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An analysis of crystal dissolution fronts in flows through porous media
Part I: Homogeneous charge distribution

Peter Knabner[1] , Cornelius J. van Duijn[2] , Sabine Hengst[3]

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[1] University of Erlangen–Nürnberg
Institute of Applied Mathematics
Martensstraße 3
D – 91058 Erlangen
Germany

[2] Delft University of Technology
Department of Mathematics
P.O. Box 5031
NL 2600 GA Delft
The Netherlands

[3] Institute of Applied Analysis
and Stochastics
Mohrenstraße 39
D – 10117 Berlin
Germany

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Edited by
Institut für Angewandte Analysis und Stochastik (IAAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2004975
e-mail (X.400): c=de;a=d400;p=iaas-berlin;s=preprint
e-mail (Internet): preprint@iaas-berlin.d400.de

An analysis of crystal dissolution fronts in flows through porous media

Part 1: Homogeneous charge distribution [1]

P. Knabner
University of Erlangen–Nürnberg, Institute of Applied Mathematics,
Martensstraße 3, D – 91058 Erlangen, Germany

C. J. van Duijn
Delft University of Technology, Department of Mathematics,
P.O. Box 5031, NL 2600 GA Delft, The Netherlands

S. Hengst
Institute for Applied Analysis and Stochastics (IAAS),
Mohrenstraße 39, D – 10117 Berlin, Germany

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Abstract

We propose a model for transport of solutes in a porous medium participating in a dissolution/precipitation reaction, in general not in equilibrium. For an unbounded spatial domain, travelling wave solutions exists, if and only if the charge distribution is constant and we deal with a dissolution situation. The travelling wave in fact exhibits a sharp dissolution front. The wave is given in a nearly explicit manner. Also for the limit cases of equilibrium reaction or no dispersion, travelling waves are established under the same conditions, but with different qualitative properties.

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1. INTRODUCTION

In this paper we consider chemistry–affected transport processes in porous media, where the transported solutes are participants in a precipitation–dissolution reaction, which in general is not in equilibrium, but kinetically controlled. The mineral phase, which occurs as a very thin layer on the grains of the porous medium and does not effect the pore geometry, may be present or completely dissolved, or not yet precipitated.

In [11] Rubin set up a model for the spatially one–dimensional situation on a bounded domain with specific inflow boundary and initial conditions, which lead to the evolution of a dissolution front. The initial–boundary value problem is formulated classically, i.e. with the explicit appearance of the dissolution front as a free boundary. A transformation of the problem is indicated, on which a numerical approximation scheme and a general analysis assuring existence, uniqueness and some general properties are based in [15] and [10], respectively. Further references to numerical approaches, also for more complex situations, may be found in [15].

We will deal with the same problem and model, also allowing for non–equilibrium, but our aim is different. We start from a model formulation without explicit reference to free boundaries in form of dissolution and precipitation fronts, i.e. from a "weak" or "variational" interpretation of the emerging conservation and rate equations. For two situations, characterized by an unbounded spatial domain and a specific condition in terms of the excess charge distribution, we establish the solutions analytically. For the situation dealt with in this first part, this will be travelling wave solutions, i.e. fixed profiles, which are translated in time with a constant wave speed. These solutions are given in a nearly explicit manner. All qualitative properties of the solution can be investigated in detail.

We will analyse two spatially one–dimensional flow regimes, with constant water content, bulk density, pore velocity $q[cm/s]$ and diffusion/dispersion coefficient $D[cm^2/s]$. The spatial domain will be represented by the whole real line, according to the goal to develop solutions, which describe the behaviour for large flow domains, e.g. large soil columns. In Section 2 we will derive the following set of equations for u and v [millimoles/cm^2], being the molar concentration of one of the participating ions in solution, say the cation, and a scaled concentration of the crystalline solid, both relative to the water volume, and a third unknown $w[-]$, which necessarily appears due to the nature of the dissolution reaction.

$$\frac{\partial}{\partial t}(u+v) + q\frac{\partial}{\partial x}u - D\frac{\partial^2}{\partial x^2}u = 0, \tag{1.1}$$

$$\frac{\partial}{\partial t}v = k\left\{g(u;c) - wK\right\}, \tag{1.2}$$

$$0 \le w(x,t) \le 1, \quad \text{and} \quad w(x,t) = 1 \qquad \text{if} \quad v(x,t) > 0, \tag{1.3}$$

$$\text{for } -\infty < x < \infty, \quad t > 0.$$

The dissolution/precipitation reaction is described by the (nonlinear) function g related to the precipitation reaction, the constant $K > 0$, being the saturation constant, and the rate parameter $k > 0$. There is a further function c appearing in (1.2), which may be interpreted as the (scaled) excess (positive) charge distribution. This function satisfies the linear diffusion–advection equation:

$$\frac{\partial}{\partial t}c + q\frac{\partial}{\partial x}c - D\frac{\partial^2}{\partial x^2}c = 0, \qquad \text{for } -\infty < x < \infty, \quad t > 0. \tag{1.4}$$

Equations (1.1), (1.2), (1.4) have to be supplemented by appropriate initial and/or boundary conditions. This will give c explicitly in the cases to be considered here.

Furthermore, there are several limit situations. If the rate parameter k is large compared to the parameters of the transport process, it is reasonable to use the local equilibrium assumption and to substitute the non–equilibrium description (1.2) by a quasistationary description, which in formally obtained by letting $k \to \infty$, i.e.

$$g(u;c) = wK. \tag{1.5}$$

We will refer to this case as "$k = \infty$".

If the dispersive transport is negligible compared to the advective transport, it is reasonable to let $D \searrow 0$, i.e. cancel the corresponding term in (1.1) and (1.4), and thus consider a hyperbolic system. We will refer to this case as "$D = 0$".

The two charge distributions c to be considered are:

$$- \text{ The function } c \text{ is constant.} \tag{1.6}$$

This situation will be analysed in the first part of the paper. It will turn out that for certain boundary conditions, which correspond to the appearance of a dissolution front, travelling wave solutions exist. In fact, (1.6) is necessary for their existence. These travelling waves will be analysed, for the original problem taking non-equilibrium and dispersion into account, in Section 3, and for both limit cases in Section 4. One can interpret these solutions as the large time behaviour of the situations, where the boundary condition does not disturb the constant initial charge distribution (see Section 3 for more details). As the appearance of a dissolution front often corresponds to the injection of a fluid with a different ionic composition, the contrary, i.e. the disturbance of the initial charge distribution, may be considered to be likely. In the formulation of (1.1) – (1.4) this can be modelled by an initial distribution of c in form of a step function, i.e. with two levels $c^*, c_* > 0$:

$$c(x,0) = c_0(x) = \begin{cases} c^* & \text{for } \; x < 0, \\ c_* & \text{for } \; x \geq 0. \end{cases} \tag{1.7}$$

We will restrict ourselves to the limit case $D = 0$, where the solution of (1.4) is then given by the piston flow profile, i.e. we have

– The function c is given by $\quad c(x,t) = c_0(x - qt)$, $\quad$ (1.8)
with c_0 from (1.7).

In Part 2 of the paper we will analyse this situation, if also the initial conditions for u and v are analogous step functions, i.e. we will consider the Riemann problem of (1.1) – (1.4) (with $D = 0$). It will turn out that the structure of solutions is more involved, exhibiting two fronts of different speed, where one is very much related to the travelling wave solutions of this Part 1.

2. MODELLING OF EQUILIBRIUM AND NON–EQUILIBRIUM CRYSTAL DISSOLUTION AND PRECIPITATION

Mass and Charge Conservation

We are going to discuss the following precipitation-dissolution reaction in a porous medium: We consider as solutes two species M_1, M_2, say M_1 being a cation and M_2 an anion. In addition there may be a crystalline solid $\overline{M}_{12}$ present (at the grain surface). Here we adopt the notation of Rubin [11] and relate to the reasoning of [5], [15], [13]. M_1, M_2 may precipitate at the surface of the porous skeleton and form $\overline{M}_{12}$ and vice versa the crystalline solid may dissolve. The stoichiometry of the reaction is supposed to be as follows

$$\overline{M}_{12} \rightleftarrows nM_1 + mM_2. \tag{2.1}$$

Here n, m denote positive numbers. A reaction like (2.1) leads to the electroneutrality of the fluid, if the valence of M_1 is m and of M_2 is n. Let c_i [millimoles/cm^3] be the molar concentration of M_i in solution relative to the water volume, and let c_{12} [millimoles/g] be the molar concentration of $\overline{M}_{12}$ relative to the mass of the porous skeleton. By its meaning only $c_1, c_2, c_{12} \geq 0$ are reasonable and will be considered. We want to describe the conservation equations for a general flow regime, where at the moment the dimension and geometry of the macroscopic spatial domain are unspecified. The underlying geology and water flow regime are described by the water content $\Theta[-]$, the bulk density $\varrho[g/cm^3]$, the diffusion/dispersion tensor $\underline{\underline{D}}[cm^2/s]$ and the specific discharge vector $\underline{q}^*[cm/s]$. Here we assume that the dispersive part in D is so prominent that we need not to distinguish between the different species. Furthermore, we assume that the pore geometry, i.e. Θ, is not effected by the reaction (2.1). As $\overline{M}_{12}$ is spatially immobile, the conservation of the corresponding total masses leads to the partial differential equations

$$\frac{\partial}{\partial t}(\Theta c_1) + n\varrho\frac{\partial}{\partial t}c_{12} - \text{ div } (\Theta\underline{\underline{D}}\nabla c_1 - \underline{q}^* c_1) = 0, \tag{2.2}$$

$$\frac{\partial}{\partial t}(\Theta c_2) + m\varrho\frac{\partial}{\partial t}c_{12} - \text{ div } (\Theta\underline{\underline{D}}\nabla c_2 - \underline{q}^* c_2) = 0. \tag{2.3}$$

Here $x[cm]$ and $t[s]$ denote points in time and space, and $\frac{\partial}{\partial t}$, div, ∇ the partial time derivative, and the divergence and gradient with respect to x.

If we define

$$c = mc_1 - nc_2 \tag{2.4}$$

then equations (2.2), (2.3) imply that the quantity c is conserved:

$$\partial_t(\Theta c) - \text{ div } (\Theta \underline{\underline{D}} \nabla c - \underline{q}^* c) = 0. \tag{2.5}$$

In general the valence of M_1 is $m\nu$ and the valence of M_2 is $n\nu$ with some positive integer ν. If M_1 is a cation and M_2 an anion, then νc denotes the total positive electric charge of the solute. It is a well-known observation (cf. e.g. [5]) that we can resolve equation (2.5) (with appropriate initial and boundary conditions) at least numerically and thus reduce the number of unknown functions by one by setting

$$c_2 = \frac{1}{n}(mc_1 - c). \tag{2.6}$$

The requirement $c_1 \geq 0, c_2 \geq 0$ for a solution of (2.2), (2.3) is then equivalent to

$$c_1 \geq \left(\frac{c}{m}\right)_+ \quad , \tag{2.7}$$

where $u_+ = u$ for $u \geq 0$ and $u_+ = 0$ for $u < 0$.

Rate description for dissolution and precipitation

Now we turn to the description of the chemical reaction. If r_d and r_p [millimoles$/cm^3$] denote the dissolution and precipitation rates relative to the water volume and k^* $[1/s]$ a rate parameter then

$$\varrho \frac{\partial}{\partial t} c_{12} = \Theta k^* (r_p - r_d). \tag{2.8}$$

Note that the appearance of the factors ϱ, Θ in (2.8) is caused by the choice of the reference systems of r_p and r_d. In [5] e.g. r_p, r_d are related to the total volume such that Θ does not appear in (2.8). The following arguments hold true independent of the reference system chosen, i.e. for another reference system of r_p, r_d equation (2.8) has to be changed correspondingly by a factor on the right hand side and analogously all the following expressions derived from (2.8). For $k^* \to \infty$ equation (2.8) reduces at least formally to

$$r_p = r_d \tag{2.9}$$

(see Section 4 for a precise statement).

I.e., r_p, r_d have to be such that (2.9) is equivalent to the description of the equilibrium of the dissolution/precipitation reaction. Because of this we remind in the following usual assumptions, which lead to the accepted equilibrium description (cf. [13]).

Independent of equilibrium or non–equilibrium, we have:

- The activity of the crystalline solid is a positive constant, say k_d [millimoles/cm^3], therefore:
$$r_d = k_d, \quad \text{if } c_{12} > 0. \tag{2.10}$$
- The dissolution rate is given by the *mass action law* or extensions with concentration dependent activity coefficients (e.g. given by the Debye–Hückel theory), therefore
$$r_p = k_p r(c_1, c_2), \tag{2.11}$$
where r is a smooth nonnegative nonlinearity (for arguments $c_1, c_2 \geq 0$); an example is, if we assume the mass action law
$$r(c_1, c_2) = c_1^n c_2^m. \tag{2.12}$$

The nonlinearity r will be discussed later on in relation to the analysis of Section 3. Thus in the presence of crystalline solid at some (x,t) we have the well–known equilibrium description by the *solubility product:*

$$\text{If } c_{12} > 0, \quad \text{then } r(c_1, c_2) = K, \tag{2.13}$$

where $K := k_d/k_p$. The condition $r(c_1, c_2) = K$ is called *saturation* and $r(c_1, c_2) < K$ or $> K$ *under–* and *oversaturation,* respectively. Due to (2.6) these (in)equalities can be reduced to a relation for c_1 alone for given c. The concentration c_1, fulfilling $r(c_1, c_2) = K$, is also called the *solubility* (for given c).

If $c_{12} = 0$ (at some (x,t)), then c_1, c_2 are no longer determined, but only have to fulfill:

In equilibrium, we have:

- Either saturation or undersaturation holds.
- If crystalline solid is present, then saturation holds.
- Under undersaturated conditions, no crystalline solid can be present.

This is equivalent to

$$\begin{aligned} &0 \leq r(c_1, c_2) \leq K \\ &c_{12} > 0 \Rightarrow r(c_1, c_2) = K \\ &r(c_1, c_2) < K \Rightarrow c_{12} = 0 \end{aligned} \tag{2.14}$$

or in more compact notation

$$\begin{aligned} &0 \leq r(c_1 c_2) \leq K, \ c_{12} \geq 0 \\ &(K - r(c_1, c_2))c_{12} = 0 \end{aligned} \qquad . \tag{2.15}$$

or

$$\begin{aligned} &0 \in k_p r(c_1, c_2) - k_d H(c_{12}) \\ &(\Leftrightarrow r(c_1, c_2) \in K H(c_{12})), \end{aligned} \tag{2.16}$$

where H is the set–valued Heaviside"function" (better: Heavisidegraph) defined by

$$H(u) = \begin{cases} \{1\} & \text{for } u > 0 \\ [0,1] & \text{for } u = 0 \\ \{0\} & \text{for } u < 0. \end{cases} \tag{2.17}$$

Operations with set–valued function are performed by applying them to each possible member of the image sets. Note that with a single–valued version of H, also if discontinuous, we would not be able to express (2.14) in the form (2.16). Each possible choice $H(0) = \alpha(\in[0,1])$ would fix $r(c_1, c_2) = \alpha K$ for $c_{12} = 0$.

We now turn to the rate description for the non–equilibrium case. Usual requirements are (cf. [13], [14]):

In non–equilibrium, we have:

- Also oversaturation is possible. (2.18)
- If there is no crystalline solid present, precipitation can only occur in case of oversaturation. (2.19)

Guided by relation (2.16), we propose the following rate description

$$\varrho \frac{\partial}{\partial t} c_{12} \in \Theta k^*(k_p r(c_1, c_2) - k_d H(c_{12})) \tag{2.20a}$$

or equivalently

$$\begin{aligned} &\varrho \frac{\partial}{\partial t} c_{12} = \Theta k^*(k_p r(c_1, c_2) - k_d w), \\ &\text{where } w \in H(c_{12}) \ \text{ or equivalently} \\ &0 \leq w \leq 1 \quad \text{and} \quad w = 1 \quad \text{for } c_{12} > 0. \end{aligned} \tag{2.20b}$$

Thus we have arrived at (2.2), (2.3), (2.20) with the unknowns c_1, c_2, c_{12} or equivalently at (2.2), (2.5), (2.6), (2.20) with the unknowns c_1, c_{12}, c, supplemented with appropriate initial and boundary conditions, as a possible model for transport and non–equilibrium dissolution and precipitation.

This means that we keep the precipitation rate of (2.11), and also the dissolution rate from (2.10) for $c_{12} > 0$, but allow for a "fictious" dissolution rate $k_d w \leq k_d$ for $c_{12} = 0$. This is necessary to be compatible with (2.16) / (2.14) at equilibrium.

If we use the elimination of c_2 by means of c, setting

$$g(c_1; c) := r\left(c_1, \frac{1}{n}(mc_1 - c)\right), \tag{2.21}$$

and furthermore, define

$$\begin{aligned} u &:= c_1, & v &:= n\varrho/\Theta c_{12}, \\ q &:= q^*/\Theta, & k &:= nk^*k_p, \\ K &:= k_d/k_p \end{aligned} \tag{2.22}$$

the equations (2.2), (2.5), (2.20) take the form, which for one space dimension is displayed in (1.1) – (1.4).

First structural properties of the model

For later purposes we analyse the properties of the function w: Let Ω be the spatial domain, where the dissolution/precipitation is supposed to take place during the time interval $[0, T]$. We subdivide the space–time domain $Q_T = \Omega \times [0, T]$ into three disjoint sets,

$$Q_T = A \cup B \cup C, \tag{2.23}$$

where

$$\begin{aligned} A &= \{(x,t) \in Q_T \mid c_{12}(x,t) > 0\}, \\ B &= \ \text{int}\ \{(x,t) \in Q_T \mid c_{12}(x,t) = 0\}, \\ C &= \ \text{bdry}\ \{(x,t) \in Q_T \mid c_{12}(x,t) = 0\} \backslash A. \end{aligned} \tag{2.24}$$

Here int and bdry denote the topological interior and boundary of the corresponding set with respect to the usual euclidean distance in Q_T, cf. e.g. Simmons [12]. Note that in the definition of C we have to exclude points where $c_{12} > 0$ (e.g. points in A) from the boundary of the set where $c_{12} = 0$, because c_{12} may be apriori discontinuous. We have by (2.20b)

$$w(x,t) = 1 \quad \text{for} \quad (x,t) \in A, \tag{2.25}$$

and as $\frac{\partial}{\partial t} c_{12}(x,t) = 0$ for $(x,t) \in B$:

$$w(x,t) = r(c_1(x,t), c_2(x,t))/K \quad \text{for} \quad (x,t) \in B. \tag{2.26}$$

The topological structure of the set C is not clear apriori, as we want to take here into account all kinds of scenarios caused by various initial and boundary conditions. In particular, we do not know the regularity of the solutions apriori, i.e. we cannot expect that $\frac{\partial}{\partial t}c_{12}$ is continuous etc. (compare the solutions developed in Section 3). In the simple one–dimensional situation of Section 3 the set C will turn out to be a straight line and thus "negligible" in Q_T (rigorously: its (Lebesgue–)measure in Q_T is zero). In general we expect that C is a collection of surfaces in the space–time domain, i.e. a "free boundary" in the problem, as it will be the dissolution front in the situation to be studied in Section 3. If the situation is more complex in the sense that the set C has positive measure, then due to a result in [8] (Lemma A.4, p. 53) we have $\frac{\partial}{\partial t}c_{12}(x,t) = 0$ for almost every $(x,t) \in C$, i.e. possibly with the exception of a set of measure zero. Therefore

$$w(x,t) = r(c_1(x,t), c_2(x,t))/K \quad \text{for almost every} \quad (x,t) \in C. \tag{2.27}$$

Loosely speaking, the function w is determined by c_1, c_2 as given by (2.25) - (2.27) up to a set of measure zero, which in general may be viewed as surfaces in the space–time domain, where the transition $c_{12} > 0$ to $c_{12} = 0$ takes place. The function w may be discontinuous there, as it will be the case for the solutions of Section 3.

In particular, we see that two different formulations based on the equations (2.2), (2.5), (2.6), (2.20) are possible. The one, which we will pursue here, may be called a weak formulation and treats the equations directly by attributing a proper meaning to (2.20a) in the form (2.20b) and requiring the equations only almost everywhere, i.e. in an integral sense. The structure of the solution has to be determined aposteriori, i.e. possible dissolution or precipitation fronts, which are contained in the set C. Especially for more space dimensions this approach allows for more general situations. For this reason we will follow it here. The classical formulation, on the other hand, guesses the appearance of dissolution and precipitation fronts, and treats them explicitly as unknown free boundaries. The set–valued nature of H or equivalently the function w do not appear explicitly in the formulation. For one space dimension and equilibrium reactions this approach has been applied by [11], [15], [10].

Whether the property (2.19) is satisfied, cannot be concluded in general: Due to (2.26) - (2.27) we have:

For a point $(x,t) \in Q_T$ with $r(c_1(x,t), c_2(x,t)) > K$ we have: (x,t) cannot belong to B and it cannot belong to C up to a set of measure 0, as otherwise $w(x,t) > 1$.

I.e., in (2.19) we are dealing with a situation at some "exceptional" $(\overline{x},\overline{t}) \in C$, which we identified as the transition points from $c_{12} = 0$ to $c_{12} > 0$. The notion of precipitation is used here in the following sense due to $c_{12}(\overline{x},\overline{t}) = 0$:

We have that $c_{12}(\overline{x},t) > 0$ and $\frac{\partial}{\partial t}c_{12}(\overline{x},t) \geq 0$ in some time interval $(\overline{t}, \overline{t} + \Delta t]$,

i.e. $r(c_1(\overline{x},\overline{t}), c_2(\overline{x},\overline{t})) \geq K$, assuming c_1, c_2 to be continuous.

In general $r(c_1(\overline{x},\overline{t}), c_2(\overline{x},\overline{t})) = K$ may be possible. In the special situation of a batch experiment, however, i.e. for spatially constant concentrations, equation (2.2) reduces to

$$n\varrho\frac{\partial}{\partial t}c_{12} = -\frac{\partial}{\partial t}(\Theta c_1), \tag{2.28}$$

and thus an increase in c_{12} in $(\bar{t}, \bar{t} + \Delta t]$ would lead to a decrease in c_1 and c_2, i.e. to undersaturation contradicting (2.20b) and $w = 1$. For this argument we need that the nonlinearity g defined by (2.21) is monotone increasing in c_1 (compare Section 3 and Appendix A for the validity of this assumption). We conclude that in the case of a batch experiment, (2.19) holds true for the proposed model. Concerning the validity of the requirement (2.19), note that in [13], [14] only batch situations are considered.

An alternative rate description

Next we propose an alternative formulation for the reaction rate: Another equivalent form of the equilibrium conditions is given by

$$0 \in H(\max(c_{12}, r(c_1, c_2) - K))\,(k_p r(c_1, c_2) - k_d). \tag{2.29}$$

This suggests as alternative to (2.20) the following rate description

$$\varrho\frac{\partial}{\partial t}c_{12} \in \Theta k^* H(\max(c_{12}, r(c_1, c_2) - K))\,(k_p r(c_1, c_2) - k_d) \tag{2.30a}$$

or equivalently

$$\begin{gathered}
\varrho\frac{\partial}{\partial t}c_{12} = \Theta k^* w(k_p r(c_1, c_2) - k_d), \\
\text{where } w \in H(\max(c_{12}, r(c_1, c_2) - K)), \\
\text{or equivalently} \\
0 \le w \le 1, \quad \text{and} \\
w = 1 \quad \text{for } c_{12} > 0 \quad \text{or} \quad r(c_1, c_2) > K.
\end{gathered} \tag{2.30b}$$

This means that the rates of (2.10) and (2.11) are kept, if crystalline solid is present or the fluid is oversaturated. Otherwise, an overall nonpositive rate, i.e. dissolution rate, is possible.

The equivalence of the different formulations

To investigate the relation between this and the model (2.20), we consider the function w in (2.30). At points, where the fluid is saturated, i.e. $r(c_1, c_2) = K$, w cannot be determined from the equation (2.30b). Thus we do not change the solution, if we select w in accordance with (2.30b).

Repeating the above discussion leads to

$$\begin{aligned} w(x,t) &= 1 \quad \text{for} \quad (x,t) \in A, \\ w(x,t) &= 0 \quad \text{for} \quad (x,t) \in B, \\ w(x,t) &= 0 \quad \text{for almost every} \quad (x,t) \in C. \end{aligned} \tag{2.31}$$

With this correspondence of the functions w, we see that the models (2.20) and (2.30) in fact are equivalent, if we compare weak solutions, where the equations (2.2), (2.3), (2.20) or (2.30) are only considered almost everywhere in Q_T.

The formulation, proposed and used in [5] is similar to (2.30), but not identical. It corresponds to (2.30), if the set–valued function is substituted by a single–valued, discontinuous one with $H(0) = 0$. If we interpret the model of [5] in a sense that the rate is not defined at the "exceptional" points of C, i.e. in a weak sense, then due to (2.31) also this formulation is equivalent to the two developed above. If, however, the rate function of [5] is required to hold everywhere, then the models are different and e.g. the solutions developed in Section 3 are excluded.

Linearized rate descriptions

There are still other variants of the rate function, usually to be found in the corresponding text books (see e.g. [13]), which may be considered as linearizations of (2.20) or (2.30). We use the elimination of c_2 by means of c, i.e. let c be given by (2.5) supplemented with appropriate initial and boundary conditions. The nonlinearity in the rate equation (2.20) is then given by $g = g(c_1; c)$. For the following discussion we will assume that there is a unique $c_S = c_S(x,t) \geq 0$, satisfying

$$g(c_S; c) = K, \tag{2.32}$$

i.e. c_S being the solubility and

$$\begin{aligned} g(c_1; c) &< K \quad \text{for} \quad 0 \leq c_1 < c_S, \\ g(c_1; c) &> K \quad \text{for} \quad c_1 > c_S. \end{aligned} \tag{2.33}$$

This property is fulfilled, if g is strictly increasing in the variable c_1. In Appendix A it will be shown that this property holds true for the expressions usually taken for r.

Consider (2.20) or (2.30) for $c_{12} > 0$, i.e.

$$\begin{aligned} \varrho \frac{\partial}{\partial t} c_{12} &= \Theta k^* k_p (g(c_1; c) - g(c_S; c)) \\ &= \Theta k^* k_p \Bigg(\frac{\partial}{\partial c_1} g(c_S; c)\,(c_1 - c_S) \\ &\quad + \text{higher order terms in} \quad (c_1 - c_S) \Bigg). \end{aligned} \tag{2.34}$$

This suggests the following alternative rate description for $c_{12} > 0$

$$\varrho \frac{\partial}{\partial t} c_{12} = \Theta k_L (c_1 - c_S), \tag{2.35}$$

where the rate k_L in this linearized version approximately corresponds to

$$k_L \sim k^* k_p \frac{\partial}{\partial c_1} g(c_S; c). \tag{2.36}$$

The rate function (2.35) can be found, e.g. in [13], [14]. For comparison one has to take into account that a batch experiment situation is considered there, i.e. (2.28) holds and thus with a water content Θ constant in time equation (2.35) is equivalent to

$$\frac{\partial}{\partial t} c_1 = k_L / n (c_S - c_1). \tag{2.37}$$

As above, we have to extend (2.35) also to vanishing crystal concentrations. Analogously to (2.20) and (2.30) we propose the two variants

$$\varrho \frac{\partial}{\partial t} c_{12} \in \Theta k_L (c_1 - H(c_{12}) c_S) \tag{2.38}$$

or

$$\varrho \frac{\partial}{\partial t} c_{12} \in \Theta k_L H(\max(c_{12}, c - c_S)) \, (c - c_S). \tag{2.39}$$

Mathematically, (2.38) and (2.39) are special cases of (2.20) and (2.30), respectively, if one substitutes

$$r(c_1, c_2) = g(c_1; c) = c_1, \ \textit{i.e.} \quad K = c_S, \quad \text{and} \quad k^* k_p = k_L. \tag{2.40}$$

Therefore the above discussion concerning the functions w and the equivalence of the two model formulations hold true here analogously.

3. TRAVELLING WAVE SOLUTIONS FOR NON–EQUILIBRIUM REACTIONS WITH DISPERSION

Significance and definition of travelling wave solutions

The most simple situation appears, if the underlying flow field $\underline{q}^*$ and, correspondingly, Θ and $\underline{\underline{D}}$ are constant. If the influence of boundary conditions etc. is such that fluid flow, solute transport and chemical reaction only take place in the direction of $\underline{q}^*$ and are homogeneous orthogonal to $\underline{q}^*$, we can reduce the model (2.2), (2.5), (2.6), (2.20) to one space dimension, what we will do in the following. In particular,

$\underline{\underline{D}} = D$, $\underline{q^*} = q^*$ are scalar values from now on. This assumption usually is justified considering soil column experiments. We will take $q^* > 0$ such that $x = -\infty$ corresponds to upstream and $x = +\infty$ to downstream. Depending on the interplay of reaction and dispersion, the concentration profiles in a *continuous feed experiment,* i.e. for a constant inflow concentration may be *self–sharpening fronts,* i.e. stabilize for large times to a fixed spatial profile, which is transported in time with a constant wave speed a. The mathematical model for this limit solution is a travelling wave solution, i.e. a solution of (2.2), (2.3), (2.20) for one space dimension, which only depends on the variable

$$\eta = x - at. \tag{3.1}$$

Setting $c_i(x,t) = c_i(\eta)$, $i = 1,2$, $c_{12}(x,t) = c_{12}(\eta)$, and as a consequence $c(x,t) = c(\eta)$, we obtain the ordinary differential equations

$$\begin{aligned} &- a(\Theta c_1 + n\varrho c_{12})' \; - \; \Theta D c_1'' + q^* c_1' = 0, &(3.2a)\\ &- a(\Theta c_2 + m\varrho c_{12})' \; - \; \Theta D c_2'' + q^* c_2' = 0, &\\ &- a\varrho c_{12}' \in \Theta k^*(k_p r(c_1, c_2) - k_d H(c_{12})), &(3.2b) \end{aligned}$$

and

$$(-a\Theta + q^*)c' - \Theta D c'' = 0 \tag{3.3}$$

for $-\infty < \eta < \infty$. To select the solutions related to the situation described above, we have to prescribe boundary conditions at $\eta = -\infty$ and $\eta = +\infty$:

$$\begin{array}{llll} c_i(-\infty) & = c_i^*, & c_i(+\infty) & = c_{i*}, \qquad i = 1,2, \\ c_{12}(-\infty) & = c_{12}^*, & c_{12}(+\infty) & = c_{12*}. \end{array} \tag{3.4}$$

If we consider a travelling wave solution as the limit profile for $t \to \infty$ for a corresponding initial–boundary value problem for $x \geq 0$, then the boundary condition at $\eta = +\infty$ corresponds to the (constant) initial condition $(x > 0, t = 0)$ and the boundary condition at $\eta = -\infty$ to the (inflow) boundary condition for $x = 0,\ t > 0$. Note that any solution of (3.2) – (3.4) can be translated in η by an arbitrary amount to give a new solution. The one, whose total mass corresponds asymptotically for large times to the total mass of the solution of the initial–boundary value problem, will appear to be the asymptotic limit.

Homogenous charge distribution as necessary condition for travelling waves

As a first observation, also the electric charge c fulfills boundary conditions at $\eta = \pm\infty$ due to (3.4) and (2.4), which is only possible, if c is constant for all $-\infty < \eta < \infty$ or equivalently the boundary conditions of c coincide:

$$mc_1^* - nc_2^* = mc_{1*} - nc_{2*} \quad . \tag{3.5}$$

Thus (3.5) is a necessary condition for the existence of a travelling wave solution and will be assumed to hold from now on in this part of the paper. It may be interpreted as the fact and the overall electric charge of the invading fluid coincides with the electric charge of the resident fluid. If this situation does not occur, different asymptotic profiles will develop, which are the subject of Part 2 of the paper.

Travelling wave formulation in transformed variables, fixed wave speed

With c being a constant given by

$$c := mc_1^* - nc_2^* \tag{3.6}$$

we reduce the problem to the variables $u = u(\eta)$, $v = v(\eta)$ defined in (2.22) and $w = w(\eta)$:

$$-au' - av' - Du'' + qu' = 0, \tag{3.7}$$

$$-av' = k(g(u;c) - Kw), \tag{3.8}$$

$$0 \le w(\eta) \le 1, \quad w(\eta) = 1, \quad \text{if } v(\eta) > 0, \tag{3.9}$$

for $-\infty < \eta < \infty$.

The boundary conditions transform to

$$\begin{aligned} u(-\infty) = u^*, \quad u(+\infty) = u_*, \\ v(-\infty) = u^*, \quad v(+\infty) = v_*, \end{aligned} \tag{3.10}$$

where $v^* = n\varrho/\Theta c_{12}^*$ etc.

Note that $q > 0$ such that $\eta = -\infty$ corresponds to upstream and $\eta = +\infty$ to downstream. We consider only nonnegative boundary conditions and require the solution to be nonnegative, which is equivalent to (see (2.7))

$$u \ge \left(\frac{c}{m}\right)_+, \quad v \ge 0. \tag{3.11}$$

We allow for jumps of w, v' and u'', and require all other appearing functions to be continuous. For (3.7), (3.8) to hold properly, the jumps must compensate each other. If $[f] := f(x+) - f(x-)$ denotes the jump of a function f at a point x from right to left, then a solution of (3.7) - (3.9) has to satisfy the *jump relation*

$$a[v'] = -D[u''] = kK[w]. \tag{3.12}$$

By integration, taking (3.10) into account, equation (3.7) can be reduced to a first order equation (see Proposition 1.2, Corollary 1.3 in [3] for details):

$$\begin{aligned} u' &= \frac{q-a}{D}(u - u_*) - \frac{a}{D}(v - v_*) \\ &= \frac{q-a}{D}(u - u^*) - \frac{a}{D}(v - v^*). \end{aligned} \tag{3.13}$$

This procedure also fixes the wave speed a to

$$a = \frac{\Delta u}{\Delta u + \Delta v}\, q, \quad \text{if} \quad \Delta u + \Delta v \ne 0, \tag{3.14}$$

where $\Delta u := u^* - u_*, \quad \Delta v := v^* - v_*$.

Excluding some trivial cases, we may assume $\Delta u + \Delta v \neq 0$, thus having a fixed wave speed. It will turn out that for the only relevant case, where a travelling wave exists $\Delta u < 0$, $\Delta v < 0$ such that $a < q$, reflecting the retardation effect of the chemical reaction, analogous to adsorption reactions (see van Duijn and Knabner [2]).

The impossibility of precipitation waves

For the further discussion, we have to assume for fixed c:

$$\begin{aligned} & g(.;c) \text{ is strictly increasing for } u \geq \left(\frac{c}{m}\right)_+, \\ & g\left(\left(\frac{c}{m}\right)_+; c\right) = 0. \end{aligned} \tag{3.15}$$

If r is given by the mass action law according to (2.12), then (3.15) is obvious, as $c_2 \geq 0$ is guaranteed by $u \geq (c/m)_+$. For the Debye–Hückel description of r in Appendix A a reasonable sufficient condition for (3.15) is developed.

Condition (3.15) assures the unique existence of the *solubility* concentration $u_S \geq 0$ (related to c and K):

$$g(u_S; c) = K \tag{3.16}$$

and u_S monotonically depends on K (for fixed c), i.e. for increasing K also u_S increases.

The first order system formulation (3.13), (3.7), (3.10) guarantees $u'(+\infty) = 0$, $u'(-\infty) = 0$. A further necessary condition is $v'(+\infty) = 0$, $v'(-\infty) = 0$. A rigorous discussion displayed in detail in [3] concludes from this:

Only for the following boundary conditions travelling waves may be possible:

- *The stationary wave* $(a = 0)$ *for a saturated fluid:*
  $u^* = u_* = u_S$
  and therefore $u(\eta) = u_S$ for $-\infty < \eta < \infty$.
  $v^* > 0, \quad v_* > 0, \quad v^* \neq v_*$
  with arbitrary $v(\eta) > 0$ for $-\infty < \eta < \infty$.
  This trivial case will not be considered furthermore.

- *The dissolution wave:*

$$\begin{aligned} & v^* = 0, \quad v_* > 0 \quad \text{arbitrary} \\ & \left(\frac{c}{m}\right)_+ \leq u^* < u_S, \quad u_* = u_S. \end{aligned} \tag{3.17}$$

- *The precipitation wave:*

$$\begin{aligned} & v^* > 0 \quad \text{arbitrary}, \quad v_* = 0 \\ & u^* = u_S, \quad \left(\frac{c}{m}\right)_+ \leq u_* < u_S. \end{aligned} \tag{3.18}$$

The next important observation, justified in detail in [3], is:

Both, dissolution and precipitation waves can only exist for undersaturated fluids:

$$u(\eta) < u_S \quad \text{for} \quad -\infty < \eta < \infty. \tag{3.19}$$

An immediate consequence from (3.8) and (3.9) is:

$$v'(\eta) > 0, \quad \text{if} \quad v(\eta) > 0, \tag{3.20}$$

and from this:

$$\text{Precipitation waves cannot exist.} \tag{3.21}$$

Hence, the only possibility left is a dissolution wave, which we will show to exist.

The dissolution front of a dissolution wave

Due to the boundary conditions the cases $v(\eta) = 0$ for each $-\infty < \eta < \infty$ and also $v(\eta) > 0$ for each $-\infty < \eta < \infty$ are excluded (see [3], Proposition 2.3 for a full argument) such that a dissolution wave must have a dissolution front in the following sense:

There exists a number L such that

$$\begin{aligned} v(\eta) = 0 \quad &\text{for} \quad -\infty < \eta \leq L, \\ v(\eta) > 0 \quad &\text{for} \quad L < \eta < \infty. \end{aligned} \tag{3.22}$$

In the variables (x,t) this front has the position $(at+L,t)$. The line $x = s(t) = at+L$ is the free boundary of the solution, relationships (3.12) together with (3.22) build the free boundary condition. A further discussion (see [3]) reveals

$$u'(\eta) > 0 \quad \text{for} \quad -\infty < \eta < \infty, \tag{3.23}$$

$$\begin{aligned} u''(\eta) > 0 \quad &\text{for} \quad -\infty < \eta < L, \\ u''(\eta) < 0 \quad &\text{for} \quad L < \eta < \infty, \end{aligned} \tag{3.24}$$

$$v''(\eta) < 0 \quad \text{for} \quad L < \eta < \infty. \tag{3.25}$$

That is, u changes curvature at the dissolution front and this is the only position, where a jump according to (3.12) really occurs, and the jump is positive, as due to (3.25) $v'(L+) > 0$ (and $v'(L-) = 0$).

Furthermore, as developed for a general situation in (2.23) – (2.27), the function w is determined uniquely by the other variables: The subdivision of (2.23) is given here by

$$\begin{aligned} A =&\{(x,t) \in Q_T | \quad x > at + L\} \\ B =&\{(x,t) \in Q_T | \quad x < at + L\} \\ C =&\{(x,t) \in Q_T | \quad x = at + L\}, \end{aligned} \tag{3.26}$$

where the domain Ω here is the whole real line. According to (2.25), (2.26)

$$\begin{aligned} w(\eta) &= 1 \quad \text{for} \quad \eta > L, \\ w(\eta) &= g(u(\eta);c)/K \quad \text{for} \quad \eta < L. \end{aligned} \tag{3.27}$$

Again we see that w has a jump discontinuity at $\eta = L$, where w is not defined. This is in accordance with (2.27), as C, being just a straight line, has measure zero in Q_T.

A shooting algorithm to compute a travelling wave

Due to its structure, the computation of u, v, w according to (3.13), (3.8) - (3.10) is equivalent to find a number $u^* < u_0 < u_* \ (= u_S)$ such that the solution of the initial value problem

$$\begin{aligned} u' &= \frac{q-a}{D}(u-u^*) - \frac{a}{D}(v-v^*) \quad \text{for} \quad \eta > L \\ v' &= \frac{k}{a}(K - g(u;c)) \qquad\qquad\quad \text{for} \quad \eta > L \\ u(L) &= u_0, \qquad v(L) = 0 \end{aligned} \tag{3.28}$$

fulfills

$$u(\infty) = u_*, \ v(\infty) = v_*, \ v(\eta) > 0 \quad \text{for} \quad \eta > L.$$

If such a value u_0 can be found then the solution of (3.28) can be prolongated to the desired solution by solving

$$\begin{aligned} u' &= \frac{q-a}{D}(u-u^*) \qquad \text{for} \quad \eta < L \\ u(0) &= u_0. \end{aligned} \tag{3.29}$$

The solution is given by

$$u(\eta) = (u_0 - u^*)\exp\left(\frac{q-a}{D}\eta\right) + u^* \quad \text{for} \quad \eta \leq L. \tag{3.30}$$

For any $u^* < u_0 < u_*$ we have

$$u'(\eta) > 0, \qquad \text{if} \quad v(\eta) < l(u(\eta)), \tag{3.31}$$

where l is the straight line connecting (u^*, v^*) and (u_*, v_*), and

$$v'(\eta) > 0, \qquad \text{if} \quad u(\eta) < u_*. \tag{3.32}$$

Therefore three cases can happen:

Case A: u, v increase, till eventually for some $\overline{\eta} > L \quad v(\overline{\eta}) = l(u(\overline{\eta})), \quad u(\overline{\eta}) < u_*$.

Case B: u, v increase, till eventually for some $\overline{\eta} > L \quad u(\overline{\eta}) = u_*, \quad v(\overline{\eta}) < l(u(\overline{\eta}))$.

Case C: Neither case A nor case B happen, such that $u(\infty) = u_*, \quad v(\infty) = v_*$, i.e. u_0 is the desired value.

As values u_0 from case A must be smaller as values from case B, the case C must occur for at least one u_0 and such a value can be found by the following *shooting algorithm.*

I. $n = 0$

Select $u^* < u_L^n < u_*$ small enough such that case A occurs for $u_0 = u_L^n$.

Select $u^* < u_R^n < u_*$ big enough such that case B occurs for $u_0 = u_R^n$.

II. $n := n + 1, \quad u^{n+1} := \frac{1}{2}(u_L^n + u_R^n)$,

compute u, v according to (3.28) and $u_0 = u^{n+1}$ and check whether case A or case B occurs.

For case A: $u_L^{n+1} := u^{n+1}, \quad u_R^{n+1} := u_R^n$.

For case B: $u_L^{n+1} := u_L^n, \quad u_R^{n+1} := u^{n+1}$.

If case C occurs or $\left|u_R^{n+1} - u_L^{n+1}\right|$ is small enough, stop, otherwise goto II.

This algorithm will at least produce sequences $u_L^n \leq u_0 < u_R^n$ such that $|u_R^n - u_L^n| \to 0$. The computation of u, v according to (3.28) in I. and in II. can be performed with any numerical procedure, if k and D are in the same range as the other parameters. If one also wants to compute the limit cases $k \to \infty$ or $D \to 0$ (see Section 4), the system (3.28) becomes very stiff, so that the use of a corresponding procedure with order and step size control as e.g. Gear's method (see e.g. [7]) becomes decisive for the correct performance of the shooting algorithm. For the following examples the parameters still allowed the use of a method for mildly stiff system with appropriately chosen fixed step size (see [1]).

The parameters of the following basic example, called the reference case and depicted in Figure 1, are assembled in Table 1 and chosen analogously to the computation of [15], p. 1568, Figure 3. The differences are the following: K is slightly smaller, in [15] only the equilibrium case $k = \infty$ is considered, and most important, the charge distribution c of [15] corresponds to the step function (1.7) with $c^* = 2.0 * 10^{-5}, c_* = 0$ rather than a constant $c = 0$. Therefore the computations of [15] are in relation to the solutions of Part 2 of the paper and will be discussed there.

$$\begin{aligned}
\Theta &= 0.32[-], \ \varrho = 1,8[g/cm^3], \\
q^* &= 0.3 * 10^{-3}[cm/s], \ \Theta D = 0.2 * 10^{-3}[cm^2/s], \\
M_1 &= Sr^{2+}, \ M_2 = SO_4^{2-}, \\
n &= m = 1, \ r(c_1, c_2) = c_1 c_2, \\
c &= 0, \\
k_p &= 1, \ k_d = K = 3.86884 * 10^{-7}, \\
k^* &= 0.1, \ \text{i.e. } k = 0.1, \\
c_1^* &= 2.0 * 10^{-5} \ [millimoles/cm^3], \\
c_{12}^* &= 0 \ [millimoles/g], \\
c_{1*} &= 6.22 * 10^{-4}, \\
c_{12*} &= 4.9 * 10^{-5}
\end{aligned}$$

Table 1: Parameters of the reference case.

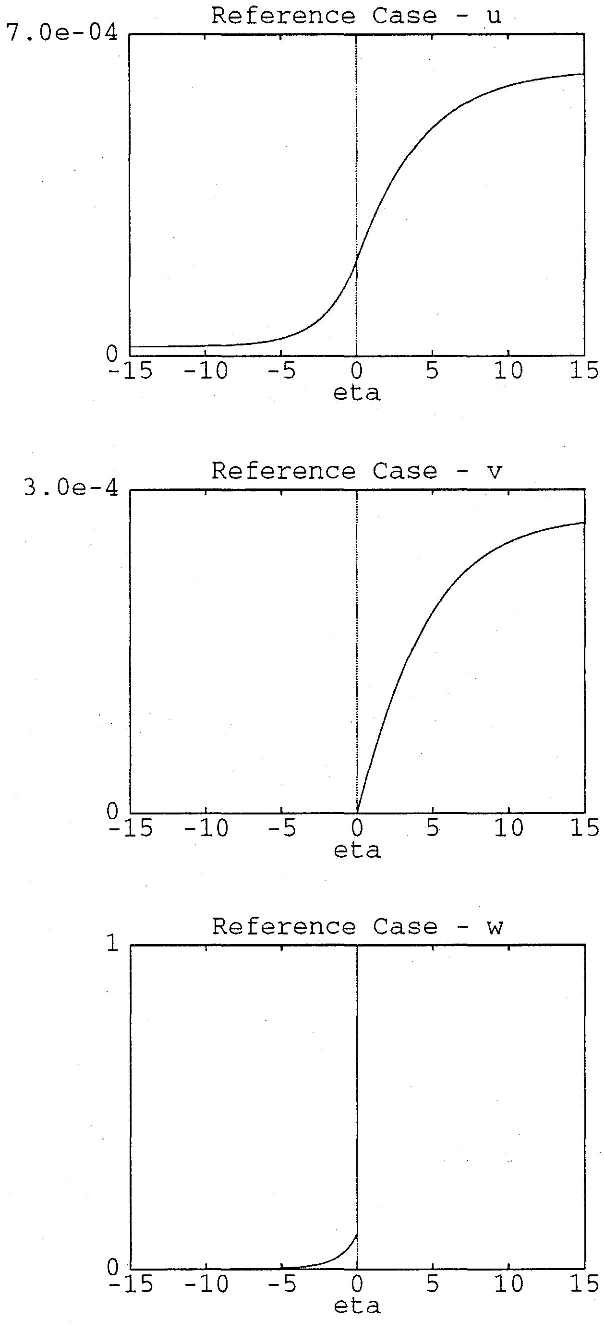

Figure 1: The travelling wave solution for the reference parameters.

The closed form solution for the linearized model

For the linearized model (2.38), i.e. $g(u;c) = u,\ u_S = K$, the initial value u_0 from (3.28) can be computed explicitly:
Rewriting (3.28) as the second order equation

$$Du'' - (q-a)u' - k(u-K) = 0 \tag{3.33}$$

we see that the general solution with the boundary condition $u(\infty) = u_* = u_S$ is given by

$$u(\eta) = u_S - (u_S - u_0)\exp(-\lambda(\eta - L)) \quad \text{for} \quad \eta \geq L, \tag{3.34a}$$

with

$$\lambda = \frac{q-a}{2D}\left(\left(1 + \frac{4Dk}{(q-a)^2}\right)^{1/2} - 1\right) \tag{3.34b}$$

and thus

$$v(\eta) = \frac{k}{a}(u_S - u_0)\frac{1}{\lambda}\left(1 - \exp(-\lambda(\eta - L))\right) \quad \text{for} \quad \eta \geq L. \tag{3.35}$$

The value u_0 is determined by the shooting requirement

$$\frac{k}{a}(u_S - u_0)\frac{1}{\lambda} = v(\infty) \overset{!}{=} v_* \tag{3.36}$$

which gives

$$u_0 = u_S - \frac{2(u_S - u^*)}{1 + \left(1 + \frac{4Dk}{(q-a)^2}\right)^{1/2}}. \tag{3.37}$$

For the reference case, a comparison with the linearized model with k chosen according to (2.36) is shown in Figure 2, which shows only slight variations.

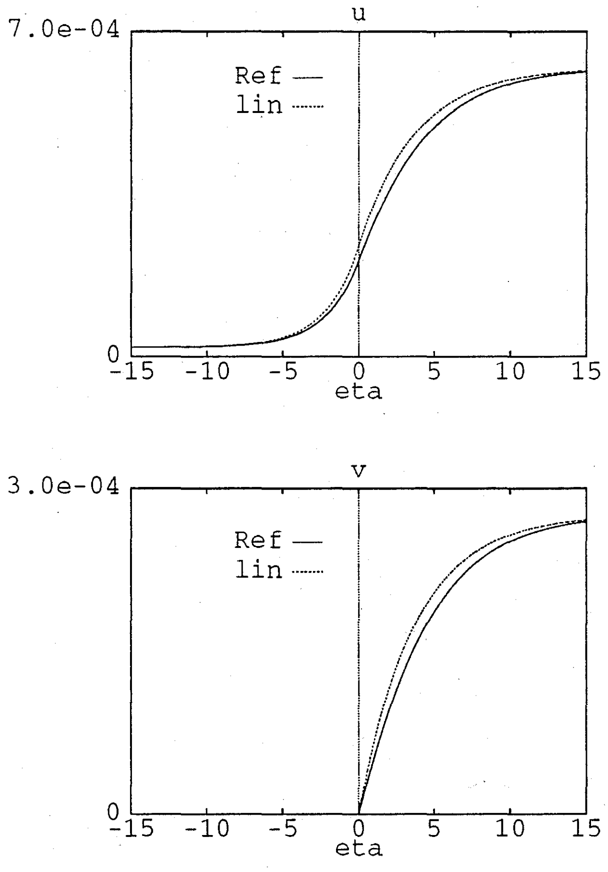

Figure 2: The travelling wave solution for the reference case (mass action law) and the linearized model.

4. TRAVELLING WAVE SOLUTIONS FOR EQUILIBRIUM REACTIONS OR NEGLIGIBLE DISPERSION

Equilibrium reactions

In Section 1 we indicated two limit cases, which we will examine in this section. The first "$k = \infty$", corresponds to the formal procedure

$$k \to \infty, \quad K > 0 \quad \text{constant} \quad (k, K \text{ defined in } (2.22))$$

and substitutes in a quasistationary manner the rate equation (2.20) by the equilibrium description (2.15) (or equivalently (2.14) or (2.16)).

Travelling wave formulation

The corresponding travelling wave formulation is given by (3.2a), (3.3), (3.4) with (3.2b) substituted by (2.15). We see that we can repeat the discussion of Section 3 leading to (3.5) as a necessary condition for travelling waves and the following formulation in the transformed variables:
We look for functions $u = u(\eta)$, $v = v(\eta)$ and $w = w(\eta)$ such that

$$-au' - av' - Du'' + qu' = 0, \tag{4.1}$$

$$g(u; c) = Kw, \tag{4.2}$$

$$0 \leq w(\eta) \leq 1, \quad w(\eta) = 1, \quad \text{if} \quad v(\eta) > 0, \tag{4.3}$$

for $-\infty < \eta < \infty$, which satisfy the boundary and sign conditions (3.10), (3.11).

Reduced continuity of solutions

As we will see later on, to make a solution possible we have to give up the continuity of the crystal dissolution v and allow for jumps in v. This is reasonable, as no ordinary differential equation has to hold for v. Similarly, we have to allow for jumps in u' and thus u'' does not exist (as a function). This requires a proper interpretation of (4.1): If the jumps of Du' and av cancel each other, then the combination

$$Du' + av \quad \text{is continuous} \tag{4.4}$$

and we can interpret (4.1) as

$$(Du' + av)' = (q - a)u' \tag{4.5}$$

such that the *jump relations* are

$$D[u'] = -a[v] = \frac{D}{q-a}\left[(Du' + av)'\right]. \tag{4.6}$$

The impossibility of precipitation waves and a closed form solution for dissolution waves

The discussion leading to the wave speed a given by (3.14) and to the possible cases of dissolution or precipitation wave (3.17) or (3.18) can be repeated. The conclusion (3.19), that the fluid is undersaturated is not true here, since it was based on the fact to have the ordinary differential equation (3.8) at one's disposal (see [3], proof of Prop. 2.1). It can only be concluded that the fluid is not oversaturated. But again a precipitation wave is impossible and the dissolution wave has a simple structure leading to an explicit solution. This can be seen as follows:

In an η–interval, where v is positive, then due to (4.2), (4.3) and the uniqueness of the solubility concentration (according to (3.16)) we have $u = u_S$ and by (3.13) $v = \tilde{v}$, where $\tilde{v} = v_*$ for a dissolution wave and $\tilde{v} = v^*$ for a precipitation wave. Thus a wave induces a sequence of η–subintervals on which either $v = \tilde{v} > 0$ and $u = u_S$ or $v = 0$. In the later subintervals u is given explicitly by (3.13) and the value $u = u_S$ at the left boundary point. From this we can conclude (compare [3], proof of Prop. 4.3 for dissolution waves):

$$\text{Precipitation waves cannot exist.} \tag{4.7}$$

Furthermore, a dissolution wave is characterized by a dissolution front $\eta = L$ and

$$v(\eta) = 0 \quad \text{for} \quad \eta < L, \quad v(\eta) = v_* \quad \text{for} \quad \eta > L, \tag{4.8}$$

$$u(\eta) = (u_S - u^*) \exp\left(\frac{q-a}{D}(\eta - L)\right) + u^* \quad \text{for} \quad \eta < L, \tag{4.9}$$

$$u(\eta) = u_S \quad \text{for} \quad \eta \geq L.$$

Thus $\eta = L$ is the only point where jumps according to (4.6) occur, where can be made more precise by $[v] = v_*$ for $\eta = L$. Again it represents the free boundary $x = s(t)$ of the solution, (4.6) together with $v(s(t), t) = 0$ being the free boundary condition. It is identical with the free boundary condition of [15] (p. 1563, (5d)) for this special case. As also $u(s(t), t) = u_S$, the transformation $\tilde{u} = u - u_S$ leads to the classical Stefan problem and is a special case of the transformation applied in [11].

Convergence (rates) for $k \to \infty$

To have a consistent modelling of the equilibrium and the non–equilibrium case, we expect that the solutions of (3.7) - (3.10) converge to (4.8), (4.9) for $k \to \infty$ and all other parameters fixed. Figure 3 displays a sequence of numerical solutions for the data of Table 1 and $k = 0.1, 1, 10, 100$ which clearly show a convergence from below to the equilibrium solution (4.8), (4.9).

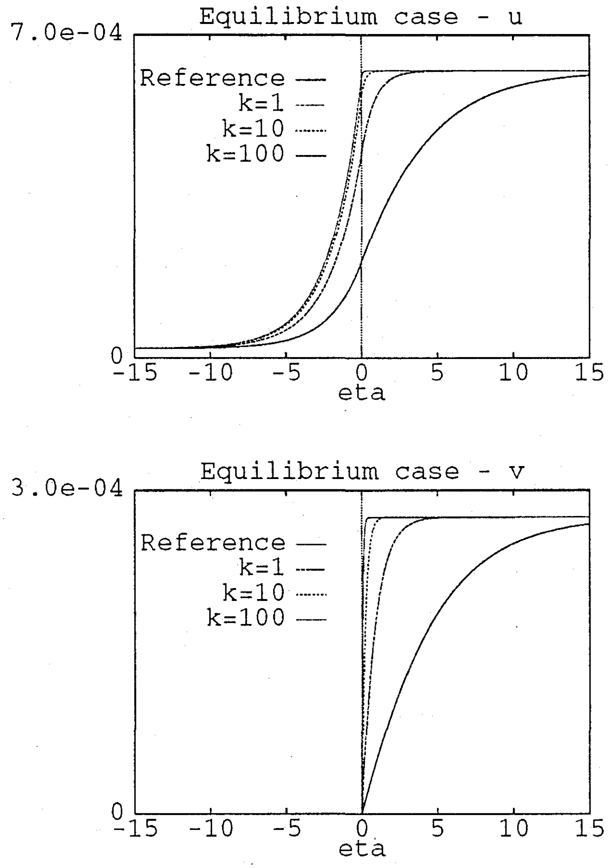

Figure 3: Convergence of the travelling wave solutions for $k \to \infty$.

This observation can also be made rigorous (see [3], Th. 4.5). Furthermore, even the rate of convergence can be detected, what we will do for the linearized model by means of its closed form solution (3.34) – (3.36). The most critical point is the front $\eta = L$ due to the jump of v for $k = \infty$: From (3.36) we see for $k < \infty$:

$$k^{1/2}(u_S - u(L)) \to \frac{(q-a)(u_S - u^*)}{D^{1/2}}, \quad \text{for} \quad k \to \infty, \tag{4.10}$$

i.e. the convergence rate is $1/k^{1/2}$ and even the coefficient of the leading term is given. By the explicit solution (3.30) and (4.9) for $\eta < L$ respectively, we see that the same rate also holds true for $\eta < L$, and for $\eta > L$ even an exponential term is added due to (3.34): A more general reasoning (see [3], Th. 5.1) reveals that for general chemistry descriptions r we can nearly justify the rate of (4.10): $k^{1/2}$ has to substituted by $k^{1/2}/\log(k)$. Numerical experiments indicate that one cannot dispense with the logarithmic factor.

Negligible Dispersion

The second limit case "$D = 0$" introduced in Section 1 corresponds to the formal procedure

$$D \to 0$$

and emphasizes situations, where advective or kinetic effects are much more important than dispersive ones.

Travelling wave formulation

We set $D = 0$ in (3.2) - (3.4), i.e.:
We look for functions $u = u(\eta), v = v(\eta)$, and $w = w(\eta)$ such that

$$-au' - av' + qu' = 0, \tag{4.11}$$
$$-av' = k(g(u;c) - Kw), \tag{4.12}$$
$$0 \leq w(\eta) \leq 1, \quad w(\eta) = 1, \quad \text{if} \quad v(\eta) > 0 \tag{4.13}$$

which satisfy the boundary and sign conditions (3.10), (3.11). As we will see later on, to make a solution possible, we have to allow for jumps in u', v' and w, which have to cancel each other due to the following jump relations

$$\frac{\Delta v}{\Delta u}[u'] = [v'] = \frac{kK}{a}[w] \tag{4.14}$$

as $(q-a)/a = \Delta v / \Delta u$ (defined in (3.14)).

The impossibility of precipitation waves

As above, we can repeat the discussion of Section 3, leading to the wave speed (3.14) and a first order system, which instead of (3.13) contains

$$(q-a)(u-u^*) = a(v-v^*), \tag{4.15}$$

i.e. the shape of u and v are the same. The whole reasoning from (3.17) to (3.22) holds true also here, even with simpler arguments. The only possible case of a dissolution wave is given in a nearly explicit form: It is given by the dissolution front $\eta = L$ and

$$v(\eta) = 0 \text{ for } \eta \leq L, \; v = \frac{\Delta v}{\Delta u}(u-u^*) \quad \text{for} \quad \eta > L, \tag{4.16}$$
$$u(\eta) = u^* \text{ for } \eta \leq L, \; u' = \frac{k}{q-a}(K - g(u;c)), \; u(L) = u^* \text{ for } \eta > L, \tag{4.17}$$

i.e. only the initial value problem in (4.17) has to be resolved.
In particular, the jumps of (4.14) only occur at $\eta = L$ and can be made more precise by $[w] = 1 - g(u^*;c)/K$. Again, with $v = 0$, they give the free boundary condition.

Convergence(rates) for $D \to 0$

The convergence of the solutions of (3.7) - (3.10) to the ones of (4.16), (4.17) for $D \to 0$ can be clearly observed numerically (see Figure 4, where the data of Table 1 are used with $D = 6.25 * 10^{-4}, 2.76 * 10^{-4}, 6.25 * 10^{-6}$). The convergence is not from one side, rather there is one intersection point of all profiles for $\eta > L$. There is a rigorous argument for this assertion ([3], Prop. 4.7). For the linearized model its closed form solution (3.34) – (3.36) reveals a rate of convergence. At $\eta = L$ we have for $D \to 0$

$$\begin{aligned} \frac{1}{D}\left(u(L) - u^*\right) &= \frac{(u_S - u_*)4k}{(q-a)^2} \Bigg/ \left(\frac{4Dk}{(q-a)^2} + 2\left(1 + \frac{4Dk}{(q-a)^2}\right)^{1/2} + 2 \right) \\ &\to \frac{(u_S - u_*)k}{(q-a)^2}, \end{aligned} \tag{4.18}$$

i.e. the convergence is linear in D. This rate, even with the leading coefficient of (4.18) can also be justified in general (see [3], Th. 5.4) and it holds true for all arguments η, for $\eta < 0$ even enhanced by an exponential term.

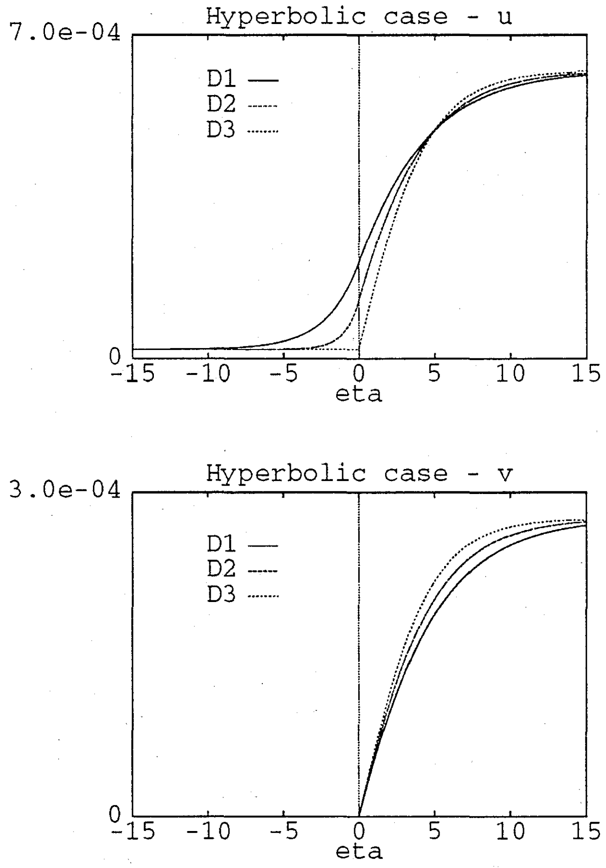

Figure 4: Convergence of the travelling wave solution for $D \to 0$.

CONCLUSIONS

We set up a model for transport and dissolution/precipitation, where the kinetics of the reaction is taken into account. Contrary to other possible approaches, our model is consistent with the corresponding model, assuming equilibrium for the reaction. The set–valuedness of the nonlinearity allows for solutions with sharp dissolution and precipitation fronts, which in fact occur. It turns out that travelling wave solutions only exist for a constant charge distribution and only as dissolution waves. These waves are given nearly explicitly: It remains to solve numerically initial value problems for ordinary differential equations, possibly within a shooting algorithm. All qualitative properties of the solutions can be investigated in detail, including the appearance of a dissolution front, the behaviour there, the convergence to limit cases of equilibrium or no dispersion, even with convergence rates. Thus these solutions enhance the understanding of the interplay advection–dispersion–dissolution–kinetics, also for more general situations, which can only be attacked numerically. In particular, they can be used to validate numerical codes for these problems.

APPENDIX A: THE RATE FUNCTION ACCORDING TO DEBYE–HÜCKEL THEORY

The purpose of this appendix is to indicate reasonable sufficient conditions under which the function $g(x;c)$ according to (2.21) is strictly monotone increasing in x for $x \geq (c/m)_+$ and a fixed real number c, i.e. satisfies (3.15), if the rate function is described according to the Debye–Hückel theory. The precipitation rate function r from (2.11) then takes the form (compare e.g. [13], [6], [9])

$$r(c_1,c_2) = \gamma_1^n c_1^n \gamma_2^m c_2^m \tag{A.1}$$

with positive integers n, m and

$$\begin{aligned}\gamma_1 &:= \exp(-m^2\sigma(I)) \ , \\ \gamma_2 &:= \exp(-n^2\sigma(I)) \ ,\end{aligned} \tag{A.2}$$

$$\sigma(I) := \frac{a_1 I^{1/2}}{1+a_2 I^{1/2}} \ , \tag{A.3}$$

where a_1, a_2 are positive parameters, and

$$I := \frac{1}{2}(m^2 c_1 + n^2 c_2 + \hat{c}). \tag{A.4}$$

Note that according to Section 1 we have taken the electric charges of M_1, M_2 to be $m, -n$ (or $-m, n$), respectively. The constant $\hat{c} \geq 0$ in the definition of the *ionic strength* I expresses the weighted sum of concentrations of all further species in solution, which do not take part in the dissolution/precipitation reaction. For the sake of simplicity we have taken the "effective diameters" of M_i to be the same, otherwise we would have to distinguish between different a_2^i in the definition of $\sigma = \sigma_i$. Due to (2.6) $f(x) := g(x;c)$ takes the form

$$f(x) = \gamma_1^n x^n \gamma_2^m \left(\frac{1}{n}(mx-c)\right)^m \tag{A.5}$$

and

$$\begin{gathered} I(x) = \alpha x + \beta, \qquad \text{where} \\ \alpha := \frac{1}{2}m(n+m), \quad \beta := \frac{1}{2}(\hat{c} - nc). \end{gathered} \tag{A.6}$$

As the problem is stated till now, there will be cases where f is not strictly increasing in x for $x \geq (c/m)_+$. Thus we have to impose a reasonable restriction on the parameters. We will show that it is sufficient to assume

$$4a_2 \geq nm(n+m)a_1. \tag{A.7}$$

This condition, which is invariant under linear scalings of x, c and $\hat{c}$, i.e. a change of concentration units, seems to be fulfilled in general (cf. e.g. [9]).

Let $K > 0$ and consider the equation

$$f(x) = K \qquad \text{or equivalently} \quad Kg(x) = h(x), \tag{A.8}$$

where

$$\begin{aligned} g(x) &:= \exp(\gamma\sigma(I(x))), \\ \text{with} \quad \gamma &:= mn(n+m), \end{aligned} \tag{A.9}$$

$$h(x) := x^n \left(\frac{1}{n}(mx - c)\right)^m. \tag{A.10}$$

Then an equivalent formulation of the assertion is:

For $K > 0$ there exists a unique solution $x = x_K \geq (c/m)_+$ of (A.8) and the solution depends monotonically on K, i.e.

$$0 < K_1 < K_2 \;\Rightarrow\; x_{K_1} < x_{K_2}. \tag{A.11}$$

The functions g, h have the following properties for $x \geq (c/m)_+$:

$$\begin{aligned} &g \text{ and } h \text{ are strictly monotone increasing and smooth for } x > (c/m)_+, \\ &g((c/m)_+) \geq 1, \quad g(\infty) = \exp(\gamma a_1/a_2), \\ &h((c/m)_+) = 0, \quad h(\infty) = \infty, \\ &h \text{ is strictly convex for } x \geq (c/m)_+. \end{aligned} \tag{A.12}$$

Therefore the existence of a solution x_K of (A.8) is clear, and $h(x_K) \geq K$ for each solution x_K. If g changes curvature, then there may be several solutions, but the solution is unique, if g is concave for $x \geq h^{-1}(K)$. In this case also (A.11) is satisfied, which can be seen as follows:

We have

$$\begin{aligned} h(x) &< Kg(x) \quad \text{for} \quad (c/m)_+ \leq x < x_K, \\ h(x) &> Kg(x) \quad \text{for} \quad x > x_K. \end{aligned} \tag{A.13}$$

Let $0 < K_1 < K_2$ and x_1, x_2 be the corresponding solutions of (A.8), then by means of (A.12), (A.13)

$$x_1 = h^{-1}(K_1 g(x_1)) < h^{-1}(K_2 g(x_1)) =: \hat{x}$$

and

$$h(\hat{x}) = K_2 g(x_1) < K_2 g(\hat{x}), \quad \text{i.e. } \hat{x} < x_2$$

and thus $\qquad x_1 < x_2.$

A sufficient condition for concavity of g (for $x \geq (c/m)_+$) is given by (A.7), which can be seen by direct computation:

$$g(x) = g_1(g_2(x)) \text{ with } g_1 = \exp,\ g_2 = \gamma\sigma(I), \text{ i.e.}$$
$$g''(x) = g(x)(g_2'^2(x) + g_2''(x)) = g(x)\left((\gamma\alpha\sigma'(I))^2 + \gamma\alpha^2\sigma''(I)\right),$$

i.e. the sign of $g''(x)$ is the sign of

$$\gamma\sigma'^2(I) + \sigma''(I) = \\ -\frac{3}{4}\,\frac{a_1 a_2^2}{I^{3/2}(1 + a_2 I^{1/2})^4}\left[I + \frac{4a_2 - \gamma a_1}{3a_2^2}\, I^{1/2} + \frac{1}{3a_2^2}\right]. \tag{A.14}$$

If (A.7) is satisfied, then the square bracket in (A.14) is strictly positive for $x \geq (c/m)_+$, i.e. $I \geq 0$, and thus $g''(x) < 0$.

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