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Boundary Element Discretization of Poincaré–Steklov Operators

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This paper is devoted to the construction of a discretization of Poincaré–Steklov (PS) operators for elliptic boundary value problems with the boundary element method (BEM). PS operators are natural mathematical tools for the investigation of boundary value problems and their numerical solution with domain decomposition (DD) methods based on the finite element (FE) solution of the subproblems (cf. [1], [9]). We will show that the discretizations of PS operators with a direct Galerkin BEM possess the same properties as the FE discretizations if the boundary elements satisfy some natural conditions. Hence the given construction provides a base for the analysis of different DD methods using the BE solution of subproblems, of the coupling of FE and BE methods and related problems.

1. Introduction

As model problem we consider the mixed boundary value problem for the Laplacian. Let $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ be a bounded domain with piecewise smooth boundary $\partial\Omega$ such that angles at corners and edges do not degenerate. The boundary $\partial\Omega$ is partitioned into 3 nonintersecting domains $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma$ and suppose that $\Gamma_D \neq \phi$, $\overline{\Gamma} \cap \overline{\Gamma}_N = \phi$. Let $\varphi \in H^{-1/2}(\Gamma)$ be given and consider the problem:
Find $u \in H^1(\Omega)$ such that

$$\left.\begin{aligned} -\Delta u &= 0\,, \quad \text{in}\, \Omega\,, \\ u|_{\Gamma_D} &= 0\,, \\ \frac{\partial u}{\partial n}|_{\Gamma_N} &= 0\,, \end{aligned}\right\} \qquad (1.1)$$

$$\frac{\partial u}{\partial n}|_{\Gamma} = \varphi\,. \qquad (1.2)$$

Here $\frac{\partial u}{\partial n}$ denotes the derivative with respect to the outer normal.
The PS operator T is defined as the mapping

$$T : \varphi \rightarrow \gamma_o u := u|_{\Gamma}\,, \qquad (1.3)$$

where u is the solution of (1.1–2). It is well known (cf. [7], [1]) that the linear operator $T : H^{-1/2}(\Gamma) \to \tilde{H}^{1/2}(\Gamma)$ is bounded and invertible, symmetric with respect to the duality between $H^{-1/2}(\Gamma)$ and $\tilde{H}^{1/2}(\Gamma)$, induced by the scalar product of $L^2(\Gamma)$

$$(T\varphi, \psi) = (\varphi, T\psi) = \int_\Gamma T\varphi(x)\psi(x)d\Gamma_x\,,$$

and positive definite

$$(T\varphi, \varphi) \geq c||\varphi||^2_{H^{-1/2}(\Gamma)}\,.$$

Here H^s denotes the usual Sobolev spaces:

$$H^s(\Omega) = \{u|_\Omega : u \in H^s(\mathbb{R}^d)\}\,, s \in \mathbb{R}\,,$$

$$H^s(\partial\Omega) = \begin{cases} \{u|_{\partial\Omega} : u \in H^{s+1/2}(\mathbb{R}^d)\} & , s > 0\,, \\ L^2(\partial\Omega) & , s = 0\,, \\ (H^{-s}(\partial\Omega))' & , s < 0\,. \end{cases}$$

For $\Gamma' \subset \partial\Omega$

$$\begin{aligned} H^s(\Gamma') &= \{u|_{\Gamma'} : u \in H^s(\partial\Omega)\} & , s \geq 0\,, \\ \tilde{H}^s(\Gamma') &= (H^{-s}(\Gamma'))' & , s < 0\,, \\ \tilde{H}^s(\Gamma') &= \{u \in H^s(\Gamma') : \tilde{u} \in H^s(\partial\Omega)\} & , s \geq 0\,, \\ H^s(\Gamma') &= (\tilde{H}^{-s}(\Gamma'))' & , s < 0\,, \end{aligned}$$

where $\tilde{u}$ denotes the extension of u by zero to $\partial\Omega$. We note that $T^{-1} = S$, where

$$S : \lambda \in \tilde{H}^{1/2}(\Gamma) \to \gamma_1 u := \frac{\partial u}{\partial n}|_\Gamma \tag{1.4}$$

and u solves the problem (1.1) together with the Dirichlet boundary condition on Γ

$$\gamma_o u = \lambda \tag{1.5}$$

The FE discretization of the PS operator T and its inverse S follows immediately from the variational formulation of (1.1). Let us suppose that we are given a space $V_h \subset H^1(\Omega)$ of finite element functions on Ω vanishing on Γ_D

$$V_h = \{v_h : v_h|_{\Gamma_D} = 0\}\,, \ \dim V_h < \infty\,.$$

Denote by

$$\begin{aligned} \overset{o}{V}_h &= \{v_h \in V_h : \gamma_o v_h = 0\}\,, \\ \tilde{X}_h &= \{\gamma_o v_h : v_h \in V_h\} \subset \tilde{H}^{1/2}(\Gamma)\,. \end{aligned}$$

For given $\lambda_h \in \widetilde{X}_h$ we determine $u_h \in V_h$ such that

$$\gamma_o u_h = \lambda_h$$

and

$$a(u_h, v_h) := \int_\Omega \nabla u_h \nabla v_h dw = 0\,, \ \forall v_h \in \overset{o}{V}_h\ . \tag{1.6}$$

Then $a(u_h, v_h)$, $v_h \in V_h$, defines a linear functional of $\gamma_o v_h \in \widetilde{X}_h$ bounded in the $\widetilde{H}^{1/2}(\Gamma)$–norm. Let $P_h : \widetilde{H}^{1/2}(\Gamma) \to \widetilde{X}_h$ be a bounded projection onto $\widetilde{X}_h$, by P_h' we denote its $L^2(\Gamma)$–adjoint projection and $Y_h := \text{im } P_h' \subset H^{-1/2}(\Gamma)$ can be identified with the dual space of $\widetilde{X}_h$. Hence we can set

$$a(u_h, v_h) = (\chi_h, \gamma_o v_h)\,,\ \chi_h \in Y_h\,,\ ||\chi_h||_{H^{-1/2}(\Gamma)} \le c\,.$$

It can be easily seen (cf. [1]) that the mapping

$$S_h : \widetilde{X}_h \to Y_h\,,\ S_h\lambda_h = \chi_h$$

is linear and invertible,

$$||S_h\lambda_h||_{H^{-1/2}(\Gamma)} \le c||\lambda_h||_{\widetilde{H}^{1/2}(\Gamma)}\,,\ \forall \lambda_h \in \widetilde{X}_h\,, \tag{1.7}$$

symmetric

$$(S_h\lambda_h, \lambda_h') = (\lambda_h, S_h\lambda_h') = a(u_h, u_h')\,,$$

where u_h' solves (1.6) with $\gamma_o u_h' = \lambda_h'$, and positive definite

$$(S_h\lambda_h, \lambda_h) = a(u_h, u_h) \ge c_1||u_h||_{H^1(\Omega)} \ge c||\lambda_h||_{\widetilde{H}^{1/2}(\Gamma)}\,.$$

Moreover

$$\begin{aligned} ||(S - S_h)\lambda_h||_{H^{-1/2}(\Gamma)} &\le ||(P_h'S - S_h)\lambda_h||_{H^{-1/2}(\Gamma)} + ||(I - P_h')S\lambda_h||_{H^{-1/2}(\Gamma)} \\ &\le c\left(\sup_{v_h\in V_h} \frac{((S - S_h)\lambda_h, \gamma_o v_h)}{||\gamma_o v_h||_{\widetilde{H}^{1/2}(\Gamma)}} + \inf_{\varphi_h \in Y_h} ||\gamma_1 u - \varphi_h||_{H^{-1/2}(\Gamma)}\right) \qquad (1.8) \\ &\le c\left(\inf_{v_h\in V_h} ||u - v_h||_{H^1(\Omega)} + \inf_{\varphi_h\in Y_h} ||\gamma_1 u - \varphi_h||_{H^{-1/2}(\Gamma)}\right)\,, \end{aligned}$$

where u is the exact solution of (1.1) with $\gamma_o u = \lambda_h$. We note that the constants (1.7) and (1.8) depend on the norm of P_h. Thus, if $||P_h||_{\widetilde{H}^{1/2}(\Gamma)}$ is uniformly bounded with respect to h, then estimates (1.7) and (1.8) hold with constants independent of λ_h and h.
Now it is clear how to define the FE discretization of T. Choose Y_h and $\chi_h \in Y_h$, find $u_h \in V_h$ such that

$$a(u_h, v_h) = (\chi_h, \gamma_o v_h)\,,\ \forall v_h \in V_h\,,$$

and define $T_h\chi_h := \gamma_o u_h$. Since $T_h = S_h^{-1}$, this operator has analogous properties as S_h and T, moreover

$$||(T - T_h)\chi_h||_{\tilde{H}^{1/2}(\Gamma)} \le c \inf_{v_h \in V_h} ||u - v_h||_{H^1(\Omega)},$$

where u solves (1.1) and $\gamma_1 u = \chi_h$.
The mentioned mapping properties of PS operators and their FE discretizations were essentially used (sometimes unknowingly) for the formulation and analysis of various DD methods, which are in fact equivalent to the iterative solution of operator equations with PS operators of different subdomains. In the following we will prove that a direct Galerkin BE solution of problem (1.1) (considered in Section 2) yields discretizations S_h and T_h of the operators S and T with the same mapping properties mentioned above (proved in Section 3). Therefore convergence results for many DD methods remain valid if the FE solution of subproblems is replaced by their BE solution whenever it is possible.

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2. A Direct Galerkin BEM

In this section we discuss some results concerning a boundary integral method for solving mixed boundary value problems for the Laplacian.
Let $u \in H^1(\Omega)$, $\Omega \subset \mathbb{R}^d$, be a solution of

$$-\Delta u = 0 \quad \text{in } \Omega . \tag{2.1}$$

Then we have the representation

$$u(x) = \frac{1}{2}\int_{\partial\Omega} G(x,y)\frac{\partial u(y)}{\partial n} d\Gamma_y - \frac{1}{2}\int_{\partial\Omega} \frac{\partial}{\partial n_y} G(x,y)u(y)d\Gamma_y \, , x \in \Omega \, , \tag{2.2}$$

with the fundamental solution

$$G(x,y) = \begin{cases} -\frac{1}{\pi}\, ln|x-y| \, , & d = 2 \\ \frac{1}{2\pi}\, |x-y|^{-1} \, , & d = 3 \, . \end{cases}$$

Let us define the boundary integral operators for $x \in \partial\Omega$

$$V\varphi(x) := \int_{\partial\Omega} G(x,y)\varphi(y)d\Gamma_y \, , \; K\varphi(x) := \int_{\partial\Omega} \frac{\partial}{\partial n_y} G(x,y)\varphi(y)d\Gamma_y \, ,$$

$$K'\varphi(x) := \frac{\partial}{\partial n_x}\int_{\partial\Omega} G(x,y)\varphi(y)d\Gamma_y \, , \; D\varphi(x) = -\frac{\partial}{\partial n_x}\int_{\partial\Omega} \frac{\partial}{\partial n_y} G(x,y)\varphi(y)d\Gamma_y \, .$$

The following properties are well known (cf. [2]):

$$V : H^{-1/2+\sigma}(\partial\Omega) \to H^{1/2+\sigma}(\partial\Omega)\,, \qquad D : H^{1/2+\sigma}(\partial\Omega) \to H^{-1/2+\sigma}(\partial\Omega)\,,$$
$$K : H^{1/2+\sigma}(\partial\Omega) \to H^{1/2+\sigma}(\partial\Omega)\,, \qquad K' : H^{-1/2+\sigma}(\partial\Omega) \to H^{-1/2+\sigma}(\partial\Omega)$$

are continuous for $|\sigma| \le 1/2$. The operator K' is the adjoint of K with respect to the duality between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$ induced by the scalar product $(\cdot,\cdot)_{L^2(\partial\Omega)}$. Moreover,

$$\begin{array}{rcl} (Dv,v)_{L^2(\partial\Omega)} & \ge & c\,|v|_{H^{1/2}(\partial\Omega)}\,, \quad c>0\,, \\ & & \text{for all } v \in H^{1/2}(\partial\Omega)\,,\ |\cdot|_{H^{1/2}(\Omega)} \text{ denotes the seminorm}\,, \\ (V\psi,\psi)_{L^2(\partial\Omega)} & \ge & c\,||\psi||_{H^{-1/2}(\partial\Omega)}\,, \quad c>0\,, \\ & & \text{for all } \psi \in H^{-1/2}(\partial\Omega) \quad \text{if } d=3 \\ & & \text{and for all } \psi \in H^{-1/2}(\partial\Omega) \text{ with } (\psi,1)_{L^2(\partial\Omega)} = 0 \quad \text{if } d=2\,. \end{array} \tag{2.3}$$

To ensure the solvability of boundary integral equations we need that the operator V is invertible. Therefore if $d = 2$ then we assume in the following that cap $(\partial\Omega) \neq 1$, i.e. $\Omega \subset \mathbb{R}^2$ has the property

(P) If $\psi \in H^{-1/2}(\partial\Omega)$ solves $V\psi = 0$ then $\psi = 0$.

Note that for any $\Omega \subset \mathbb{R}^2$ the domains $m\Omega = \{mx : x \in \Omega\}$, $m > 0$, satisfy (P) with the exception of one value m_e (cf. [6]).

Now the representation (2.2) and the jump relations for single and double layer potentials lead to the equalities on $\partial\Omega$

$$u = \frac{1}{2}\left((I-K)u + V\frac{\partial u}{\partial n}\right)\,, \quad \frac{\partial u}{\partial n} = \frac{1}{2}\left(Du + (I+K')\frac{\partial u}{\partial n}\right)\,. \tag{2.4}$$

Hence, if we consider the mixed boundary value problem for (2.1)

$$\begin{array}{l} u|_{\Gamma_1} = g_1\,, \\ \dfrac{\partial u}{\partial n}|_{\Gamma_2} = g_2\,, \end{array} \tag{2.5}$$

where $g_1 \in H^{1/2}(\Gamma_1)$, $g_2 \in H^{-1/2}(\Gamma_2)$, $\overline{\Gamma_1} \cup \overline{\Gamma_2} = \partial\Omega$, $\Gamma_1 \neq \phi$, and take the limits of $u(x)$ for $x \in \Gamma_2$ and of the normal derivative $\frac{\partial u}{\partial n}$ for $x \in \Gamma_1$, then we get the equation

$$\begin{pmatrix} D_{22} & K'_{21} \\ -K_{12} & V_{11} \end{pmatrix} \begin{pmatrix} v \\ \psi \end{pmatrix} = \begin{pmatrix} -D_{21} & I - K'_{22} \\ I + K_{11} & -V_{12} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}\,, \tag{2.6}$$

where the subscripts in D_{jk}, etc., mean integration over Γ_k and evaluation on Γ_j. Here $v := u|_{\Gamma_2}$, $\psi := \frac{\partial u}{\partial n}|_{\Gamma_1}$ denote the unknown boundary values of the solution u of (2.1), (2.5). If we substitute in (2.6) $v = v^* + lg_1$, $\psi = \psi^* + lg_2$ with arbitrary extensions $lg_1 \in H^{1/2}(\partial\Omega)$, $lg_2 \in H^{-1/2}(\partial\Omega)$ then we obtain a system of boundary integral equations

$$A\begin{pmatrix} v^* \\ \psi^* \end{pmatrix} := \begin{pmatrix} D_{22} & K'_{21} \\ -K_{12} & V_{11} \end{pmatrix} \begin{pmatrix} v^* \\ \psi^* \end{pmatrix} = \begin{pmatrix} -D_{\Gamma_2} & I - K'_{\Gamma_2} \\ I + K_{\Gamma_1} & -V_{\Gamma_1} \end{pmatrix} \begin{pmatrix} lg_1 \\ lg_2 \end{pmatrix} =: \begin{pmatrix} f_2 \\ f_1 \end{pmatrix}\,. \tag{2.7}$$

Here D_{Γ_2} for example denotes integration over $\partial\Omega$ and evaluation on Γ_2. If $g_1 \in H^s(\Gamma_1)$, $g_2 \in H^{s-1}(\Gamma_2)$ then $f_2 \in H^{s-1}(\Gamma_2)$, $f_1 \in H^s(\Gamma_1)$ for $s \in (0,1)$. Moreover, the operator

$$A: \begin{array}{c} \tilde{H}^s(\Gamma_2) \\ \times \\ \tilde{H}^{s-1}(\Gamma_1) \end{array} \rightarrow \begin{array}{c} H^{s-1}(\Gamma_2) \\ \times \\ H^s(\Gamma_1) \end{array}$$

is continuous and satisfies a Gårding inequality in $\tilde{H}^{1/2}(\Gamma_2)\times \tilde{H}^{-1/2}(\Gamma_1)$, for $U = (w,\varphi) \in \tilde{H}^{1/2}(\Gamma_2)\times \tilde{H}^{-1/2}(\Gamma_1)$ we have

$$\begin{aligned}\langle AU, U\rangle &= (D_{22}w + K'_{21}\varphi, w)_{L^2(\Gamma_2)} + (-K_{12}w + V_{11}\varphi, \varphi)_{L^2(\Gamma_1)} \\ &= (D_{22}w, w)_{L^2(\Gamma_2)} + (V_{11}\varphi, \varphi)_{L^2(\Gamma_1)} \\ &= (D\,\tilde{w}, \tilde{w})_{L^2(\partial\Omega)} + (V\,\tilde{\varphi}, \tilde{\varphi})_{L^2(\partial\Omega)},\end{aligned}$$

where $\tilde{w}, \tilde{\varphi}$ denote the extension by zero to $\partial\Omega$. Relations (2.3) imply the existence of a compact operator C such that

$$\langle (A+C)U, U\rangle \geq c\left(||w||^2_{\tilde{H}^{1/2}(\Gamma_2)} + ||\varphi||^2_{\tilde{H}^{-1/2}(\Gamma_1)}\right). \tag{2.8}$$

Since for $d = 3$ we can set in (2.8) $C = 0$ and for $d = 2$ because of property (P) the system (2.7) has no eigensolutions we obtain

Theorem 2.1. *[4], [8]. Let $g_1 \in H^{1/2}(\Gamma_1)$, $g_2 \in H^{-1/2}(\Gamma_2)$ with arbitary extensions $lg_1 \in H^{1/2}(\partial\Omega)$, $lg_2 \in H^{-1/2}(\partial\Omega)$. Then there exists exactly one solution $(v^*, \psi^*) \in \tilde{H}^{1/2}(\Gamma_2) \times \tilde{H}^{-1/2}(\Gamma_1)$ of (2.7) and $v := v^* + lg_1|_{\Gamma_2} \in H^{1/2}(\Gamma_2)$, $\psi := \psi^* + lg_2|_{\Gamma_2} \in H^{-1/2}(\Gamma_1)$ solve (2.6).*

Corollary 2.1. *Let $g_1 \in \tilde{H}^{1/2}(\Gamma_1)$, $g_2 \in \tilde{H}^{-1/2}(\Gamma_2)$. Then (2.6) is uniquely solvable and*

$$\begin{aligned} c_1\left(||g_1||_{\tilde{H}^{1/2}(\Gamma_1)} + ||g_2||_{\tilde{H}^{-1/2}(\Gamma_2)}\right) &\leq ||v||_{\tilde{H}^{1/2}(\Gamma_2)} + ||\psi||_{\tilde{H}^{-1/2}(\Gamma_1)} \leq \\ &\leq c_2\left(||g_1||_{\tilde{H}^{1/2}(\Gamma_1)} + ||g_2||_{\tilde{H}^{-1/2}(\Gamma_2)}\right), \end{aligned}$$

where c_1, c_2 do not depend on g_1 and g_2.

The Gårding inequality (2.8) yields the convergence of Galerkin methods for the approximate solution of system (2.7). We choose on Γ_2 and Γ_1 finite dimensional sets of approximating functions $M_h \subset H^{1/2}(\Gamma_2)$; $N_h \subset \tilde{H}^{-1/2}(\Gamma_1)$ with $\lim_{h\to 0} \dim M_h = \lim_{h\to 0} \dim N_h = \infty$.
Furthermore we suppose that g_1 can be extended by some $lg_1|_{\Gamma_2} \in M_h$ and g_2 can be extended by some $lg_2|_{\Gamma_1} \in N_h$ which implies that $g_2 \in \tilde{H}^{-1/2}(\Gamma_2)$. We denote $\tilde{M}_h = M_h \cap \tilde{H}^{1/2}(\Gamma_2)$ and consider the Galerkin method: Find $(v_h^*, \psi_h^*) \in \tilde{M}_h \times N_h$ such that

$$\left\langle A\begin{pmatrix} v_h^* \\ \psi_h^* \end{pmatrix}, \begin{pmatrix} w_h \\ \varphi_h \end{pmatrix}\right\rangle = \left\langle \begin{pmatrix} f_2 \\ f_1 \end{pmatrix}, \begin{pmatrix} w_h \\ \varphi_h \end{pmatrix}\right\rangle, \forall (w_h, \varphi_h) \in \tilde{M}_h \times N_h. \tag{2.9}$$

Under the assumption that $\bigcup_h \tilde{M}_h$ and $\bigcup_h N_h$ are dense in $\tilde{H}^{1/2}(\Gamma_2)$ and $\tilde{H}^{-1/2}(\Gamma_1)$, resp., by standard methods the following results can be obtained.

Theorem 2.2. *[4]. For any $g_1 \in H^{1/2}(\Gamma_1)$, $g_2 \in \widetilde{H}^{-1/2}(\Gamma_2)$ and all (sufficently small if $d = 2$) h the Galerkin equations (2.9) are uniquely solvable and*

$$||v_h^*||_{\widetilde{H}^{1/2}(\Gamma_2)} + ||\psi_h^*||_{\widetilde{H}^{-1/2}(\Gamma_1)} \le c\left(||lg_1||_{H^{1/2}(\partial\Omega)} + ||lg_2||_{H^{-1/2}(\partial\Omega)}\right) .$$

Moreover, the functions $v_h := v_h^* + lg_1|_{\Gamma_2}$ and $\psi_h := \psi_h^* + lg_2|_{\Gamma_1}$ approximate the unknown boundary values quasioptimally

$$\begin{aligned} &||v - v_h||_{H^{1/2}(\Gamma_2)} + ||\psi - \psi_h||_{\widetilde{H}^{-1/2}(\Gamma_1)} \le \\ &\le c\left(\inf_{\varphi_h \in N_h} ||\psi - \varphi_h||_{\widetilde{H}^{-1/2}(\Gamma_1)} + \inf_{w_h \in \widetilde{M}_h} ||v^* - w_h||_{\widetilde{H}^{1/2}(\Gamma_2)}\right) \end{aligned}$$

where the constant c does not depend on h.

Using the regularity of the solution (v^*, ψ^*) for the case $d = 2$ (cf. [3]), the structure of the mapping A and the Aubin–Nitsche Lemma one can estimate the convergence of the approximate solutions in Sobolev norms of lower order.

Theorem 2.3. *Let $d = 2$. Then there exist $\delta > 1/4$ and $c > 0$ such that*

$$\begin{aligned} &||v - v_h||_{H^{1/2-\delta}(\Gamma_2)} + ||\psi - \psi_h||_{\widetilde{H}^{-1/2-\delta}(\Gamma_1)} \le \\ &\qquad \le c\varepsilon_\delta(h)\left(||v - v_h||_{H^{1/2}(\Gamma_2)} + ||\psi - \psi_h||_{\widetilde{H}^{-1/2}(\Gamma_1)}\right) , \end{aligned}$$

where

$$\varepsilon_\delta(h) := \sup_{\substack{w \in \widetilde{H}^{1/2+\delta}(\Gamma_2) \\ \varphi \in \widetilde{H}^{-1/2+\delta}(\Gamma_1)}} \inf_{\substack{w_h \in \widetilde{M}_h \\ \varphi_h \in N_h}} \left(\frac{||w - w_h||_{\widetilde{H}^{1/2}(\Gamma_2)}}{||w||_{\widetilde{H}^{1/2+\delta}(\Gamma_2)}} + \frac{||\varphi - \varphi_h||_{\widetilde{H}^{-1/2}(\Gamma_1)}}{||\varphi||_{\widetilde{H}^{-1/2+\delta}(\Gamma_1)}}\right) .$$

From the density of $\bigcup_h \widetilde{M}_h$ and $\bigcup_h N_h$ we have $\varepsilon_\delta(h) \to 0$ as $h \to 0$. Note that δ depends on the inner angles at the corners of $\partial\Omega$ and at the points where boundary conditions change ([3]).

We remark that under the condition $g_2 \in \widetilde{H}^{-1/2}(\Gamma_2)$ one can choose $lg_2|_{\Gamma_1} = 0$, but for some considerations in Section 3 it is useful to admit $lg_2|_{\Gamma_1} \neq 0$.

In the following we will use an equivalent formulation of the Galerkin method (2.9): Find $v_h \in M_h$ and $\psi_h \in N_h$ such that

$$\widetilde{v}_h := \begin{pmatrix} g_1 \\ v_h \end{pmatrix} \in H^{1/2}(\partial\Omega),\ \widetilde{\psi}_h := \begin{pmatrix} \psi_h \\ g_2 \end{pmatrix} \in H^{-1/2}(\partial\Omega) \tag{2.10}$$

and the boundary values of the function

$$u_h(x) := \frac{1}{2}\int_{\partial\Omega} G(x,y)\, \widetilde{\psi}_h(y) d\Gamma_y - \frac{1}{2}\int_{\partial\Omega} \frac{\partial}{\partial n_y} G(x,y)\, \widetilde{v}_h(y) d\Gamma_y\,,\ x \in \Omega\,, \tag{2.11}$$

satisfy the equations

$$\int_{\Gamma_1}(u_h - g_1)\varphi_h d\Gamma + \int_{\Gamma_2}\left(\frac{\partial u_h}{\partial n} - g_2\right) w_h d\Gamma = 0\,,\ \forall (w_h, \varphi_h) \in \widetilde{M}_h \times N_h\,. \tag{2.12}$$

Note that u_h solves (2.1), on $\partial\Omega$ we have

$$u_h = \frac{1}{2}((I-K)\,\widetilde{v}_h + V\,\widetilde{\psi}_h)\,,\ \frac{\partial u_h}{\partial n} = \frac{1}{2}(D\,\widetilde{v}_h + (I+K')\,\widetilde{\psi}_h) \tag{2.13}$$

and Theorem 2.2 yields the estimate

$$\begin{aligned} &||u_h - u||_{H^{1/2}(\partial\Omega)} + ||\frac{\partial u_h}{\partial n} - \frac{\partial u}{\partial n}||_{H^{-1/2}(\partial\Omega)} \le \\ &\le c\left(\inf_{w_h \in \widetilde{M}_h} ||v^* - w_h||_{\widetilde{H}^{1/2}(\Gamma_2)} + \inf_{\varphi_h \in N_h} ||\psi^* - \varphi_h||_{\widetilde{H}^{-1/2}(\Gamma_1)}\right)\,. \end{aligned} \tag{2.14}$$

For the definition of the BE discretizations of PS operators in Section 3 we need a symmetric bilinear form. Let us suppose that u_h, u'_h, have the form (2.11) with densities $(\widetilde{v}_h, \widetilde{\psi}_h)$ and $(\widetilde{v}'_h, \widetilde{\psi}'_h)$, resp., and are solutions of equations (2.12) for corresponding boundary data $g_1, g'_1 \in \widetilde{H}^{1/2}(\Gamma_1)$, $g_2, g'_2 \in \widetilde{H}^{-1/2}(\Gamma_2)$. Then we may choose $lg_1|_{\Gamma_2} = lg'_1|_{\Gamma_2} = 0$, whereas $lg_2|_{\Gamma_1} \in N_h$, $lg'_2|_{\Gamma_1} \in N_h$ are arbitrary. We consider

$$\begin{aligned} &\int_{\Gamma_1} g_1 \frac{\partial u'_h}{\partial n} d\Gamma + \int_{\Gamma_2} g'_2 u_h d\Gamma = \int_{\partial\Omega} u_h \frac{\partial u'_h}{\partial n} d\Gamma + \\ &+ \int_{\Gamma_1}(g_1 - u_h)\frac{\partial u'_h}{\partial n} d\Gamma + \int_{\Gamma_2}\left(g'_2 - \frac{\partial u'_h}{\partial n}\right) u_h d\Gamma = \\ &= \int_{\partial\Omega} u_h \cdot \frac{\partial u'_h}{\partial n} d\Gamma + \int_{\Gamma_1}(g_1 - u_h)\left(\frac{\partial u'_h}{\partial n} - \varphi_h\right) d\Gamma + \int_{\Gamma_2}\left(g'_2 - \frac{\partial u'_h}{\partial n}\right)(u_h - w_h) d\Gamma \end{aligned}$$

for all $(w_h, \varphi_h) \in \widetilde{M}_h \times N_h$ in view of (2.12). If we choose $w_h = v^*_h$, $\varphi_h = \psi'_h$ (cf. (2.10)) as solutions of the corresponding equations (2.9) then

$$\begin{aligned} \int_{\Gamma_1} g_1 \frac{\partial u'_h}{\partial n} d\Gamma + \int_{\Gamma 2} g'_2 u_h d\Gamma &= \int_{\partial\Omega} u_h \frac{\partial u'_h}{\partial n} d\Gamma - \int_{\partial\Omega}(\widetilde{v}_h - u_h)\left(\widetilde{\psi}'_h - \frac{\partial u'_h}{\partial n}\right) d\Gamma = \\ &= \frac{1}{2}\left((D\,\widetilde{v}_h, \widetilde{v}'_h)_{L^2(\partial\Omega)} + (V\,\widetilde{\psi}_h, \widetilde{\psi}'_h)_{L^2(\partial\Omega)}\right) = \\ &= \int_{\Gamma_1} g'_1 \frac{\partial u_h}{\partial n} d\Gamma + \int_{\Gamma_2} g_2 u'_h d\Gamma\,, \end{aligned} \tag{2.15}$$

here we used (2.13) and the symmetry of D and V. We remark that

$$\begin{aligned} \widetilde{v}_h - u_h &= \frac{1}{2}\left((I+K)\,\widetilde{v}_h - V\,\widetilde{\psi}_h)\right)\,, \\ \widetilde{\psi}_h - \frac{\partial u_h}{\partial n} &= \frac{1}{2}\left(-D\,\widetilde{v}_h + (I-K')\,\widetilde{\psi}_h)\right) \end{aligned} \tag{2.16}$$

are the boundary values on $\partial\Omega$ of the function

$$u_h^a(x) := -\frac{1}{2}\int_{\partial\Omega} G(x,y)\,\tilde{\psi}_h(y)d\Gamma_y + \frac{1}{2}\int_{\partial\Omega} \frac{\partial}{\partial n_y} G(x,y)\,\tilde{v}_h(y)d\Gamma_y\,,\ x \in \mathbb{R}^d\backslash\overline{\Omega}\,.$$

Now we use Theorem 2.2, Corollary 2.1 and (2.3) to estimate for the solution u_h of (2.11–2.12)

$$(D\,\tilde{v}_h, \tilde{v}_h)_{L^2(\partial\Omega)} \le c_1 ||\,\tilde{v}_h\,||^2_{H^{1/2}(\partial\Omega)} \le c_2\left(||v_h^*||^2_{\tilde{H}^{1/2}(\Gamma_2)} + ||g_1||^2_{\tilde{H}^{1/2}(\Gamma_1)}\right) \le$$
$$\le c\left(||g_1||^2_{\tilde{H}^{1/2}(\Gamma_1)} + ||lg_2||^2_{H^{-1/2}(\partial\Omega)}\right),$$
$$(V\,\tilde{\psi}_h, \tilde{\psi}_h)_{L^2(\partial\Omega)} \le c_1 ||\,\tilde{\psi}_h\,||^2_{H^{-1/2}(\partial\Omega)} \le c\left(||g_1||^2_{\tilde{H}^{1/2}(\Gamma_1)} + ||lg_2||^2_{H^{-1/2}(\partial\Omega)}\right),$$
$$(D\,\tilde{v}_h, \tilde{v}_h)_{L^2(\partial\Omega)} \ge c_1 |\,\tilde{v}_h\,|^2_{H^{1/2}(\partial\Omega)} \ge c\left(|v_h^*|^2_{H^{1/2}(\Gamma_2)} + |g_1|^2_{H^{1/2}(\Gamma_1)}\right) \ge c|g_1|^2_{H^{1/2}(\Gamma_1)},$$

and if $d = 3$

$$(V\,\tilde{\psi}_h, \tilde{\psi}_h)_{L^2(\partial\Omega)} \ge c_1 ||\,\tilde{\psi}_h\,||^2_{H^{-1/2}(\partial\Omega)} \ge c||g_2||^2_{H^{-1/2}(\Gamma_2)}\,.$$

If $d = 2$ then we represent $\tilde{\psi}_h = \overline{\psi}_h + \omega e$, where $Ve = 1$ and $\omega = (\tilde{\psi}_h, 1)_{L^2(\partial\Omega)}/(e,1)_{L^2(\partial\Omega)}$. Then $(\overline{\psi}_h, 1)_{L^2(\partial\Omega)} = 0$ and

$$(V\,\tilde{\psi}_h, \tilde{\psi}_h)_{L^2(\partial\Omega)} = \left(V\overline{\psi}_h, \overline{\psi}_h\right)_{L^2(\partial\Omega)} + \omega^2 \cdot (e,1)_{L^2(\partial\Omega)}\,.$$

ω^2 can be estimated using Theorem 2.3, since

$$|\int_{\partial\Omega} \tilde{\psi}_h\,d\Gamma| = |\int_{\partial\Omega}\left(\tilde{\psi}_h - \frac{\partial u}{\partial n}\right)d\Gamma| = |\int_{\Gamma_1}(\psi_h - \psi)d\Gamma| \le$$
$$\le ||\psi_h - \psi||_{\tilde{H}^{-1/2-\delta}} \cdot \int_{\Gamma_1} d\Gamma \le c\varepsilon_\delta(h)\left(||g_1||_{\tilde{H}^{1/2}(\Gamma_1)} + ||lg_2||_{H^{-1/2}(\partial\Omega)}\right).$$

Hence, if $d = 2$ then

$$(V\,\tilde{\psi}_h, \tilde{\psi}_h)_{L^2(\partial\Omega)} \ge c\left(||g_2||^2_{H^{-1/2}(\Gamma_2)} - \varepsilon_\delta^2(h)\left(||g_1||^2_{\tilde{H}^{1/2}(\Gamma_1)} + ||lg_2||^2_{H^{-1/2}(\partial\Omega)}\right)\right).$$

Theorem 2.4. *Let u_h, u_h' be the BE solutions of (2.1), (2.4) for the boundary data $g_1, g_1' \in \tilde{H}^{1/2}(\Gamma_1)$ and $g_2, g_2' \in \tilde{H}^{-1/2}(\Gamma_2)$, resp., obtained via (2.11–2.12) Then*

$$b(u_h, u_h') := \int_{\Gamma_1} g_1 \frac{\partial u_h'}{\partial n} d\Gamma + \int_{\Gamma_2} g_2' u_h d\Gamma$$

is a symmetric bilinear form. Moreover, for any extension lg_2 of g_2 with $lg_2|_{\Gamma_1} \in N_h$

$$b(u_h, u_h) \le c_1\left(||g_1||^2_{\tilde{H}^{1/2}(\Gamma_1)} + ||lg_2||^2_{H^{-1/2}(\partial\Omega)}\right)$$

and

$$b(u_h,u_h) \geq c_2\Big(\big(|g_1|^2_{H^{1/2}(\Gamma_1)} + ||g_2||^2_{H^{-1/2}(\Gamma_2)}\big) -$$
$$- \varepsilon^2_\delta(h)\big(||g_1||^2_{\widetilde{H}^{1/2}(\Gamma_1)} + ||lg_2||^2_{H^{-1/2}(\partial\Omega)}\big)\Big),$$

with constants independent of g_1, g_2 and h. The second term of the right hand side in the last inequality appears only if $d=2$.

3. BE Discretization of PS Operators

Now we are in the position to construct discretizations of the operators S and T using the Galerkin BEM. According to the definition of this method we are given spaces of trial functions $M_h \subset H^{1/2}(\Gamma_N)$ and $N_h \in \widetilde{H}^{-1/2}(\Gamma_D)$, $\widetilde{M}_h = M_h \cap \widetilde{H}^{1/2}(\Gamma_N)$. Furthermore, on Γ we have finite dimensional spaces $\widetilde{X}_h \subset \widetilde{H}^{1/2}(\Gamma)$ to approximate the boundary values $\gamma_o u$ and $Y_h \subset \widetilde{H}^{-1/2}(\Gamma)$ for the approximation of the normal derivatives $\gamma_1 u$. In order to obtain mappings between the BE approximations of $\gamma_o u$ and $\gamma_1 u$ providing the same properties as the FE discretizations of S and T we need the following relations between the spaces of trial functions:

1. For any h there exists a bounded projection $P_h : \widetilde{H}^{1/2}(\Gamma) \to \widetilde{X}_h$ onto $\widetilde{X}_h$, such that the $L^2(\Gamma)$–adjoint projection P_h' maps onto Y_h, im $P_h' = Y_h$, and
$$||P_h' f||_{H^{-1/2}(\Gamma)} \leq c||f||_{H^{-1/2}(\Gamma)}, \forall f \in H^{-1/2}(\Gamma). \tag{3.1}$$
Hence $\widetilde{X}_h^{\perp} \cap Y_h = \phi$.

2. Since $\overline{\Gamma} \cap \overline{\Gamma}_N = \phi$ it is natural to introduce $Z_h := \{\zeta_h : \zeta_h|_{\Gamma_D} \in N_h,\ \zeta_h|_\Gamma \in Y_h\}$. We require that $Y_h \subset Z_h$ possesses an extension property:
$\exists E_h \in L(Y_h, Z_h) : E_h\varphi_h|_\Gamma = \varphi_h$ and
$$||E_h\varphi_h||_{\widetilde{H}^{-1/2}(\Gamma\cup\Gamma_D)} \leq c||\varphi_h||_{H^{-1/2}(\Gamma)}. \tag{3.2}$$

Note that in general we do not assume that the constants in (3.1) and (3.2) are independent of h. But it will be clear that for some estimates concerning the BE discretizations S_h and T_h as $h \to 0$ we need the uniform boundedness of $||P_h||_{\widetilde{H}^{1/2}(\Gamma)}$ and $||E_h||_{H^{-1/2}(\Gamma)\to\widetilde{H}^{-1/2}(\Gamma\cup\Gamma_D)}$. Due to the boundary conditions $u|_{\Gamma_D} = 0$, $\frac{\partial u}{\partial n}|_{\Gamma_N} = 0$ of (1.1) the BE solutions u_h are of the form (2.11) with densities

$$\begin{aligned} &\widetilde{v}_h \in H^{1/2}(\partial\Omega) \text{ such that } \widetilde{v}_h\,|_{\Gamma_D} = 0\,,\ \widetilde{v}_h\,|_{\Gamma_N} \in \widetilde{M}_h\,,\ \widetilde{v}_h\,|_\Gamma \in \widetilde{X}_h\,,\\ &\widetilde{\psi}_h \in H^{-1/2}(\partial\Omega) \text{ such that } \widetilde{\psi}_h\,|_{\Gamma_D} \in N_h\,,\ \widetilde{\psi}_h\,|_{\Gamma_N} = 0\,,\ \widetilde{\psi}_h\,|_\Gamma \in Y_h\,. \end{aligned} \tag{3.3}$$

Because of $\overline{\Gamma}\cap\overline{\Gamma}_N=\phi$ the conditions $\widetilde{v}_h|_{\Gamma_N}\in\widetilde{M}_h$, $\widetilde{v}_h|_{\Gamma}\in\widetilde{X}_h$ are natural. Hence the solutions u_h of (1.1) with the direct Galerkin BEM belong to the linear space

$$V_h := \Bigg\{ u_h \text{ of the form (2.11), the densities } \widetilde{v}_h, \widetilde{\psi}_h \text{ satisfy (3.3)}$$

$$\text{and} \int\limits_{\Gamma_D} u_h\varphi_h d\Gamma + \int\limits_{\Gamma_N} \frac{\partial u_h}{\partial n} w_h d\Gamma = 0\,,\ \forall (w_h,\varphi_h)\in\widetilde{M}_h\times N_h \Bigg\}.$$

BE solutions of (1.1) with Dirichlet data $\lambda_h\in\widetilde{X}_h$ we denote by $R_{D,h}\lambda_h$, with Neumann data $\chi_h\in Y_h$ by $R_{N,h}\chi_h$, i.e.

$$\begin{aligned} &\lambda_h\in\widetilde{X}_h\to R_{D,h}\lambda_h\in V_h \text{ with} \int\limits_{\Gamma}(\gamma_o(R_{D,h}\lambda_h)-\lambda_h)\varphi_h=0\,,\ \forall\varphi_h\in Y_h\,,\\ &\chi_h\in Y_h\to R_{N,h}\chi_h\in V_h \text{ with} \int\limits_{\Gamma}(\gamma_1(R_{N,h}\chi_h)-\chi_h)w_h=0\,,\ \forall w_h\in\widetilde{X}_h\,. \end{aligned} \tag{3.4}$$

From Theorem 2.2 we deduce that for all (sufficiently small if $d=2$) h the functions $R_{D,h}\lambda_h$ and $R_{N,h}\chi_h$ are uniquely determined.
Now we construct the BE discretization of the operator S. Let $\lambda_h\in\widetilde{X}_h$, then due to Theorem 2.4

$$b(R_{D,h}\lambda_h,R_{D,h}\lambda_h)=\int\limits_{\Gamma\cup\Gamma_D}\widetilde{\lambda}_h\,\frac{\partial R_{D,h}\lambda_h}{\partial n}\,d\Gamma$$

with

$$\widetilde{\lambda}_h(x)=\begin{cases}\lambda_h(x)\,,\ x\in\Gamma\,,\\ 0\,,\ x\in\Gamma_D\,,\end{cases}$$

and

$$\begin{aligned} &b(R_{D,h}\lambda_h,R_{D,h}\lambda_h)\le c_1\|\,\widetilde{\lambda}_h\,\|^2_{\widetilde{H}^{1/2}(\Gamma\cup\Gamma_D)}\,,\\ &b(R_{D,h}\lambda_h,R_{D,h}\lambda_h)\ge c_2\left(|\,\widetilde{\lambda}_h\,|^2_{\widetilde{H}^{1/2}(\Gamma\cup\Gamma_D)}-\varepsilon_\delta^2(h)\|\,\widetilde{\lambda}_h\,\|^2_{\widetilde{H}^{1/2}(\Gamma\cup\Gamma_D)}\right)\,, \end{aligned}$$

where in the last inequality the second term of the right hand side appears only for $d=2$. But

$$\|\,\widetilde{\lambda}_h\,\|_{\widetilde{H}^{1/2}(\Gamma\cup\Gamma_D)}\le c_3\|\lambda_h\|_{\widetilde{H}^{1/2}(\Gamma)}$$

and

$$|\lambda_h|_{\widetilde{H}^{1/2}(\Gamma\cup\Gamma_D)}\ge c_4\|\lambda_h\|_{\widetilde{H}^{1/2}(\Gamma)}$$

with constants independent of λ_h and h. Hence for all (sufficiently small if $d=2$) h we have

$$c_1\|\lambda_h\|^2_{\widetilde{H}^{1/2}(\Gamma)}\le b(R_{D,h}\lambda_h,R_{D,h}\lambda_h)\le c_2\|\lambda_h\|^2_{\widetilde{H}^{1/2}(\Gamma)} \tag{3.5}$$

Using (3.1) we define

$$S_h\lambda_h := P_h'\gamma_1(R_{D,h}\lambda_h) \tag{3.6}$$

such that

$$b(R_{D,h}\lambda_h, R_{D,h}\phi_h) = \int_\Gamma \lambda_h S_h \phi_h d\Gamma = (\lambda_h, S_h\phi_h)\,, \quad \lambda_h, \phi_h \in \widetilde{X}_h\ . \tag{3.7}$$

Theorem 3.1. *For all h ($< h_o$ if $d = 2$) the BE discretization S_h (3.6) of the operator S has the following properties:*

(i) $S_h : \widetilde{X}_h \to Y_h$ *is bounded,*

$$||S_h\lambda_h||_{H^{-1/2}(\Gamma)} \le c||\lambda_h||_{\widetilde{H}^{1/2}(\Gamma)}\,, \ \forall \lambda_h \in \widetilde{X}_h\ . \tag{3.8}$$

If the projections P_h are uniformly bounded then the constant c in (3.8) is independent of h.

(ii) $c_1||\lambda_h||^2_{\widetilde{H}^{1/2}(\Gamma)} \le (S_h\lambda_h, \lambda_h) \le c_2||\lambda_h||^2_{\widetilde{H}^{1/2}(\Gamma)}$, *$c_1$, c_2 do not depend on $\lambda_h \in \widetilde{X}_h$ and h.*

(iii) $(S_h\lambda_h, \phi_h) = (\lambda_h, S_h\phi_h)\,, \ \forall \lambda_h, \phi_h \in \widetilde{X}_h$.

(iv) $||(S_h - S)\lambda_h||_{H^{-1/2}(\Gamma)} \le c\Big(\inf\limits_{w_h \in \widetilde{M}_h} ||u - w_h||_{\widetilde{H}^{1/2}(\Gamma_N)} + \inf\limits_{\zeta_h \in Z_h} ||\frac{\partial u}{\partial n} - \zeta_h||_{\widetilde{H}^{-1/2}(\Gamma \cup \Gamma_D)}\Big)$, *where u solves (1.1) with the Dirichlet condition $\gamma_o u = \lambda_h$ on Γ. If the projections P_h are uniformly bounded, then c is independent of h.*

Proof.

(i) From (3.1) and Theorem 2.3 we have

$$\begin{aligned} ||S_h\lambda_h||_{H^{-1/2}(\Gamma)} &= ||P_h'\gamma_1(R_{D,h}\lambda_h)||_{H^{-1/2}(\Gamma)} \le \\ &\le c_1||\gamma_1(R_{D,h}\lambda_h)||_{H^{-1/2}(\Gamma)} \le c||\lambda_h||_{\widetilde{H}^{1/2}(\Gamma)}\,. \end{aligned}$$

(ii) and (iii) follow immediately from (3.5), (3.7) and Theorem 2.4.

(iv) $||(S_h - S)\lambda_h||_{H^{-1/2}(\Gamma)} \le ||(S_h - P_h'S)\lambda_h||_{H^{-1/2}(\Gamma)} + ||(I - P_h')S\lambda_h||_{H^{-1/2}(\Gamma)}$
$\le c\Big(||\gamma_1(R_{D,h}\lambda_h - u)||_{H^{-1/2}(\Gamma)} + \inf\limits_{\varphi_h \in Y_h} ||\gamma_1 u - \varphi_h||_{H^{-1/2}(\Gamma)}\Big)$
The application of Theorem 2.2 and the estimate

$$\inf_{\varphi_h \in Y_h} ||\gamma_1 u - \varphi_h||_{H^{-1/2}(\Gamma)} \le \inf_{\zeta_h \in Z_h} ||\frac{\partial u}{\partial n} - \zeta_h||_{\widetilde{H}^{-1/2}(\Gamma \cup \Gamma_D)}$$

yield the assertion. ∎

Next we define the BE discretization of T. Let $\chi_h \in Y_h$, $R_{N,h}\chi_h \in V_h$ exists for all (sufficiently small if $d = 2$) h and we set

$$T_h\chi_h := P_h R_{N,h}\chi_h \,, \tag{3.9}$$

such that from Theorem 2.4 and (3.1)

$$(T_h\chi_h, \phi_h) = b(R_{N,h}\chi_h, R_{N,h}\phi_h)\,, \; \chi_h, \phi_h \in Y_h \,. \tag{3.10}$$

But in general $T_h \neq S_h^{-1}$, as can be seen from the following.
Let $\lambda_h \in \widetilde{X}_h$. If we denote the densities defining $R_{D,h}\lambda_h$ via (2.11) by $\widetilde{v}_h$, $\widetilde{\psi}_h$ and the corresponding densities of $R_{N,h}(S_h\lambda_h)$ by $\widetilde{v}'_h$, $\widetilde{\psi}'_h$ then

$$\begin{aligned}
b(R_{D,h}\lambda_h, R_{N,h}(S_h\lambda_h)) &= \int_{\partial\Omega} R_{D,h}\lambda_h \frac{\partial R_{N,h}(S_h\lambda_h)}{\partial n} d\Gamma - \\
&- \int_{\partial\Omega} \left(\widetilde{v}_h - R_{D,h}\lambda_h\right)\left(\widetilde{\psi}'_h - \frac{\partial R_{N,h}(S_h\lambda_h)}{\partial n}\right) d\Gamma = \\
&= \int_{\partial\Omega} \widetilde{v}_h \frac{\partial R_{N,h}(S_h\lambda_h)}{\partial n} d\Gamma + \int_{\partial\Omega} \widetilde{\psi}'_h R_{D,h}\lambda_h d\Gamma - \int_{\partial\Omega} \widetilde{v}_h \widetilde{\psi}'_h \, d\Gamma = \\
&= \int_\Gamma \lambda_h \gamma_1(R_{N,h}(S_h\lambda_h)) d\Gamma + \int_\Gamma S_h\lambda_h \gamma_o(R_{D,h}\lambda_h) d\Gamma - \int_\Gamma \lambda_h S_h \lambda_h d\Gamma = \\
&= (S_h\lambda_h, \lambda_h) = b(R_{D,h}\lambda_h, R_{D,h}\lambda_h)
\end{aligned}$$

Hence

$$\begin{aligned}
&b(R_{D,h}\lambda_h - R_{N,h}(S_h\lambda_h), R_{D,h}\lambda_h - R_{N,h}(S_h\lambda_h)) = \\
&\quad = (S_h\lambda_h, \lambda_h) - 2(S_h\lambda_h, \lambda_h) + (T_h S_h \lambda_h, S_h\lambda_h)
\end{aligned}$$

and

$$\begin{aligned}
((T_h S_h - I)\lambda_h, S_h\lambda_h) &= \frac{1}{2}\left((D(\widetilde{v}_h - \widetilde{v}'_h), \widetilde{v}_h - \widetilde{v}'_h)_{L^2(\partial\Omega)} + (V(\widetilde{\psi}_h - \widetilde{\psi}'_h), \widetilde{\psi}_h - \widetilde{\psi}'_h)_{L^2(\partial\Omega)}\right) \\
&\geq c\left(|\, \widetilde{v}_h - \widetilde{v}'_h \,|^2_{H^{1/2}(\partial\Omega)} + ||\, \widetilde{\psi}_h - \widetilde{\psi}'_h \,||^2_{H^{-1/2}(\partial\Omega)}\right) .
\end{aligned} \tag{3.11}$$

But using the extension property (3.2) we can prove that T_h is spectrally equivalent to S_h^{-1}.

Theorem 3.2. *Suppose that estimates (3.1) and (3.2) are satisfied uniformly with respect to h. Then for any $h(< h_o$ if $d = 2)$ the BE discretization T_h (3.9) of the PS operator T has the following properties:*

(i) $T_h : Y_h \to \widetilde{X}_h$ is bounded,

$$||T_h\chi_h||_{\widetilde{H}^{1/2}(\Gamma)} \leq c||\chi_h||_{H^{-1/2}(\Gamma)}\,, \; \forall \chi_h \in Y_h \,.$$

(ii) $c_1||\chi_h||^2_{H^{-1/2}(\Gamma)} \leq (T_h\chi_h, \chi_h) \leq c_2||\chi_h||^2_{H^{-1/2}(\Gamma)}$, *the constants do not depend on* $\chi_h \in Y_h$ *and* h.

(iii) $(T_h\chi, \phi_h) = (\chi_h, T_h\phi_h)$, $\chi_h, \psi_h \in Y_h$.

(iv) $((T_hS_h - I)\lambda_h, S_h\lambda_h) \geq 0$, $\forall\lambda_h \in \tilde{X}_h$.

(v) $||(T_h - T)\chi_h||_{\tilde{H}^{1/2}(\Gamma)} \leq c\left(\inf_{w_h\in\tilde{X}_h} ||u - w_h||_{\tilde{H}^{1/2}(\Gamma)} + \right.$

$\left. + \inf_{\varphi_h\in N_h} ||\frac{\partial u}{\partial n} - \varphi_h||_{\tilde{H}^{-1/2}(\Gamma_D)} + \inf_{w_h\in\tilde{M}_h} ||u - w_h||_{\tilde{H}^{1/2}(\Gamma_N)}\right)$,

where u *solves (1.1) with the Neumann condition* $\gamma_1 u = \chi_h$.

Proof.

(i)

$$||T_h\chi_h||_{\tilde{H}^{1/2}(\Gamma)} = \sup_{\varphi} \frac{|(T_h\chi_h, \varphi)|}{||\varphi||_{H^{-1/2}(\Gamma)}} =$$
$$= \sup_{\varphi} \frac{|(T_h\chi_h, P_h'\varphi)|}{||P_h'\varphi||_{H^{-1/2}(\Gamma)}} \frac{||P_h'\varphi||_{H^{-1/2}(\Gamma)}}{||\varphi||_{H^{-1/2}(\Gamma)}}.$$

Using (3.2), Theorem 2.4 and (3.10) we obtain

$$|(T_h\chi_h, P_h'\varphi)| = |b(R_{N,h}\chi_h, R_{N,h}(P_h'\varphi)| \leq$$
$$\leq c_1||E_h\chi_h||_{\tilde{H}^{-1/2}(\Gamma\cup\Gamma_D)}||E_h(P_h'\varphi)||_{\tilde{H}^{-1/2}(\Gamma\cup\Gamma_D)} \leq$$
$$\leq c||\chi_h||_{H^{-1/2}(\Gamma)}||P_h'\varphi||_{H^{-1/2}(\Gamma)}.$$

From (3.1) we get the assertion.

(ii) Theorem 2.4 and (3.2) imply

$$(T_h\chi_h, \chi_h) = b(R_{N,h}\chi_h, R_{N,h}\chi_h) \leq c_1||E_h\chi_h||^2_{\tilde{H}^{-1/2}(\Gamma\cup\Gamma_D)} \leq c||\chi_h||^2_{H^{-1/2}(\Gamma)}.$$

If $d = 2$ then

$$(T_h\chi_h, \chi) \geq c_1\left(||\chi_h||^2_{H^{-1/2}(\Gamma)} - \varepsilon_\delta^2(h)||E_h\chi_h||^2_{\tilde{H}^{-1/2}(\Gamma\cup\Gamma_D)}\right)$$
$$\geq c_1(1 - \varepsilon_\delta^2(h))||\chi_h||^2_{H^{-1/2}(\Gamma)}.$$

Note that the boundedness of P_h is not used.

(iii) and (iv) follow from Theorem 2.4, (3.10) and (3.11).

(v)

$$\begin{aligned}||(T_h - T)\chi_h||_{\tilde{H}^{1/2}(\Gamma)} &\le ||(T_h - P_hT)\chi_h||_{\tilde{H}^{1/2}(\Gamma)} + ||(I - P_h)T\chi_h||_{\tilde{H}^{1/2}(\Gamma)} \\ &\le \sup_{\varphi} \frac{|((T_h - P_hT)\chi_h, \varphi))}{||\varphi||_{H^{-1/2}(\Gamma)}} + c \inf_{w_h \in \tilde{X}_h} ||w_h - T\chi_h||_{\tilde{H}^{1/2}(\Gamma)} .\end{aligned}$$

We have

$$\frac{|((T_h - P_hT)\chi_h, \varphi)|}{||\varphi||_{H^{-1/2}(\Gamma)}} \le c \frac{|((T_h - T)\chi_h, P_h'\varphi)|}{||P_h'\varphi||_{H^{-1/2}(\Gamma)}}$$

and

$$((T_h - T)\chi_h, P_h'\varphi) = \int_{\Gamma\cup\Gamma_D} (R_{N,h}\chi_h - u)E_h(P_h'\varphi)d\Gamma$$

since $\int_{\Gamma_D} R_{N,h}\chi_h \cdot \varphi_h d\Gamma = 0$, $\forall\varphi_h \in N_h$, and u denotes the solution of (1.1) with $\gamma_1 u = \chi_h$.

Therefore, using (3.2)

$$\begin{aligned}||(T_h - T)\chi_h||_{\tilde{H}^{1/2}(\Gamma)} &\le c \sup_{\varphi} \frac{|\int_{\Gamma\cup\Gamma_D} (R_{N,h}\chi_h - u) \cdot E_h(P_h'\varphi)d\Gamma|}{||E_h(P_h'\varphi)||_{\tilde{H}^{-1/2}(\Gamma\cup\Gamma_D)}} \\ &\le c||R_{N,h}\chi_h - u||_{H^{1/2}(\Gamma\cup\Gamma_D)}\end{aligned}$$

and Theorem 2.2 yields the assertion. ∎

4. Remarks

In these concluding remarks we give an example of BE functions satisfying (3.1) and (3.2) and we consider the BE discretization of PS operators in the case $\Gamma = \partial\Omega$.

Let $d = 2$ and suppose that $\gamma_o u$ and $\gamma_1 u$ shall be approximated by linear boundary elements. The natural choice of $\tilde{X}_h$ is the set of continuous piecewise linear functions subordinate to a mesh $\Delta = \{x_k\}_{k=0}^n$ on Γ and vanishing at the endpoints x_o and x_n of Γ, $\tilde{X}_h = \overset{o}{S}_1(\Delta)$. There are many different possibilities to choose a projection P_h onto $\tilde{X}_h$ bounded in $\tilde{H}^{1/2}(\Gamma)$ and so to determine $Y_h = \text{im } P_h'$. For example, denote by w_k, $k = 1, \cdots, n-1$, the hat functions, i.e. $w_k \in \overset{o}{S}_1(\Delta)$, $w_k(x_j) = \delta_{kj}$, $k, j = 1, \cdots, n-1$, and let $w_o(x) = 1 - \sum_{k=1}^{n-1} w_k(x)$, $x \in \Gamma$. Then $S_1(\Delta) = \text{span } (w_k, k = 0, \cdots n-1)$ is the set of periodic linear splines, by $\overline{P}_\Delta$ we denote the orthoprojection $\overline{P}_\Delta : L^2(\Gamma) \to S_1(\Delta)$ and by $\{\varphi_k\}_{k=0}^{n-1} \subset S_1(\Delta)$ the biorthogonal base of $\{w_k\}_{k=0}^{n-1}$. We introduce $P_\Delta u := \sum_{k=1}^{n-1}(u, \varphi_k)w_k$, clearly P_Δ is bounded in $\tilde{H}^{1/2}(\Gamma)$, im $P_\Delta = \overset{o}{S}_1(\Delta)$ and im $P_\Delta' = \text{span } (\varphi_k, k = 1, \cdots, n-1) \subset S_1(\Delta)$.

The projections P_Δ are uniformly bounded in $\tilde{H}^{1/2}(\Gamma)$ for any sequence of quasiuniform meshes, i.e.

$$\overline{\Delta} := \max_{1\le k\le n_\Delta} |x_k - x_{k-1}| \le c \min_{1\le k\le n_\Delta} |x_k - x_{k-1}|$$

for any mesh $\Delta = \{x_k\}_{k=0}^{n_\Delta}$ with a constant c independent of Δ.
Indeed, for quasiuniform meshes we have

$$||\varphi_o||_{L^2(\Gamma)} \cdot ||w_o||_{L^2(\Gamma)} \leq c$$

and

$$||P_\Delta u||_{L^2(\Gamma)} \leq ||\overline{P}_\Delta u||_{L^2(\Gamma)} + ||(u, \varphi_o)w_o||_{L^2(\Gamma)} \leq c||u||_{L^2(\Gamma)}.$$

Moreover, let $u \in \widetilde{H}^1(\Gamma)$, then for the interpolating spline $Q_\Delta u = \sum_{k=1}^{n-1} u(x_k)w_k$ we have

$$||u - Q_\Delta u||_{L^2(\Gamma)} \leq c\overline{\Delta}||u||_{\widetilde{H}^1(\Gamma)}.$$

Then

$$|(u, \varphi_o)| = |(u - Q_\Delta u, \varphi_o)| \leq c\overline{\Delta}||u||_{\widetilde{H}^1(\Gamma)} \cdot ||\varphi_o||_{L^2(\Gamma)}$$

and the uniform boundedness of $\overline{P}_\Delta$ and the inverse property of splines yield

$$\begin{aligned} ||P_\Delta u||_{\widetilde{H}^1(\Gamma)} &\leq ||\overline{P}_\Delta u||_{\widetilde{H}^1(\Gamma)} + ||(u, \varphi_o)w_o||_{H^1(\Gamma)} \\ &\leq ||u||_{\widetilde{H}^1(\Gamma)} + c\overline{\Delta}||w_o||_{H^1(\Gamma)} \cdot ||u||_{\widetilde{H}^1(\Gamma)} \cdot ||\varphi_o||_{L^2(\Gamma)} \\ &\leq c||u||_{\widetilde{H}^1(\Gamma)}. \end{aligned}$$

Hence by interpolation $||P_\Delta u||_{\widetilde{H}^{1/2}(\Gamma)} \leq c||u||_{\widetilde{H}^{1/2}(\Gamma)}$, $\forall u \in \widetilde{H}^{1/2}(\Gamma)$.
Furthermore,the extension property (3.2) holds for a wide class of piecewise polynomial functions, as shown in [10] for the case of Sobolev spaces $H^m(\Gamma)$, m nonnegative integer, and mentioned in [5] for arbitrary $H^s(\Gamma)$. Especially, (3.2) holds for the piecewise linear functions in $H^{-1/2}(\Gamma)$.
Now we apply the results of Sections 2 and 3 in order to construct S_h and T_h if $\Gamma = \partial\Omega$. Since the Neumann problem is not uniquely solvable some modifications are necessary. It can be easily seen that the mentioned properties of the PS operators remain valid if T and S are considered as mappings acting between the factor space $H^{1/2}(\partial\Omega)/R(\partial\Omega)$ and $H_o^{-1/2}(\partial\Omega) = \{\psi \in H^{-1/2}(\partial\Omega): (\psi, 1) = 0\}$, $R(\partial\Omega)$ denotes the set of constant functions on $\partial\Omega$. Moreover $T = D^{-1}(I - K') : H_o^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)/R(\partial\Omega)$ is bounded and invertible with $T^{-1} = S = V^{-1}(I + K)$. Remark that the two spaces are in duality with respect to the $L^2(\partial\Omega)$–scalar product, $(H_o^{-1/2}(\partial\Omega))' = H^{1/2}(\partial\Omega)/R(\partial\Omega)$, and that the FE discretizations of T and S also are isomorphisms between the factor space of traces of finite elements $X_h/R(\partial\Omega)$ and its dual. We show that the BE discretizations S_h and T_h constructed analogously to Section 3 preserve this property.
First we have to choose spaces of trial functions $X_h \subset H^{1/2}(\partial\Omega)$ for the approximation of $u|_{\partial\Omega}$ and $Y_h \subset H^{-1/2}(\partial\Omega)$ for the approximation of $\frac{\partial u}{\partial n}|_{\partial\Omega}$. We assume that $\dim X_h = \dim Y_h$, $R(\partial\Omega) \subset X_h$ and $X_h^\perp \cap Y_h = \phi$. Then there exists a bounded projection onto X_h, $P_h : H^{1/2}(\partial\Omega) \to X_h$ such that im $P_h' = Y_h$, i.e. Y_h can be identified with the dual of X_h.
For given $\lambda_h \in X_h$ the density $\widetilde{\psi}_h$ of (2.11) is determined from

$$(V\,\widetilde{\psi}_h, \varphi_h) = ((I + K)\lambda_h, \varphi_h)\,, \ \forall \varphi_h \in Y_h\,,$$

or, equivalently, from the projection equation

$$P_h V \tilde{\psi}_h = P_h (I + K) \lambda_h \,. \tag{4.1}$$

Since $R(\partial\Omega) = ker\,(I + K) \subset X_h$ it is clear that $ker\, P_h'(I + K')|_{Y_h} = \{e_h\} \neq \phi$ and the solution $\tilde{\psi}_h$ of (4.1) belongs to $\{\chi_h \in Y_h : (\chi_h, Ve_h) = 0\}$. Therefore it suffices to restrict to $\lambda_h \in \tilde{X}_h := X_h/R(\partial\Omega)$. But in general $P_h V e_h \notin R(\partial\Omega)$ such that $\tilde{\psi}_h \notin \tilde{X}_h' = \{\chi_h \in Y_h : (\chi_h, 1) = 0\} =: \tilde{Y}_h$.
Defining as in Section 3

$$\begin{aligned} S_h \lambda_h &= P_h'(D\lambda_h + (I + K') \tilde{\psi}_h) = \\ &= (P_h' D + P_h'(I + K') P_h' (P_h V P_h')^{-1} P_h (I + K)) \lambda_h \end{aligned}$$

we see that

$$S_h \tilde{X}_h = \tilde{Y}_h = \tilde{X}_h' \,.$$

Applying the ideas of the proof of Theorem 3.1 we conclude that for $S_h : \tilde{X}_h \to \tilde{Y}_h$ the assertions of this theorem are valid.
For $\chi_h \in \tilde{Y}_h$ the construction analogous to Section 3 leads to

$$T_h \chi_h = (P_h V + P_h (I - K) P_h (P_h' D P_h)^{-1} P_h' (I - K')) \chi_h \,,$$

which can be considered as mapping from $\tilde{Y}_h$ to $\tilde{X}_h$ and the assertions of Theorem 3.2 remain true.
Finally we mention the matrix representation for S_h and T_h. Let $\{w_k\}_{k=0}^{n-1} \subset X_h$, $\{\varphi_k\}_{k=0}^{n-1} \subset Y_h$ be biorthogonal bases, $(w_k, \varphi_j) = \delta_{kj}$, then

$$\begin{aligned} S_h &= D_h + (I + K_h') V_h^{-1} (I + K_h) \,, \\ T_h &= V_h + (I - K_h) D_h^{-1} (I - K_h') \,, \end{aligned}$$

where

$$\begin{aligned} (V_h)_{j,k} &= (V\varphi_j, \varphi_k) \\ (D_h)_{j,k} &= (Dw_j, w_k) \\ (K_h)_{j,k} &= (Kw_j, \varphi_k) = (K_h')_{k,j} \,. \end{aligned}$$

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