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Stability of bifurcating periodic solutions of differential inequalities in $\mathbb{R}^3$

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Abstract

A bifurcation problem for the inequality

$$\begin{cases} U(t) \in K \\ (\dot{U}(t) - A_\lambda U(t) - G(\lambda, U(t)), V - U(t)) \geq 0 \text{ for all } V \in K, \text{ a.a. } t \in [0,T) \end{cases}$$

is considered, where K is a closed convex cone in $\mathbb{R}^3$, A_λ a real 3×3 matrix, λ a real parameter, G a small perturbation. We investigate small periodic solutions bifurcating at λ_0 from the branch of trivial solutions and corresponding to parameters λ for which the trivial solution is unstable. It is proved that these solutions are stable or they are contained in a certain attracting set $\mathbf{A}_\lambda$ if zero is stable as the solution of our inequality with $\lambda = \lambda_0$.

0. Introduction

Let K be a closed convex cone with its vertex at the origin in $\mathbb{R}^3$, A_λ a real 3×3 matrix, λ a real parameter, G a small perturbation (precisely see assumptions below). Consider the inequality

$$\text{(I)} \qquad \begin{cases} U(t) \in K \\ (\dot{U}(t) - A_\lambda U(t) - G(\lambda, U(t)), V - U(t)) \geq 0 \text{ for all } V \in K, \text{ a.a. } t \in [0,T). \end{cases}$$

It is proved in [2], [3], [5] that under certain assumptions there exists a bifurcation point λ_0 of (I) at which periodic solutions of (I) bifurcate from the branch of trivial solutions. While an elementary approach based on geometrical ideas is used in [2] for inequalities in $\mathbb{R}^3$, more complicated methods describing the situation in $\mathbb{R}^n$ are given in [3], [5].

The aim of this paper is to show that in some cases the bifurcating solutions lying in the domain of instability of the trivial solution are stable in a certain sense. We shall consider a situation when the trivial solution of the linearized inequality

$$\text{(LI)} \qquad \begin{cases} U(t) \in K \\ (\dot{U}(t) - A_\lambda U(t), V - U(t)) \geq 0 \text{ for all } V \in K, \text{ a.a. } t \geq 0 \end{cases}$$

losses its stability at λ_0 (the assumption (λ_+) below) and zero is attracting for (I) with $\lambda = \lambda_0$ (the asumption (A)). We shall show that if there is unique small (nontrivial) periodic solution of (I) for a given λ lying in the domain of instability of the trivial solution of (LI) (and close to λ_0) then this unique solution is asymptotically stable (Theorem 1.1). This is a certain analogy of the situation in case of the classical Hopf bifurcation for equations (see e.g. [4], [6]). Unfortunately, the set of bifurcating solutions can be more complicated. In the general case, bifurcating periodic solutions are contained in a certain attracting set $\mathbf{A}_\lambda$ (Theorem 1.2) which can be described precisely in some special situations (Remark 1.4). In fact, our assumptions do not guarantee the existence of bifurcating solutions near λ_0, but they can be supplemented such that λ_0 is really a bifurcation point (Section 2).

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1. Main Results

Notation 1.1.

$(U,V)=\sum_{i=1}^{3}u_iv_i, |U|^2=(U,U)$ for $U=[u_1,u_2,u_3], V=[v_1,v_2,v_3]$,

$W_j(\lambda)$ - eigenvectors of $A_\lambda, W_{1,2}(\lambda)=U_1(\lambda)\pm iU_2(\lambda), W_3(\lambda)=U_3(\lambda)$ with $U_j(\lambda)\in\mathbb{R}^3$ $(j=1,2,3)$,

$U_\lambda(\cdot,V), U_{0,\lambda}(\cdot,V)$ - the solution of (I), (LI), respectively, (i.e. the absolutely continuous function satisfying (I) or(LI)) with the initial condition V for $t=0$,

$x^j_\lambda(\cdot,V)$ - the coordinates of $U_\lambda(\cdot,V)$ with respect to $U_j(\lambda)$,

$S_\lambda=Lin\{U_3(\lambda)\}$,

$P_\lambda V=x^1_\lambda(V)U_1(\lambda)+x^2_\lambda(V)U_2(\lambda)$ for any $V=\sum_{j=1}^{3}x^j_\lambda(V)U_j(\lambda)$ (the projection onto the plane $Lin\{U_1,(\lambda),U_2(\lambda)\}$)

$\rho_\lambda(\cdot,V),\varphi_\lambda(\cdot,V)$ - continuous functions defined by

$$P_\lambda U_\lambda(t,V)=\rho_\lambda(t,V)[\cos(\varphi_V-\varphi_\lambda(t,V))U_1(\lambda)+\sin(\varphi_V-\varphi_\lambda(t,V))U_2(\lambda)]$$

for $t\in[0,t_0)$ if $U_\lambda(t,V)\notin S_\lambda$ on $[0,t_0)$, where φ_V satisfies

$$P_\lambda V=\rho_\lambda(0,V)[\cos\varphi_V\cdot U_1(\lambda)+\sin\varphi_V\cdot U_2(\lambda)]$$

(i.e. polar coordinates of $U_\lambda(t,V)$ but with the angle measured from the ray given by V),

$\rho_{0,\lambda}(\cdot,V),\varphi_{0,\lambda}(\cdot,V)$ are defined analogously but by using $U_{0,\lambda}(\cdot,V)$ instead of $U_\lambda(\cdot,V)$,

$$t_\lambda(V)=\inf\{t_0;U_\lambda(t,V)\notin S_\lambda \text{ for } t\in[0,t_0],\varphi_\lambda(t_0,V)=2\pi\},$$

$$t_{0,\lambda}(V)=\inf\{t_0;U_{0,\lambda}(t,V)\notin S_\lambda \text{ for } t\in[0,t_0],\varphi_{0,\lambda}(t_0,V)=2\pi\},$$

$$t^k_\lambda(V)=\inf\{t_0;U_\lambda(t,V)\notin S_\lambda \text{ for } t\in[0,t_0],\varphi_\lambda(t_0,V)=2k\pi\},$$

$$g_\lambda(V)=\frac{x^3_\lambda(V)}{\sqrt{(x^1_\lambda(V))^2+(x^2_\lambda(V))^2}} \text{ for } V\in\mathbb{R}^3\setminus S_\lambda, V=\sum_{j=1}^{3}x^j_\lambda(V)U_j(\lambda).$$

We shall consider a fixed λ_0 such that the eigenvalues $\mu_1(\lambda),\mu_2(\lambda),\mu_3(\lambda)$ and the corresponding eigenvectors $W_1(\lambda),W_2(\lambda),W_3(\lambda)$ of A_λ depend continuously on $\lambda\in[\lambda_0,\lambda_0+\varepsilon]$ (for some $\varepsilon>0$) and satisfy the following assumptions:

(μ) $\quad\mu_{1,2}(\lambda)=\alpha(\lambda)\pm i\beta(\lambda),\beta(\lambda)>0,\alpha(\lambda)>\mu_3(\lambda),\ \mu_3(\lambda)<0$ for all $\lambda\in[\lambda_0,\lambda_0+\varepsilon]$.

Particularly, it follows from (μ) that the solutions of the equation $\dot U(t)=A_\lambda U(t)$ circulate around the axis $U_3(\lambda)$ (with the exception of those starting at this axis).

Further, we shall suppose that for any fixed λ, our cone K can be described by a finite number N of functions f^j_λ as follows (cf. [2]):

(K)

$$\begin{cases} \text{for any } \lambda \in [\lambda_0, \lambda_0+\varepsilon] \text{ there are continuous convex functions } f^j_\lambda : \mathbb{R}^2 \to \mathbb{R}\ (j=1,\dots,N) \\ \text{continuously differentiable on } \mathbb{R}^2 \setminus [0,0] \text{ such that } f^j_\lambda(rx_1, rx_2) = r f^j_\lambda(x_1,x_2) \\ \text{for all } r>0, [x_1,x_2] \in \mathbb{R}^2, \\ K = \{U = \sum\limits_{i=1}^{3} x_i U_i(\lambda); x_3 \geq f^j_\lambda(x_1,x_2), j=1,\dots,N\} \text{ for any } \lambda \in [\lambda_0,\lambda_0+\varepsilon], \\ K \neq \{U = \sum\limits_{i=1}^{3} x_i U_i(\lambda); x_3 \geq 0\} \text{ for any } \lambda \in [\lambda_0,\lambda_0+\varepsilon], \\ \text{for any } V \in K, V \neq 0, \lambda \in [\lambda_0,\lambda_0+\varepsilon] \text{ there is a pair of indices } 1 \leq k \leq j \leq N \\ \text{and a neighbourhood } \mathbb{U}(V) \text{ of } V \text{ such that} \\ \mathbb{U}(V) \cap K = \{U \in \mathbb{U}(V); \\ U = \sum\limits_{i=1}^{3} x_i U_i(\lambda), x_3 \geq f^k_\lambda(x_1,x_2), x_3 \geq f^j_\lambda(x_1,x_2)\} \text{ for } \lambda \in [\lambda_0,\lambda_0+\varepsilon]. \end{cases}$$

(The last condition means that in a neighbourhood of a given point, K can be described by one or two functions.)

It will be always supposed that the nonlinearity G satisfies the conditions

$$\text{(G)} \qquad \lim_{|U|\to 0} \frac{|G(\lambda,U)|}{|U|} = 0 \text{ uniformly for } \lambda \in [\lambda_0,\lambda_0+\varepsilon],$$

$$\text{(L)} \qquad \begin{cases} \text{for any } R>0 \text{ there exists } C>0 \text{ such that} \\ |G(\lambda,U_1)-G(\lambda,U_2)| \leq C|U_1-U_2| \text{ for all } \lambda \in [\lambda_0,\lambda_0+\varepsilon], |U_1|, |U_2| \leq R. \end{cases}$$

We shall assume that it is possible to choose V_λ such that

$$\text{(V)} \qquad \begin{cases} V_\lambda \in \partial K, |V_\lambda| = 1, g_\lambda(V_\lambda) = \max\limits_{0\neq V \in \partial K} g_\lambda(V) \text{ for all } \lambda \in [\lambda_0,\lambda_0+\varepsilon], \\ V_\lambda \text{ depend continuously on } \lambda \in [\lambda_0,\lambda_0+\varepsilon]. \end{cases}$$

Notice that for any fixed $\lambda \in [\lambda_0,\lambda_0+\varepsilon]$, there exists always at least one V_λ satisfying the conditions from the first line of (V). This V_λ represents the ray in ∂K which is the closest one to S_λ with respect to the inner product

$$\langle U,V\rangle_\lambda = \sum_{j=1}^{3} x^j_\lambda(U) x^j_\lambda(V) \text{ for } U = \sum_{j=1}^{3} x^j_\lambda(U) U_j(\lambda),\ V = \sum_{j=1}^{3} x^j_\lambda(V) U_j(\lambda).$$

Hence, (V) is fulfilled automatically if the eigenvectors are independent of λ. But in the general case, the continuity condition need not to be fulfilled.

Further, we shall suppose that the solutions of the linearized inequality (LI) circulate around the axis $U_3(\lambda)$ for $\lambda \in [\lambda_0,\lambda_0+\varepsilon]$, there is a periodic solution of (LI) for $\lambda=\lambda_0$, the trivial solution of (I) for $\lambda=\lambda_0$ is asymptotically stable and the trivial solution of (LI) for $\lambda \in (\lambda_0,\lambda_0+\varepsilon]$ is unstable. Precisely, we demand

(λ_0)

$U_{0,\lambda_0}(\cdot, V_{\lambda_0})$ is periodic , $t_{0,\lambda}(V) < T_0$ for all $V \in \partial K,\ \lambda \in [\lambda_0,\lambda_0+\varepsilon]$ (with some $T_0>0$),

(λ_+) $$\lim_{t\to\infty} |U_{0,\lambda}(t, V_\lambda)| = +\infty \text{ for all } \lambda \in (\lambda_0, \lambda_0 + \varepsilon],$$

(A) there is $R > 0$ such that $\lim\limits_{t\to+\infty} |U_{\lambda_0}(t, V)| = 0$ for all $V \in K, |V| < R$.

Remark 1.1. Particularly, it follows from (K) that $U_3(\lambda) \in intK$ for all $\lambda \in [\lambda_0, \lambda_0+\varepsilon]$. Recall that if $U_\lambda(t, V) \in intK$ for all $t \in (t_1, t_2)$ then $U_\lambda(\cdot, V)$ is simultaneously a solution of the equation

$$\dot{U}(t) - A_\lambda U(t) - G(\lambda, U(t)) = 0 \tag{1.1}$$

on this interval. Hence, it follows from (μ) that there is $\rho_0 > 0$ such that any solution of (I) with $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$ and an initial condition $V \in K, |V| < \rho_0$ circulates around $U_3(\lambda)$ as long as it lies in $intK$ and it intersects ∂K after some time. For the linearized equation $\dot{U}(t) - A_\lambda U(t) = 0$, this follows from the explicit form of solutions expressed by $\mu_j(\lambda), U_j(\lambda)$, the behaviour of solutions of (1.1) is similar on a small neigbourhood of the origin under the assumption (G).

Remark 1.2. It is easy to see that if (λ_0) is fulfilled then the assumption (λ_+) is equivalent to

$$|U_{0,\lambda}(t_{0,\lambda}(V_\lambda), V_\lambda)| > 1.$$

Remark 1.3. It will follow from Consequence 3.2 below that for the investigation of stability, it is sufficient to consider only periodic solutions starting at $rV_\lambda, r \in (0, \rho_0)$ because any periodic solution with a sufficiently small initial condition must intersect such a point. Further, it will follow from Consequence 3.2 that any periodic solution $U_\lambda(\cdot, rV_\lambda)$ with $r \in (0, \rho_0), \lambda \in (\lambda_0, \lambda_0+\varepsilon]$ must satisfy

$$U_\lambda(t_\lambda^k(rV_\lambda), rV_\lambda) = rV_\lambda$$

with some $k \in \mathbb{N}$. Particularly, $t_\lambda(V_\lambda)$ is finite if $U_\lambda(\cdot, rV_\lambda)$ is periodic.

Theorem 1.1. Let the assumptions $(\mu), (K), (G), (L), (V), (\lambda_0), (\lambda_+), (A)$ be fulfilled. Then there exist $\rho_0 > 0, \varepsilon_0 > 0$ with the following property:

If for some $\lambda \in (\lambda_0, \lambda_0 + \varepsilon_0)$ there exists precisely one $r_\lambda \in (0, \rho_0)$ such that $U_\lambda(\cdot, r_\lambda V_\lambda)$ is periodic then this solution is asymptotically stable. Further,

(1.3) if $V \in K,\ U_\lambda(t, V) = sV_\lambda$ with some $t \geq 0, s \in (0, \rho_0)$ then

$$\lim_{t\to+\infty} \operatorname{dist}(U_\lambda(t, V), \mathbb{A}_\lambda) = 0,$$

where $\mathbb{A}_\lambda = \{W; W = U_\lambda(t, r_\lambda V_\lambda), t \geq 0\}$.

Notation 1.2. Let $\rho_0 > 0$ be given. Introduce the following numbers (depending on ρ_0) and the set of all ω-limit points of all solutions $U_\lambda(\cdot, rV_\lambda), r \in [\underline{\rho}_\lambda, \overline{\rho}_\lambda]$:

$$\begin{aligned}
&\underline{\rho}_\lambda = \inf\{r \in (0, \rho_0); U_\lambda(\cdot, rV_\lambda) \text{ is periodic }\},\\
&\overline{\rho}_\lambda = \sup\{r \in (0, \rho_0); U_\lambda(\cdot, rV_\lambda) \text{ is periodic }\},\\
&\mathbb{A}_\lambda = \{W; W = \lim_{n\to\infty} U_\lambda(t_n, rV_\lambda) \text{ for some } t_n \to +\infty, r \in [\underline{\rho}_\lambda, \overline{\rho}_\lambda]\}.
\end{aligned}$$

We shall say that $\mathbb{A}_\lambda$ is attracting if there is a neighbourhood $\mathbb{U}$ of $\mathbb{A}_\lambda$ such that $\lim\limits_{t\to+\infty} \operatorname{dist}(U_\lambda(t, V), \mathbb{A}_\lambda) = 0$ for all $V \in \mathbb{U}$.

(Of course, under the assumptions of Theorem 1.1, the set $\mathbb{A}_\lambda$ coincides with that introduced above.)

Remark 1.4. If $U_\lambda(t_\lambda(rV_\lambda), rV_\lambda) = rV_\lambda$ for any periodic solution $U_\lambda(\cdot, rV_\lambda)$ with $r \in (0, \rho_0)$ (i.e. if any periodic solution $U_\lambda(\cdot, rV_\lambda)$ with $r \in (0, \rho_0)$ has the period $t_\lambda(rV_\lambda)$) then $\mathbb{A}_\lambda$ consists of all periodic trajectories $U_\lambda(\cdot, rV_\lambda)$, $r \in (0, \rho_0)$. (In this case $U_\lambda(\cdot, \underline{\rho}_\lambda V_\lambda)$, $U_\lambda(\cdot, \overline{\rho}_\lambda V_\lambda)$ are periodic.) For the proof of this assertion, similar ideas as those leading to the proof of Theorem 1.2 below can be used. Unfortunately, it is not clear how to show when this situation really occurs.

Theorem 1.2. Let the assumptions $(\mu), (K), (G), (L), (V), (\lambda_0), (\lambda_+), (A)$ be fulfilled. Then there exist $\rho_0 > 0, \varepsilon_0 > 0$ with the following property:

If for some $\lambda \in (\lambda_0, \lambda_0 + \varepsilon_0)$ there is at least one periodic solution $U_\lambda(\cdot, rV_\lambda)$ with $r \in (0, \rho_0)$ then $0 < \underline{\rho}_\lambda \leq \overline{\rho}_\lambda < \rho_0$ and the set $\mathbb{A}_\lambda$ is attracting. Further,

(1.4)

$$\text{if } V \in K,\ U_\lambda(t, V) = sV_\lambda \text{ with some } t \geq 0, s \in (0, \rho_0) \text{ then } \lim_{t \to +\infty} \operatorname{dist}(U_\lambda(t, V), \mathbb{A}_\lambda) = 0.$$

Remark 1.5. The statements of Theorems 1.1, 1.2 remain valid if we replace $[\lambda_0, \lambda_0 + \varepsilon]$ and $(\lambda_0, \lambda_0 + \varepsilon_0)$ by $[\lambda_0 - \varepsilon, \lambda_0]$ and $(\lambda_0 - \varepsilon_0, \lambda_0)$ in all assumptions as well as in the assertions.

Remark 1.6. Denote by $T_K(U)$ the contingent cone to K at a point $U \in K$, i.e.

$$T_K(U) = \overline{\bigcup_{h>0} \bigcup_{V \in K} h(V - U)}.$$

For any $U \in K$, $W \in \mathbb{R}^3$, denote by $P_U W$ the projection of W onto $T_K(U)$, i.e. the unique element from $T_K(U)$ satisfying

$$|P_U W - W| = \min_{Z \in T_K(U)} |Z - W|.$$

It is known (see [1]) that an absolutely continuous function $U : [0, T) \to K$ is a solution of (I) if and only if

$$\dot{U}(t) = P_{U(t)}\big(A_\lambda U(t) + G(\lambda, U(t))\big) \quad \text{for a.a. } t \in [0, T).$$

Further, any solution of (I) is right differentiable and its right derivative is right continuous in $[0, T)$ (see [7]). Particularly, the last equation holds for all $t \in [0, T)$ if $\dot{U}(t)$ is understood as the right derivative.

2. Relation to the existence result

Suppose that

$$\text{(U)} \qquad U_j(\lambda) = U_j \text{ are independent of } \lambda$$

and recall Theorem 1 from [2], where (U) is supposed and therefore also $g = g_\lambda$ is independent of λ. Let V_0 be such that

$$V_0 \in \partial K, |V_0| = 1, g(V_0) = \max_{0 \neq V \in \partial K} g(V).$$

Theorem 2.1. (See [2], Theorem 1.) Let (U) be fulfilled. Suppose that $\lambda_1 < \lambda_2$ are such that

$$t_{0,\lambda}(V_0) < +\infty \text{ for } \lambda_1 \le \lambda \le \lambda_2, \tag{2.1}$$

$$\alpha(\lambda) > \mu_3(\lambda), \beta(\lambda) > 0 \text{ for } \lambda_1 \le \lambda \le \lambda_2, \tag{2.2}$$

$$|U_{0,\lambda}(t_{0,\lambda}(V_0), V_0)| < |V_0| (= 1) \text{ for } \lambda = \lambda_1, \tag{2.3}$$

$$|U_{0,\lambda}(t_{0,\lambda}(V_0), V_0)| > |V_0| \text{ for } \lambda = \lambda_2. \tag{2.4}$$

Then for any sufficiently small $r > 0$ there exists $\lambda \in (\lambda_1, \lambda_2)$ such that $U_\lambda(\cdot, rV_0)$ is a periodic solution of (I). There is at least one bifurcation point $\lambda_I \in (\lambda_1, \lambda_2)$ in which periodic solutions of (I) bifurcate from the branch of trivial solutions.

Remark 2.1. The assertion of Theorem 2.1 remains valid if we replace the assumption (U) by (V). In this case, it is possible to repeat all considerations from the proof of Theorem 1 in [2]. Of course, it is necessary to replace the fixed v from (8) in [2] by V_λ and to reformulate all Lemmas and Remarks used. But all proofs remain without any essential change.

If the assumptions of Theorem 1.1 or 1.2 are fulfilled and

$$|U_{0,\lambda}(t_{0,\lambda}(V_0), V_0)| < |V_0| \text{ for } \lambda \in (\lambda_0 - \varepsilon, \lambda_0)$$

then it follows from Theorem 2.1 and Remark 2.1 that periodic solutions of (I) bifurcate from the branch of trivial solutions at λ_0. Theorems 1.1 and 1.2 ensure that if these bifurcating solutions lie on the right of λ_0 then they are asymptotically stable or the corresponding set $\mathbb{A}_\lambda$ is attracting. Notice that in [2], there are given sufficient conditions under which the assumptions of Theorem 2.1 are fulfilled and there is also an example for which the theory can be applied.

3. Proof of Theorem 1.2.

In the following Lemmas, we shall always suppose that $(\mu), (K), (G), (L), (V), (\lambda_0), (\lambda_+), (A)$ are fulfilled. We set $T = 2T_0$ (with the exception of Lemma 3.1), where T_0 is from (λ_0).

Lemma 3.1. For any $T > 0$ there exists $\rho_0 > 0$ such that the solution $U_\lambda(\cdot, V)$ exists on $[0, T+1)$ for any $\lambda \in [\lambda_0, \lambda_0 + \varepsilon], V \in K, |V| \le \rho_0$ and that the following conditions are fulfilled for any $\lambda_n \in [\lambda_0, \lambda_0 + \varepsilon], V_n \in K, |V_n| < \rho_0, \lambda_n \to \lambda, V_n \to V$:

$$U_{\lambda_n}(\cdot, V_n) \to U_\lambda(\cdot, V) \text{ in } C([0,T]), \tag{3.1}$$

$$\text{if } U_\lambda(t, V) \notin S_\lambda \text{ for } t \in [0,T] \text{ then } \varphi_{\lambda_n}(\cdot, V_n) \to \varphi_\lambda(\cdot, V) \text{ in } C([0,T]), \tag{3.2}$$

$$\text{if } V \notin S_\lambda, t_\lambda(V) < T, \dot{\varphi}_\lambda(t_\lambda(V), V) > 0 \text{ then } t_{\lambda_n}(V_n) \to t_\lambda(V). \tag{3.3}$$

Further, let $\lambda_n \in [\lambda_0, \lambda_0 + \varepsilon], V_n \in K, \lambda_n \to \lambda, 0 \ne |V_n| \to 0, \frac{V_n}{|V_n|} \to W$, let $T > 0$ be arbitrary. Then

$$\frac{U_\lambda(\cdot, V_n)}{|V_n|} \to U_{0,\lambda}(\cdot, W) \text{ in } C([0,T]), \tag{3.4}$$

$$\text{if } W \notin S_\lambda \text{ then } \varphi_{\lambda_n}(\cdot, V_n) \to \varphi_{0,\lambda}(\cdot, W) \text{ in } C([0,T]), \tag{3.5}$$

$$\text{if } W \notin S_\lambda, t_{0,\lambda}(W) < +\infty \text{ and } \dot{\varphi}_{0,\lambda}(t_{0,\lambda}(W), W) > 0 \text{ then } t_{\lambda_n}(V_n) \to t_{0,\lambda}(W). \tag{3.6}$$

Proof. For the case of U_j independent of λ see [5], Lemma 2.1, Theorems 2.1, 2.2 and Consequence 2.2, for the general case see [3]. (See also Remark 2.1.)

Lemma 3.2. Let $\lambda \in \mathbb{R}, V \in K \setminus S$. Then

$$U_{0,\lambda}(t,V) \notin S \quad \text{for all } t \geq 0, \tag{3.7}$$

$$\text{if } \dot{\varphi}_{0,\lambda}(t_0,V) \leq 0 \text{ for some } t_0 \geq 0 \text{ then } \dot{\varphi}_{0,\lambda}(t,V) \leq 0 \text{ for all } t \geq t_0. \tag{3.8}$$

Proof. See [2], Lemma 2. Note that the independence of the eigenvectors of λ considered in [2] plays no role here because we consider a fixed λ. (See also Remark 2.1.)

Lemma 3.3. There exists $\rho_0 > 0$ such that

if $\lambda \in [\lambda_0, \lambda_0 + \varepsilon], V \in K \backslash S_\lambda, |V| < \rho_0, g(V) \leq g(V_\lambda)$ then $g(U_\lambda(t,V)) \leq g(V_\lambda)$ for all $t \in [0,T]$,

if $\lambda \in [\lambda_0, \lambda_0 + \varepsilon], V \in K \backslash S_\lambda, g(V) \leq g(V_\lambda)$ then $g(U_{0,\lambda}(t,V)) \leq g(V_\lambda)$ for all $t \geq 0$.
Particularly, if $r \in (0,\rho_0), \lambda \in [\lambda_0, \lambda_0 + \varepsilon], t_\lambda(rV_\lambda) < +\infty$ then $U_\lambda(t_\lambda(rV_\lambda), rV_\lambda) = sV_\lambda$ with some $s > 0$. If $t_{0,\lambda}(V_\lambda) < +\infty$ then $U_{0,\lambda}(t_{0,\lambda}(V_\lambda), V_\lambda) = sV_\lambda$ with some $s > 0$.

Proof. See [2], Lemma 3 and Remark 2.1.

Consequence 3.1. There exists $\rho_0 > 0$ such that

$$\dot{\varphi}_{0,\lambda}(t_{0,\lambda}(V_\lambda), V_\lambda) > 0 \quad \text{for all } \lambda \in [\lambda_0, \lambda_0 + \varepsilon], \tag{3.9}$$

$$\dot{\varphi}_\lambda(0, rV_\lambda) > 0 \quad \text{for all } \lambda \in [\lambda_0, \lambda_0 + \varepsilon], \;\; r \in (0, \rho_0], \tag{3.10}$$

$$t_\lambda(rV_\lambda) < T, \; \dot{\varphi}_\lambda(t_\lambda(rV_\lambda), rV_\lambda) > 0 \quad \text{for all } \lambda \in [\lambda_0, \lambda_0 + \varepsilon], \;\; r \in (0, \rho_0]. \tag{3.11}$$

Proof. (Cf. also parts II, III, V of the proof of Theorem 1 in [2].) It follows from the assumtion (λ_0) and (3.8) from Lemma 3.2 that

$$\dot{\varphi}_{0,\lambda}(0, V_\lambda) > 0 \quad \text{for all } \lambda \in [\lambda_0, \lambda_0 + \varepsilon].$$

Further, it follows from Lemma 3.3 that

$$U_{0,\lambda}(t_{0,\lambda}(V_\lambda), V_\lambda) = k(\lambda) V_\lambda \quad \text{for all } \lambda \in [\lambda_0, \lambda_0 + \varepsilon]$$

with some $k(\lambda) > 0$. Hence,

$$\dot{\varphi}_{0,\lambda}(t_{0,\lambda}(V_\lambda), V_\lambda) = \dot{\varphi}_{0,\lambda}(0, k(\lambda)V_\lambda) = \dot{\varphi}_{0,\lambda}(0, V_\lambda) > 0 \quad \text{for all } \lambda \in [\lambda_0, \lambda_0 + \varepsilon],$$

i.e. (3.9) holds.

Suppose that (3.10) in not true, i.e. there exists $\lambda_n \in [\lambda_0, \lambda_0 + \varepsilon]$, $r_n > 0$ such that $\lambda_n \to \lambda$, $r_n \to 0$,

$$\dot{\varphi}_{\lambda_n}(0, r_n V_{\lambda_n}) \leq 0.$$

Remark 1.6 together with (G) and the fact that the cones $T_K(V_\lambda)$, $T_K(rV_\lambda)$ coincide, imply

$$\frac{1}{r_n} \dot{U}_{\lambda_{r_n}}(0, r_n V_{\lambda_n}) = \frac{1}{r_n} P_{r_n V_{\lambda_n}} \left(A_{\lambda_n} r_n V_{\lambda_n} + G(\lambda_n, r_n V_{\lambda_n}) \right) =$$
$$= P_{V_{\lambda_n}} \left(A_{\lambda_n} V_{\lambda_n} + \frac{1}{r_n} G(\lambda_n, r_n V_{\lambda_n}) \right) \to P_{V_\lambda} A_\lambda V_\lambda = \dot{U}_{\lambda,0}(0, V_\lambda).$$

If follows also $\dot{\varphi}_{\lambda_n}(0, r_n V_{\lambda_n}) \to \dot{\varphi}_{\lambda,0}(0, V_\lambda)$ and we obtain $\dot{\varphi}_{0,\lambda}(0, V_\lambda) \leq 0$ which contradicts (3.9) and (3.10) is proved.

Suppose that (3.11) is not true. Then there exist $\lambda_n \in [\lambda_0, \lambda_0 + \varepsilon]$, $r_n \searrow 0$ such that at least one of the inequalities in (3.11) is not fulfilled for $r = r_n$. But it follows from (λ_0), (3.10) and (3.6) in Lemma 3.1 that $t_{\lambda_n}(r_n V_{\lambda_n}) < T$ for r_n small enough and therefore

$$\dot{\varphi}_{\lambda_n}(t_{\lambda_n}(r_n V_{\lambda_n}), r_n V_{\lambda_n}) \leq 0.$$

We have $U_\lambda(t, 0) = 0$ for all $t \geq 0$ and Lemma 3.1 implies $U_{\lambda_n}(t_{\lambda_n}(r_n V_{\lambda_n}), r_n V_{\lambda_n}) \to 0$. We can write $U_{\lambda_n}(t_{\lambda_n}(r_n V_{\lambda_n}), r_n V_{\lambda_n}) = k(\lambda_n, r_n) V_{\lambda_n}$ by Lemma 3.3 again and it follows $k(\lambda_n, r_n) \to 0$. Hence, we obtain

$$0 \geq \dot{\varphi}_{\lambda_n}(t_{\lambda_n}(r_n V_{\lambda_n}), r_n V_{\lambda_n}) = \dot{\varphi}_{\lambda_n}(0, k(\lambda_n, r_n) V_{\lambda_n})$$

which contradicts (3.10).

Remark 3.1. Let $r_1, r_2 \in (0, \rho_0)$,

$$|U_\lambda(t_\lambda(r_1 V_\lambda), r_1 V_\lambda)| \geq r_1, \tag{3.12}$$

$$|U_\lambda(t_\lambda(r_2 V_2), r_2 V_\lambda)| \leq r_2 \tag{3.13}$$

for some $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_0]$. Then there exists r lying in the closed interval I bounded by r_1, r_2 and such that $U_\lambda(t_\lambda(r V_\lambda), r V_\lambda) = r V_\lambda$, i.e. $U_\lambda(\cdot, r V_\lambda)$ is periodic. Indeed, the function $h(r) = |U_\lambda(t_\lambda(r V_\lambda), r V_\lambda)|$ is continuous on I by Lemma 3.1, Consequence 3.1 and $h(r_1) \geq r_1, h(r_2) \leq r_2$. It follows that $h(r) = r$ for some $r \in I$.

Remark 3.2. For any $\lambda \in (\lambda_0, \lambda_0 + \varepsilon]$ there exists ρ^0_λ such that

$$|U_\lambda(t_\lambda(r V_\lambda), r V_\lambda)| > r \text{ for all } r \in (0, \rho^0_\lambda).$$

Indeed, otherwise $r_n \searrow 0$ would exist such that $r_n^{-1} |U_\lambda(t_\lambda(r_n V_\lambda), r_n V_\lambda)| \leq 1$ and the limiting process by using (3.4), (3.6) from Lemma 3.1 (and Consequence 3.1) would give $|U_{0,\lambda}(t_{0,\lambda}(V_\lambda), V_\lambda)| \leq 1$ which contradicts the assumption (λ_+) (see Remark 1.2).

Lemma 3.4. There is $\rho_0 > 0$ such that

$$|U_{\lambda_0}(t_{\lambda_0}(r V_{\lambda_0}), r V_{\lambda_0})| < r \text{ for all } r \in (0, \rho_0]. \tag{3.14}$$

Further, if $\rho_1 \in (0, \rho_0)$ then there exists $\varepsilon_0 > 0$ such that

$$|U_\lambda(t_\lambda(r V_\lambda), r V_\lambda)| < r \text{ for all } \lambda \in [\lambda_0, \lambda_0 + \varepsilon_0],\ r \in (\rho_1, \rho_0). \tag{3.15}$$

Proof. Let R be from the assumption (A). It follows from Lemma 3.1 and Consequence 3.1 (by using the fact that $U_{\lambda_0}(\cdot, 0) = 0$) that there exists ρ_0 such that

$$|U_{\lambda_0}(t_{\lambda_0}(r V_{\lambda_0}), r V_{\lambda_0})| < R \text{ for all } r \in (0, \rho_0].$$

Suppose that there is $r_1 \in (0, \rho_0]$ satisfying (3.12) with $\lambda = \lambda_0$. Simultaneously, it follows from (A) that (3.13) holds for $\lambda = \lambda_0$ with some $r_2 > 0$ small enough. Remark 3.1 ensures the existence of $r \in (r_2, r_1) \subset (0, \rho_0)$ such that $U_{\lambda_0}(\cdot, r V_{\lambda_0})$ is periodic which contradicts (A). Hence, (3.14) is proved. If (3.15) were not true then λ_n, r_n would exist such that

$$\lambda_n \searrow \lambda_0,\ r_n \to r \in [\rho_1, \rho_0],\ |U_{\lambda_n}(t_{\lambda_n}(r_n V_{\lambda_n}), r_n V_{\lambda_n})| \geq r_n.$$

The limiting process by using (3.1),(3.3) (Lemma 3.1) and (3.11) (Consequence 3.1) would give

$$|U_{\lambda_0}(t_{\lambda_0}(r V_{\lambda_0}), r V_{\lambda_0})| \geq r$$

which is impossible by (3.14).

Notation 3.1.

$$\underline{r}_\lambda = \sup\{r_\lambda \in (0,\rho_0); |U_\lambda(t_\lambda(rV_\lambda), rV_\lambda)| > r \text{ for all } r \in (0, r_\lambda)\},$$
$$\bar{r}_\lambda = \inf\{r_\lambda \in (0,\rho_0); |U_\lambda(t_\lambda(rV_\lambda), rV_\lambda)| < r \text{ for all } r \in (r_\lambda, \rho_0)\}.$$

Remark 3.3. We have $\underline{r}_\lambda > 0$ for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon]$ by Remark 3.2. Further, if $\rho_1 \in (0, \rho_0)$ and ε_0 is the corresponding number from Lemma 3.4 then $\bar{r}_\lambda < \rho_1 < \rho_0$ by Lemma 3.4 for $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_0]$. It follows from Lemma 3.1 and Consequence 3.1 that $U_\lambda(t_\lambda(\bar{r}_\lambda V_\lambda), \bar{r}_\lambda V_\lambda) = \bar{r}_\lambda V_\lambda$, $U_\lambda(t_\lambda(\underline{r}_\lambda V_\lambda), \underline{r}_\lambda, V_\lambda) = \underline{r}_\lambda V_\lambda$ and therefore $\underline{\rho}_\lambda \le \underline{r}_\lambda \le \bar{r}_\lambda \le \bar{\rho}_\lambda$. Of course, if there is only one periodic solution $U_\lambda(\cdot, rV_\lambda)$ with $r \in (0, \rho_0)$ for some $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_0]$ as in the assumptions of Theorem 1.1 then $\underline{\rho}_\lambda = \underline{r}_\lambda = \bar{r}_\lambda = \bar{\rho}_\lambda$.

Remark 3.4. Setting successively $V = 2U(t), V = 0$, we obtain that any solution of (I) satisfies $(\dot{U}(t) - A_\lambda U(t) - G(\lambda, U(t)), U(t)) = 0$.

Remark 3.5. Let ρ_0 be from Lemma 3.1. If $V \in K, |V| \neq 0, \lambda \in [\lambda_0, \lambda_0 + \varepsilon]$ then $|U_\lambda(t, V)| \neq 0$ for all $t \in [0, T]$. Indeed, if we set $U(t) = U_\lambda(t, V)$ then Remark 3.4 implies $\frac{d}{dt}(|U(t)|^2) = 2(\dot{U}(t), U(t)) = 2(A_\lambda U(t) + G(\lambda, U(t)), U(t)) \ge -C|U(t)|^2$ for all $t \in [0, T]$, where C depends on $\max\limits_{t\in[0,T]} |U(t)|$ (see the assumption (L)). It follows $|U(t)|^2 \ge e^{-Ct}|V|^2$ for all $t \in [0, T]$.

Lemma 3.5. Let ρ_0 be such that the assertions of the previous Lemmas hold. Then there exists $\rho_1 > 0$ such that

$$|U_\lambda(t'_\lambda(V), V)| < \rho_0 \text{ for all } \lambda \in [\lambda_0, \lambda_0 + \varepsilon], V \in K, |V| < \rho_1$$

where we denote $t'_\lambda(V) = \min\{t; t > 0, U_\lambda(t, V) = rV_\lambda \text{ with some } r > 0\}$.

Proof. First, recall that $t'_\lambda(V)$ is well-defined by Remark 1.1 and Lemma 3.3. Set

$$\tau = \min_{\lambda\in[\lambda_0,\lambda_0+\varepsilon]} \{[g_\lambda(V_\lambda)]^{-2} + 1\}^{-\frac{1}{2}}.$$

It follows from Lemma 3.1 that there is $\rho_2 > 0$ such that

$$\text{if } W \in K, x^3_\lambda(W) < \rho_2 \text{ then } x^3_\lambda(t, W) < \tau\rho_0 \text{ for all } t \in [0, T]. \tag{3.16}$$

Further, there is $\rho_1 > 0$ such that if $V \in K, |V| < \rho_1$ then $x^3_\lambda(V) < \rho_2$ and it follows from (μ) (see also Remark 1.1) that

$$x^3_\lambda(t, V) < \rho_2 \text{ for all } t \in [0, t_0] \text{ with } t_0 = \min\{t; U_\lambda(t, V) \in \partial K\}.$$

Hence, (3.16) implies

$$x^3_\lambda(t, V) < \tau\rho_0 \text{ for all } t \in [0, t_0 + T]. \tag{3.17}$$

It follows from Consequence 3.1 and Lemma 3.3 that $t'_\lambda(V) \in [t_0, t_0 + T]$ and therefore (3.17) together with the definition of $t'_\lambda(V)$ and τ imply

$$|U_\lambda(t'_\lambda(V), V)| = [1 + (g_\lambda(V_\lambda))^{-2}]^{\frac{1}{2}} |x^3_\lambda(t'_\lambda(V), V)| < \rho_0.$$

Consequence 3.2. Let $\rho_1 > 0$ be from Lemma 3.5 and let ε_0 be the corresponding number from Lemma 3.4 such that (3.15) holds. Then it follows from Consequence 3.1, (3.15) from Lemma 3.4 and Lemma 3.5 that $U_\lambda(t_\lambda^k(rV_\lambda), rV_\lambda)$ is defined and $|U_\lambda(t_\lambda^k(rV_\lambda), rV_\lambda)| \in (0, \rho_0)$ for all $r \in (0, \rho_0), k \in \mathbb{N}, \lambda \in [\lambda_0, \lambda_0 + \varepsilon_0]$. Further, $U_\lambda(\cdot, rV_\lambda)$ with some $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_0], r \in (0, \rho_0)$ is periodic if and only if $U_\lambda(t_\lambda^k(rV_\lambda), rV_\lambda) = rV_\lambda$ for some $k \in \mathbb{N}$. If $U_\lambda(\cdot, V)$ is periodic and $|V|$ is small enough then there is $t_0 > 0, k \in \mathbb{N}$ such that $U_\lambda(t_0, V) = rV_\lambda$, $U_\lambda(t_\lambda^k(rV_\lambda), rV_\lambda) = rV_\lambda$. Indeed, $U_\lambda(t_1, V) \in \partial K$ for some $t_1 > 0$ according Remark 1.1, Lemma 3.3 ensures that $U_\lambda(t_0, V) = rV_\lambda$ for some $t_0 \geq 0, r \in (0, \rho_0)$ and the assertion follows.

Lemma 3.6. Let ρ_0 be such that the assertions of the previous Lemmas hold. Then

$$\lim_{\lambda \to \lambda_{0+}} \overline{\rho}_\lambda = 0. \tag{3.18}$$

Proof. First, let us show that

$$\lim_{\lambda \to \lambda_{0+}} \overline{r}_\lambda = 0.$$

Indeed, otherwise λ_n would exist such that

$$\lambda_n \searrow \lambda_0, \overline{r}_{\lambda_n} \to r \in (0, \rho_0], \; U_{\lambda_n}(t_{\lambda_n}(\overline{r}_{\lambda_n} V_{\lambda_n}), \overline{r}_{\lambda_n} V_{\lambda_n}) = \overline{r}_{\lambda_n} V_{\lambda_n}$$

(see Remark 3.3).The limiting process (by using Consequence 3.1 and (3.1), (3.3) from Lemma 3.1) would give $U_{\lambda_0}(t_{\lambda_0}(rV_{\lambda_0}), rV_{\lambda_0}) = rV_{\lambda_0}$ which constradicts Lemma 3.4. Further, suppose that there are λ_n such that $\lambda_n \searrow \lambda_0, \overline{\rho}_{\lambda_n} \to \overline{\rho} \in (0, \rho_0]$. It follows from the definition of $\overline{\rho}_\lambda$ and Consequence 3.2 that there are also $r_n \leq \overline{\rho}_{\lambda_n}, r_n \to \overline{\rho}, k_n \in \mathbb{N}$ such that $U_{\lambda_n}(t_{\lambda_n}^{k_n}, (r_n V_{\lambda_n}), r_n V_{\lambda_n}) = r_n V_{\lambda_n}$. We have

$$|U_{\lambda_n}(t_{\lambda_n}(rV_{\lambda_n}), rV_{\lambda_n})| < r \text{ for all } r \in (\overline{r}_{\lambda_n}, \rho_0)$$

and it follows that r_n could be chosen simultaneously such that there exist $s_n < \overline{r}_{\lambda_n}$ satisfying

$$U_{\lambda_n}(t_{\lambda_n}^{k_n - 1}(r_n V_{\lambda_n}), r_n V_{\lambda_n}) = s_n V_{\lambda_n}, \; U_{\lambda_n}(t_{\lambda_n}(s_n V_{\lambda_n}), s_n V_{\lambda_n}) = r_n V_{\lambda_n}.$$

But $s_n \to 0$ and therefore the left hand side in the last equation converges to zero by Consequence 3.1 and Lemma 3.1 while the right hand side tends to $\overline{\rho} V_\lambda \neq 0$ which is the contradiction.

Lemma 3.7. Let ρ_0 be such that the assertions of the previous Lemmas hold. Then there exists $\varepsilon_0 > 0$ with the following property. If $\lambda \in (\lambda_0, \lambda_0 + \varepsilon_0]$ and there is $r \in (0, \rho_0)$ such that $U_\lambda(\cdot, rV_\lambda)$ is periodic then

$$0 < \underline{\rho}_\lambda \leq \overline{\rho}_\lambda < \rho_0, \tag{3.19}$$

$$|U_\lambda(t_\lambda(rV_\lambda), rV_\lambda)| > r \text{ for all } r \in (0, \underline{\rho}_\lambda), \tag{3.20}$$

$$|U_\lambda(t_\lambda(rV_\lambda), rV_\lambda)| < r \text{ for all } r \in (\overline{\rho}_\lambda, \rho_0). \tag{3.21}$$

Proof. Let $\rho_1 \in (0, \rho_0)$ be from Lemma 3.5, let $\varepsilon_0 > 0$ be such that (3.15) in Lemma 3.4 holds. Suppose that $\underline{\rho}_\lambda = 0$ for some $\lambda \in (\lambda_0, \lambda_0 + \varepsilon_0]$. According Consequence 3.2, there exist $r_n > 0, k_n \in \mathbb{N}$, such that $r_n \searrow 0$, $U_\lambda(t_\lambda^{k_n}(r_n V_\lambda), r_n V_\lambda) = r_n V_\lambda$. It follows from the definition and positiveness of $\underline{r}_\lambda$ (see Remark 3.2) that r_n could be chosen simultaneously such that there are s_n satisfying

$$s_n > \underline{r}_\lambda,\ U_\lambda(t_\lambda^{k_n-1}(r_nV_\lambda), r_nV_\lambda) = s_nV_\lambda,\ U_\lambda(t_\lambda(s_nV_\lambda), s_nV_\lambda) = r_nV_\lambda.$$

We have $s_n < \rho_0$ by Lemma 3.5 and (3.15). Therefore we can suppose $s_n \to s > 0$. Hence, the left hand side in the last equation should tend to $U_\lambda(t_\lambda(sV_\lambda), sV_\lambda) \neq 0$ by Consequence 3.1, Lemma 3.1 and Remark 3.5 while the right hand side tends to zero. This is the contradiction and $0 < \underline{\rho}_\lambda$ is proved. The remaining inequalities in (3.19) follow from the definition of $\underline{\rho}_\lambda, \overline{\rho}_\lambda$ and from Lemma 3.6 for ε_0 small enough. The inequalities (3.20), (3.21) follow directly from Remark 3.3 and Notation 3.1.

Lemma 3.8. There exists $\delta > 0$ such that $t_\lambda(rV_\lambda) \geq \delta$ for all $\lambda \in [\lambda_0, \lambda_0+\varepsilon]$, $r \in (0, \rho_0]$.

Proof. Suppose the contrary. Then there are $\lambda_n \in [\lambda_0, \lambda_0+\varepsilon]$, $r_n \in (0, \rho_0]$ such that $\lambda_n \to \lambda$, $r_n \to r$, $t_{\lambda_n}(r_nV_{\lambda_n}) \to 0$. Set $t_n = t_{\lambda_n}(r_nV_{\lambda_n})$, $U_n(t) = U_{\lambda_n}(t, r_nV_{\lambda_n})$. According Remark 3.4, we obtain

$$0 = \big(U_n(t_n) - r_nV_{\lambda_n}, Z\big) = \int_0^{t_n} \big(\dot{U}_n(t), Z\big)\, dt \geq \int_0^{t_n} \big(A_{\lambda_n}U_n(t) + G\big(\lambda_n, U_n(t)\big), Z\big) dt,$$

$$0 = |U_n(t_n)|^2 - r_n^2 = 2\int_0^{t_n} \big(\dot{U}_n(t), U_n(t)\big) dt = 2\int_0^{t_n} \big(A_{\lambda_n}U_n(t) + G\big(\lambda_n, U_n(t)\big), U_n(t)\big) dt.$$

First, let $r > 0$. Dividing the equations by t_n, we obtain by the limiting process (by using (3.1) from Lemma 3.1) that

$$0 \geq \big(A_\lambda rV_\lambda + G(\lambda, rV_\lambda), Z\big) \quad \text{for any } Z \in K,\ 0 = \big(A_\lambda rV_\lambda + G(\lambda, rV_\lambda), rV_\lambda\big) = 0.$$

It follows that $U_\lambda(t, rV_\lambda) = rV_\lambda$ is a stationary solution of (I) which is impossible by Consequence 3.1. If $r = 0$ then we can divide the above equations by r_nt_n and use (3.4) from Lemma 3.1 to obtain

$$0 \geq (A_\lambda V_\lambda, Z) \quad \text{for any } Z \in K,\ 0 = (A_\lambda V_\lambda, V_\lambda) = 0.$$

That means V_λ is a stationary solution of (LI) which contradicts the assumption (λ_0).

Lemma 3.9. Let ρ_0 be such that the assertions of the previous Lemmas hold. Then

$$\lim_{\lambda\to\lambda_0+} \sup_{\substack{r\in[\underline{\rho}_\lambda, \overline{\rho}_\lambda] \\ t\geq 0}} |U_\lambda(t, rV_\lambda)| = 0.$$

Proof. Suppose the contrary. According Lemma 3.6, there exist $\lambda_n \searrow \lambda_0, r_n \leq \overline{\rho}_{\lambda_n} \searrow 0, t_n > 0$ such that

$$|U_{\lambda_n}(t_n, r_nV_{\lambda_n})| \geq \delta \tag{3.22}$$

with some $\delta > 0$. It follows from Consequence 3.2 and Lemma 3.8 that there are k_n such that

$$t_n \in [t_{\lambda_n}^{k_n}(r_nV_{\lambda_n}), t_{\lambda_n}^{k_n+1}(r_nV_{\lambda_n})].$$

Setting $V_n = U_{\lambda_n}(t^{k_n}_{\lambda_n}(r_n V_{\lambda_n}), r_n V_{\lambda_n})$ we can write (3.22) as

$$|U_{\lambda_n}(t'_n, V_n)| \geq \delta, t'_n = t_n - t^{k_n}_{\lambda_n}(r_n V_{\lambda_n}) \in [0, T]$$

(see also Consequence 3.1). According Consequence 3.1 and Lemma 3.3, we can define s^j_n by

$$U_{\lambda_n}(t_{\lambda_n}(s^j_n V_{\lambda_n}), s^j_n V_{\lambda_n}) = s^{j+1}_n V_{\lambda_n}, s^0_n = r_n.$$

We have $V_n = U_{\lambda_n}(t_{\lambda_n}(s^{k_n}_n V_{\lambda_n}), s^{k_n}_n V_{\lambda_n})$. Let us show that $s^{k_n}_n \to 0$ (for $n \to +\infty$). If follows from Lemma 3.1 and Consequence 3.1 that for any $\xi > 0$ there is $r_0 > 0$ such that $|U_\lambda(t_\lambda(rV_\lambda), rV_\lambda)| < \xi$ for any $r \in (0, r_0)$, Lemma 3.6 ensures the existence of n_0 such that $\overline{\rho_{\lambda_n}} \in (0, r_0)$ for $n \geq n_0$. This together with (3.21) (Lemma 3.7) implies that $s^j_n < \xi$ for $n \geq n_0$, $j = 1, 2, \ldots$. But $\xi > 0$ was arbitrary and, particularly, $s^{k_n}_n \to 0$. Hence, it follows by using Consequence 3.1 and Lemma 3.1 that $|V_n| \to 0$ and therefore $U_{\lambda_n}(\cdot, V_n) \to 0$ in $C([0, T])$ by Lemma 3.1. This is the contradiction.

Proof of Theorem 1.2. Consider a fixed small ρ_0 such that all the assertions of Lemmas and Remarks given above hold. Let ρ_1 be from Lemma 3.5, ε_0 the corresponding number from Lemma 3.4 such that also the assertions of the other Lemmas and Remarks above hold. It follows from Lemma 3.9 and the definition of $\mathbb{A}_\lambda$ that ε_0 can be taken simultaneously such that

$$|V| < \frac{\rho_1}{2} \text{ for all } V \in \mathbb{A}_\lambda, \lambda \in (\lambda_0, \lambda_0 + \varepsilon_0]. \tag{3.23}$$

Lemma 3.5 implies that

$$|U_\lambda(t'_\lambda(V), V)| < \rho_0 \text{ for all } V \in \mathbb{U}_{\frac{\rho_1}{2}}(\mathbb{A}_\lambda), \lambda \in (\lambda_0, \lambda_0 + \varepsilon_0],$$

where $\mathbb{U}_{\frac{\rho_1}{2}}(\mathbb{A}_\lambda)$ is the $\frac{\rho_1}{2}$-neighbourhood of $\mathbb{A}_\lambda$, i.e.

$$\text{for any } V \in \mathbb{U}_{\frac{\rho_1}{2}}(\mathbb{A}_\lambda), \lambda \in (\lambda_0, \lambda_0 + \varepsilon_0] \text{ there is } s \in (0, \rho_0) \text{ such that } U_\lambda(t'_\lambda(V), V) = sV_\lambda.$$

Hence, the attractivity of $\mathbb{A}_\lambda$ is a consequence of the assertion (1.4) of Theorem 1.2. Of course, if $U_\lambda(t_0, V) = rV_\lambda$ for some $t_0 \geq 0, r \in (0, \rho_0)$ then
$\lim_{t \to +\infty} \operatorname{dist}(U_\lambda(t, V), \mathbb{A}_\lambda) = \lim_{t \to +\infty} \operatorname{dist}(U_\lambda(t, rV_\lambda), \mathbb{A}_\lambda)$. Hence, for the proof of (1.4) it is sufficient to show that

$$\lim_{t \to +\infty} \operatorname{dist}(U_\lambda(t, r_0 V_\lambda), \mathbb{A}_\lambda) = 0 \text{ for all } r_0 \in (0, \rho_0), \lambda \in (\lambda_0, \lambda_0 + \varepsilon_0]. \tag{3.24}$$

If this is not true for some r_0 then we have $\operatorname{dist}(U_\lambda(t_n, r_0 V_\lambda, \mathbb{A}_\lambda) > \delta > 0$ with some $t_n \to +\infty$. In the case $r_0 \in [\underline{\rho}_\lambda, \overline{\rho}_\lambda]$, $U_\lambda(t_n, r_0 V_\lambda)$ are bounded by Lemma 3.9 and therefore we can suppose $U_\lambda(t_n, r_0 V_\lambda) \to W$ for some W. But then $W \in \mathbb{A}_\lambda$ by the definition and this is the contradiction.

Hence, it remains to prove (3.24) for $r_0 \in (0, \underline{\rho}_\lambda) \cup (\overline{\rho}_\lambda, \rho_0)$. Consider a fixed $r_0 \in (0, \underline{\rho}_\lambda) \cup (\overline{\rho}_\lambda, \rho_0)$ and define r_n by the equations

$$U_\lambda(t_\lambda(r_{n-1} V_\lambda), r_{n-1} V_\lambda) = r_n V_\lambda, n = 1, 2, \ldots. \tag{3.25}$$

The sequence r_n is well defined by Consequence 3.2. We can suppose that ε_0 is such that $\overline{\rho}_\lambda < \rho_1$ for $\lambda \in [\lambda_0, \lambda_0 + \varepsilon_0]$ (see Lemma 3.6). Then we have $r_n \in (0, \rho_0)$ by (3.21) from Lemma 3.7 and Lemma 3.5. Suppose without loss of generality that also

$$r_n \in (0, \underline{\rho}_\lambda) \cup (\overline{\rho}_\lambda, \rho_0) \text{ for } n = 0, 1, 2, \ldots. \tag{3.26}$$

We shall prove that

$$\text{if } r_{n_0} < \underline{\rho}_\lambda \text{ for some } n_0 \text{ then } r_n > r_{n_0} \text{ for all } n > n_0, \tag{3.27}$$

$$\text{if } r_{n_0} > \overline{\rho}_\lambda \text{ for some } n_0 \text{ then } r_n < r_{n_0} \text{ for all } n > n_0. \tag{3.28}$$

Suppose that (3.27) does not hold. Then it follows from (3.20) in Lemma 3.7 that we can find n_0, k such that

$$r_{n_0} < \underline{\rho}_\lambda, r_{n_0+1} > \underline{r}_\lambda, r_{n_0+k} < r_{n_0}. \tag{3.29}$$

Set $\underline{r} = \inf\{r; |U_\lambda(t_\lambda(rV_\lambda), rV_\lambda)| = \underline{r}_\lambda\}$. The continuous dependence on initial condition (see Lemma 3.1) together with (3.29) imply

$$0 < \underline{r} < r_{n_0}, U_\lambda(t_\lambda(\underline{r}V_\lambda), \underline{r}V_\lambda) = \underline{r}_\lambda V_\lambda. \tag{3.30}$$

Then (3.29), (3.30) together with the periodicity of $U_\lambda(\cdot, \underline{r}_\lambda V_\lambda)$ (Remark 3.3) imply

$$|U_\lambda(t_\lambda^k(r_{n_0}V_\lambda), r_{n_0}V_\lambda)| < r_{n_0}, |U_\lambda(t_\lambda^k(\underline{r}V_\lambda), \underline{r}V_\lambda)| = \underline{r}_\lambda > \underline{r}.$$

The function $h(r) = |U_\lambda(t_\lambda^k(rV_\lambda), rV_\lambda)|$ is defined and continuous on $[\underline{r}, r_{n_0}]$ with respect to Consequence 3.2 and Lemma 3.1. The same considerations as in Remark 3.1 imply the existence of $r \in [\underline{r}, r_{n_0}]$ such that $U_\lambda(r_\lambda^k(rV_\lambda), rV_\lambda) = rV_\lambda$. That means $U_\lambda(\cdot, rV_\lambda)$ is periodic, $r < \underline{\rho}_\lambda$ and this is the contradiction with the definition of $\underline{\rho}_\lambda$. The implication (3.28) can be proved analogously (by changing the sign in all inequalities discussed and replacing $\underline{\rho}_\lambda$ and inf by $\overline{\rho}_\lambda$ and sup, respectively).

Denote by $\{s_n^1\}$ and $\{s_n^2\}$ the subsequence of $\{r_n\}$ of all $r_n \in (0, \underline{\rho}_\lambda)$ and $r_n \in (\overline{\rho}_\lambda, \rho_0)$, respectively. It follows from (3.26) that $\{r_n\} = \{s_n^1\} \cup \{s_n^2\}$ and (3.27), (3.28) ensure that

$$\{s_n^1\} \text{ is increasing }, \{s_n^2\} \text{ is decreasing }. \tag{3.31}$$

Hence,

$$s_n^1 \nearrow s_1 \text{ and } s_n^2 \searrow s_2 \text{ if } \{s_n^1\} \text{ is infinite and } \{s_n^2\} \text{ is infinite, respectively }. \tag{3.32}$$

Suppose for a moment that

$$s_1 = \underline{\rho}_\lambda \text{ if } \{s_n^1\} \text{ is infinite and } s_2 = \overline{\rho}_\lambda \text{ if } \{s_n^2\} \text{ is infinite, respectively }. \tag{3.33}$$

First, suppose that $\{s_n^1\}$ is infinite, $s_1 = \underline{\rho}_\lambda$ and $\{s_n^2\}$ is finite. Then

$$s_n^1 = r_{k+n} \text{ for } n \geq n_0 (\text{ with some } n_0, k)$$

and the limiting proces in (3.25) gives

$$U_\lambda(t_\lambda(\underline{\rho}_\lambda V_\lambda, \underline{\rho}_\lambda V_\lambda) = \underline{\rho}_\lambda V_\lambda.$$

It follows from Lemma 3.1 that $U_\lambda(\cdot, s_n^1 V_\lambda) \to U_\lambda(\cdot, \underline{\rho}_\lambda V_\lambda)$ in $C([0,T])$ and therefore

$$\lim_{t\to+\infty} |U_\lambda(t, r_0 V_\lambda) - U_\lambda(t, \underline{\rho}_\lambda V_\lambda)| = 0.$$

Simultaneously $U_\lambda(t, \underline{\rho}_\lambda V_\lambda) \in \mathbf{A}_\lambda$ for all $t \geq 0$ because of $U_\lambda(\cdot, \underline{\rho}_\lambda V_\lambda)$ is periodic. Hence, the equality in (3.24) holds. (Note that in this case $s_1 = \underline{\rho}_\lambda = \underline{r}_\lambda$.) Analogously we obtain

$$\lim_{t\to+\infty} |U_\lambda(t, r_0 V_\lambda) - U_\lambda(t, \bar{\rho}_\lambda V_\lambda)| = 0, \; U_\lambda(t, \bar{\rho}_\lambda V_\lambda) \in \mathbf{A}_\lambda \text{ for all } t \geq 0$$

if $\{s_n^1\}$ is finite, $\{s_n^2\}$ is infinite, $s_2 = \bar{\rho}_\lambda$. (In this case $s_2 = \bar{\rho}_\lambda = \bar{r}_\lambda$.) Further, suppose that both $\{s_n^1\}, \{s_n^2\}$ are infinite and (3.33) holds. Then there exist increasing sequences $\{k_n\}, \{\ell_n\}, \{m_n\}, \{p_n\}$ of indices such that

$$U_\lambda(t_\lambda(s_{k_n}^1 V_\lambda), s_{k_n}^1 V_\lambda) = s_{\ell_n}^2 V_\lambda, \; U_\lambda(t_\lambda(s_{m_n}^2 V_\lambda), s_{m_n}^2 V_\lambda) = s_{p_n}^1 V_\lambda \text{ for all } n \geq n_0 \tag{3.34}$$

and the limiting process by Lemma 3.1 gives

$$U_\lambda(t_\lambda(\underline{\rho}_\lambda V_\lambda), \underline{\rho}_\lambda V_\lambda) = \bar{\rho}_\lambda V_\lambda, U_\lambda(t_\lambda(\bar{\rho}_\lambda V_\lambda), \bar{\rho}_\lambda V_\lambda) = \underline{\rho}_\lambda V_\lambda, \tag{3.35}$$

that means

$$U_\lambda(t_\lambda^2(\underline{\rho}_\lambda V_\lambda), \underline{\rho}_\lambda V_\lambda) = \underline{\rho}_\lambda V_\lambda.$$

Hence, $U_\lambda(\cdot, \underline{\rho}_\lambda V_\lambda)$ is periodic with the period $t_\lambda^2(\underline{\rho}_\lambda V_\lambda)$ and therefore

$$U_\lambda(t, \underline{\rho}_\lambda V_\lambda) \in \mathbf{A}_\lambda \text{ for all } t \geq 0.$$

(In this case $s^1 = \underline{\rho}_\lambda = \underline{r}_\lambda$, $s_2 = \bar{r}_\lambda = \bar{\rho}_\lambda$). Now, the equality in (3.24) follows by the limiting process from (3.34), (3.35) similarly as in the former situations.

Hence, for the proof of (3.24) it remains to show that (3.33) holds. First, suppose that $\{s_n^1\}$ is infinite and $s_1 < \underline{\rho}_\lambda$. We shall show that then

$$|U_\lambda(t_\lambda(s_n^1 V_\lambda), s_n^1 V_\lambda)| > \bar{\rho}_\lambda \text{ for all } n \geq n_0. \tag{3.36}$$

If (3.36) were not true then a subsequence $\{s_{k_n}^1\}$ would exist such that

$$U_\lambda(t_\lambda(s_{k_n}^1 V_\lambda), s_{k_n}^1 V_\lambda) = s_{k_n+1}^1 V_\lambda, \; n = 1, 2, \ldots.$$

The limiting process by using Lemma 3.1 and Consequence 3.1 would give $U_\lambda(t_\lambda(s_1 V_\lambda), s_1 V_\lambda) = s_1 V_\lambda, s_1 < \underline{\rho}_\lambda$. This is impossible by the definition of $\underline{\rho}_\lambda$ and (3.36) is proved. Particularly, $\{s_n^2\}$ is infinite. It follows that there exist subsequences $s_{k_n}^2 \to s_2, s_{\ell_n}^2 \to s_2$ such that

$$U_\lambda(t_\lambda(s_n^1 V_\lambda), s_n^1 V_\lambda) = s_{k_n}^2 V_\lambda, U_\lambda(t_\lambda(s_{\ell_n}^2 V_\lambda), s_{\ell_n}^2 V_\lambda) = s_n^1 V_\lambda \text{ for all } n \geq n_0.$$

The limiting process gives

$$U_\lambda(t_\lambda(s_1 V_\lambda), s_1 V_\lambda) = s_2 V_\lambda, \; U_\lambda(t_\lambda(s_2 V_\lambda), s_2 V_\lambda) = s_1 V_\lambda,$$

that means

$$U_\lambda(t_\lambda^2(s_1 V_\lambda), s_1 V_\lambda) = s_1 V_\lambda, \; s_1 \in (0, \underline{\rho}_\lambda).$$

This contradicts the definition of $\underline{\rho}_\lambda$ and the first equality in (3.33) is proved. The second one can be proved analogously.

Remark 3.6. We obtained $\underline{\rho_\lambda} = \underline{r_\lambda}$, $\overline{r_\lambda} = \overline{\rho_\lambda}$ in all cases precisely discussed in the proof of Theorem 1.2. But this is not clear if for instance $|U_\lambda(t_\lambda(r_0 V_\lambda), rV_\lambda)| \in (\underline{\rho_\lambda}, \overline{\rho_\lambda})$ for some $r_0 \in (0, \underline{\rho_\lambda}) \cup (0, \overline{\rho_\lambda})$. (We did not need study this situation precisely for the proof of Theorem 1.2.)

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