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Multi-scale clustering for a non-Markovian spatial branching process

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Abstract

Consider a system of particles which move in R^d according to a symmetric α–stable motion, have a lifetime distribution of finite mean, and branch with an offspring law of index $1+\beta$. In case of the critical dimension $d = \alpha/\beta$, the phenomenon of multi-scale clustering occurs. This is expressed in an fdd scaling limit theorem, where initially we start with an increasing localized population or with an increasing homogeneous Poissonian population. The limit state is uniform, but its intensity varies in line with the scaling index according to a continuous-state branching process of index $1+\beta$. Our result generalizes the case $\alpha = 2$ of Brownian particles of Klenke (1998), where pde methods had been used which are not available in the present setting.

Contents

1 Introduction and statement of results

1.1 Motivation and purpose

Multi-scale clustering phenomena had been exposed by several models as the voter model (e.g. Cox and Griffeath ([CG86]) and interacting diffusions (e.g. Fleischmann and Greven [FG94]). They occur in the critical dimension. Here "multi-scale" means that clusters grow on different *macroscopic* scales. For spatial branching processes this was dealt with in Klenke [Kle97, Kle98].

In the latter two papers, Markov branching processes in R^d had been considered in a particle model as well as in a superprocess setting. These models are based on two driving components: migration and branching. For the particle model this means, that first of all particles move independently according to (standard) Brownian motions in R^d. But additionally, at a fixed rate, that is after independent identically exponentially distributed lifetimes, branching occurs. In such a branching event, a particle is independently replaced by a random number of offspring in a critical way. Here "critical" means that the expected number of offspring of a particle equals one. Moreover, the common offspring law is assumed to be of index $1+\beta\in(1,2]$ (see Hypothesis 1(c) below). In the special case $\beta=1$, the number of offspring is maximally two. For $\beta<1$ instead, the offspring law has infinite variance. The offspring evolve independently according to the same rules. The only dependence assumption in the model is that offspring start from their "parents'" position.

In this model, the driving effects compete to each other: The critical branching leads to extinction if started from a finite population, and the spatial spread has a smoothing effect in space. But the latter is dimension dependent: As higher the dimension is, as more smoothing occurs. Thus, if the dimension is high enough, even steady states for infinite populations are possible. "High enough" here means, that $d>2/\beta$, and these dimensions are called supercritical.

In non-supercritical dimensions $d\leq 2/\beta$ instead, the system locally dies as time tends to infinity. That is, the extinction features of critical branching dominates the spatial dispersion by the independent Brownian motions. But by the criticality of branching, the system is mean mass preserving, hence the overall density of particles is conserved at all finite times. Therefore, starting with an infinite population, besides the local extinction, huge clumps of particles are present at rare escaping places.

In the critical dimension $d=2/\beta$ (that is $d=2$ in the finite variance case $\beta=1$), there is an additional effect: clumps grow at a whole range of macroscopic scales. To expose this, the population system is spatially contracted in a time dependent way. In addition, the initial system is fed with more and more particles also in a time dependent way.

By the mentioned maximal independence assumptions in the model, log-Laplace functionals are a basic technical tool. In fact, they connect the stochastic system to initial value problems of the reaction-diffusion equation

$$\frac{\partial}{\partial t}u \;=\; \frac{1}{2}\Delta u - c\,u^{1+\beta} \quad \text{on } (0,\infty)\times \mathsf{R}^d. \tag{1}$$

Here the d–dimensional Laplacian Δ stands for the Brownian migration and the non-linear term for the branching. The multiple clustering behavior is related to asymptotic properties of solutions to (1). To get hands on them, the main method in [Kle97, Kle98] was to construct sub- and super-solutions to equation (1). Here the explicit form of the heat kernel helped to find such semi-solutions. (See also Samarski et al. [SGKM87, Section 1.2] and Bramson et al. [BCG93]).

Our *purpose* is twofold. Mainly we want to pass from Brownian motions to symmetric stable processes of index $\alpha \in (0,2]$. That is, to replace in equation (1) the differential operator $\frac{1}{2}\Delta$ by the fractional Laplacian $\Delta_\alpha := -(-\Delta)^{\alpha/2}$. The critical dimension is then $d = \alpha/\beta$. If $\alpha < 2$, the pde tools mentioned above break down since Δ_α is not a differential operator. But we also want to give up the Markovian nature of the process in the particle setting: We replace the exponential life times by i.i.d. life times with a finite mean (in the spirit of classical Bellman-Harris branching processes or age-dependent branching processes). By this finite mean assumption, the critical dimension will not be changed. The model is available from the literature, we essentially take it from Fleischmann et al. [FVW03].

As in the latter paper, the main tool is an integral equation first studied by Kaj and Sagitov [KS98], for which we have to investigate asymptotic properties of its scaled solutions. If the lifetimes of particles are exponentially distributed, the mentioned integral equation is related to the function-valued ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u \;=\; \Delta_\alpha u - c\,u^{1+\beta} \quad \text{on } (0,\infty)\times \mathsf{R}^d. \tag{2}$$

Our approach covers the case $\alpha = 2$, so that in particular we give an alternative proof for results of [Kle98].

1.2 The (d,α,β,G)–branching particle system

The model we are dealing with is a spatial generalization of Bellman-Harris branching process. This is based on the following ingredients, for convenience we put it in a hypothesis.

Hypothesis 1 (Ingredients of the branching particle system)

(a) (Particles' motion process ξ) For a fixed constant $\alpha \in (0,2]$, consider the symmetric α–*stable* process $(\xi, P_x,\ x \in \mathsf{R}^d)$ in R^d, (cf. Breiman [Bre68, p.317] or Bertoin [Ber96, Ch. VIII]). This is the (time-homogeneous) Markov process with generator $\Delta_\alpha = -(-\Delta)^{\alpha/2}$, the fractional Laplacian (Yosida [Yos74, p.260]), and with càdlàg paths. We denote by $\mathrm{p} = \{\mathrm{p}_t(y) : t > 0,\ y \in \mathsf{R}^d\}$ the continuous transition densities of this *particle motion process* (migration process) ξ.

(b) (Particles' lifetime τ) Introduce the non-lattice *lifetime distribution function* G of a random variable $\tau > 0$ with finite expectation $E\tau =: \kappa > 0$.

(c) (Critical branching mechanism) Consider the *offspring generating function*

$$f(s) \ := \ Es^{\zeta} \ = \ s + c_f \ (1-s)^{1+\beta} \ =: \ s + \Psi(1-s), \tag{3}$$

$0 \le s \le 1$, of the random number ζ of offspring of a particle, with constants $\beta \in (0,1]$ and $c_f \in \left(0, \frac{1}{1+\beta}\right]$. Consequently,

$$P(\zeta = k) \ = \ \delta_{1,k} \ + \ c_f \, (-1)^k \binom{1+\beta}{k}, \qquad k \ge 0,$$

where $\delta_{1,k}$ is the Kronecker symbol. Clearly, $E\zeta = 1$ (*criticality*), and we are dealing with a branching mechanism in the normal domain of attraction of a stable law of index $1+\beta$. Of course, $E\zeta^2 < \infty$ if and only if $\beta = 1$.

(d) (Test functions) Pick a constant $p \in (d, d+\alpha]$ (recall that α is the motion index), and introduce the *reference function*

$$\phi_p(x) \ := \ \left(1 + |x|^2\right)^{-p/2}, \qquad x \in \mathsf{R}^d. \tag{4}$$

Let $\mathcal{C}_p = \mathcal{C}_p(\mathsf{R}^d)$ denote the set of all continuous functions $\varphi : \mathsf{R}^d \to \mathsf{R}$ such that

$$\|\varphi\| \ := \ \sup_{x \in \mathsf{R}^d} \frac{|\varphi(x)|}{\phi_p(x)} \ < \ \infty, \tag{5}$$

and such that the map $x \mapsto \varphi(x)/\phi_p(x)$ can continuously be extended to a function on $\dot{\mathsf{R}}^d$, where $\dot{\mathsf{R}}^d$ is the one-point compactification of R^d. Then $\left(\mathcal{C}_p, \|\cdot\|\right)$ is a separable Banach space.

(e) (State space $\mathcal{N}_p$) Let $\mathcal{M}_p = \mathcal{M}_p(\mathsf{R}^d)$ denote the set of all *p–tempered measures* on R^d, that is (non-negative) measures μ on R^d such that the integral $\int_{\mathsf{R}^d} \mu(\mathrm{d}x)\, \phi_p(x)$ is finite. Introduce the weakest topology in $\mathcal{M}_p$ such that for each $\varphi \in \mathcal{C}_p$ the mapping

$$\mu \ \mapsto \ \langle \mu, \varphi \rangle \ := \ \int_{\mathsf{R}^d} \mu(\mathrm{d}x)\, \varphi(x)$$

is continuous. Note that the (normed) Lebesgue measure ℓ on R^d belongs to $\mathcal{M}_p$. Write $\mathcal{N}_p = \mathcal{N}_p(\mathsf{R}^d)$ for the subset of all counting measures μ in $\mathcal{M}_p$, that is, measures with values in $\{0, 1, ..., \infty\}$.

The set $\mathcal{N}_p$ inherits the topology of $\mathcal{M}_p$. It serves as the *state space* of the branching particle system we will introduce. Especially, the Dirac delta measure $\delta_x \in \mathcal{N}_p$ describes a *single particle* with position $x \in \mathsf{R}^d$. ◇

Recall that a random counting measure $\pi = \pi_\mu$ on R^d is called a *Poissonian* particle field with intensity measure $\mu \in \mathcal{M}_p$ if it has log-Laplace transform

$$-\log \mathbf{E} \exp \langle \pi, -\varphi \rangle \;=\; \big\langle \mu, 1 - \mathrm{e}^{-\varphi} \big\rangle, \qquad \varphi \in \mathcal{C}_p^+.$$

(As with R_+, an index $+$ on a set refers to all of its non-negative members.) In particular, $\pi_{i_0\,\ell}$ denotes the homogeneous Poissonian particle field with intensity $i_0 > 0$.

Here now is our basic model. (In order to get a Markovian setting, include residual life times in the description of the phase space, see [FVW03].)

Definition 2 (Branching particle system Z) The (in general non-Markovian) process $Z = \big\{ Z_t : t \geq 0 \big\}$ we are dealing with can be described by the following properties:

- At time $t = 0$, start with a measure $Z_0 = \mu \in \mathcal{N}_p(\mathsf{R}^d)$.
- Each particle $\delta_x \leq \mu$ starts, independently of the other particles of μ, a path ξ in R^d with law P_x.
- But it lives only a finite time (with probability one) which is an independent copy of τ.
- In the moment of its death, it produces offspring which number is an independent copy of ζ.
- Newly born particles get paths, which are independent copies of ξ starting at the parents' death time from the parents' position.
- And they get lifetimes, which are independent copies of τ.
- Write $\mathbf{P}_\mu$ for the law of Z. This is considered as a measure on the set $\mathcal{D}(\mathsf{R}_+, \mathcal{N}_p)$ of all $\mathcal{N}_p$–valued *càdlàg* paths.

For convenience, this process $(Z, \mathbf{P}_\mu)$ is said to be a (d, α, β, G)*–branching particle system.* Write $\mathbf{P}_{i_0}$ instead, if Z_0 is the homogeneous Poissonian particle field $\pi_{i_0\,\ell}$ with intensity $i_0 > 0$. ◇

Note that we imposed maximal independence assumptions in defining Z. The main dependence assumption is that newly born particles start from the ancestor's death place. Clearly, Z is Markovian if and only if the lifetime distribution G is an exponential law.

1.3 Main result: Multi-scale clustering of Z

We are interested in the long-time behavior of the spatial correlations of the (d,α,β,G)–branching particle system Z of Definition 2 in the *critical dimension* $d=\alpha/\beta$ (meaning always that this is assumed to be an integer). Recall that α and $1+\beta$ are the motion and branching indices, respectively. Here is the more precise setting: For each constant $h<1$, introduce the following *time-dependent mass-space scaling*

$$Z_t^h(B) \;:=\; (t^h \log t)^{-1/\beta}\, Z_t(t^{h/\alpha}B), \qquad t>0, \quad \text{Borel } B\subseteq \mathsf{R}^d. \tag{6}$$

Consequently, space is contracted and mass renormalized, both in a t–dependent way. Moreover, we will feed the initial state of Z additionally with particles, also in a t–dependent way, that is to look at $Z_t^h\in\mathcal{M}_p$ under the laws $\mathbf{P}_{\left[i_0\,(t\log t)^{1/\beta}\right]\delta_{t^{1/\alpha}x}}$ and $\mathbf{P}_{i_0\,(\log t)^{1/\beta}}$, $i_0>0$, respectively.

To describe our main result, we also need to introduce a "classical" object.

Definition 3 (Continuous-state branching of index $1+\beta$) For a positive constant γ, denote by $\eta=\{\eta_t:\, t\geq 0\}$ the *continuous-state branching process* with index $1+\beta$ and branching rate γ. That is, η is the (time-homogeneous) non-negative Markov process with càdlàg paths having log-Laplace transition function

$$-\log E\left\{\mathrm{e}^{-\theta\eta_t}\,\middle|\,\eta_0\right\} \;=\; \eta_0\, v(t;\theta), \qquad t,\theta\geq 0, \tag{7}$$

where, for θ fixed, $v=v(\,\cdot\,;\theta)=\{v(t;\theta):\, t\geq 0\}$ is the unique solution to the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}v \;=\; -\gamma\, v^{1+\beta} \quad \text{with initial condition} \quad v(0;\theta)=\theta. \tag{8}$$

Consequently,

$$v(t;\theta) \;=\; \theta\left(1+\gamma\,\beta\, t\,\theta^{\beta}\right)^{-1/\beta}, \qquad t,\theta\geq 0. \tag{9}$$

◇

Recall that under suitable scalings, η arises as a limiting process from Galton-Watson processes with offspring generating function f from Hypothesis 1(c) (see, for instance, Lamperti [Lam67]).

Here is our *main result*:

Theorem 4 (Multi-scale clustering for Z) *Let $d=\alpha/\beta$. Consider the $\mathcal{M}_p(\mathsf{R}^d)$–valued processes*

$$\left\{Z_t^h:\; 0\leq h<1\right\}, \quad t>1, \quad \textit{and} \quad \{\eta_{1-h}\,\ell:\; 0\leq h<1\},$$

with Z_t^h *defined in* (6) *and where* η *is the continuous-state branching process of Definition* 3, *but with branching rate*

$$\gamma := c_f D \quad where \quad D := \frac{1}{\kappa}\int_{\mathsf{R}^d} \mathrm{d}y\, \mathrm{p}_1^{1+\beta}(y). \tag{10}$$

Under laws of Z *and* η *which still have to be described, we ask for the convergence*

$$\left\{Z_t^h :\ 0 \le h < 1\right\} \underset{t\uparrow\infty}{\overset{\text{fdd}}{\longrightarrow}} \left\{\eta_{1-h}\,\ell :\ 0 \le h < 1\right\} \tag{11}$$

in the sense of convergence of finite-dimensional distributions. Fix $i_0 > 0$.

(a) (Localized initial state) *Fix a point* $x \in \mathsf{R}^d$. *Claim* (11) *holds under the distributions* $\mathbf{P}_{\left[i_0\,(t\log t)^{1/\beta}\right]\delta_{t^{1/\alpha}x}}$ *of* Z *and if* $\eta_0 = i_0\,\mathrm{p}_1(x)$.

(b) (Homogeneous initial state) *Claim* (11) *also holds under the distributions* $\mathbf{P}_{i_0\,(\log t)^{1/\beta}}$ *of* Z *and if* $\eta_0 = i_0$.

Consequently, the limit state is uniform, and its intensity varies in dependence on the multi-scale index h and according to the continuous-state branching process η. In the infinite population case of (b), as $t \uparrow \infty$, for each fixed scaling index $h \in [0,1)$, clusters grow at scale $(t^h \log t)^{1/\beta}$ as $t \uparrow \infty$. Recall that we started Z with a t–dependent initial intensity $i_0\,(\log t)^{1/\beta}$. In particular, if $h = 0$, for t large, $Z_t(B)$ is of order $(\log t)^{1/\beta}\,\eta_1\,\ell(B)$ with $\eta_1 \ge 0$ the (random) state of the continuous-state branching process η at time 1 if started at time 0 at $\eta_0 = i_0$.

Remark 5 (Tightness) Unfortunately, it remains *open* whether the fdd convergence statement (11) can be lifted up to convergence of laws on Skorohod path space. ◇

1.4 Refined asymptotics

Theorem 4 is based on some refined asymptotic statements we now want to describe. For this purpose, we introduce the following notations. Fix $\varphi \in \mathcal{C}_p^+(\mathsf{R}^d)$, and set

$$Q_t\varphi\,(x) := \mathbf{E}_{\delta_x}\left(1 - \mathrm{e}^{-\langle Z_t\,,\varphi\rangle}\right), \qquad t \ge 0, \quad x \in \mathsf{R}^d, \tag{12}$$

and

$$\varphi_{h,t}(x) := (t^h \log t)^{-1/\beta}\,\varphi(t^{-h/\alpha}x), \qquad 0 \le h < 1,\ t > 0,\ x \in \mathsf{R}^d. \tag{13}$$

Theorem 6 (Refined asymptotics for Q**)** *Assume* $d = \alpha/\beta$. *Then, for fixed* $x \in \mathsf{R}^d$ *and* $h \in [0,1)$,

$$(t\log t)^{1/\beta}\,Q_t(\varphi_{h,t})(t^{1/\alpha}x) \underset{t\uparrow\infty}{\longrightarrow} \mathrm{p}_1(x)\,v\big(1-h;\langle \ell,\varphi\rangle\big), \tag{14}$$

with the "macroscopic" log-Laplace function v *from* (9), *but with branching rate* γ *as in* (10).

The proof of this theorem is postponed to Subsection 2.9 below.

1.5 Multi-scale clustering for the (d, α, β)–superprocess

In order to pass to a superprocess setting via a high density limit, we consider a whole *family* $\{Z^{(\varepsilon)} : 0 < \varepsilon \leq 1\}$ of $(d, \alpha, \beta, G^{\varepsilon})$–branching particle systems. On them we assume that

- the (deterministic) initial states $Z_0^{(\varepsilon)}$ satisfy $\varepsilon Z_0^{(\varepsilon)} \to \mu$ in $\mathcal{M}_p(\mathsf{R}^d)$ as $\varepsilon \downarrow 0$,
- the lifetime distributions G^{ε} are given by $G^{\varepsilon}(s) := G(\varepsilon^{-\beta} s)$, $s \geq 0$, with G as before (with mean κ).

Then the $\varepsilon Z^{(\varepsilon)}$ converge in law on Skorohod space $\mathcal{D}(\mathsf{R}_+, \mathcal{M}_p)$ to a limit process denoted by $X = \{X_t : t \geq 0\}$. Here $(X, \mathbb{P}_\mu)$ is the famous (d, α, β)–*superprocess* with initial state $X_0 = \mu$ and with branching rate $\varrho = c_f / \kappa$ with c_f from (3). Recall that the (time-homogeneous) Markov process X is determined by its log-Laplace functional

$$-\log \mathbb{P}_\mu \mathrm{e}^{-\langle X_t, \varphi \rangle} \;=\; \big\langle \mu, u(t, \cdot) \big\rangle, \qquad t \geq 0, \quad \varphi \in \mathcal{C}_p^+,$$

where $u = u(\cdot, \cdot; \varphi) = \big\{u(t, x; \varphi) : t \geq 0,\, x \in \mathsf{R}^d\big\}$ is the unique non-negative solution of the log-Laplace equation

$$u(t, x; \varphi) \;=\; E_x\left(\varphi(\xi_t) \;-\; \varrho \int_0^t \mathrm{d}s\; u^{1+\beta}(t-s, \xi_s\,; \varphi)\right), \tag{15}$$

$t \geq 0$, $x \in \mathsf{R}^d$, which is a more detailed version of (2). For the convergence statement, see, for instance, [KS98].

For this (d, α, β)–superprocess X the following result holds analogously to Theorem 4. Here the scaled quantities X_t^h are defined just as in (6).

Theorem 7 (Multi-scale clustering for X) *Let $d = \alpha/\beta$. Consider the $\mathcal{M}_p(\mathsf{R}^d)$–valued processes*

$$\big\{X_t^h : -\infty < h < 1\big\}, \quad t > 1, \quad \textit{and} \quad \{\eta_{1-h}\, \ell : -\infty < h < 1\}, \tag{16}$$

where η is the continuous-state branching process of Definition 3, but with branching rate

$$\gamma \;:=\; c_f\, D \quad \textit{where} \quad D \;:=\; \frac{1}{\kappa} \int_{\mathsf{R}^d} \mathrm{d}y\; \mathrm{p}_1^{1+\beta}(y).$$

Under laws of X and η which still have to be described, we ask for the convergence

$$\big\{X_t^h : -\infty < h < 1\big\} \;\underset{t\uparrow\infty}{\overset{\text{fdd}}{\longrightarrow}}\; \{\eta_{1-h}\, \ell : -\infty < h < 1\} \tag{17}$$

in the sense of convergence of finite-dimensional distributions. Fix $i_0 > 0$.

(a) (Localized initial state) *Fix a point* $x \in \mathsf{R}^d$. *Claim* (17) *holds under the distributions* $\mathbb{P}_{\left[i_0\,(t\log t)^{1/\beta}\right]\delta_{t^{1/\alpha}x}}$ *of* X *and if* $\eta_0 = i_0\,\mathrm{p}_1(x)$.

(b) (Homogeneous initial state) *Claim* (17) *also holds under the distributions* $\mathbb{P}_{i_0\,(\log t)^{1/\beta}\ell}$ *of* X *and if* $\eta_0 = i_0$.

Note that in the superprocess setting also negative scaling indices are allowed. This multi-scale clustering of X is based on the following analogy of Theorem 6.

Theorem 8 (Refined asymptotics for u**)** *Assume* $d = \alpha/\beta$. *Then, for fixed* $x \in \mathsf{R}^d$ *and* $-\infty < h < 1$,

$$(t\log t)^{1/\beta}\,u\big(t, t^{1/\alpha}x; \varphi_{h,t}\big) \underset{t\uparrow\infty}{\longrightarrow} \mathrm{p}_1(x)\,v\big(1-h; \langle \ell, \varphi\rangle\big), \tag{18}$$

with the macroscopic log-Laplace function v *from* (9), *but with branching rate* γ *as in* (10).

The proofs of Theorems 8 and 7 are easier than the ones concerning the statements in the (non-Markovian) particle model case, and we will indicate them in Subsection 3.6 below.

2 Refined asymptotics for Q

The purpose of this section is to prove the refined asymptotics for Q as stated in Theorem 6. A key step will be an approximate renewal equation (Proposition 12) and an L^1–convergence statement (Proposition 16).

2.1 On the renewal function

The symbol c will always denote a positive constant which may vary from place to place. Notation $c_{(\#)}$ and $c_{\#}$ instead will refer to such a constant which first occurred in formula line (#) and, for instance, Lemma #, respectively.

For convenience, here we collect some properties of the renewal function, say N, related to the lifetime distribution G from Hypothesis 1(b):

$$N_t := \sum_{i=1}^{\infty} G^{*i}(t), \qquad t \ge 0. \tag{19}$$

Lemma 9 (A renewal function weighted increment) *There is a constant* $c_9 = c_9(G)$ *such that*

$$0 \le \int_q^r N^-_{t-\mathrm{d}s}\,\frac{1}{s} \le c_9\Big(2 + \log\frac{r}{q}\Big), \qquad 1 \le q < r \le t, \tag{20}$$

where $N^-_{t-\mathrm{d}s}$ *refers to a Stieltjes integration with respect to the non-decreasing function* $s \mapsto N^-_{t-s} := -N_{t-s}$.

Proof By the key renewal theorem (see e.g. Feller [Fel71, Chapter XI, § 1]),

$$N_s - N_{s-1} \underset{s\uparrow\infty}{\longrightarrow} \frac{1}{\kappa}. \tag{21}$$

Combined with the fact that $s \mapsto N_s$ is non-decreasing, we get

$$0 < \sup_{s\geq 0}\,(N_s - N_{s-1}) =: c_9 < \infty,$$

where we use the convention $N_s := 0$ if $s < 0$. Let $1 \leq q < r \leq t$. With this constant c_9,

$$\begin{aligned}\int_q^r N^-_{t-\mathrm{d}s}\,\frac{1}{s} &\leq \sum_{i=[q]}^{[r]} \int_i^{i+1} N^-_{t-\mathrm{d}s}\,\frac{1}{s} \leq \sum_{i=[q]}^{[r]} \frac{1}{i}\,(N_{t-i} - N_{t-i-1}) \\ &\leq c_9 \sum_{i=[q]}^{[r]} \frac{1}{i} \leq c_9\Big(2 + \log\frac{r}{q}\Big),\end{aligned} \tag{22}$$

as desired. ■

Lemma 10 (A renewal measure asymptotics) *Let* $g : [0,1] \to \mathsf{R}_+$ *denote a non-increasing function and* $0 \leq h < 1$. *Then*

$$\lim_{\varepsilon\downarrow 0}\limsup_{t\uparrow\infty}\sup_{r\in[t^{h+\varepsilon},t]} \frac{1}{\log t}\left|\frac{1}{\kappa}\int_{t^h}^{r}\mathrm{d}s\,\frac{1}{s}\,g\big(\tfrac{\log s}{\log t}\big) - \int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^-_{r-\mathrm{d}s}\,\frac{1}{s}\,g\big(\tfrac{\log s}{\log t}\big)\right| = 0.$$

Note that this statement (and also several later ones) becomes trivial if G is the exponential distribution, since here $N_r = r/\kappa,\ r \geq 0$.

Proof Let $0 \leq h < 1,\ 0 < \varepsilon \leq (1-h)/4$, and $1 < t^{h+\varepsilon} \leq r \leq t$. Because g is monotone, by (21), for $t \geq t_0 = t_0(\varepsilon)$,

$$\begin{aligned}&\int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^-_{r-\mathrm{d}s}\,\frac{1}{s}\,g\big(\tfrac{\log s}{\log t}\big) \leq \sum_{i=[t^{h+\varepsilon}]}^{[r^{1-\varepsilon}]} \int_i^{i+1} N^-_{r-\mathrm{d}s}\,\frac{1}{s}\,g\big(\tfrac{\log s}{\log t}\big) \\ &\leq \sum_{i=[t^{h+\varepsilon}]}^{[r^{1-\varepsilon}]} \frac{1}{i}\,g\big(\tfrac{\log i}{\log t}\big)\,(N_{r-i} - N_{r-i-1}) \leq \frac{1+\varepsilon}{\kappa} \sum_{i=[t^{h+\varepsilon}]}^{[r^{1-\varepsilon}]} \frac{1}{i}\,g\big(\tfrac{\log i}{\log t}\big) \\ &\leq \frac{1+\varepsilon}{\kappa}\int_{t^{h+\varepsilon/2}}^{r^{1-\varepsilon/2}} \mathrm{d}s\,\frac{1}{s}\,g\big(\tfrac{\log s}{\log t}\big).\end{aligned} \tag{23}$$

By similar arguments,

$$\int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^-_{r-\mathrm{d}s}\,\frac{1}{s}\,g\big(\tfrac{\log s}{\log t}\big) \geq \frac{1-\varepsilon}{\kappa}\int_{t^{h+2\varepsilon}}^{r^{1-2\varepsilon}} \mathrm{d}s\,\frac{1}{s}\,g\big(\tfrac{\log s}{\log t}\big). \tag{24}$$

On the other hand, by boundedness of the g,

$$\Big(\int_{t^h}^{t^{h+\varepsilon}} + \int_{r^{1-\varepsilon}}^{r}\Big)\mathrm{d}s\;\frac{1}{s}\,g\big(\tfrac{\log s}{\log t}\big) \;\le\; c\,\varepsilon\log t. \tag{25}$$

Estimates (23)–(25) together imply the claim. ■

2.2 The scaled renewal equation

From now on we *fix for a while* $\varphi \in \mathcal{C}_p^+$ *and* $0 \le h < 1$. Also, we only pay attention to the *critical parameter constellation*

$$d \;=\; \alpha/\beta. \tag{26}$$

To prepare for the proof of the refined asymptotics, it will be advantageous to introduce some additional parameters at the left hand side of (14). In fact, we pass to

$$(r\log t)^{1/\beta}\,Q_r(\theta\varphi_{h,t})\,(r^{1/\alpha}x) \;=:\; F_{r,t,\theta}\big(a_h(r,t),x\big), \tag{27}$$

$1 \le t^h < r \le t,\ \theta \ge 0,\ 0 \le h < 1,\ x \in \mathsf{R}^d,$ with

$$a_h(r,t) \;:=\; \frac{\log r}{\log t} - h \;=\; \frac{\log(rt^{-h})}{\log t} \;\in\; (0,1] \tag{28}$$

and

$$F_{r,t,\theta}(a,x) \;:=\; (r\log t)^{1/\beta}\,Q_r\Big(\tfrac{(a\log t)^{1/\beta}}{(\log(rt^{-h}))^{1/\beta}}\,\theta\varphi_{h,t}\Big)(r^{1/\alpha}x), \tag{29}$$

$0 < a \le 1.$

Recall from [KS98, Lemma 3] that the following *"renewal equation"* holds:

$$Q_t\varphi\,(x) \;=\; E_x\big[1-\mathrm{e}^{-\varphi(\xi_t)}\big] - \int_0^t N^-_{t-\mathrm{d}s}\,E_x\Psi\big(Q_s\varphi\,(\xi_{t-s})\big), \tag{30}$$

$t \ge 0,\ x \in \mathsf{R}^d$. It implies the *expectation formula*

$$\mathbf{E}_{\delta_x}\langle Z_t\,,\varphi\rangle \;=\; E_x\varphi(\xi_t), \qquad x\in\mathsf{R}^d,\quad t\ge 0,\quad \varphi\in\mathcal{C}_p(\mathsf{R}^d), \tag{31}$$

(for instance, pass from $\varphi \ge 0$ to $\theta\varphi$ and differentiate to $\theta > 0$ at $\theta = 0+$), and the *domination*

$$0 \;\le\; Q_t\varphi\,(x) \;\le\; E_x\varphi(\xi_t), \qquad x\in\mathsf{R}^d,\quad t\ge 0,\quad \varphi\in\mathcal{C}_p(\mathsf{R}^d). \tag{32}$$

We want to use equation (30) to study some asymptotic properties of $F_{r,t,\theta}\big(a_h(r,t),x\big)$ from (27). For this aim, in (30) replace the pair t,x by $r, r^{1/\alpha}x$, and φ by $\theta\varphi_{h,t}$, as well as multiply the equation by $(r\log t)^{1/\beta}$. Then we get the *scaled renewal equation*

$$\begin{aligned} &F_{r,t,\theta}\big(a_h(r,t),x\big)\\ &=\; L_{r,t,\theta}(x) - (r\log t)^{1/\beta}\int_0^r N^-_{r-\mathrm{d}s}\,E_{r^{1/\alpha}x}\Psi\big(Q_s(\theta\varphi_{h,t})\,(\xi_{r-s})\big), \end{aligned} \tag{33}$$

$1 \leq t^h < r \leq t,\ \theta \geq 0,\ 0 \leq h < 1,\ x \in \mathsf{R}^d,$ where

$$L_{r,t,\theta}(x) \ := \ (r \log t)^{1/\beta} E_{r^{1/\alpha}x}\Big[1 - \exp\big[-\theta\varphi_{h,t}(\xi_r)\big]\Big]. \tag{34}$$

2.3 Dominations concerning the scaled equation

Clearly,

$$F_{r,t,\theta}\big(a_h(r,t),x\big) \ \leq \ L_{r,t,\theta}(x) \ \leq \ (r \log t)^{1/\beta} E_{r^{1/\alpha}x}\, \theta\varphi_{h,t}(\xi_r). \tag{35}$$

But from definition (13) of $\varphi_{h,t}$ and the *self-similarity*

$$b^{d/\alpha}\, \mathrm{p}_{bs}(b^{1/\alpha}y) \ = \ \mathrm{p}_s(y), \qquad b,s > 0, \quad y \in \mathsf{R}^d, \tag{36}$$

of the α–stable kernel p we obtain

$$(r \log t)^{1/\beta} E_{r^{1/\alpha}x}\, \varphi_{h,t}(\xi_r) \ = \ \int_{\mathsf{R}^d} \mathrm{d}y\ \mathrm{p}_1(t^{h/\alpha}r^{-1/\alpha}y - x)\, \varphi(y), \tag{37}$$

since $d/\alpha = 1/\beta$ by criticality (26). Hence

$$(r \log t)^{1/\beta} E_{r^{1/\alpha}x}\, \varphi_{h,t}(\xi_r) \ \leq \ \mathrm{p}_1(0)\, \langle \ell, \varphi \rangle. \tag{38}$$

Combining (35) and (38),

$$0 \ \leq \ F_{r,t,\theta}\big(a_h(r,t),x\big) \ \leq \ L_{r,t,\theta}(x) \ \leq \ \mathrm{p}_1(0)\, \langle \ell, \theta\varphi \rangle. \tag{39}$$

In particular,

$$\theta^{-1} F_{r,t,\theta}\big(a_h(r,t),x\big) \quad \text{is uniformly bounded} \tag{40}$$

in the considered r,t,h,θ and x. On the other hand, integrating the right hand side of equation (33) with respect to $\mathrm{d}x$, from its non-negativity we get the estimate

$$\begin{aligned} 0 \ &\leq \ (\log t)^{1/\beta} \int_0^r N^-_{r-\mathrm{d}s} \int_{\mathsf{R}^d} \mathrm{d}x\ \Psi\big[Q_s(\theta\varphi_{h,t})\,(x)\big] \\ &\leq \int_{\mathsf{R}^d} \mathrm{d}x\ L_{r,t,\theta}(x) \ \leq \ \int_{\mathsf{R}^d} \mathrm{d}x\ (r \log t)^{1/\beta} E_{r^{1/\alpha}x}\, \theta\varphi_{h,t}(\xi_r) \ = \ \langle \ell, \theta\varphi \rangle, \end{aligned} \tag{41}$$

where we used twice the criticality (26) as well as (35).

Lemma 11 (Convergence of $L_{r,t,\theta}$) *For $0 < \varepsilon \leq 1 - h$ and $\theta \geq 0$,*

$$\lim_{t \uparrow \infty} \ \sup_{r \in [t^{h+\varepsilon},t]} \big| L_{r,t,\theta}(x) - \mathrm{p}_1(x)\, \langle \ell, \theta\varphi \rangle \big| \ = \ 0, \qquad x \in \mathsf{R}^d, \tag{42}$$

and

$$\lim_{t \uparrow \infty} \ \sup_{r \in [t^{h+\varepsilon},t]} \int_{\mathsf{R}^d} \mathrm{d}x\ \big| L_{r,t,\theta}(x) - \mathrm{p}_1(x)\, \langle \ell, \theta\varphi \rangle \big| \ = \ 0. \tag{43}$$

Proof From definition (34) of $L_{r,t,\theta}(x)$, similarly to (37),

$$L_{r,t,\theta}(x) \;=\; (t^h \log t)^{1/\beta} \int_{\mathsf{R}^d} \mathrm{d}y \; \mathrm{p}_1(t^{h/\alpha} r^{-1/\alpha} y - x) \;\times \Big[1 - \exp\big[-(t^h \log t)^{-1/\beta} \theta \varphi(y)\big]\Big]. \tag{44}$$

But

$$0 \;\le\; t^{h/\alpha} r^{-1/\alpha} \;\le\; t^{-\varepsilon/\alpha} \underset{t\uparrow\infty}{\longrightarrow} 0$$

in the considered range of r. Then the extended dominated convergence theorem implies (42).

Distinguishing between $|y| \le K$ and $|y| > K$ in (44), and letting $K \uparrow \infty$, also (43) follows. This finishes the proof. ■

2.4 Approximate renewal equation

A crucial tool in our development is the following asymptotic equation. Recall that we fixed $\varphi \in \mathcal{C}_p^+$ and $0 \le h < 1$, and that $F_{r,t,\theta}$ and $L_{r,t,\theta}$ had been defined in (27) and (34).

Proposition 12 (Approximate renewal equation) *Let* $\theta \ge 0$, $0 < \varepsilon \le (1-h)/2$, *and* $1 < t^{h+\varepsilon} \le r^{1-\varepsilon} \le t^{1-\varepsilon}$. *Then, for each* $x \in \mathsf{R}^d$,

$$F_{r,t,\theta}\big(a_h(r,t), x\big) \;=\; L_{r,t,\theta}(x) - S^{\varepsilon}_{r,t,\theta}(x) - \mathrm{p}_1(x) \frac{1}{\log t} \int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^-_{r-\mathrm{d}s} \, \frac{1}{s} \int_{\mathsf{R}^d} \mathrm{d}y \; \Psi\big(F_{s,t,\theta}\big(a_h(s,t), y\big)\big). \tag{45}$$

Here $S^{\varepsilon}_{r,t,\theta}(x)$ *is an error term satisfying*

$$\lim_{\varepsilon\downarrow 0} \limsup_{t\uparrow\infty} \sup_{r\in[t^{h+\varepsilon},t]} \left(\big|S^{\varepsilon}_{r,t,\theta}(x)\big| + \int_{\mathsf{R}^d} \mathrm{d}z \; \big|S^{\varepsilon}_{r,t,\theta}(z)\big| \right) \;=\; 0. \tag{46}$$

As a preparation for the proof we expose the following estimate.

Lemma 13 (A partial bound) *There is a constant* $c_{13} = c_{13}(\varphi)$ *such that*

$$E_x \Psi\big(Q_s(\theta\varphi_{h,t})\,(\xi_{r-s})\big) \;\le\; c_{13}\, \theta^{1+\beta} \, (\log t)^{-1} \, \big(s^{-1} \wedge t^{-h}\big) \, E_x \varphi_{h,t}(\xi_r)$$

for $t > 1$, $0 \le s \le r \le t$, $\theta \ge 0$, *and* $x \in \mathsf{R}^d$.

Proof First of all,

$$E_x \varphi_{h,t}(\xi_s) \;\le\; \big(\mathrm{p}_s(0)\, \langle \ell, \varphi_{h,t} \rangle\big) \wedge \|\varphi_{h,t}\|_\infty . \tag{47}$$

Moreover,

$$\mathrm{p}_s(s^{1/\alpha} x) \;=\; s^{-1/\beta}\, \mathrm{p}_1(x) \tag{48}$$

by self-similarity (36), as well as

$$\langle \ell, \varphi_{h,t} \rangle \;\equiv\; (\log t)^{-1/\beta} \langle \ell, \varphi \rangle, \quad \|\varphi_{h,t}\|_\infty \;=\; (\log t)^{-1/\beta}\, t^{-h/\beta}\, \|\varphi\|_\infty . \quad (49)$$

Combining with the critical parameter constellation $d = \alpha/\beta$ yields

$$\big(E_x \varphi_{h,t}(\xi_s)\big)^\beta \;\le\; c_{13}\, \big(s^{-1} \wedge t^{-h}\big)\, (\log t)^{-1} \quad (50)$$

for some constant $c_{13} = c_{13}(\varphi)$. Then domination (32) with φ replaced by $\theta\varphi_{h,t}$ gives

$$E_x \Psi\big(Q_s(\theta\varphi_{h,t})\,(\xi_{r-s})\big) \;\le\; \theta^{1+\beta}\, E_x \Psi\big(E_{\xi_{r-s}} \varphi_{h,t}(\xi'_s)\big)$$

with ξ' an independent copy of ξ. Now (50) and the Markov property of ξ imply the claim. ■

2.5 Some error terms

Related to the expectation expression occurring in the scaled renewal equation (33) we introduce *six error terms*: For the fixed $\varphi \in \mathcal{C}_p^+$ and $0 \le h < 1$, as well as $\theta \ge 0$, $0 < \varepsilon \le (1-h)/2$, $1 < t^{h+\varepsilon} \le r^{1-\varepsilon} \le t^{1-\varepsilon}$, $K \ge 0$, and $x \in \mathsf{R}^d$, set

$$\begin{aligned}
{}^1I(x) \;&=\; {}^1I_{r,t,\theta}(x) \;:=\; \int_0^{t^h} N^-_{r-\mathrm{d}s}\, E_{r^{1/\alpha}x} \Psi\big(Q_s(\theta\varphi_{h,t})\,(\xi_{r-s})\big), \\
{}^2I(x) \;&=\; {}^2I^\varepsilon_{r,t,\theta}(x) \;:=\; \int_{t^h}^{t^{h+\varepsilon}} N^-_{r-\mathrm{d}s}\, E_{r^{1/\alpha}x} \Psi\big(Q_s(\theta\varphi_{h,t})\,(\xi_{r-s})\big), \\
{}^3I(x) \;&=\; {}^3I^\varepsilon_{r,t,\theta}(x) \;:=\; \int_{r^{1-\varepsilon}}^{r} N^-_{r-\mathrm{d}s}\, E_{r^{1/\alpha}x} \Psi\big(Q_s(\theta\varphi_{h,t})\,(\xi_{r-s})\big), \\
{}^4I(x) \;&=\; {}^4I^{\varepsilon,K}_{r,t,\theta}(x) \\
&:= \int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^-_{r-\mathrm{d}s}\, E_{r^{1/\alpha}x} 1_{\{|\xi_{r-s}| > K s^{1/\alpha}\}} \Psi\big(Q_s(\theta\varphi_{h,t})\,(\xi_{r-s})\big), \\
{}^5I(x) \;&=\; {}^5I^{\varepsilon,K}_{r,t,\theta}(x) \;:=\; \int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^-_{r-\mathrm{d}s} \int_{|y| \le K s^{1/\alpha}} \mathrm{d}y \;\times \\
&\qquad \big[\mathrm{p}_{r-s}(y - r^{1/\alpha}x) - \mathrm{p}_r(-r^{1/\alpha}x)\big]\, \Psi\big(Q_s(\theta\varphi_{h,t})\,(y)\big), \\
{}^6I(x) \;&=\; {}^6I^{\varepsilon,K}_{r,t,\theta}(x) \;:=\; \mathrm{p}_r(-r^{1/\alpha}x) \int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^-_{r-\mathrm{d}s} \;\times \\
&\qquad \int_{|y| > K s^{1/\alpha}} \mathrm{d}y\; \Psi\big(Q_s(\theta\varphi_{h,t})\,(y)\big).
\end{aligned}$$

Lemma 14 (Error terms) *Let $0 < \delta \le 1$ and $0 < \varepsilon < (1-h)/2$. Then there exists a constant $c_{14} = c_{14}(\varphi)$, a $t_0 = t_0(\varphi, h, \varepsilon, \delta)$, and a $K_0 =$*

$K_0(\varphi,\delta)\ge 1$ *such that for all* $t\ge t_0$, $\theta\ge 0$, $1<t^{h+\varepsilon}\le r^{1-\varepsilon}\le t^{1-\varepsilon}$, $x\in \mathsf{R}^d$, *and* $K\ge K_0$,

$$\begin{aligned} 0 \;&\le\; \sum_{1\le i\le 6}\Big(\big|{}^{i}I(x)\big| \,+\, \int_{\mathsf{R}^d}\mathrm{d}z\;\big|{}^{i}I(z)\big|\Big) \\ &\le\; c_{14}\left(\varepsilon+\delta+\tilde{\delta}_t(\varepsilon,K,h,t_0)\right)\left(\theta+\theta^{1+\beta}\right)(r\log t)^{-1/\beta}, \end{aligned} \tag{51}$$

where, for ε,h,t_0 *fixed,*

$$\lim_{K\uparrow\infty}\,\limsup_{t\uparrow\infty}\,\tilde{\delta}_t(\varepsilon,K,h,t_0) \;=\; 0. \tag{52}$$

Proof 1° $\left({}^{1}I(x)\right)$ By Lemma 13,

$$0 \;\le\; {}^{1}I(x) \;\le\; c_{13}\,\theta^{1+\beta}\,(\log t)^{-1}\,E_{r^{1/\alpha}x}\varphi_{h,t}(\xi_r)\,t^{-h}\int_0^{t^h} N^{-}_{r-\mathrm{d}s}\,. \tag{53}$$

From the key renewal theorem follows that

$$t^{-h}\left(N_r-N_{r-t^h}\right)\;\underset{r,t\uparrow\infty}{\longrightarrow}\;\frac{1}{\kappa}\,, \tag{54}$$

while $(\log t)^{-1}\le\varepsilon$ for $t\ge t_0(\varepsilon)>1$. Therefore

$${}^{i}I(x)\;\le\;c\,\varepsilon\,\theta^{1+\beta}\,E_{r^{1/\alpha}x}\varphi_{h,t}(\xi_r) \tag{55}$$

holds for $i=1$.

2° $\left({}^{2}I(x)+{}^{3}I(x)\right)$ Again by Lemma 13,

$$\begin{aligned} 0 \;\le\; {}^{2}I(x)+{}^{3}I(x) \;&\le\; c_{13}\,\theta^{1+\beta}\,(\log t)^{-1}\times \\ &E_{r^{1/\alpha}x}\varphi_{h,t}(\xi_r)\left(\int_{t^h}^{t^{h+\varepsilon}}+\int_{r^{1-\varepsilon}}^{r}\right)N^{-}_{r-\mathrm{d}s}\,\frac{1}{s}\,. \end{aligned}$$

But by Lemma 9 the latter integral expressions are bounded by

$$2\,c_9\,(2/\log t+\varepsilon)\log t \;\le\; c\,\varepsilon\log t.$$

This yields (55) also for $i=2,3$. Now by inequality (38) and the last identity in the array (41),

$$E_{r^{1/\alpha}x}\varphi_{h,t}(\xi_r)+\int_{\mathsf{R}^d}\mathrm{d}z\;E_{r^{1/\alpha}z}\varphi_{h,t}(\xi_r)\;\le\;c\,(r\log t)^{-1/\beta}. \tag{56}$$

Therefore, from (55) for $i\le 3$, the assertion concerning $i\le 3$ within estimate (51) follows.

3° $\left({}^{4}I(x)\right)$ By domination (32), for an independent copy ξ' of ξ,

$$0\;\le\;\Psi\big(Q_s(\theta\varphi_{h,t})\,(\xi_{r-s})\big)\;\le\;\theta^{1+\beta}\,\Psi\big(E_{\xi_{r-s}}\varphi_{h,t}(\xi'_s)\big) \tag{57}$$

(given ξ). But by definition (13), substitution, self-similarity (36) of p, and critical parameter constellation (26),

$$E_y \varphi_{h,t}(\xi'_s) \;=\; (s\log t)^{-1/\beta} \int_{\mathsf{R}^d} \mathrm{d}z \;\mathrm{p}_1\big(t^{h/\alpha}s^{-1/\alpha}z - s^{-1/\alpha}y\big)\,\varphi(z). \qquad (58)$$

Decompose the latter integration range into $|z| > K/2$ and $|z| \le K/2$. In the first case,

$$\int_{|z|>K/2} \mathrm{d}z \;\mathrm{p}_1\big(t^{h/\alpha}s^{-1/\alpha}z - s^{-1/\alpha}y\big)\,\varphi(z) \;\le\; \mathrm{p}_1(0)\int_{|z|>K/2} \mathrm{d}z \;\varphi(z) \;\le\; \delta^{1/\beta}$$

for $K \ge K_0 = K_0(\varphi,\delta)$. In the remaining case, with $y = \xi_{r-s}$,

$$\int_{|z|\le K/2} \mathrm{d}z \;\mathrm{p}_1\big(t^{h/\alpha}s^{-1/\alpha}z - s^{-1/\alpha}\xi_{r-s}\big)\,\varphi(z) \;\le\; \delta^{1/\beta}$$

for $K \ge K_0$, enlarging $K_0 = K_0(\varphi,\delta)$ if needed. In fact, $s \ge t^h$ implies $\big|t^{h/\alpha}s^{-1/\alpha}z\big| \le |z| \le K/2$, whereas $\big|s^{-1/\alpha}\xi_{r-s}\big| > K$. Consequently, for all $K \ge K_0$ we use the estimate

$$\big(E_{\xi_{r-s}}\varphi_{h,t}(\xi'_s)\big)^{\beta}\; 1_{\{|\xi_{r-s}| > Ks^{1/\alpha}\}} \;\le\; \delta\,(s\log t)^{-1} \qquad (59)$$

in (57) and then go back to the definition of ${}^4I(x)$:

$$0 \;\le\; {}^4I(x) \;\le\; \delta\,(\log t)^{-1}\,\theta^{1+\beta}\,E_{r^{1/\alpha}x}\varphi_{h,t}(\xi_r)\int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^-_{r-\mathrm{d}s}\,\frac{1}{s}\,,$$

for $K \ge K_0$. But

$$0 \;<\; \log\frac{r^{1-\varepsilon}}{t^{h+\varepsilon}} \;\le\; (1-h-2\varepsilon)\log t \;\le\; \log t,$$

hence Lemma 9 yields

$$\int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^-_{r-\mathrm{d}s}\,\frac{1}{s} \;\le\; c_9\,(2+\log t) \;\le\; c\log t. \qquad (60)$$

Thus,

$$0 \;\le\; {}^4I(x) \;\le\; c\,\delta\,\theta^{1+\beta}\,E_{r^{1/\alpha}x}\varphi_{h,t}(\xi_r).$$

Using again (56), we see that ${}^4I(x)$ contributes to (51) as claimed.

4° $\big({}^5I(x)\big)$ By self-similarity (36) and critical parameter constellation we have

$$\begin{aligned}&\big|\mathrm{p}_{r-s}(y - r^{1/\alpha}x) - \mathrm{p}_r(-r^{1/\alpha}x)\big|\\ &=\; r^{-1/\beta}\,\big|\mathrm{p}_{1-s/r}(r^{-1/\alpha}y - x) - \mathrm{p}_1(-x)\big| \;\le\; \tilde{\delta}^{(1)}_t(\varepsilon,K,h,t_0)\,r^{-1/\beta}. \qquad (61)\end{aligned}$$

In fact, $1-s/r \geq 1-r^{-\varepsilon} \geq 1-t^{-\varepsilon} \geq 1-t_0^{-\varepsilon} > 0,$ as well as

$$\left|r^{-1/\alpha}y\right| \;\leq\; K\,(s/r)^{1/\alpha} \;\leq\; K\,t^{-h\varepsilon/\alpha} \underset{t\uparrow\infty}{\longrightarrow} 0, \tag{62}$$

since the assumption $t^{h+\varepsilon} \leq r^{1-\varepsilon}$ implies $t^h \leq r$, and because p is jointly uniformly continuous for the time variable away form the origin. Here $\tilde{\delta}_t^{(1)}(\varepsilon,K,h,t_0) \to 0$ as $t \uparrow \infty$, for fixed ε,K,h,t_0. At the same time, again by (61),

$$\begin{aligned}&\int_{\mathsf{R}^d}\mathrm{d}z\;\left|\mathrm{p}_{r-s}(y-r^{1/\alpha}z)-\mathrm{p}_r(-r^{1/\alpha}z)\right| \\ &= \; r^{-1/\beta}\int_{\mathsf{R}^d}\mathrm{d}z\;\left|\mathrm{p}_{1-s/r}(r^{-1/\alpha}y-z)-\mathrm{p}_1(-z)\right|.\end{aligned} \tag{63}$$

Decomposing the latter integral concerning $|z| \leq K$ and $|z| > K$. In the first case, we use once more jointly uniform continuity to bound the restricted integral expression as before, whereas in the second one we exploit that the restricted integral converges to 0 as $K \uparrow \infty$, uniformly in the other variables. Altogether, (63) can be bounded by $\tilde{\delta}_t^{(2)}(\varepsilon,K,h,t_0)\,r^{-1/\beta}$, where

$$\tilde{\delta}_t(\varepsilon,K,h,t_0) \;:=\; \tilde{\delta}_t^{(1)}(\varepsilon,K,h,t_0)+\tilde{\delta}_t^{(2)}(\varepsilon,K,h,t_0)$$

has the required property (52). Consequently, from the inequality in array (61) and the derived bound for (63),

$$\begin{aligned}&\left|{}^5I(x)\right| + \int_{\mathsf{R}^d}\mathrm{d}z\;\left|{}^5I(z)\right| \\ &\leq\; \tilde{\delta}_t(\varepsilon,K,h,t_0)\,r^{-1/\beta}\int_0^r N_{r-\mathrm{d}s}^{-}\int_{\mathsf{R}^d}\mathrm{d}y\;\Psi\big(Q_s(\theta\varphi_{h,t})\,(y)\big) \\ &\leq\; \tilde{\delta}_t(\varepsilon,K,h,t_0)\,(r\log t)^{-1/\beta}\,\theta\,\langle \ell,\varphi\rangle,\end{aligned}$$

where we used (41) in the last step. Thus, ${}^5I(x)$ enters into (51) in the desired way.

$5°$ $\big({}^6I(x)\big)$ Using (57) and (59), for $K \geq K_0$,

$$\begin{aligned}0 \;\leq\; &\int_{|y|>Ks^{1/\alpha}}\mathrm{d}y\;\Psi\big(Q_s(\theta\varphi_{h,t})\,(y)\big) \\ \leq\; &c_f\,\delta\,(\log t)^{-1}\,\theta^{1+\beta}\,\frac{1}{s}\int_{|y|>Ks^{1/\alpha}}\mathrm{d}y\;E_y\varphi_{h,t}(\xi_s).\end{aligned}$$

Delete the restriction in the integration domain and apply the first identity of (49). Moreover, exploit self-similarity (48). Then,

$$0 \;\leq\; {}^6I(x) \;\leq\; c\,\delta\,(\log t)^{-(1+\beta)/\beta}\,r^{-1/\beta}\,\theta^{1+\beta}\,\mathrm{p}_1(x)\int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N_{r-\mathrm{d}s}^{-}\,\frac{1}{s}\,.$$

With (60) we finish the proof. ∎

2.6 Proof of Proposition 12

Starting from the scaled renewal equation (33), we have to rewrite the second term at its right hand side. It equals

$$(r\log t)^{1/\beta}\Big(\sum_{1\le i\le 6}{}^{i}I(x)\;+\;{}^{0}I(x)\Big)$$

with the error terms ${}^{1}I(x),\dots,{}^{6}I(x)$ defined in the beginning of the previous subsection, and the main term

$$\begin{aligned}{}^{0}I(x) \;&=\; {}^{0}I^{\varepsilon}_{r,t,\theta}(x)\\ &:=\; \mathrm{p}_r(r^{1/\alpha}x)\int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^{-}_{r-\mathrm{d}s}\int_{\mathsf{R}^d}\mathrm{d}y\;\Psi\big(Q_s(\theta\varphi_{h,t})\,(y)\big). \qquad (64)\end{aligned}$$

By a simple substitution,

$$\begin{aligned}(r\log t)^{1/\beta}\;{}^{0}I(x) \;=\; (r\log t)^{1/\beta}\,\mathrm{p}_r(r^{1/\alpha}x)\int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^{-}_{r-\mathrm{d}s}\times\\ \int_{\mathsf{R}^d}\mathrm{d}y\;s^{1/\beta}\,\Psi\big(Q_s(\theta\varphi_{h,t})\,(s^{1/\alpha}y)\big).\end{aligned}$$

Using definition (27) of $F_{s,t,\theta}\big(a_h(s,t),y\big)$, and identity (48), we arrive at the desired last term in (45).

It remains to show that

$$S^{\varepsilon}_{r,t,\theta}(x) \;=\; (r\log t)^{1/\beta}\sum_{1\le i\le 6}{}^{i}I(x) \qquad (65)$$

satisfies (46). Note that $S^{\varepsilon}_{r,t,\theta}(x)$ does not depend on K, despite K occurs implicitly at the right hand side of (65) via the ${}^{i}I(x),\ 4\le i\le 6$. From decomposition (65) and estimate (51),

$$\begin{aligned}&\big|S^{\varepsilon}_{r,t,\theta}(x)\big| + \int_{\mathsf{R}^d}\mathrm{d}z\;\big|S^{\varepsilon}_{r,t,\theta}(z)\big|\\ &\le\; c\,\big(\varepsilon+\delta+\tilde{\delta}_t(\varepsilon,K,h,t_0)\big)\,(\theta+\theta^{1+\beta}), \qquad (66)\end{aligned}$$

for $K\ge K_0=K_0(\varphi,\delta)$ and $t\ge t_0$. First we built the supremum on r in the range as required in (46), and then we let $t\uparrow\infty$. Since the left hand side in inequality (66) does not depend on K, we now let $K\uparrow\infty$, which gives

$$\limsup_{t\uparrow\infty}\;\sup_{r\in[t^{h+\varepsilon},t]}\Big(\big|S^{\varepsilon}_{r,t,\theta}(x)\big| + \int_{\mathsf{R}^d}\mathrm{d}z\;\big|S^{\varepsilon}_{r,t,\theta}(z)\big|\Big) \;\le\; c\,(\varepsilon+\delta)\,(\theta+\theta^{1+\beta}).$$

Then first $\varepsilon\downarrow 0$ and afterwards $\delta\downarrow 0$ finishes the proof. ■

2.7 Approximate limiting equation

Here we want to derive a certain limiting counterpart to the approximate renewal equation of Proposition 12. For the fixed $\varphi \in \mathcal{C}_p^+$, set

$$F_\theta(a,x) \;:=\; \mathrm{p}_1(x)\, v\big(a; \langle \ell, \theta\varphi\rangle\big), \qquad \theta, a \geq 0, \quad x \in \mathsf{R}^d, \tag{67}$$

with v the log-Laplace function of η as in (9), with branching rate γ from (10). Note that

$$F_\theta(a,x) \;=\; \mathrm{p}_1(x)\,\langle \ell, \theta\varphi\rangle - \mathrm{p}_1(x)\,\frac{1}{\kappa}\int_0^a \mathrm{d}s \int_{\mathsf{R}^d} \mathrm{d}y\;\Psi\big(F_\theta(s,y)\big) \tag{68}$$

and, for $\theta > 0$,

$$0 \;\leq\; \theta^{-1}\, F_\theta(a,x) \;\leq\; \mathrm{p}_1(x)\,\langle \ell, \varphi\rangle \;\leq\; \mathrm{p}_1(0)\,\langle \ell, \varphi\rangle. \tag{69}$$

Recall that besides $\varphi \in \mathcal{C}_p^+$ also $0 \leq h < 1$ are fixed.

Lemma 15 (Approximate limiting equation) *Let $\theta \geq 0$, $0 < \varepsilon \leq (1-h)/2$, and $1 < t^{h+\varepsilon} \leq r^{1-\varepsilon} \leq t^{1-\varepsilon}$. Then for each $x \in \mathsf{R}^d$,*

$$\begin{aligned} F_\theta\big(a_h(r,t), x\big) \;&=\; \mathrm{p}_1(x)\,\langle \ell, \theta\varphi\rangle - {}'\!S^{\varepsilon}_{r,t,\theta}(x) \\ &\quad -\; \mathrm{p}_1(x)\,\frac{1}{\log t}\int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^-_{r-\mathrm{d}s}\,\frac{1}{s}\int_{\mathsf{R}^d} \mathrm{d}y\;\Psi\Big(F_\theta\big(a_h(s,t), y\big)\Big). \end{aligned}$$

Here ${}'\!S^{\varepsilon}_{r,t,\theta}(x)$ is an error term satisfying a statement as in (46).

Proof Let $\theta, \varepsilon, r, t, x$ as in the lemma. From (68) and substitution

$$s \;\mapsto\; a_h(s,t) \;=\; \frac{\log s}{\log t} - h \tag{70}$$

[recall (28)], we obtain

$$\begin{aligned} F_\theta\big(a_h(r,t), x\big) \;&=\; \mathrm{p}_1(x)\,\langle \ell, \theta\varphi\rangle \\ &\quad -\; \mathrm{p}_1(x)\,\frac{1}{\log t}\,\frac{1}{\kappa}\int_{t^h}^{r} \mathrm{d}s\;\frac{1}{s}\int_{\mathsf{R}^d} \mathrm{d}y\;\Psi\Big(F_\theta\Big(\tfrac{\log s}{\log t} - h, y\Big)\Big). \end{aligned} \tag{71}$$

By Lemma 10, the second term at the right hand side of (71) equals

$$\mathrm{p}_1(x)\,\frac{1}{\log t}\int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^-_{r-\mathrm{d}s}\,\frac{1}{s}\int_{\mathsf{R}^d} \mathrm{d}y\;\Psi\Big(F_\theta\Big(\tfrac{\log s}{\log t} - h, y\Big)\Big)$$

except the error term ${}'\!S^{\varepsilon}_{r,t,\theta}(x) \;=\; \mathrm{p}_1(x)\,\rho^{\varepsilon}_{r,t,\theta}$ satisfying (46). With the definition of $a_h(s,t)$, the proof is then finished. ∎

2.8 Convergence in $L^1(\mathrm{d}x)$

We will use Proposition 12 to derive the following result.

Proposition 16 (Convergence in $L^1(\mathrm{d}x)$) *There exists a positive* $\theta_0 = \theta_0(\varphi, h)$ *such that*

$$\lim_{t\uparrow\infty} \sup_{r\in[t^{h+\varepsilon},t]} \int_{\mathsf{R}^d} \mathrm{d}x \, \Big| F_{r,t,\theta}\big(a_h(r,t),x\big) - F_\theta\big(a_h(r,t),x\big) \Big| \;=\; 0$$

for $0 < \varepsilon \le 1-h$ *and* $0 \le \theta \le \theta_0$.

Proof Set

$$J^{(\beta)}_{r,t,\theta} \;:=\; \int_{\mathsf{R}^d} \mathrm{d}x \, \Big| F^{1+\beta}_{r,t,\theta}\big(a_h(r,t),x\big) - F^{1+\beta}_\theta\big(a_h(r,t),x\big) \Big|, \tag{72}$$

where for the purpose of this notation we also allow $\beta = 0$. In virtue of Proposition 12 and Lemma 15,

$$\Big| F_{r,t,\theta}\big(a_h(r,t),x\big) - F_\theta\big(a_h(r,t),x\big) \Big| \;\le\; \Big| L_{r,t,\theta}(x) - \mathrm{p}_1(x)\,\langle \ell, \theta\varphi\rangle \Big|$$
$$+\; \Big| S^\varepsilon_{r,t,\theta}(x) + {}'S^\varepsilon_{r,t,\theta}(x) \Big| \;+\; \mathrm{p}_1(x)\,\frac{c_f}{\log t} \int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^-_{r-\mathrm{d}s}\,\frac{1}{s}\, J^{(\beta)}_{s,t,\theta}\,. \tag{73}$$

From the elementary inequality

$$\big| a^{1+\beta} - b^{1+\beta} \big| \;\le\; (1+\beta)\,|a-b|\,\big(a^\beta + b^\beta\big), \qquad a,b \ge 0, \tag{74}$$

and since $\theta^{-1}F_{s,t,\theta}$ and $\theta^{-1}F_\theta$ are uniformly bounded [recall (40) and (69)], there is a constant $c_{(75)} = c_{(75)}(\varphi)$ such that

$$J^{(\beta)}_{s,t,\theta} \;\le\; c_{(75)}\,\theta^\beta\, J^{(0)}_{s,t,\theta}\,. \tag{75}$$

By Lemma 11, there is a $t_0 = t_0(\varphi,\theta) > 1$ such that for all $t \ge t_0$ the $L^1(\mathrm{d}x)$–norm of the first term at the right hand side of inequality (73) is bounded from above by $\varepsilon = \varepsilon(\theta)$, uniformly in the considered r. Therefore, integrating inequality (73) with $\mathrm{d}x$, using (75) we get

$$J^{(0)}_{r,t,\theta} \;\le\; \varepsilon + \sigma^\varepsilon_{r,t,\theta} + \frac{c_f}{\log t} \int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^-_{r-\mathrm{d}s}\,\frac{1}{s}\, c_{(75)}\,\theta^\beta\, J^{(0)}_{s,t,\theta}\,, \qquad t \ge t_0\,,$$

where

$$\sigma^\varepsilon_{r,t,\theta} \;:=\; \int_{\mathsf{R}^d} \mathrm{d}x \, \big| S^\varepsilon_{r,t,\theta}(x) + {}'S^\varepsilon_{r,t,\theta}(x) \big|\,.$$

Introduce

$$J^{(0)}_{t,\theta} \;:=\; \sup_{r\in[t^{h+\varepsilon},t]} J^{(0)}_{r,t,\theta}\,. \tag{76}$$

Then the previous estimate yields

$$J_{t,\theta}^{(0)} \;\le\; \varepsilon + \sup_{r\in[t^{h+\varepsilon},t]} \sigma^{\varepsilon}_{r,t,\theta}$$
$$+\; J_{t,\theta}^{(0)}\,\theta_0^{\beta}\,c_{(75)}\,\frac{c_f}{\log t}\sup_{r\in[t^{h+\varepsilon},t]}\int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^{-}_{r-\mathrm{d}s}\,\frac{1}{s}\,, \qquad t\ge t_0\,.$$

But by Lemma 9,

$$\int_{t^{h+\varepsilon}}^{r^{1-\varepsilon}} N^{-}_{r-\mathrm{d}s}\,\frac{1}{s} \;\le\; c\,(1-h)\,\log t, \qquad t\ge t_0\,, \tag{77}$$

enlarging t_0 if needed. Altogether,

$$J_{t,\theta}^{(0)} \;\le\; \varepsilon + \sup_{r\in[t^{h+\varepsilon},t]} \sigma^{\varepsilon}_{r,t,\theta} + J_{t,\theta}^{(0)}\,\theta^{\beta}\,c_{(75)}\,c_9\,c_f\,(1-h), \qquad t\ge t_0\,.$$

We choose $\theta_0>0$ so small that

$$\theta_0^{\beta}\,c_{(75)}\,c_9\,c_f\,(1-h) \;=:\; \iota \;<\; 1.$$

Then, for $\theta\in[0,\theta_0]$ and $t\ge t_0$,

$$(1-\iota)J_{t,\theta}^{(0)} \;\le\; \varepsilon + \sup_{r\in[t^{h+\varepsilon},t]} \sigma^{\varepsilon}_{r,t,\theta}\,.$$

Letting $t\uparrow\infty$ and then $\varepsilon\downarrow 0$, by (46) [applied to $S^{\varepsilon}_{r,t,\theta}(x)$ and ${}'S^{\varepsilon}_{r,t,\theta}(x)$] we obtain $\lim_{t\uparrow\infty} J_{t,\theta}^{(0)}=0$. This finishes the proof. ■

2.9 Refined asymptotics for Q (proof of Theorem 6)

Recall definition (72) of $J^{(\beta)}_{r,t,\theta}$. Exploit the elementary inequality (74) and use that $F_{r,t,\theta}$ and F_θ are uniformly bounded (for the fixed θ). Then by Proposition 16, for $0<\varepsilon\le 1-h$, there exists a $\theta_0=\theta_0(\varphi,h)$ such that

$$\lim_{t\uparrow\infty}\sup_{r\in[t^{h+\varepsilon},t]} J^{(\beta)}_{r,t,\theta} \;=\; 0, \qquad 0<\varepsilon\le 1-h, \quad 0\le\theta\le\theta_0\,. \tag{78}$$

From Proposition 12 and Lemma 15,

$$\Big|F_{t,t,\theta}(1-h,x)-F_\theta(1-h,x)\Big| \;\le\; \Big|L_{t,t,\theta}(x)-\mathrm{p}_1(x)\,\langle\ell,\theta\varphi\rangle\Big|$$
$$+\;\Big|S^{\varepsilon}_{t,t,\theta}(x)+{}'S^{\varepsilon}_{t,t,\theta}(x)\Big| \;+\; \mathrm{p}_1(x)\,\frac{c_f}{\log t}\int_{t^{h+\varepsilon}}^{t^{1-\varepsilon}} N^{-}_{t-\mathrm{d}s}\,\frac{1}{s}\,J^{(\beta)}_{s,t,\theta}\,. \tag{79}$$

By Lemma 11, the first term at the right hand side converges to 0 as $t\uparrow\infty$. Also, for a given $\delta>0$, by (78), there is a $t_0=t_0(\delta,\theta)>1$ such that for all $t\ge t_0$

$$\int_{t^{h+\varepsilon}}^{t^{1-\varepsilon}} N^{-}_{t-\mathrm{d}s}\,\frac{1}{s}\,J^{(\beta)}_{s,t,\theta} \;\le\; \delta\int_{t^{h+\varepsilon}}^{t^{1-\varepsilon}} N^{-}_{t-\mathrm{d}s}\,\frac{1}{s}\,.$$

Hence, recalling (77), the third term at the right hand side of (79) will vanish, too. Finally, the middle term will disappear by (46). Consequently,

$$F_{t,t,\theta}(1-h,x) \underset{t\uparrow\infty}{\longrightarrow} F_\theta(1-h,x), \qquad 0<\theta\le\theta_0\,,$$

which by definitions (29) and (67) can be rewritten as

$$(t\log t)^{1/\beta}\,Q_t(\theta\varphi_{h,t})\,(t^{1/\alpha}x) \underset{t\uparrow\infty}{\longrightarrow} \mathrm{p}_1(x)\,v\big(1-h;\langle\ell,\theta\varphi\rangle\big). \tag{80}$$

Assume for the moment, both sides are analytic functions in $\theta\ge 0$ (or $\Re\theta>0$). Then (80) holds for all $\theta\ge 0$. Then we can specialize to $\theta=1$ to finish the proof.

To get this analyticity, for later use we put additionally a factor $i_0>0$. Then, for any $\theta\ge 0$, by (7),

$$i_0\,\mathrm{p}_1(x)\,v\big(1-h;\langle\ell,\theta\varphi\rangle\big) \;=\; -\log E\Big\{\mathrm{e}^{-\langle\ell,\varphi\rangle\theta\eta_{1-h}}\;\Big|\;\eta_0=i_0\,\mathrm{p}_1(x)\Big\}, \tag{81}$$

which is a log-Laplace function, hence analytic in the considered θ–domain. On the other hand, by definitions (12), (13), and (6),

$$Q_t(\theta\varphi_{h,t})\,(t^{1/\alpha}x) \;=\; \Big(1-\mathbf{E}_{\delta_{t^{1/\alpha}x}}\exp\big[-\theta\langle Z_t^h,\varphi\rangle\big]\Big), \tag{82}$$

and we reduced it to a Laplace function, implying again analyticity. This completes the proof. ∎

3 Multi-scale clustering

The purpose of this section is to verify the multi-scale clustering as stated in Theorem 4. With the refined asymptotics for Q established in the previous section, convergence of one-dimensional distributions can easily be proven. More efforts are needed for the multi-dimensional case.

3.1 Convergence of one-dimensional distributions

Proof of Theorem 4(a) Fix again $\varphi\in\mathcal{C}_p^+$ and $0\le h<1$. By (82),

$$\mathbf{E}_{\delta_{t^{1/\alpha}x}}\exp\langle Z_t^h,-\varphi\rangle \;=\; 1-Q_t(\varphi_{h,t})\,(t^{1/\alpha}x).$$

Thus, by the branching property,

$$\begin{aligned}&\log\mathbf{E}_{\left[i_0\,(t\log t)^{1/\beta}\right]\delta_{t^{1/\alpha}x}}\exp\langle Z_t^h,-\varphi\rangle\\ &=\;\left[i_0\,(t\log t)^{1/\beta}\right]\,\log\big(1-Q_t(\varphi_{h,t})\,(t^{1/\alpha}x)\big).\end{aligned}$$

Now we can apply the refined asymptotics of Theorem 6 to get

$$\lim_{t\uparrow\infty}\log\mathbf{E}_{\left[i_0\,(t\log t)^{1/\beta}\right]\delta_{t^{1/\alpha}x}}\exp\langle Z_t^h,-\varphi\rangle \;=\; -i_0\,\mathrm{p}_1(x)\,v\big(1-h;\langle\ell,\varphi\rangle\big).$$

Then (81) gives statement (a). ■

Proof of Theorem 4(b) Recall that the initial population Z_0 is here assumed to be a homogeneous Poisson point field with intensity $i_0\,(\log t)^{1/\beta}$. Then this time we get

$$-\log \mathbf{E}_{i_0\,(\log t)^{1/\beta}} \exp\langle Z_t^h, -\varphi\rangle \;=\; i_0\,(\log t)^{1/\beta}\,\big\langle \ell, Q_t(\varphi_{h,t})\big\rangle. \qquad (83)$$

Since we are in the critical dimension $d=\alpha/\beta$, the right hand side of identity (83) can be rewritten as

$$i_0\,(t\log t)^{1/\beta}\int_{\mathsf{R}^d}\mathrm{d}x\;Q_t(\varphi_{h,t})\,(t^{1/\alpha}x) \;=\; i_0\int_{\mathsf{R}^d}\mathrm{d}x\;F_{t,t,1,h}(1-h,x). \qquad (84)$$

But from (35) and Lemma 11,

$$F_{t,t,1,h}(1-h,x) \;\le\; L_{t,t,1,h}(x) \underset{t\uparrow\infty}{\longrightarrow} \mathrm{p}_1(x)\,\langle \ell,\varphi\rangle.$$

Thus, by the extended dominated convergence theorem and again by Theorem 6, in (84) we may pass to the limit as $t\uparrow\infty$ to arrive at

$$i_0\,v\big(1-h;\langle\ell,\varphi\rangle\big) \;=\; -\log E\left\{\mathrm{e}^{-\langle\ell,\varphi\rangle\eta_{1-h}}\;\Big|\;\eta_0=i_0\right\},$$

giving statement (b). ■

3.2 Approximate multi-variate limiting equation

Here we want to generalize Lemma 15 to the multi-variate case. To prepare for this, recall that the *finite-dimensional distributions* of the continuous-state branching process η of Definition 3 satisfy

$$-\log E\Big\{\exp\Big[-\sum_{1\le i\le n} b_i\,\eta_{a_i}\Big]\;\Big|\;\eta_0\Big\} \;=\; \eta_0\,v^{(n)}(\mathbf{a};\mathbf{b}), \qquad (85)$$

where $n\ge 1$ is fixed, $\mathbf{a}=(a_n,\dots,a_1)$ with $0<a_n<\cdots<a_1<\infty$, $\mathbf{b}=(b_n,\dots,b_1)\ge 0$, and where $v^{(1)}:=v$ from (9), and for $n\ge 2$,

$$v^{(n)}(\mathbf{a};\mathbf{b}) \;:=\; v^{(n-1)}\big(a_n,\dots,a_3,a_2;\,b_n,\dots,b_3,b_2+v(a_1-a_2;b_1)\big). \qquad (86)$$

(This follows simply from the Markov and branching property.) Since v solves (8), one can show that

$$v^{(n)}(\mathbf{a};\mathbf{b}) \;=\; \sum_{1\le i\le n} b_i \;-\;\gamma\int_0^{a_1}\mathrm{d}s\;\big[v^{(n)}(\mathbf{a}-s;\mathbf{b})\big]^{1+\beta}$$

with the conventions that $\mathbf{a}-s:=(a_n-s,\dots,a_1-s)$ and that for $n\ge 2$,

$$v^{(n)}(\mathbf{a}-s;\mathbf{b}) \;:=\; v^{(n-1)}(a_{n-1}-s,\dots,a_1-s;b_{n-1},\dots,b_1)$$

if $a_n - s < 0$, that is, if the minus operation leaves the non-negatives.

In analogy with (67), for fixed functions $\boldsymbol{\varphi} = (\varphi^1, \dots, \varphi^n) \in (\mathcal{C}_p^+)^n$, with ordered $\mathbf{a}$ as before and $n \geq 2$, we introduce

$$F_\theta^{(n)}(\mathbf{a}, x) \;:=\; \mathrm{p}_1(x)\, v^{(n)}\big(\mathbf{a}; \langle \ell, \theta\boldsymbol{\varphi}\rangle\big), \qquad \theta \geq 0, \quad x \in \mathsf{R}^d. \tag{87}$$

Here we abbreviated $\langle \ell, \theta\boldsymbol{\varphi}\rangle \;:=\; \sum_{1\leq i\leq n} \langle \ell, \theta\varphi^i\rangle$. Also, from now on we take the branching rate γ from (10). Note that the $F_\theta^{(n)}$ solve

$$F_\theta^{(n)}(\mathbf{a}, x) \;=\; \mathrm{p}_1(x)\langle \ell, \theta\boldsymbol{\varphi}\rangle - \mathrm{p}_1(x)\,\frac{1}{\kappa}\int_0^{a_1} \mathrm{d}s \int_{\mathsf{R}^d} \mathrm{d}y\; \Psi\big(F_\theta^{(n)}(\mathbf{a} - s, y)\big).$$

(Opposed to (68), here we cannot provide a substitution as $s \mapsto a - s$.)

Besides the φ^i, fix now $\mathbf{h} = (h_1, \dots, h_n)$ satisfying $0 \leq h_1 < \cdots < h_n < 1 =: h_{n+1}$. Recalling notation $a_h(r,t)$ from (28), put

$$\mathbf{a_h}(r,t) \;:=\; (a_{h_1}(r,t), \dots, a_{h_n}(r,t)).$$

Analogously to (71), for $t^{h_n} \leq r \leq t$,

$$\begin{aligned}
&F_\theta^{(n)}\big(\mathbf{a_h}(r,t), x\big)\\
&= \mathrm{p}_1(x)\langle \ell, \theta\boldsymbol{\varphi}\rangle \;-\; \mathrm{p}_1(x)\,\frac{1}{\log t}\,\frac{1}{\kappa}\int_{t^{h_1}}^{r} \mathrm{d}s\,\frac{1}{s}\int_{\mathsf{R}^d}\mathrm{d}y\;\Psi\Big(F_\theta^{(n)}\big(\tfrac{\log s}{\log t} - \mathbf{h}, y\big)\Big)\\
&= \mathrm{p}_1(x)\langle \ell, \theta\boldsymbol{\varphi}\rangle \;-\; \mathrm{p}_1(x)\,\frac{1}{\log t}\,\frac{1}{\kappa}\int_{t^{h_1}}^{t^{h_n}} \mathrm{d}s\,\frac{1}{s}\int_{\mathsf{R}^d}\mathrm{d}y\;\Psi\Big(F_\theta^{(n-1)}\big(\mathbf{a_h}(s,t), y\big)\Big)\\
&\qquad\qquad -\; \mathrm{p}_1(x)\,\frac{1}{\log t}\,\frac{1}{\kappa}\int_{t^{h_n}}^{r} \mathrm{d}s\,\frac{1}{s}\int_{\mathsf{R}^d}\mathrm{d}y\;\Psi\Big(F_\theta^{(n)}\big(\mathbf{a_h}(s,t), y\big)\Big).
\end{aligned}$$

Here in the last step we have applied the equality

$$F_\theta^{(n)}(\mathbf{a_h}(s,t), x) \;=\; F_\theta^{(n-1)}(\mathbf{a_h}(s,t), x) \quad \text{if} \quad t^{h_1} < s \leq t^{h_n}.$$

Using Lemma 10 this gives the following analogy with Lemma 15:

Lemma 17 (Approximate multi-variate limiting equation) *Let $\theta \geq 0$, $0 < \varepsilon \leq \min_{1\leq i\leq n}(h_{i+1} - h_i)/2$, and $1 < t^{h_n+\varepsilon} \leq r^{1-\varepsilon} \leq t^{1-\varepsilon}$. Then for each $x \in \mathsf{R}^d$,*

$$\begin{aligned}
F_\theta^{(n)}\big(\mathbf{a_h}(r,t), x\big) &= \mathrm{p}_1(x)\,\langle \ell, \theta\varphi\rangle - {'\mathrm{S}}^{\varepsilon}_{r,t,\theta}(x)\\
&\quad -\; \mathrm{p}_1(x)\,\frac{1}{\log t}\int_{t^{h_1+\varepsilon}}^{t^{h_n-\varepsilon}} N^-_{r-\mathrm{d}s}\,\frac{1}{s}\int_{\mathsf{R}^d}\mathrm{d}y\;\Psi\Big(F_\theta^{(n-1)}\big(\mathbf{a_h}(s,t), y\big)\Big)\\
&\quad -\; \mathrm{p}_1(x)\,\frac{1}{\log t}\int_{t^{h_n+\varepsilon}}^{r^{1-\varepsilon}} N^-_{r-\mathrm{d}s}\,\frac{1}{s}\int_{\mathsf{R}^d}\mathrm{d}y\;\Psi\Big(F_\theta^{(n)}\big(\mathbf{a_h}(s,t), y\big)\Big).
\end{aligned} \tag{88}$$

Here ${'\mathrm{S}}^{\varepsilon}_{r,t,\theta}(x)$ is an error term satisfying a statement as in (46) [with h replaced by h_n].

3.3 Scaled multi-variate renewal equation

We use the abbreviation

$$\boldsymbol{\varphi}_{\mathbf{h},t} \;=\; \bigl(\varphi^1_{h_1,t}\,, \ldots, \varphi^n_{h_n,t}\bigr)$$

with $\varphi^i_{h_i,t}$ as in (13). Set

$$Q^{(n)}_t\bigl(\boldsymbol{\varphi}_{\mathbf{h},t}\bigr)(x) \;:=\; \mathbb{E}_{\delta_x}\biggl(1-\exp\Bigl[-\sum_{1\le i\le n}\bigl\langle Z_t\,,\varphi^i_{h_i,t}\bigr\rangle\Bigr]\biggr), \qquad x\in\mathsf{R}^d.$$

From (30),

$$Q^{(n)}_t\boldsymbol{\varphi}_{\mathbf{h},t}(x) \;=\; E_x\biggl(1-\exp\Bigl[-\sum_{1\le i\le n}\varphi^i_{h_i,t}(\xi_t)\Bigr]\biggr) \\ -\int_0^t N^-_{t-\mathrm{d}s}\;E_x\,\Psi\bigl(Q^{(n)}_s\boldsymbol{\varphi}_{\mathbf{h},t}\,(\xi_{t-s})\bigr). \qquad (89)$$

Moreover, define

$$F^{(n)}_{r,t,\theta}(\mathbf{a_h}(r,t),x) \;:=\; (r\log t)^{1/\beta}\,Q^{(i)}_r\bigl(\theta\boldsymbol{\varphi}_{\mathbf{h},t}\bigr)\,(r^{1/\alpha}x) \qquad (90)$$
$$\text{if}\quad t^{h_i}<r\le t^{h_{i+1}},\quad 1\le i\le n.$$

We want to use the multi-variate version of the renewal equation (89) to investigate the asymptotic behavior of $F^{(n)}_{r,t,\theta}(\mathbf{a_h}(r,t),x)$. This leads to the *scaled multi-variate renewal equation*

$$F^{(n)}_{r,t,\theta}(\mathbf{a_h}(r,t),x) \;=\; L^{(n)}_{r,t,\theta}(x) - R^{(n)}_{r,t,\theta}(x) \\ -\;(r\log t)^{1/\beta}\int_{t^{h_n}}^{r} N^-_{r-\mathrm{d}s}\,E_{r^{1/\alpha}x}\Psi\bigl(Q^{(n)}_s(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(\xi_{r-s})\bigr), \qquad (91)$$

$1\le t^{h_n}<r\le t,\ \theta\ge 0,\ 0\le h_n<1,\ x\in\mathsf{R}^d,$ where

$$L^{(n)}_{r,t,\theta}(x) \;:=\; (r\log t)^{1/\beta}E_x\biggl[1-\exp\Bigl[-\theta\sum_{1\le i\le n}\varphi^i_{h_i,t}(\xi_t)\Bigr]\biggr] \qquad (92)$$

and

$$R^{(n)}_{r,t,\theta}(x) \;:=\; (r\log t)^{1/\beta}\int_0^{t^{h_n}} N^-_{r-\mathrm{d}s}\,E_{r^{1/\alpha}x}\Psi\bigl(Q^{(n)}_s(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(\xi_{r-s})\bigr). \qquad (93)$$

In analogy with Lemma 11 we have the following statement.

Lemma 18 (Convergence of $L^{(n)}_{r,t,\theta}$) *For $0<\varepsilon\le 1-h_n$ and $\theta\ge 0$,*

$$\lim_{t\uparrow\infty}\ \sup_{r\in[t^{h_n+\varepsilon},t]}\Bigl|L^{(n)}_{r,t,\theta}(x)-\mathrm{p}_1(x)\langle\ell,\theta\boldsymbol{\varphi}\rangle\Bigr| \;=\; 0, \qquad x\in\mathsf{R}^d,$$

and

$$\lim_{t\uparrow\infty}\ \sup_{r\in[t^{h_n+\varepsilon},t]}\int_{\mathsf{R}^d}\mathrm{d}x\,\Bigl|L^{(n)}_{r,t,\theta}(x)-\mathrm{p}_1(x)\langle\ell,\theta\boldsymbol{\varphi}\rangle\Bigr| \;=\; 0.$$

3.4 Approximate multi-variate renewal equation

Similarly to Proposition 12 one needs the following key result.

Proposition 19 (Approximate multi-variate renewal equation) *Let* $\theta \ge 0,\ 0 < \varepsilon \le \min_{1\le i\le n}(h_{i+1}-h_i)/2,$ *and* $1 < t^{h_n+\varepsilon} \le r^{1-\varepsilon} \le t^{1-\varepsilon}$. *Then, for each* $x \in \mathsf{R}^d$,

$$
\begin{aligned}
F^{(n)}_{r,t,\theta}\big(\mathbf{a_h}(r,t),x\big) \;=\; & L^{(n)}_{r,t,\theta}(x) - \mathrm{S}^{\varepsilon}_{r,t,\theta}(x)\\
& -\ \mathrm{p}_1(x)\,\frac{1}{\log t}\int_{t^{h_1+\varepsilon}}^{t^{h_n}} N^-_{r-\mathrm{d}s}\,\frac{1}{s}\int_{\mathsf{R}^d}\mathrm{d}y\ \Psi\big(F^{(n-1)}_{s,t,\theta}\big(\mathbf{a_h}(s,t),y\big)\big)\\
& -\ \mathrm{p}_1(x)\,\frac{1}{\log t}\int_{t^{h_n+\varepsilon}}^{r^{1-\varepsilon}} N^-_{r-\mathrm{d}s}\,\frac{1}{s}\int_{\mathsf{R}^d}\mathrm{d}y\ \Psi\Big(F^{(n)}_{s,t,\theta}\big(\mathbf{a_h}(s,t),y\big)\Big). \qquad (94)
\end{aligned}
$$

Here $\mathrm{S}^{\varepsilon}_{r,t,\theta}(x)$ *is an error term satisfying a statement as in* (46).

Note that $F^{(n)}_{r,t,\theta}\big(\mathbf{a_h}(s,t),x\big) = F^{(n-1)}_{r,t,\theta}\big(\mathbf{a_h}(s,t),x\big)$ if $t^{h_1} < s \le t^{h_n}$. However, it will be convenient for us to keep two integral terms at the right-hand side of (94).

The proof of Proposition 19 splits into three lemmas.

Lemma 20 (Representation of the second integral) *Let* $\theta \ge 0,\ 0 < \varepsilon \le (1-h_n)/2,$ *and* $1 < t^{h_n+\varepsilon} \le r^{1-\varepsilon} \le t^{1-\varepsilon}$. *Then, for each* $x \in \mathsf{R}^d$,

$$
\begin{aligned}
&(r\log t)^{1/\beta}\int_{t^{h_n}}^{r} N^-_{r-\mathrm{d}s}\,E_{r^{1/\alpha}x}\Psi\big(Q^{(n)}_s(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(\xi_{r-s})\big)\\
&=\ \mathrm{p}_1(x)\,\frac{1}{\log t}\int_{t^{h_n+\varepsilon}}^{r^{1-\varepsilon}} N^-_{r-\mathrm{d}s}\,\frac{1}{s}\int_{\mathsf{R}^d}\mathrm{d}y\ \Psi\Big(F^{(n)}_{s,t,\theta}\big(\mathbf{a_h}(s,t),y\big)\Big) + {}''\mathrm{S}^{\varepsilon}_{r,t,\theta}(x).
\end{aligned}
$$

Here ${}''\mathrm{S}^{\varepsilon}_{r,t,\theta}(x)$ *is an error term satisfying a statement as in* (46).

Proof Applying the elementary inequalities

$$
1-\exp\Big[-\sum_{i=1}^{n}x_i\Big] \;\le\; \sum_{i=1}^{n}\big(1-\mathrm{e}^{-x_i}\big), \qquad (x_1,\dots,x_n)\ge 0, \qquad (95)
$$

and

$$
\Big(\sum_{i=1}^{n}b_i\Big)^{1+\beta} \;\le\; n^{1+\beta}\sum_{i=1}^{n}b_i^{1+\beta}, \qquad (b_1,\dots,b_n)\ge 0, \qquad (96)
$$

with $x_i = \big\langle Z_t\,,\varphi^i_{h_i,t}\big\rangle$ and $b_i = Q_s(\theta\varphi^i_{h_i,t})\,(y)$, we get

$$
\Psi\big(Q^{(n)}_s(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(y)\big) \;\le\; n^{1+\beta}\sum_{i=1}^{n}\Psi\big(Q_s(\theta\varphi^i_{h_i,t})\,(y)\big). \qquad (97)
$$

This inequality shows that in order to evaluate from above the integrals and quantities involving $\Psi\big(Q^{(n)}_s(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(y)\big)$ we may deal separately with the summands entering the right-hand side of (97). Using this fact in combination with Lemma 14 and Proposition 12, and taking into account that $[t^{h_n}, r] \subseteq [t^{h_j}, r],\ j = 1, 2, ..., n,$ we will be able to establish the needed representation. ■

Lemma 21 (A further representation) *Let* $\theta \geq 0,\ n \geq 2,$ *and* $i \in \{1, 2, ..., n-1\}$ *be fixed. Take* $0 < \varepsilon \leq (1-h_n)/2,$ *and* $1 < t^{h_n+\varepsilon} \leq r^{1-\varepsilon} \leq t^{1-\varepsilon}$. *Then, for each* $x \in \mathsf{R}^d$,

$$(r \log t)^{1/\beta} \int_{t^{h_i}}^{t^{h_{i+1}}} N^{-}_{r-\mathrm{d}s}\, E_{r^{1/\alpha}x} \Psi\big(Q_s^{(n)}(\theta \boldsymbol{\varphi}_{\mathbf{h},t})\,(\xi_{r-s})\big) \;=\; \hat{\mathrm{S}}_{r,t,\theta}(x,i)$$

$$+\,(r \log t)^{1/\beta} \int_{t^{h_i}}^{t^{h_{i+1}}} N^{-}_{r-\mathrm{d}s}\, E_{r^{1/\alpha}x} \Psi\big(Q_s^{(i)}(\theta \boldsymbol{\varphi}_{\mathbf{h},t})\,(\xi_{r-s})\big) \qquad (98)$$

where

$$\limsup_{t\uparrow\infty}\ \sup_{r\in[t^{h_n},\,t^{1-\varepsilon}]} \left(\hat{\mathrm{S}}_{r,t,\theta}(x,i) + \int_{\mathsf{R}^d} \mathrm{d}z\ \hat{\mathrm{S}}_{r,t,\theta}(z,i)\right) = 0. \qquad (99)$$

Proof Using inequality (74) and the estimate

$$\left(1 - \exp\Big[-\sum_{j=1}^{n} x_j\Big]\right) - \left(1 - \exp\Big[-\sum_{j=1}^{i} x_j\Big]\right) \;\leq\; \sum_{j=i+1}^{n} x_j\,, \qquad (100)$$

$(x_1, \ldots, x_n) \geq 0,$ with $a = Q_s^{(n)} \ldots \geq b = Q_s^{(i)} \ldots$ and $x_i = \big\langle Z_t\,, \varphi^i_{h_i,t} \big\rangle$, and applying (32) we get

$$\begin{aligned}
0 \;&\leq\; \Psi\big(Q_s^{(n)}(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(y)\big) - \Psi\big(Q_s^{(i)}(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(y)\big) \\
&\leq\; c_f\,(1+\beta)\,\Big(Q_s^{(n)}(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(y)\Big)^{\beta}\,\Big(Q_s^{(n)}(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(y) - Q_s^{(i)}(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(y)\Big) \\
&\leq\; c_f\,(1+\beta)\,\theta^{1+\beta}\Big(\sum_{k=1}^{n} E_y \varphi^k_{h_k,t}(\xi_{s,k})\Big)^{\beta} \sum_{j=i+1}^{n} E_y \varphi^j_{h_j,t}(\xi_{s,j}) \\
&\leq\; c_f\,(1+\beta)\,\theta^{1+\beta}\,n^{\beta} \sum_{k=1}^{n} \big(E_y \varphi^k_{h_k,t}(\xi_{s,k})\big)^{\beta} \sum_{j=i+1}^{n} E_y \varphi^j_{h_j,t}(\xi_{s,j}), \qquad (101)
\end{aligned}$$

where $\xi_{s,k},\ k = 1, 2, ..., n,$ are independent copies of ξ_s. In view of (50),

$$\sum_{k=i+1}^{n} \big(E_y \varphi^k_{h_k,t}(\xi_{s,k})\big)^{\beta} \;\leq\; c_{13}\,(\log t)^{-1} \sum_{k=i+1}^{n} t^{-h_k} \;\leq\; c_{13} n\,(\log t)^{-1}\, t^{-h_{i+1}}. \qquad (102)$$

On the other hand, by (50) and (57)–(59) we see that for any $\delta > 0$ there exist $t_0 = t_0\,(\delta)$ and $K_0 = K_0\,(\boldsymbol{\varphi}, \delta)$ such that for all $t \geq t_0$ and $K \geq K_0$,

$$\begin{aligned}
\sum_{k=1}^{i} \big(E_y \varphi^k_{h_k,t}(\xi_{s,k})\big)^{\beta} \;&\leq\; c_{(103)}\,(s \log t)^{-1}\,\Big(\delta\, 1_{\{|y|>Ks^{1/\alpha}\}} + 1_{\{|y|\leq Ks^{1/\alpha}\}}\Big) \\
&\leq\; c_{(103)}\,(s \log t)^{-1}\,\Big(\delta + 1_{\{|y|\leq Ks^{1/\alpha}\}}\Big) \qquad (103)
\end{aligned}$$

if $s \geq \max_{k \leq i} \{t^{h_k}\} = t^{h_i}$. Using (102) and (103) in (101) with $y = \xi_{r-s}$ we obtain

$$\begin{aligned} & E_{r^{1/\alpha}x}\Big(\Psi\big(Q_s^{(n)}(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(\xi_{r-s})\big) - \Psi\big(Q_s^{(i)}(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(\xi_{r-s})\big)\Big) \\ & \leq\; c_f\,(1+\beta)\,(\log t)^{-1}\,\theta^{1+\beta}\,\big(c_9 n t^{-h_{i+1}} + \delta c_{(103)} s^{-1}\big) \;\times \\ & \qquad \sum_{j=i+1}^{n} E_{r^{1/\alpha}x}\big(E_{\xi_{r-s}}\varphi^j_{h_j,t}(\xi_{s,j})\big) \;+\; c_f\,(1+\beta)\,(s\log t)^{-1}\,\theta^{1+\beta} \;\times \\ & \qquad\qquad \sum_{j=i+1}^{n} E_{r^{1/\alpha}x}\Big(1_{\{|\xi_{r-s}|\leq K s^{1/\alpha}\}} E_{\xi_{r-s}}\varphi^j_{h_j,t}(\xi_{s,j})\Big). \end{aligned} \tag{104}$$

For subsequent arguments we need to evaluate the right-hand side of inequality (104). By the Markov property and (38) we see that

$$(r\log t)^{1/\beta} \sum_{j=i+1}^{n} E_{r^{1/\alpha}x}\big(E_{\xi_{r-s}}\varphi^j_{h_j,t}(\xi_{s,j})\big) \;\leq\; \mathrm{p}_1(0) \sum_{j=i+1}^{n} \langle \ell, \varphi^j\rangle. \tag{105}$$

On the other hand, by (58) for any $j \geq i+1$ we have

$$\begin{aligned} & (r\log t)^{1/\beta} E_{r^{1/\alpha}x}\Big(1_{\{|\xi_{r-s}|\leq K s^{1/\alpha}\}} E_{\xi_{r-s}}\varphi^j_{h_j,t}\,(\xi_{s,j})\Big) \;=\; (r/s)^{-1/\beta} \;\times \\ & \qquad E_{r^{1/\alpha}x} 1_{\{|\xi_{r-s}|\leq K s^{1/\alpha}\}} \int_{\mathsf{R}^d} \mathrm{d}z\;\varphi^j\,(z)\,\mathrm{p}_1\big(t^{h_j/\alpha}s^{-1/\alpha}z - s^{-1/\alpha}\xi_{r-s}\big) \\ & =\; r^{1/\beta}\int_{|w|\leq K} \mathrm{d}w\;\mathrm{p}_{r-s}\big(ws^{1/\alpha} - xr^{1/\alpha}\big)\int_{\mathsf{R}^d} \mathrm{d}z\;\varphi^j\,(z)\,\mathrm{p}_1\big(t^{h_j/\alpha}s^{-1/\alpha}z - w\big) \\ & =\; \int_{|w|\leq K} \mathrm{d}w\;\mathrm{p}_{1-sr^{-1}}\big(ws^{1/\alpha}r^{-1/\alpha} - x\big)\int_{\mathsf{R}^d} \mathrm{d}z\;\varphi^j\,(z)\,\mathrm{p}_1\big(t^{h_j/\alpha}s^{-1/\alpha}z - w\big) \\ & \leq\; \mathrm{p}_{1-sr^{-1}}\,(0)\int_{\mathsf{R}^d} \mathrm{d}z\;\varphi^j\,(z)\int_{|w|\leq K} \mathrm{d}w\;\mathrm{p}_1\big(t^{h_j/\alpha}s^{-1/\alpha}z - w\big). \end{aligned} \tag{106}$$

Combining (105), (106), and (104) we finally get

$$\begin{aligned} & (r\log\; t)^{1/\beta} E_{r^{1/\alpha}x}\Big(\Psi\big(Q_s^{(n)}(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(y)\big) - \Psi\big(Q_s^{(i)}(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(y)\big)\Big) \\ & \leq c\,\mathrm{p}_1\,(0)\,(\log t)^{-1}\,\theta^{1+\beta}\,\big(t^{-h_{j+1}} + \delta s^{-1}\big) \sum_{j=i+1}^{n} \langle \ell, \varphi^j\rangle \;+\; c\,\mathrm{p}_{1-sr^{-1}}\,(0) \;\times \\ & \qquad (s\log t)^{-1}\,\theta^{1+\beta} \sum_{j=i+1}^{n} \int_{\mathsf{R}^d} \mathrm{d}z\;\varphi^j\,(z)\int_{|w|\leq K} \mathrm{d}w\;\mathrm{p}_1\big(t^{h_j/\alpha}s^{-1/\alpha}z - w\big) \end{aligned}$$

for all $t \geq t_0$ and $K \geq K_0$. Hence, we conclude that

$$0 \leq \hat{\mathrm{S}}_{r,t,\theta}(x,i) \leq c\,\theta^{1+\beta}\,(\log t)^{-1}\Big(t^{-h_{i+1}}\int_{t^{h_i}}^{t^{h_{i+1}}} N^-_{r-\mathrm{d}s} + \delta\int_{t^{h_i}}^{t^{h_{i+1}}} N^-_{r-\mathrm{d}s}\frac{1}{s}\Big)$$
$$+\; c\,\theta^{1+\beta}\,(\log t)^{-1}\int_{t^{h_i}}^{t^{h_{i+1}}} N^-_{r-\mathrm{d}s}\frac{1}{s}\sum_{j=i+1}^{n}\int_{\mathsf{R}^d}\mathrm{d}z\;\varphi^j\,(z)\;\times$$
$$\int_{|w|\leq K}\mathrm{d}w\;\mathrm{p}_1\big(t^{h_j/\alpha}s^{-1/\alpha}z-w\big) \tag{107}$$

for all $t \geq t_0\,(\delta)$, $K \geq K_0\,(\boldsymbol{\varphi},\delta)$, and a constant $c = c(t_0, K_0, \mathbf{h})$. From the convergence statement (54) and Lemma 9,

$$(\log t)^{-1}\left(t^{-h_{i+1}}\int_{t^{h_i}}^{t^{h_{i+1}}} N^-_{r-\mathrm{d}s} \;+\; \delta\int_{t^{h_i}}^{t^{h_{i+1}}} N^-_{r-\mathrm{d}s}\frac{1}{s}\right) \;\leq\; c_{(108)}\delta \tag{108}$$

for $t \geq t_0$ (by enlarging t_0 if needed). Again by Lemma 9, for any $\varepsilon_1 \in (0, h_{j+1}-h_j)$,

$$(\log t)^{-1}\int_{t^{h_{i+1}-\varepsilon_1}}^{t^{h_{i+1}}} N^-_{r-\mathrm{d}s}\frac{1}{s}\sum_{j=i+1}^{n}\int_{|w|\leq K}\mathrm{d}w\int_{\mathsf{R}^d}\mathrm{d}z\;\varphi^j\,(z)\,\mathrm{p}_1(t^{h_j/\alpha}s^{-1/\alpha}z-w)$$
$$\leq\; (\log t)^{-1}\int_{t^{h_{i+1}-\varepsilon_1}}^{t^{h_{i+1}}} N^-_{r-\mathrm{d}s}\frac{1}{s}\sum_{j=i+1}^{n}\int_{\mathsf{R}^d}\mathrm{d}z\;\varphi^j\,(z)\;\leq\; c_{(109)}\varepsilon_1 \tag{109}$$

for all $t \geq t_0 = t_0\,(\delta,\varepsilon_1)$ (again by enlarging t_0 if needed). Thus, it remains to deal with

$$(\log t)^{-1}\int_{t^{h_i}}^{t^{h_{i+1}-\varepsilon_1}} N^-_{r-\mathrm{d}s}\frac{1}{s}\sum_{j=i+1}^{n}\int_{|w|\leq K}\mathrm{d}w\int_{\mathsf{R}^d}\mathrm{d}z\;\varphi^j\,(z)\,\mathrm{p}_1\big(t^{h_j/\alpha}s^{-1/\alpha}z-w\big).$$

Since for $j \geq i+1$ and $s \in \big[t^{h_i}, t^{h_{i+1}-\varepsilon_1}\big]$,

$$t^{h_j/\alpha}s^{-1/\alpha} \;\geq\; t^{h_{i+1}/\alpha}t^{-(h_{i+1}-\varepsilon_1)/\alpha} \;=\; t^{\varepsilon_1/\alpha} \underset{t\uparrow\infty}{\longrightarrow} \infty,$$

there exists a $t_1 = t_1\,(\delta,\varepsilon_1,K) \geq t_0$ such that for all $t \geq t_1$,

$$\sup_{s\in\left[t^{h_i},t^{h_{i+1}-\varepsilon_1}\right]} \mathrm{p}_1\big(t^{h_j/\alpha}s^{-1/\alpha}z-w\big) \;\leq\; \varepsilon_1 K^{-d}$$

for all $|z| \geq t^{-\varepsilon_1/(2\alpha)}$ and $|w| \leq K$, and, in addition,

$$\sum_{j=i+1}^{n}\int_{|z|<t^{-\varepsilon_1/(2\alpha)}}\mathrm{d}z\;\varphi^j\,(z) \;\leq\; \varepsilon_1\,.$$

Hence, for all $t \geq t_1$,

$$\begin{aligned}
&\sup_{s\in\left[t^{h_i},t^{h_{i+1}-\varepsilon_1}\right]} \sum_{j=i+1}^{n} \int_{\mathsf{R}^d} \mathrm{d}z\ \varphi^j(z) \int_{|w|\leq K} \mathrm{d}w\ \mathrm{p}_1\big(t^{h_j/\alpha}s^{-1/\alpha}z-w\big) \\
&\leq \sum_{j=i+1}^{n} \Big(\int_{|z|<t^{-\varepsilon_1/(2\alpha)}} \mathrm{d}z\ \varphi^j(z) + \varepsilon_1 K^{-d} \int_{|z|\geq t^{-\varepsilon_1/(2\alpha)}} \mathrm{d}z\ \varphi^j(z) \int_{|w|\leq K} \mathrm{d}w\Big) \\
&\leq\ c_{(110)}\varepsilon_1 . \qquad (110)
\end{aligned}$$

Combining (108)–(110) with (107) we conclude that there exists a $t_1 = t_1(\delta, \varepsilon_1, K)$ such that for all $t \geq t_1$,

$$0 \ \leq\ \hat{\mathrm{S}}_{r,t,\theta}(x,i)\ \leq\ c_{(108)}\delta + (c_{(109)} + c_{(110)})\varepsilon_1 .$$

Thus, letting first $t \uparrow \infty$ and than $\delta \to 0$ and $\varepsilon_1 \to 0$, we get the statement of Lemma 21 for $\hat{\mathrm{S}}_{r,t,\theta}(x,i)$. To obtain the desired statement for $\int_{\mathsf{R}^d} \mathrm{d}z\, \hat{\mathrm{S}}_{r,t,\theta}(z,i)$ one should apply the same lines of arguments with the difference that instead of (105) and (106) one needs to use the equalities [recall (41)]

$$\sum_{j=i+1}^{n} \int_{\mathsf{R}^d} \mathrm{d}x\ \ (r\log t)^{1/\beta}\, E_{r^{1/\alpha}x}\varphi^j_{h_j,t}(\xi_t) \ =\ \sum_{j=i+1}^{n} \langle \ell, \varphi^j \rangle$$

and

$$\begin{aligned}
&\int_{\mathsf{R}^d} \mathrm{d}x\ \ (r\log t)^{1/\beta}\, E_{r^{1/\alpha}x}\Big(1_{\{|\xi_{r-s}|\leq Ks^{1/\alpha}\}} E_{\xi_{r-s}}\varphi^j_{h_j,t}(\xi_{s,j})\Big) \\
&= \int_{|w|\leq K} \mathrm{d}w \int_{\mathsf{R}^d} \mathrm{d}x\, \mathrm{p}_{1-sr^{-1}}(ws^{1/\alpha}r^{-1/\alpha}-x) \int_{\mathsf{R}^d} \mathrm{d}z\, \varphi^j(z)\, \mathrm{p}_1(t^{h_j/\alpha}s^{-1/\alpha}z-w) \\
&= \int_{\mathsf{R}^d} \mathrm{d}z\ \varphi^j(z) \int_{|w|\leq K} \mathrm{d}w\ \mathrm{p}_1(t^{h_j/\alpha}s^{-1/\alpha}z-w),
\end{aligned}$$

respectively. ∎

The next lemma deals with $R^{(n)}$ defined in (93).

Lemma 22 (Asymptotic representation of $R^{(n)}$) *Let $\theta \geq 0,\ 0 < \varepsilon \leq \min_{1\leq i\leq n}(h_{i+1} - h_i)/2$, and $1 < t^{h_n+\varepsilon} \leq r^{1-\varepsilon} \leq t^{1-\varepsilon}$. Then, for each $x \in \mathsf{R}^d$,*

$$\begin{aligned}
R^{(n)}_{r,t,\theta}(x) \ &=\ \mathrm{p}_1(x)\, \frac{1}{\log t} \int_{t^{h_1+\varepsilon}}^{t^{h_n}} N^-_{r-\mathrm{d}s}\, \frac{1}{s} \int_{\mathsf{R}^d} \mathrm{d}y\ \Psi\Big(F^{(n-1)}_{s,t,\theta}\big(\mathbf{a_h}(s,t), y\big)\Big) \\
&\quad +\ \hat{\mathrm{S}}^{\varepsilon}_{r,t,\theta}(x),
\end{aligned}$$

where $\hat{\mathrm{S}}^{\varepsilon}_{r,t,\theta}(x)$ is an error term satisfying a statement as in (46).

Proof In order to establish the desired representation, we first write

$$R^{(n)}_{r,t,\theta}(x) \;=\; \sum_{i=1}^{n-1}(r\log t)^{1/\beta}\int_{t^{h_i}}^{t^{h_{i+1}}} N^-_{r-\mathrm{d}s}\, E_{r^{1/\alpha}x}\Psi\big(Q^{(n)}_s(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(\xi_{r-s})\big)$$

$$+\;(r\log t)^{1/\beta}\int_0^{t^{h_1}} N^-_{r-\mathrm{d}s}\, E_{r^{1/\alpha}x}\Psi\big(Q^{(n)}_s(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(\xi_{r-s})\big).$$

Applying the previous lemma, we get

$$R^{(n)}_{r,t,\theta}(x) \;=\; \sum_{i=1}^{n-1}(r\log t)^{1/\beta}\int_{t^{h_i}}^{t^{h_{i+1}}} N^-_{r-\mathrm{d}s}\, E_{r^{1/\alpha}x}\Psi\big(Q^{(i)}_s(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(\xi_{r-s})\big)$$

$$+\;{}''\hat{\mathrm{S}}^{\varepsilon}_{r,t,\theta}(x),$$

where ${}''\hat{\mathrm{S}}^{\varepsilon}_{r,t,\theta}(x)$ is an error term satisfying a statement as in (46). Now to each of the integrals we can apply the arguments used to establish Lemma 14 (note that $[t^{h_i},t^{h_{i+1}}]\subseteq[t^{h_j},r^{1-\varepsilon}]$ and $[0,t^{h_1}]\subseteq[0,t^{h_j}]$ for each $j\le i\le n-1$, so we can deduce the desired estimates for the counterparts of integrals ${}^1I(x),,\dots,{}^6I(x)$ from Lemma 14) and obtain [recall (90)]

$$R^{(n)}_{r,t,\theta}(x) \;=\; {}''\hat{\mathrm{S}}^{\varepsilon}_{r,t,\theta}(x) \;+\; \mathrm{p}_1(x)\,\frac{1}{\log t}\sum_{i=1}^{n-1}\int_{t^{h_i}}^{t^{h_{i+1}}} N^-_{r-\mathrm{d}s}\frac{1}{s}\;\times$$

$$\int_{\mathsf{R}^d}\mathrm{d}y\;\Psi\big((s\log t)^{\beta}\,Q^{(i)}_s(\theta\boldsymbol{\varphi}_{\mathbf{h},t})\,(s^{1/\alpha}y)\big) \;=\; \hat{\mathrm{S}}^{\varepsilon}_{r,t,\theta}(x) \;+\; \mathrm{p}_1(x)\,\frac{1}{\log t}\;\times$$

$$\int_{t^{h_1+\varepsilon}}^{t^{h_n}} N^-_{r-\mathrm{d}s}\frac{1}{s}\int_{\mathsf{R}^d}\mathrm{d}y\;\Psi\Big(F^{(n-1)}_{s,t,\theta}\big(\mathbf{a}_{\mathbf{h}}(s,t),y\big)\Big),$$

where, to fit the form (94), at the last step we included the integral

$$\int_{t^{h_1}}^{t^{h_1+\varepsilon}} N^-_{r-\mathrm{d}s}\frac{1}{s}\int_{\mathsf{R}^d}\mathrm{d}y\;\Psi\Big(F^{(n-1)}_{s,t,\theta}\big(\mathbf{a}_{\mathbf{h}}(s,t),y\big)\Big)$$

into $\hat{\mathrm{S}}^{\varepsilon}_{r,t,\theta}(x)$. This finishes the proof. ■

Combining Lemmas 20 and 22 proves Proposition 19.

3.5 Completion of the proof of Theorem 4

We prove Theorem 4 by induction. For this reason the following statement is important.

Proposition 23 (Convergence of $F^{(k)}$ in $L^1(\mathrm{d}x)$) *For* $k=1,\dots,n$, *there exists a positive* $\theta_0=\theta_0(\boldsymbol{\varphi},h_1,h_2,...,h_k)$ *such that*

$$\lim_{t\uparrow\infty}\;\sup_{r\in[t^{h_k+\varepsilon},t]}\int_{\mathsf{R}^d}\mathrm{d}x\;\Big|F^{(k)}_{r,t,\theta}\big(\mathbf{a}_{\mathbf{h}}(r,t),x\big)-F^{(k)}_{\theta}\big(\mathbf{a}_{\mathbf{h}}(r,t),x\big)\Big| \;=\; 0 \qquad (111)$$

for $0<\varepsilon\le 1-h_k$ *and* $0\le\theta\le\theta_0$.

Proof For $k = 1$ this is just Proposition 16. Assume that the desired statement is proven for some $k \le n-1$. Replacing n by $k+1$ in representation (94), making the same trick with (88), taking the difference of the obtained relations and integrating it with respect to $\mathrm{d}x$, it is not difficult to establish (111) using induction hypothesis and applying the arguments similar to those exploited to prove Proposition 16. ■

Having Proposition 23, it is a straightforward procedure to prove the following statement using the arguments applied to verify Theorem 6.

Theorem 24 (Refined asymptotics for $Q^{(n)}$) *Assume $d = \alpha/\beta$. Then, for fixed x and $0 \le h_1 < h_2 < \ldots < h_n < 1$,*

$$(t \log t)^{1/\beta}\, Q_t^{(n)}(\boldsymbol{\varphi}_{\mathbf{h},t})(t^{1/\alpha}x) \underset{t\uparrow\infty}{\longrightarrow} \mathrm{p}_1(x)\, v^{(n)}\big(\mathbf{1}-\mathbf{h}; \langle \ell, \boldsymbol{\varphi}\rangle\big) = F_1^{(n)}(\mathbf{1}-\mathbf{h}, x)$$

with the macroscopic log-Laplace function $v^{(n)}$ from (86), and branching rate γ as in (10).

Theorem 24 then implies the convergence of finite-dimensional distributions as claimed in Theorem 4.

3.6 To the proofs of Theorems 7 and 8

Instead of a detailed proof, here we only indicate some key steps. In the special case if G is the exponential distribution, renewal equation (30) reads as follows:

$$Q_t\varphi\,(x) \;=\; E_x\big[1-\mathrm{e}^{-\varphi(\xi_t)}\big] - \frac{1}{\kappa}\int_0^t \mathrm{d}s\; E_x\Psi\big(Q_s\varphi\,(\xi_{t-s})\big).$$

Comparing with log-Laplace equation (15) and using uniqueness of its solutions gives

$$u\big(t,x;1-\mathrm{e}^{-\varphi}\big) \;=\; Q_t\varphi(x).$$

For $0 < \varepsilon \le 1$ there is a $b_0 = b_0(\varepsilon)$ such that

$$1-\mathrm{e}^{-b} \;\le\; b \;\le\; 1-\mathrm{e}^{-(1+\varepsilon)b}, \qquad 0 \le b \le b_0\,.$$

By the monotonicity of u in the initial data, this will enable us to transfer the refined asymptotics of Theorem 6 into the one in Theorem 8. In fact, for sufficiently large t,

$$Q_t(\varphi_{0,t})(t^{1/\alpha}x) \;\le\; u\big(t, t^{1/\alpha};\, (\log t)^{-1/\beta}\varphi\big) \;\le\; Q_t\big((1+\varepsilon)\varphi_{0,t}\big)(t^{1/\alpha}x),$$

since $\varphi_{0,t} = (\log t)^{-1/\beta}\varphi$. Therefore Theorem 7 implies (18) in the case $h = 0$. To pass to arbitrary $h < 1$, we use the following scaling identity:

$$u\,(t,x;\varphi) \;=\; b^{1/\beta}\, u\big(bt, b^{1/\alpha}x; b^{-1/\beta}\varphi(b^{-1/\alpha}\cdot)\big), \qquad b,t>0, \quad x \in \mathsf{R}^d,$$

and specialize to $b = t^h, \ -\infty < h < 1$.

To come to Theorem 7(b), exploit the identity

$$-\log \mathbb{E}_{i_0 \, (\log t)^{1/\beta} \ell} \exp \left\langle X_t^h, -\varphi \right\rangle \ = \ i_0 \, (\log t)^{1/\beta} \left\langle \ell, u(t, \, \cdot \, ; \varphi_{h,t}) \right\rangle$$

and dominated convergence.

The case of finite dimensional distributions is treated in the same way by applying Theorem 24.

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