi\|_0^2 + K\int_0^t \|\psi_s^\epsilon\|_0^2 ds + 2\int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\partial_x\psi_s^\epsilon, \psi_s^\epsilon\rangle_0 W(ds,dy)\\ &\quad +\rho(0)\int_0^t\langle\partial_x^2\psi_s^\epsilon, \psi_s^\epsilon\rangle_0 ds + \rho(0)\int_0^t ds\int_{\mathbb{R}}(\partial_x\psi_s^\epsilon(z))^2 dz.\end{aligned}$$

Similar to (4.11), we have

$$\langle\partial_x^2\psi_s^\epsilon, \psi_s^\epsilon\rangle_0 + \int_{\mathbb{R}}(\partial_x\psi_s^\epsilon(z))^2 dz \leq K\|\psi_s^\epsilon\|_0^2,$$

and hence

$$\|\psi_t^\epsilon\|_0^2 \leq \|\phi\|_0^2 + K\int_0^t \|\psi_s^\epsilon\|_0^2 ds + 2\int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\partial_x\psi_s^\epsilon, \psi_s^\epsilon\rangle_0 W(ds,dy).$$

By Burkholder's inequality, we get

$$\begin{aligned}\boldsymbol{E}\left\{\sup_{0\leq r\leq t}\|\psi_r^\epsilon\|_0^4\right\} &\leq 4\|\phi\|_0^2 + K\boldsymbol{E}\int_0^t \|\psi_s^\epsilon\|_0^4 ds + K\int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\partial_x\psi_s^\epsilon, \psi_s^\epsilon\rangle_0^2 dyds\\ &\leq 4\|\phi\|_0^2 + K\boldsymbol{E}\int_0^t \|\psi_s^\epsilon\|_0^4 ds. \qquad (4.12)\end{aligned}$$

where the last inequality follows from the same arguments as those leading to (4.11). By Rozovskii (1990), we have $\boldsymbol{E}\{\|\psi_t^\epsilon\|_0^4\} < \infty$ for each $t \geq 0$. Then we obtain (4.9) by Gronwall's inequality. ∎

Lemma 4.3 *For the solution $\{\psi_t^\epsilon : t \geq 0\}$ of (4.5) we have*

$$\boldsymbol{E}\left\{\sup_{0\leq r\leq t}\|\psi_r^\epsilon\|_1^4\right\} \leq K(t) \qquad (4.13)$$

for a locally bounded function $K(\cdot)$ on $[0,\infty)$.

Proof. We shall omit some details since they are similar to those in the proof of Lemma 4.2. From (4.5) we have

$$\begin{aligned}\partial_x\psi_t^\epsilon(x) &= \partial_x T_\epsilon\phi(x) + \int_0^t\Big[\frac{1}{2}a'(x)\partial_x^2\psi_s^\epsilon(x) + \frac{1}{2}a(x)\partial_x^3\psi_s^\epsilon(x) - b'(x)\psi_s^\epsilon(x) - b(x)\partial_x\psi_s^\epsilon(x)\\ &\quad -\frac{1}{2}\sigma'(x)\psi_s(x)d_\epsilon(\psi_s^\epsilon)T_\epsilon\psi_s^\epsilon(x) - \frac{1}{2}\sigma(x)\partial_x\psi_s^\epsilon(x)d_\epsilon(\psi)T_\epsilon\psi_s^\epsilon(x)\\ &\quad -\frac{1}{2}\sigma(x)\psi_s^\epsilon(x)d_\epsilon(\psi)T_\epsilon\partial_\cdot\psi_s^\epsilon(x)\Big]ds\\ &\quad +\int_0^t\int_{\mathbb{R}}[h(y-x)\partial_x^2\psi_s(x) - h'(y-x)\partial_x\psi_s(x)W(ds,dy).\end{aligned}$$

Then we have

$$\begin{aligned}\|\partial_x\psi_t^\epsilon\|_0^2 &= \|T_\epsilon\partial_x\phi\|_0^2+\int_0^t\Big[\langle\partial_x\psi_s^\epsilon,a'\partial_x^2\psi_s^\epsilon+a\partial_x^3\psi_s^\epsilon\rangle_0-2\langle\partial_x\psi_s^\epsilon,b'\psi_s^\epsilon+b\partial_x\psi_s^\epsilon\rangle_0\\ &\quad -d_\epsilon(\psi_s^\epsilon)\langle\partial_x\psi_s^\epsilon,\sigma'\psi_sT_\epsilon\psi_s^\epsilon+\sigma\partial_x\psi_s^\epsilon T_\epsilon\psi_s^\epsilon+\sigma\psi_s^\epsilon T_\epsilon\partial_x\psi_s^\epsilon\rangle_0\Big]ds\\ &\quad +2\int_0^t\int_{\mathbb{R}}\langle\partial_x\psi_s^\epsilon,h(y-\cdot)\partial_x^2\psi_s^\epsilon-h'(y-\cdot)\partial_x\psi_s^\epsilon\rangle_0W(ds,dy)\\ &\quad +\int_0^t ds\int_{\mathbb{R}}\|h(y-\cdot)\partial_x^2\psi_s^\epsilon-h'(y-\cdot)\partial_x\psi_s^\epsilon\|_0^2dy.\end{aligned}$$

Similar to the previous lemma, we have that

$$\boldsymbol{E}\left\{\sup_{0\le r\le t}\|\partial_x\psi_t^\epsilon\|_0^4\right\}\le 4\|\partial_x\phi\|_0^4+K\boldsymbol{E}\int_0^t\left(\|\psi_s^\epsilon\|_0^4+\|\partial_x\psi_s^\epsilon\|_0^4\right)ds. \tag{4.14}$$

By Rozovskii (1990), $\boldsymbol{E}\left\{\sup_{0\le r\le t}\|\partial_x\psi_r^\epsilon\|_0^4\right\}<\infty$ for all $t\ge 0$. Then we obtain (4.13) by Gronwall's inequality. ■

Theorem 4.1 *Suppose that $\{a,b\}\subset C_b^2(\mathbb{R}),\sigma\in C_b^1(\mathbb{R})$, $h\in L_2(\mathbb{R})\cap C_b^1(\mathbb{R})$ and $h'\in L_2(\mathbb{R})\cap C_b^1(\mathbb{R})$. For $\phi\in W_2^1(\mathbb{R})\cap C_b(\mathbb{R})^+$, equation (4.1) has a unique $L^2(\mathbb{R})^+$-valued strong solution $\{\psi_t : t\ge 0\}$. We have a.s. $\|\psi_t\|\le e^{-b_0t}\|\phi\|$ for all $t\ge 0$, where $b_0=\inf_x b(x)$. Moreover, there is a locally bounded function $K(\cdot)$ on $[0,\infty)$ such that*

$$\boldsymbol{E}\left\{\sup_{0\le r\le t}\|\psi_r\|_1^4\right\}\le K(t), \tag{4.15}$$

and so $\{\psi_t(\cdot):t\ge 0\}$ has an $W_2^1(\mathbb{R})\cap C_b(\mathbb{R})^+$-valued version.

Proof. Let $z_t(x)=\psi_t^\epsilon(x)-\psi_t^\eta(x)$. As for (4.12), by the same arguments leading to (2.12) of Xiong (2003) we have

$$\begin{aligned}\boldsymbol{E}\sup_{0\le s\le t}\|z_s\|_0^4 &\le K\int_0^t\boldsymbol{E}\|z_r\|_0^4dr+3\|\phi\|_\infty^4\boldsymbol{E}\int_0^t\left(\int|T_\epsilon\psi_r^\eta(x)-T_\eta\psi_r^\eta(x)|^2dx\right)^2dr\\ &\quad +K\boldsymbol{E}\int_0^t|d_\epsilon(\psi_r^\epsilon)-d_\eta(\psi_r^\eta)|^4dr\\ &\quad +K\boldsymbol{E}\sum_{j=[1/\eta]+1}^{[1/\epsilon]}\int_0^t\left(\int_{\mathbb{R}}\langle h(y-\cdot)\partial_x\psi_s^\eta,z_s\rangle h_j(y)dy\right)^2ds\end{aligned} \tag{4.16}$$

As in Section 2.4 in Xiong (2003), the second and third terms on the right hand side of (4.16) converge to 0 as $\epsilon,\ \eta\to 0$. On the other hand, the last term is bounded by

$$\int_0^t\int_{\mathbb{R}}\sum_{j=[1/\eta]+1}^{[1/\epsilon]}\left(\int_{\mathbb{R}}h_j(y)h(y-x)dy\right)^2\boldsymbol{E}\{z_s(x)^2\}dx\int_{\mathbb{R}}\boldsymbol{E}\{(\partial_x\psi_s^\eta)^2\}dxds,$$

which tends to zero as $\epsilon,\eta\to 0$. As in Section 2.4 of Xiong (2003) we can show that ψ^ϵ is a Cauchy sequence whose limit ψ is the unique $L^2(\mathbb{R})^+$-valued strong solution for equation (4.1). The second assertion follows from Lemma 4.1. The last assertion follows by Lemma 4.3 and Sobolev's result. ∎

Based on Theorem 4.1, let us consider the following more useful backward SPDE:

$$\begin{aligned}\psi_{r,t}(x) &= \phi(x) + \int_r^t \left[\frac{1}{2}a(x)\partial_x^2\psi_{s,t}(x) - b(x)\psi_{s,t}(x) - \frac{1}{2}\sigma(x)\psi_{s,t}(x)^2\right]ds \\ &\quad + \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x\psi_{s,t}(x)\cdot W(ds,dy), \qquad t \geq r \geq 0, \end{aligned} \tag{4.17}$$

where "·" denotes the backward stochastic integral.

Theorem 4.2 *Suppose that $\{a,b\} \subset C_b^2(\mathbb{R}), \sigma \in C_b^1(\mathbb{R})$, $h \in L_2(\mathbb{R}) \cap C_b^1(\mathbb{R})$ and $h' \in L_2(\mathbb{R}) \cap C_b^1(\mathbb{R})$. Then for $\phi \in W_2^1(\mathbb{R}) \cap C_b(\mathbb{R})^+$, the backward equation (4.17) has a unique $W_2^1(\mathbb{R}) \cap C_b(\mathbb{R})^+$-valued strong solution $\{\psi_{r,t} : t \geq r \geq 0\}$. Further, we have a.s. $\|\psi_{r,t}\| \leq e^{-b_0(t-r)}\|\phi\|$ for all $t \geq r \geq 0$.*

Proof. For fixed $t > 0$, define the white noise

$$W_t([0,s]\times B) = -W([t-s,t]\times B), \qquad 0 \leq s \leq t, B \in \mathcal{B}(\mathbb{R}). \tag{4.18}$$

By Theorem 4.1, there is a unique strong solution $\{\phi_{r,t} : 0 \leq r \leq t\}$ of the equation

$$\begin{aligned}\phi_{r,t}(x) &= \phi(x) + \int_0^r \left[\frac{1}{2}a(x)\partial_x^2\phi_{s,t}(x) - b(x)\phi_{s,t}(x) - \frac{1}{2}\sigma(x)\phi_{s,t}(x)^2\right]ds \\ &\quad + \int_0^r \int_{\mathbb{R}} h(y-x)\partial_x\phi_{s,t}(x)W_t(ds,dy). \end{aligned} \tag{4.19}$$

Setting $\psi_{r,t}(x) := \phi_{t-r,t}(x)$, we have

$$\begin{aligned}\psi_{r,t}(x) &= \phi(x) + \int_0^{t-r} \left[\frac{1}{2}a(x)\partial_x^2\psi_{t-s,t}(x) - b(x)\psi_{t-s,t}(x) - \frac{1}{2}\sigma(x)\psi_{t-s,t}(x)^2\right]ds \\ &\quad + \int_0^{t-r} \int_{\mathbb{R}} h(y-x)\partial_x\psi_{t-s,t}(x)W_t(ds,dy) \\ &= \phi(x) + \int_r^t \left[\frac{1}{2}a(x)\partial_x^2\psi_{s,t}(x) - b(x)\psi_{s,t}(x) - \frac{1}{2}\sigma(x)\psi_{s,t}(x)^2\right]ds \\ &\quad + \int_r^t \int_{\mathbb{R}} h(y-x)\partial_x\psi_{s,t}(x)\cdot W(ds,dy). \end{aligned}$$

That is, $\{\psi_{r,t} : t \geq r \geq 0\}$ solves (4.17). The remaining assertions are immediate by Theorem 4.1. ∎

5 Conditional log-Laplace functionals

Let (c,h,σ,b,m) be given as in the introduction and assume that the conditions of Theorems 3.1 and 4.2 are satisfied. Let $\{X_t : t \geq 0\}$ be a continuous solution of the SPDE:

$$\begin{aligned}\langle\phi,X_t\rangle &= \langle\phi,\mu\rangle + t\langle\phi,m\rangle + \frac{1}{2}\int_0^t\langle a\phi'',X_s\rangle ds - \int_0^t\langle b\phi,X_s\rangle ds \\ &\quad + \int_0^t\int_{\mathbb{R}}\phi(y)Z(ds,dy) + \int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\phi',X_s\rangle W(ds,dy), \end{aligned} \tag{5.1}$$

where $W(ds, dx)$ is a time-space white noise and $Z(ds, dy)$ is an orthogonal martingale measure which is orthogonal to $W(ds, dy)$ and has covariation measure $\sigma(y)X_s(dy)ds$. Let $(\mathcal{F}_t)_{t\geq 0}$ denote the filtration generated by $\{W(ds, dy)\}$, $\{Z(ds, dy)\}$ and $\{X_s(dy)\}$. By Theorem 4.2, for $\phi \in W_2^1(\mathbb{R})^+$ the equation

$$\begin{aligned}\psi_{r,t}(x) &= \phi(x) + \int_r^t \Big[\frac{1}{2}a(x)\psi''_{s,t}(x) - b(x)\psi_{s,t}(x) - \frac{1}{2}\sigma(x)\psi_{s,t}(x)^2\Big]ds \\ &\quad + \int_r^t \int_{\mathbb{R}} h(y-x)\psi'_{s,t}(x)\cdot W(ds, dy), \qquad t \geq r \geq 0, \end{aligned} \tag{5.2}$$

has a unique strong solution $\psi_{r,t} = \psi^W_{r,t}$. The main result of this section is the following

Theorem 5.1 *Let $\boldsymbol{E}^W$ denote the conditional expectation of $\{X_t : t \geq 0\}$ given the white noise $\{W(ds, dy)\}$. Then for $t \geq r \geq 0$ and $\phi \in W_2^1(\mathbb{R}) \cap C_b(\mathbb{R})^+$ we have a.s.*

$$\boldsymbol{E}^W\{e^{-\langle \phi, X_t\rangle}|\mathcal{F}_r\} = \exp\Big\{-\langle \psi^W_{r,t}, X_r\rangle - \int_r^t \langle \psi^W_{s,t}, m\rangle ds\Big\}, \tag{5.3}$$

where $\psi^W_{r,t}$ is defined by (5.2). Consequently, $\{X_t : t \geq 0\}$ is a diffusion process with Feller transition semigroup $(Q_t)_{t\geq 0}$ given by

$$\int_{M(\mathbb{R})} e^{-\langle \phi, \nu\rangle} Q_t(\mu, d\nu) = \boldsymbol{E}\exp\Big\{-\langle \psi^W_{0,t}, \mu\rangle - \int_0^t \langle \psi^W_{s,t}, m\rangle ds\Big\}. \tag{5.4}$$

We shall give a proof of the theorem by direct calculations based on (5.1) and (5.2). This argument is different from that of Xiong (2003), where the Wong-Zakai approximation was used to get the result. Let a and b be bounded measurable functions on $[0, \infty) \times \mathbb{R}$ such that

$$\int_0^t \int_{\mathbb{R}} a(s, y)^2 ds dy < \infty.$$

For $t \geq r \geq 0$, define

$$\theta(r, t) = \exp\Big\{\int_r^t \int_{\mathbb{R}} a(s, y)W(ds, dy) - \frac{1}{2}\int_r^t \int_{\mathbb{R}} a(s, y)^2 ds dy\Big\}, \tag{5.5}$$

and

$$\zeta(r, t) = \exp\Big\{\int_r^t \int_{\mathbb{R}} b(s, y)Z(ds, dy) - \frac{1}{2}\int_r^t \langle \sigma b(s, \cdot)^2, X_s\rangle ds\Big\}. \tag{5.6}$$

Note that $\theta(r, t)$ and $\zeta(r, t)$ are both martingales in $t \geq r$. By the property of independent increments of the white noise $\{W(ds, dy)\}$ we have

$$\xi_{r,t}(x) := \boldsymbol{E}\{\psi_{r,t}(x)\theta(r, t)\} = \boldsymbol{E}\{\psi_{r,t}(x)\theta(r, t)|\mathcal{F}_r\} \tag{5.7}$$

and

$$\eta_{r,t}(x) := \boldsymbol{E}\{\psi_{r,t}(x)^2\theta(r, t)\} = \boldsymbol{E}\{\psi_{r,t}(x)^2\theta(r, t)|\mathcal{F}_r\}. \tag{5.8}$$

Lemma 5.1 *For $t \geq r \geq 0$, we have a.s.*

$$\boldsymbol{E}\{\langle \psi_{r,t}, X_r\rangle\theta(0, t)\zeta(0, t)|\mathcal{F}_r\} = \langle \xi_{r,t}, X_r\rangle\theta(0, r)\zeta(0, r) \tag{5.9}$$

and

$$\boldsymbol{E}\{\langle \sigma\psi^2_{r,t}, X_r\rangle\theta(0, t)\zeta(0, t)|\mathcal{F}_r\} = \langle \sigma\eta_{r,t}, X_r\rangle\theta(0, r)\zeta(0, r). \tag{5.10}$$

Proof. Since $\boldsymbol{E}^W\{\zeta(r,t)|\mathcal{F}_r\}=1$, by properties of conditional probabilities we have

$$\begin{aligned}\boldsymbol{E}\{\langle\psi_{r,t},X_r\rangle\theta(0,t)\zeta(0,t)|\mathcal{F}_r\} &= \boldsymbol{E}\{\langle\psi_{r,t},X_r\rangle\theta(r,t)\zeta(r,t)|\mathcal{F}_r\}\theta(0,r)\zeta(0,r)\\ &= \boldsymbol{E}\{\langle\psi_{r,t},X_r\rangle\theta(r,t)\boldsymbol{E}^W[\zeta(r,t)|\mathcal{F}_r]|\mathcal{F}_r\}\theta(0,r)\zeta(0,r)\\ &= \boldsymbol{E}\{\langle\psi_{r,t},X_r\rangle\theta(r,t)|\mathcal{F}_r\}\theta(0,r)\zeta(0,r)\\ &= \langle\xi_{r,t},X_r\rangle\theta(0,r)\zeta(0,r).\end{aligned}$$

A similar calculation gives (5.10). ∎

Lemma 5.2 *For $t\geq r\geq 0$ and $x\in\mathbb{R}$, we have a.s.*

$$\begin{aligned}\xi_{r,t}(x)-\phi(x) &= \int_r^t\left[\frac{1}{2}a(x)\xi''_{s,t}(x)-b(x)\xi_{s,t}(x)-\frac{1}{2}\sigma(x)\eta_{s,t}(x)\right]ds\\ &\quad+\int_r^t\langle h(\cdot-x),a(s,\cdot)\rangle\xi'_{s,t}(x)ds,\end{aligned}\tag{5.11}$$

where the derivatives are taken in the classical sense.

Proof. Note that the backward and forward integrals coincide for deterministic integrands. Then we may fix $t>0$ and apply Itô's formula to the process $\{\theta(r,t):r\in[0,t]\}$ to get

$$\theta(r,t)=1-\int_r^t\int_{\mathbb{R}}\theta(s,t)a(s,y)\cdot W(ds,dy).\tag{5.12}$$

By (5.2), (5.12) and backward Itô formula, for any $f\in C_b^\infty(\mathbb{R})$ we have

$$\begin{aligned}\langle\psi_{r,t},f\rangle\theta(r,t) &= \langle\phi,f\rangle+\int_r^t\left[\frac{1}{2}\langle a\psi''_{s,t},f\rangle-\langle b\psi_{s,t},f\rangle-\frac{1}{2}\langle\sigma\psi^2_{s,t},f\rangle\right]\theta(s,t)ds\\ &\quad+\int_r^t\int_{\mathbb{R}}[\langle h(y-\cdot)\psi'_{s,t},f\rangle-\langle\psi_{s,t},f\rangle a(s,y)]\theta(s,t)\cdot W(ds,dy)\\ &\quad+\int_r^t\int_{\mathbb{R}}\langle h(y-\cdot)\psi'_{s,t},f\rangle\theta(s,t)a(s,y)dsdy.\end{aligned}\tag{5.13}$$

Observe that for fixed $t>0$, the process

$$\int_r^t\int_{\mathbb{R}}[\langle h(y-\cdot)\psi'_{s,t},f\rangle-\langle\psi_{s,t},f\rangle a(s,y)]\theta(s,t)\cdot W(ds,dy)$$

is a backward martingale in $r\leq t$. Taking the expectation in (5.13) we obtain

$$\begin{aligned}\langle\xi_{r,t},f\rangle-\langle\phi,f\rangle &= \int_r^t\left[\frac{1}{2}\langle a\xi''_{s,t},f\rangle-\langle b\xi_{s,t},f\rangle-\frac{1}{2}\langle\sigma\eta_{s,t},f\rangle\right]ds\\ &\quad+\int_r^t\int_{\mathbb{R}}\langle h(y-\cdot)\xi'_{s,t},f\rangle a(s,y)dsdy.\end{aligned}$$

Then $\{\xi_{r,t}\}$ must coincides with the classical solution of the parabolic equation (5.11). ∎

Lemma 5.3 *For any $t\geq r\geq 0$, we have a.s.*

$$\langle\phi,X_t\rangle=\langle\psi_{r,t},X_r\rangle+\int_r^t\int_{\mathbb{R}}\psi_{s,t}(x)Z(ds,dx)+\frac{1}{2}\int_r^t\langle\sigma\psi^2_{s,t},X_s\rangle ds+\int_r^t\langle\psi_{s,t},m\rangle ds.\tag{5.14}$$

Proof. By (5.1) and (5.11),

$$\begin{aligned} d\langle \xi_{s,t}, X_s\rangle &= \frac{1}{2}\langle \sigma\eta_{s,t}, X_s\rangle ds + \langle \xi_{s,t}, m\rangle ds - \int_{\mathbb{R}} \langle h(y-\cdot)\xi'_{s,t}, X_s\rangle a(s,y) ds dy \\ &\quad + \int_{\mathbb{R}} \langle h(y-\cdot)\xi'_{s,t}, X_s\rangle W(ds,dy) + \int_{\mathbb{R}} \xi_{s,t}(y) Z(ds,dy). \end{aligned}$$

Since the two martingale measures $\{W(ds,dy)\}$ and $\{Z(ds,dy)\}$ are orthogonal, by Itô's formula we have

$$\begin{aligned} d\langle \xi_{s,t}, X_s\rangle\theta(0,s)\zeta(0,s) &= \frac{1}{2}\langle \sigma\eta_{s,t}, X_s\rangle\theta(0,s)\zeta(0,s)ds + \langle \xi_{s,t}, m\rangle\theta(0,s)\zeta(0,s)ds \\ &\quad + \int_{\mathbb{R}} \langle h(y-\cdot)\xi'_{s,t}, X_s\rangle\theta(0,s)\zeta(0,s)W(ds,dy) \\ &\quad + \int_{\mathbb{R}} \xi_{s,t}(y)\theta(0,s)\zeta(0,s)Z(ds,dy) \\ &\quad + \int_{\mathbb{R}} \langle \xi_{s,t}, X_s\rangle\theta(0,s)\zeta(0,s)a(s,x)W(ds,dy) \\ &\quad + \int_{\mathbb{R}} \langle \xi_{s,t}, X_s\rangle\theta(0,s)\zeta(0,s)b(s,y)Z(ds,dy) \\ &\quad + \langle \sigma\xi_{s,t}b(s,\cdot), X_s\rangle\theta(0,s)\zeta(0,s)ds. \end{aligned}$$

It then follows that

$$\begin{aligned} &\boldsymbol{E}\{\langle \phi, X_t\rangle\theta(0,t)\zeta(0,t)\} - \boldsymbol{E}\{\langle \xi_{r,t}, X_r\rangle\theta(0,r)\zeta(0,r)\} \\ =\ & \frac{1}{2}\boldsymbol{E}\Big\{\int_r^t \langle \sigma\eta_{s,t}, X_s\rangle\theta(0,s)\zeta(0,s)ds\Big\} + \boldsymbol{E}\Big\{\int_r^t \langle \xi_{s,t}, m\rangle\theta(0,s)\zeta(0,s)ds\Big\} \\ & + \boldsymbol{E}\Big\{\int_r^t \langle \sigma\xi_{s,t}b(s,\cdot), X_s\rangle\theta(0,s)\zeta(0,s)ds\Big\}. \end{aligned} \tag{5.15}$$

From (5.6) it is easy to see that

$$\zeta(0,t) = 1 + \int_0^t \int_{\mathbb{R}} \zeta(0,s)b(s,y)Z(ds,dy),$$

and hence

$$\begin{aligned} &\boldsymbol{E}\Big\{\int_r^t \int_{\mathbb{R}} \psi_{s,t}(y)Z(ds,dy)\theta(0,t)\zeta(0,t)\Big\} \\ =\ & \boldsymbol{E}\Big\{\boldsymbol{E}^W\Big[\int_r^t \int_{\mathbb{R}} \psi_{s,t}(y)Z(ds,dy)\zeta(0,t)\Big]\theta(0,t)\Big\} \\ =\ & \boldsymbol{E}\Big\{\boldsymbol{E}^W\Big[\int_r^t \langle \sigma\psi_{s,t}b(s,\cdot), X_s\rangle\zeta(0,s)ds\Big]\theta(0,t)\Big\} \\ =\ & \int_r^t \boldsymbol{E}\Big[\langle \sigma\psi_{s,t}b(s,\cdot), X_s\rangle\theta(0,t)\zeta(0,s)\Big]ds \\ =\ & \int_r^t \boldsymbol{E}\Big[\langle \sigma\xi_{s,t}b(s,\cdot), X_s\rangle\theta(0,s)\zeta(0,s)\Big]ds. \end{aligned} \tag{5.16}$$

By a calculation similar to the proof of Lemma 5.1 we get

$$\boldsymbol{E}\{\langle \psi_{s,t}, m\rangle\theta(0,t)\zeta(0,t)|\mathcal{F}_s\} = \langle \xi_{s,t}, m\rangle\theta(0,s)\zeta(0,s). \tag{5.17}$$

Combining (5.10) and (5.15) – (5.17) gives

$$\begin{aligned}
&\boldsymbol{E}\{\langle\phi, X_t\rangle\theta(0,t)\zeta(0,t)\} - \boldsymbol{E}\{\langle\xi_{r,t}, X_r\rangle\theta(0,r)\zeta(0,r)\} \\
= \; & \frac{1}{2}\boldsymbol{E}\bigg\{\int_r^t \langle\sigma\psi_{s,t}^2, X_s\rangle\theta(0,t)\zeta(0,t)ds\bigg\} + \boldsymbol{E}\bigg\{\int_r^t \langle\psi_{s,t}, m\rangle\theta(0,t)\zeta(0,t)ds\bigg\} \\
& + \boldsymbol{E}\bigg\{\int_r^t \int_{\mathbb{R}} \psi_{s,t}(y)Z(ds,dy)\theta(0,t)\zeta(0,t)\bigg\}.
\end{aligned}$$

But by (5.9) we have

$$\begin{aligned}
&\boldsymbol{E}\{[\langle\phi, X_t\rangle - \langle\psi_{r,t}, X_r\rangle]\theta(0,t)\zeta(0,t)\} \\
= \; & \boldsymbol{E}\{\langle\phi, X_t\rangle\theta(0,t)\zeta(0,t)\} - \boldsymbol{E}\{\langle\xi_{r,t}, X_r\rangle\theta(0,r)\zeta(0,r)\}
\end{aligned}$$

It follows that

$$\begin{aligned}
\boldsymbol{E}\bigg\{\bigg[\langle\phi, X_t\rangle - \langle\psi_{r,t}, X_r\rangle - \frac{1}{2}\int_r^t \langle\sigma\psi_{s,t}^2, X_s\rangle ds - \int_r^t \langle\psi_{s,t}, m\rangle ds \\
- \int_r^t \int_{\mathbb{R}} \psi_{s,t}(x)Z(ds,dx)\bigg]\theta(0,t)\zeta(0,t)\bigg\} = 0.
\end{aligned}$$

Then we have the desired equation. ∎

Proof of Theorem 5.1. Recall that $Z(ds,dy)$ is an orthogonal martingale measure with covariation measure $\sigma(y)X_s(dy)ds$. Then for fixed t_1,

$$\exp\bigg\{-\int_r^t \int_{\mathbb{R}} \psi_{s,t_1}(y)Z(ds,dy) - \frac{1}{2}\int_r^t \langle\sigma\psi_{s,t_1}^2, X_s\rangle ds\bigg\}$$

is a martingale in $t \geq r$ with respect to P^W. By Lemma 5.3 we get a.s.

$$\begin{aligned}
&\boldsymbol{E}^W\{e^{-\langle\phi, X_t\rangle}|\mathcal{F}_r\} \\
= \; & \boldsymbol{E}^W\bigg[\exp\bigg\{-\langle\psi_{r,t}, X_r\rangle - \int_r^t \int_{\mathbb{R}} \psi_{s,t}(y)Z(ds,dy) \\
& \qquad -\frac{1}{2}\int_r^t \langle\sigma\psi_{s,t}^2, X_s\rangle ds - \int_r^t \langle\psi_{s,t}, m\rangle ds\bigg\}\bigg|\mathcal{F}_r\bigg] \\
= \; & \exp\bigg\{-\langle\psi_{r,t}, X_r\rangle - \int_r^t \langle\psi_{s,t}, m\rangle ds\bigg\},
\end{aligned}$$

giving (5.3). In particular, we have

$$\boldsymbol{E}\{e^{-\langle\phi, X_t\rangle}\} = \boldsymbol{E}\exp\bigg\{-\langle\psi_{0,t}, \mu\rangle - \int_0^t \langle\psi_{s,t}, m\rangle ds\bigg\}. \tag{5.18}$$

The distribution of X_t is uniquely determined by (5.18) and the uniqueness of solution of (5.1) follows. This in turn implies the strong Markov property of $\{X_t : t \geq 0\}$. Since $\psi_{r,t}(x)$ is continuous in $x \in \mathbb{R}$, the transition semigroup $(Q_t)_{t\geq 0}$ defined by (5.4) is Feller. ∎

6 Some properties of the SDSMI

We here investigate some properties of the SDSMI. Let (c, h, σ, b, m) be given as in the introduction and assume that the conditions of Theorems 3.1 and 4.2 are satisfied. By Theorem 5.1, for $t \geq r \geq 0$,

$$\int_{M(E)} e^{-\langle \phi,\nu\rangle} Q^W_{r,t}(\mu, d\nu) = \exp\bigg\{ -\langle \psi^W_{r,t}, \mu\rangle - \int_r^t \langle \psi^W_{s,t}, m\rangle ds\bigg\} \tag{6.1}$$

a.s. defines a probability kernel $Q^W_{r,t}(\mu, d\nu)$ on $M(\mathbb{R})$. Indeed, conditioned upon $\{W(ds, dy)\}$, the SDSMI is an inhomogeneous immigration process with transition semigroup $(Q^W_{r,t})_{t\geq r\geq 0}$; see Li (2002). As a special case of the above formula,

$$\int_{M(E)} e^{-\langle \phi,\nu\rangle} Q^0_{r,t}(\mu, d\nu) = \exp\{ -\langle \psi^W_{r,t}, \mu\rangle\} \tag{6.2}$$

a.s. defines a kernel $Q^0_{r,t}(\mu, d\nu)$. Let $N^W_{r,t} = Q^W_{r,t}(0, \cdot)$. We have a.s.

$$N^W_{r,t} = (N^W_{r,s} Q^0_{s,t}) * N^W_{s,t}, \qquad t \geq s \geq r \geq 0, \tag{6.3}$$

and

$$Q^W_{r,t}(\mu, \cdot) = Q^0_{r,t}(\mu, \cdot) * N^W_{r,t}, \qquad t \geq r \geq 0. \tag{6.4}$$

The two equations (6.3) and (6.4) uncover some connections between the SDSM and the SDSMI and suggest there might be a decomposition of the sample paths of the SDSMI into excursions of the SDSM in the lines set up in Dawson and Li (2003) and Li (2002). A systematic investigation of this phenomenon is left to future research.

As another application of the conditional Laplace functionals, we prove the following ergodicity property of the SDSMI.

Theorem 6.1 *Suppose that there is a constant $\epsilon > 0$ such that $b(x) \geq \epsilon$ for all $x \in \mathbb{R}$. Then the SDSMI has a unique stationary distribution Q_∞ given by*

$$\int_{M(\mathbb{R})} e^{-\langle f,\nu\rangle} Q_\infty(d\nu) = \boldsymbol{E} \exp\bigg\{ -\int_0^\infty \langle \psi^W_t, m\rangle dt\bigg\}, \tag{6.5}$$

where $\psi^W_t(x)$ is the solution of (4.1). Moreover, we have $\lim_{t\to\infty} Q_t(\mu, \cdot) = Q_\infty(\cdot)$ by weak convergence for each $\mu \in M(\mathbb{R})$.

Proof. Using the notation of the proof of Theorem 4.2, for any $t \geq r \geq 0$ we have

$$\begin{aligned}
\boldsymbol{E}\bigg\{\int_{M(\mathbb{R})} e^{-\langle \phi,\nu\rangle} N^W_{r,t}(d\nu)\bigg\} &= \boldsymbol{E} \exp\bigg\{ -\int_r^t \langle \psi^W_{s,t}, m\rangle ds\bigg\} \\
&= \boldsymbol{E} \exp\bigg\{ -\int_r^t \langle \phi^W_{t-s,t}, m\rangle ds\bigg\} \\
&= \boldsymbol{E} \exp\bigg\{ -\int_0^{t-r} \langle \phi^W_{s,t}, m\rangle ds\bigg\} \\
&= \boldsymbol{E} \exp\bigg\{ -\int_0^{t-r} \langle \psi^W_{s}, m\rangle ds\bigg\}.
\end{aligned}$$

By Theorem 4.2 we have $\|\psi^W_{s,t}\| \le e^{-\epsilon(t-s)}\|\phi\|$ for $s \le t$. It follows that

$$\begin{aligned}\lim_{t\to\infty}\int_{M(\mathbb{R})} e^{-\langle\phi,\nu\rangle}Q_t(\mu,d\nu) &= \lim_{t\to\infty}\boldsymbol{E}\exp\bigg\{-\langle\psi^W_{0,t},\mu\rangle - \int_0^t\langle\psi^W_{s,t},m\rangle ds\bigg\}\\ &= \lim_{t\to\infty}\boldsymbol{E}\exp\bigg\{-\int_0^t\langle\psi^W_{s,t},m\rangle ds\bigg\}\\ &= \boldsymbol{E}\exp\bigg\{-\int_0^\infty\langle\psi^W_s,m\rangle ds\bigg\}.\end{aligned}$$

On the other hand, by Theorem 4.1 it is easy to get

$$\lim_{\|\phi\|\to 0}\boldsymbol{E}\exp\bigg\{-\int_0^\infty\langle\psi^W_s,m\rangle ds\bigg\} = 1.$$

Then (6.5) defines a probability measure Q_∞ on $M(\mathbb{R})$ and $\lim_{t\to\infty}Q_t(\mu,\cdot) = Q_\infty(\cdot)$ by weak convergence; see e.g. Li (2002, Lemma 2.1). ∎

The properties of the SDSMI varies sharply for different choices of the parameters. The special case where $b(\cdot)\equiv 0$ and $\langle 1,m\rangle = 0$ was discussed in Dawson *et al* (2000, 2001) and Wang (1997, 1998). In this case, we have

$$\begin{aligned}\langle\phi,X_t\rangle &= \langle\phi,\mu\rangle + \frac{1}{2}\int_0^t\langle a\phi'',X_s\rangle ds + \int_0^t\int_{\mathbb{R}}\phi(y)Z(ds,dy)\\ &\quad + \int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\phi',X_s\rangle W(ds,dy).\end{aligned}\tag{6.6}$$

The solution of (6.6) is a critical branching SDSM without immigration. In particular, if $c(\cdot)$ is bounded away from zero, then $\{X_t : t > 0\}$ is absolutely continuous for any initial state X_0; see Dawson *et al* (2000, 2001) and Wang (1997). On the other hand, if $c(\cdot)\equiv 0$, then $\{X_t : t > 0\}$ is purely atomic for any initial state X_0; see Dawson and Li (2003) and Wang (1997, 2002).

Another special case is where $\sigma(\cdot)\equiv 0$ and $\langle 1,m\rangle = 0$. In this case, we get from (6.6) the linear equation

$$\langle\phi,X_t\rangle = \langle\phi,\mu\rangle + \frac{1}{2}\int_0^t\langle a\phi'',X_s\rangle ds - \int_0^t\langle b\phi,X_s\rangle ds + \int_0^t\int_{\mathbb{R}}\langle h(y-\cdot)\phi',X_s\rangle W(ds,dy)\tag{6.7}$$

The process defined in this way is closely related with the superprocesses arising from isotropic stochastic flows investigated by Ma and Xiang (2001). The following theorem shows that $\{X_t : t \ge 0\}$ is absolutely continuous for a large class of absolutely continuous initial states.

Theorem 6.2 *If $\{X_t : t \ge 0\}$ is a solution of (6.7) with $X_0(dx) = v_0(x)dx$ for some $v_0 \in L^2(\mathbb{R})$, then there is an $L^2(\mathbb{R})$-valued process $\{v_t : t \ge 0\}$ such that $X_t(dx) = v_t(x)dx$ a.s. holds.*

Proof. By Kurtz and Xiong (1999, Theorem 3.5), the equation

$$v_t(x) = v_0(x) + \int_0^t\bigg[\frac{1}{2}(av_s)''(x) - b(x)v_s(x)\bigg]ds - \int_0^t\int_{\mathbb{R}}(h(y-\cdot)v_s)'(x)W(ds,dy)\tag{6.8}$$

has a unique $L^2(\mathbb{R})$-valued solution $\{v_t : t \ge 0\}$. Let $X_t(dx) = v_t(x)dx$. Clearly, $\{X_t : t \ge 0\}$ solves (6.7). ∎

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