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Aging in two-dimensional Bouchaud's model

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ABSTRACT. Let E_x be a collection of i.i.d. exponential random variables. Symmetric Bouchaud's model on $\mathbb{Z}^2$ is a Markov chain $X(t)$ whose transition rates are given by $w_{xy} = \nu \exp(-\beta E_x)$ if x, y are neighbours in $\mathbb{Z}^2$. We study the behaviour of two correlation functions: $\mathbb{P}[X(t_w + t) = X(t_w)]$ and $\mathbb{P}\big[X(t') = X(t_w) \forall t' \in [t_w, t_w + t]\big]$. We prove the (sub)aging behaviour of these functions when $\beta > 1$.

1. INTRODUCTION

We explore in this paper a mechanism for aging of Markovian dynamics in complex random media proposed by J. P. Bouchaud. This mechanism is based on trapping. More precisely if a Markov process moves in a very complex landscape of energy, it should spend most of its time in the deep valleys of this landscape; its long time behaviour should be essentially ruled by three features: the (short) transits between these valleys, the relative positions of these valleys, and the (long) exit times from these valleys (which are usually exponentially distributed with parameters proportional to the (random) depth of the valley). Aging would then simply be the consequence of the following mechanism: the older the system is the more space it has explored, the deeper valley it is stuck in. So that the process essentially stays put for longer and longer periods of time.

In order to capture the core of this appealing picture, Bouchaud proposed a very simple effective model of trapping on a graph. Let us describe it here. Let $G = (\mathcal{V}, \mathcal{E})$ be a connected graph. The vertices of G should be seen as valleys and the graph structure as the description of the communication between these valleys. The random landscape is now given by a collection of i.i.d. random variables $\boldsymbol{E} = \{E_x\}_{x \in \mathcal{V}}$, exponentially distributed with mean 1. E_x should be seen as the depth of the valley at x. We consider a random "Gibbs" measure $\boldsymbol{\tau}$ on $\mathcal{V}$ with mass of vertex x given by

$$\tau_x = e^{\beta E_x}, \tag{1}$$

where $\beta > 0$ is the inverse temperature. We define Bouchaud's trap model as a continuous time Markov chain $X(t)$ on $\mathcal{V}$ by the jump rates

$$w_{xy} = \nu \exp\big[-\beta((1-a)E_x - aE_y)\big] \quad \text{if } x \sim y, \tag{2}$$

and zero otherwise. The constant ν fixes only the time scale and will be fixed later, and $a \in [0,1]$ tunes the influence of neighbouring valleys on jumping rates. The simplest case is $a = 0$, then it is clear that the

Markov chain X is a random time change of the simple random walk on the graph. Notice that the random measure $\boldsymbol{\tau}$ is reversible for all values of a and β.

This model has been introduced in the physics literature (see [Bou92, MB96, BM97]) on a large complete graph initially as an ansatz for the dynamics of the Random Energy Model (see [BBG03a, BBG03b] for a rigorous study). It was then considered on the graph $\mathbb{Z}^d$ in [RMB00] and first time rigorously studied by [FIN02] for $d = 1$ and $a = 0$, then by [BČ02] for $d = 1$ and general a. We study here Bouchaud's trap model on the lattice $\mathbb{Z}^2$ in $a = 0$ case. This case has been sketched in [RMB00] with a concept of partial equilibrium which is rather difficult to justify. For convenience we will choose $\nu = 1/(2d) = 1/4$.

Let us state our aging result. We consider the following two-point function:

$$R(t_w, t_w + t) = \mathbb{P}[X(t_w + t) = X(t_w)|\boldsymbol{E}], \tag{3}$$

which is the probability that the process is at the same site at time $t_w + t$ as it was at time t_w. We prove the aging behaviour for the function R.

Theorem 1.1. *For all $\beta > 1$ and $a = 0$ there exists a function $R(\theta)$ such that for $\mathbb{P}$-a.e. realisation of the environment $\boldsymbol{E}$*

$$\lim_{t_w \to \infty} R(t_w, t_w + \theta t_w) = R(\theta). \tag{4}$$

Moreover, the function $R(\theta)$ can be explicitly calculated (see Proposition 7.1) and it satisfies

$$\lim_{\theta \to 0} R(\theta) = 1 \quad \textit{and} \quad \lim_{\theta \to \infty} R(\theta) = 0. \tag{5}$$

We further study the following two-point function:

$$\Pi(t_w, t_w + t) = \mathbb{P}[X(t') = X(t_w) \forall t' \in [t_w, t_w + t]|\boldsymbol{E}], \tag{6}$$

which is the probability that the process does not jump between the times t_w and $t + t_w$. We show sub-aging behaviour for this two-point function.

Theorem 1.2. *For all $\beta > 1$ and $a = 0$ there exists a function $\Pi(\theta)$ such that for $\mathbb{P}$-a.e. realisation of the environment $\boldsymbol{E}$*

$$\lim_{t_w \to \infty} \Pi\left(t_w, t_w + \theta \frac{t_w}{\log t_w}\right) = \Pi(\theta). \tag{7}$$

The function $\Pi(\theta)$ can be again made explicit (see Proposition 8.2) and it satisfies the same relations (5) *as $R(\theta)$.*

Remarks: 1. Our results stay valid if we replace the condition that E_x is exponentially distributed with mean 1 by weaker condition

$$\mathbb{P}[\tau_x > 0] = 1 \qquad \text{and} \qquad \lim_{u \to \infty} u^\alpha \mathbb{P}(\tau_0 \geq u) = K \quad \text{with } \alpha \in (0, 1), \tag{8}$$

and $K \in (0, \infty)$. This condition is easy to verify for the original distribution with $\alpha = 1/\beta$, $K = 1$. For the rest of the paper the condition (8) is in force, we also assume for simplicity that $K = 1$. The limiting functions $R(\theta)$ and $\Pi(\theta)$ do not depend on the choice of distribution of τ_x verifying these conditions.

2. Unlike as in $d = 1$ case [FIN02, BČ02] we study here the so called quenched two-point functions, that means that we obtain (sub)aging for a.e. environment. The averaged results are an easy consequence of our theorems. To complete the picture, we recall the results of [FIN02, BČ02]. It was proved that averaged two-point functions satisfy

$$\begin{aligned} &\lim_{t_w \to \infty} \mathbb{E} R(t_w, t_w + \theta t_w) = R_1(\theta) \\ &\lim_{t_w \to \infty} \mathbb{E} \Pi(t_w, t_w + \theta t_w^{(1-a)/(1+\alpha)}) = \Pi_{1,a}(\theta). \end{aligned} \tag{9}$$

Note also that in $d = 1$ the analogous quenched results are not valid.

3. The $d \geq 3$, $a = 0$ case is treated in [Čer03]. Very similar results are obtained there. Theorem 1.1 stays valid without any change, even the function $R(\theta)$ is the same. Theorem 1.2 should be modified slightly. For the two-point function Π the same type of limit as for R should be considered, to reflect the fact that the simple random walk in $d \geq 3$ visits any site only finitely many times. It was shown there $\lim_{t_w \to \infty} \Pi(t_w, t_w + \theta t_w)$ exists.

4. The case $d = 2$, $a > 0$ is much harder and will be treated elsewhere. In this case Bouchaud's trap model is not longer time change of the simple random walk but some type of reversible random walk in random environment.

5. The choice $\nu = 1/4$ assures, that the mean waiting time of X at site x is equal to τ_x. Therefore, the process $X(t)$ stays at the site x an exponentially distributed time with mean τ_x and then it jumps with the equal probability to one of the four neighbouring sites. Formally, let $X_d(i)$, $i = 0, 1, \ldots$, denote a discrete time simple random walk on $\mathbb{Z}^2$ started at origin, and let e_i be a collection of i.i.d. exponential random variables with mean one. We use $S(n)$ to denote the "time change" of the simple random walk

$$S(n) = \sum_{i=0}^{n-1} e_i \tau_{X_d(i)}. \tag{10}$$

Then $X(t) = X_d(j)$ if $S(j) \leq t < S(j+1)$. Since the random variables τ_x and E_x are directly related by (1) we abuse terminology slightly, and call τ_x also depth of the trap at x. Actually, we do not use E_x later in this paper and the word "depth" refers always to τ.

The results of both theorems can be described heuristically in the following way. After time t_w the system is typically in a trap whose mean waiting time is of order $t_w / \log t_w$ (as can be seen from Theorem 1.2). After passing a time of that order in the trap the process X

makes excursions from it and returns there of order $\log t_w$ times before time $(1+\theta)t_w$. Then X leaves the neighbourhood of this trap and continues to explore the lattice.

We describe here the strategy that will be used to prove both the theorems. Let $n \in \mathbb{N}$. We consider the process $X(t)$ only before the exit from the disk $\mathbb{D}(n)$ with the area $m2^n n^{1-\alpha}$ around the origin. The constant m will be chosen later in order that the walk can stay a sufficiently long time inside $\mathbb{D}(n)$. We are interested mainly in the time that the walk spends in traps that are deeper than $\varepsilon 2^{n/\alpha}/n$, for $\varepsilon \ll 1$ to be fixed later (such traps will be referred to as deep traps). In the disk $\mathbb{D}(n)$ there are approximately mn/ϵ^α such traps. Since the probability of hitting a particular point in $\mathbb{D}(n)$, that is sufficiently far from the walk initial point, before the exit from $\mathbb{D}(n)$ is of order n^{-1}, the walk has a reasonable chance to hit at least one deep trap. The constant ε will be chosen small enough to ensure that the walk spends a negligible proportion of time in shallower traps.

We cut the trajectory of the process X into short parts. Every part is finished when X exits for the first time the disk of area $2^n n^\gamma$ around the initial point of the part. At this moment a new part is started. Clearly, we should take $\gamma < 1-\alpha$. For every such part we look at the time that the walk spends in the traps which we have specified in the previous paragraph. It will be proved that, with overwhelming probability, the walk hits at most one such trap in every part. Moreover, the same trap is almost never hit again in the next parts before the exit from $\mathbb{D}(n)$. To the i-th part of the trajectory we associate a random variable s_i that we call score of that part, and that is roughly the time spent by X in the deep trap that was hit during this part (the score will be defined in Section 2). It will be proved that for n sufficiently large the random variables s_i are essentially independent and the well rescaled trajectory of the sum $\sum s_i$ converges to a pure jump, increasing Lévy process. It will be also shown that this sum is a good approximation for the well rescaled time change $S(n)$.

The proof of both theorems relies on the fact that the events that we are interested in, that is the probabilities of staying a long time at the same place, mainly occur if well rescaled values of times t_w and $t+t_w$ falls into one jump of the Lévy process from the previous paragraph, or more precisely if the intersection of the range of the Lévy process with the rescaled interval $[t_w, t+t_w]$ is empty. Probability of such an event is easy to calculate using arcsine law for Lévy processes (see [Ber96]).

The theorems are proved in Sections 7 and 8 where the reader can also find the explicit expressions for functions $R(\theta)$ and $\Pi(\theta)$. The proof of the convergence of well rescaled sums of scores occupies Sections 2–6.

Throughout the paper we will deal with typically non integer objects such as $\sqrt{n2^n}$ or $2^n n^\gamma$ for discrete valued processes. In these contexts it is to be understood that the quantity referred to is the integer part.

2. The coarse-graining of $X(t)$

We introduce some notations needed later. We use $D_x(m)$, and $B_x(m)$ to denote the disk, resp. the box, with area m around the site x. If x is omitted the disk (box) is centred around the origin. Both these objects are understood as subsets of $\mathbb{Z}^2$. In the following we will very often use the claim that the disk $D(m)$ contains m sites from $\mathbb{Z}^2$, although it is not precisely true. Precisely $D(m)$ will be the disc of radius r, where r is the infimum of the radii of discs centred at the origin containing at least m lattice points. Any error we introduce by this consideration will be negligible for m large enough.

Let $n \in \mathbb{N}$ large. We consider the process $X(t)$ before the first exit from the disk $\mathbb{D}(n) \equiv D(m2^n n^{1-\alpha})$. We write

$$\begin{aligned} \Lambda_d(n) &= \inf\{i \in \mathbb{N} : X_d(i) \notin \mathbb{D}(n)\}, \\ \Lambda(n) &= \inf\{t \in \mathbb{R} : X(t) \notin \mathbb{D}(n)\} \end{aligned} \tag{11}$$

for the exit times of discrete, resp. continuous, time process from $\mathbb{D}(n)$. We will often skip the dependence on n in our notation.

We use $T_\varepsilon^M(n)$ to denote the set

$$T_\varepsilon^M(n) = \left\{ x \in \mathbb{D}(n) : \frac{\varepsilon 2^{n/\alpha}}{n} \le \tau_x < \frac{M 2^{n/\alpha}}{n} \right\}. \tag{12}$$

If M or ε are omitted, it is understood $M = \infty$, resp. $\varepsilon = 0$. The constants ε and M will be chosen later. However, we always suppose that $\varepsilon \ll 1 \ll M$. We call the traps from T^ε shallow traps, T_ε^M is the set of deep traps, and T_M is the set of very deep traps. We will show that as M becomes large the probability of hitting a point in T_M before time of order $2^{n/\alpha}$ (which is the time that X typically spends in $\mathbb{D}(n)$) will be negligible, while as ε becomes small the amount of time spent by process X before time of order $2^{n/\alpha}$ in sites of T^ε will be very small.

We write $\mathcal{E}(n)$ for the set of sites that are sufficiently far from the set $T_\varepsilon^M(n)$,

$$\mathcal{E}(n) = \mathbb{D}(n) \setminus \bigcup_{y \in T_\varepsilon^M(n)} D_y(2^n n^{-\kappa}). \tag{13}$$

The constant $\kappa = \kappa(\alpha)$ can be taken arbitrarily large, but will be fixed while $n \to \infty$. The value $\kappa = 5/(1-\alpha)$ is sufficient for our purposes. The role of the set $\mathcal{E}(n)$ will be clarified later.

Further, we introduce a function $L(u)$ satisfying

$$\mathbb{P}[\tau_0 \ge u] = u^{-\alpha} L(u). \tag{14}$$

From (8) we know that $\lim_{u\to\infty} L(u) = 1$. It is also not difficult to see that L is bounded.

We write $\chi(A)$ for the indicator function of the set A. We use the letters C, c to denote positive constants that have no particular importance. The value of these constants can change during computations.

On the other hand, the letter K is reserved for constants with particular meaning.

We define now the coarse-graining of the trajectory of the process X. Let $\gamma < 1 - \alpha$. We set $j_0^n = 0$, and then we define recursively

$$j_i^n = \min\{k > j_{i-1}^n : X_d(k) \notin D_{X_d(j_{i-1}^n)}(2^n n^\gamma)\}, \tag{15}$$

with the convention that the minimum of an empty set is equal to infinity. We use x_i^n to denote the starting points of the parts of trajectory, $x_i^n = X_d(j_i^n)$. The range of X_d between the times j and k is denoted by $X_d[j,k)$, i.e. $X_d[j,k) = \{X_d(l) : j \le l < k\}$.

We will now define the score s_i^n of the part $X_d[j_i^n, j_{i+1}^n)$. Let λ_1 be the first time when X_d hits a deep trap after the start of this part,

$$\lambda_1 = \min\{k \ge j_i^n : X_d(k) \in T_\varepsilon^M\}. \tag{16}$$

Let $y = X_d(\lambda_1)$ be the first visited deep trap after time j_i^n. Further, let λ_2 be the exit time from the disk $D_y(2^n n^{-\kappa})$,

$$\lambda_2 = \min\{k > \lambda_1 : X_d(k) \notin D_y(2^n n^{-\kappa})\}. \tag{17}$$

The last time that we need is

$$\lambda_3 = \min\left(\left\{k > \lambda_1 : X_d(k) \in T_\varepsilon^M \setminus y\right\} \cup \left\{k \ge \lambda_2 : X_d(k) \in T_\varepsilon^M\right\}\right). \tag{18}$$

It is the first time after λ_1 when X_d hits a deep trap, but we do not consider the successive hits of the trap y before the time λ_2, so it is possible that $X_d(\lambda_3) = y$

If $\lambda_1 < \lambda_2 \le j_{i+1}^n \le \lambda_3$, $j_{i+1}^n \le \Lambda_d$, and y is farther than $\sqrt{\pi^{-1} 2^n n^{-\kappa}}$ from the border of $D_{x_i^n}(2^n n^\gamma)$, we define the score associated with interval $[j_i^n, j_{i+1}^n)$ by

$$s_i^n = \sum_{k=\lambda_1}^{\lambda_2} e_k \tau_y \chi(X_d(k) = y). \tag{19}$$

The last condition assures that the movement of X inside $D_y(2^n n^{-\kappa})$ is not influenced by the border of $D_{x_i^n}(2^n n^\gamma)$. If $\lambda_1 \ge j_{i+1}^n$ and $j_{i+1}^n \le \Lambda_d$, we set $s_i^n = 0$. In both previous cases the score is simply the time spent in the first visited deep trap. In all other cases we set $s_i^n = \infty$. This value has no particular meaning, it only marks the parts of trajectory where something "unusual" happens. By unusual we mean essentially that

(a) $X_d[j_i^n, j_{i+1}^n)$ contains two deep traps, and so $\lambda_3 < j_{i+1}^n$,
(b) X_d exits $\mathbb{D}(n)$ before j_{i+1}^n, and so $\Lambda_d < j_{i+1}^n$,
(c) X_d returns to the first deep trap after exiting a disk of area $2^n n^{-\kappa}$ around it, i.e. again $\lambda_3 < j_{i+1}^n$,
(d) Disk $D_y(2^n n^{-\kappa})$ intersects the complement of $D_{x_i^n}(2^n n^\gamma)$, i.e. X hits a deep trap that is too close to the border of $D_{x_i^n}(2^n n^\gamma)$.

We will study the behaviour of the trajectory of the process

$$Y^n(t) = \frac{1}{2^{n/\alpha}} \sum_{i=0}^{\lfloor tn^{1-\alpha-\gamma} \rfloor} s_i^n. \tag{20}$$

The value of this process becomes infinite if any of the possibilities from the previous paragraph happen. Therefore, we will redefine Y^n. Let $J_1(n)$ be the index of the first part of trajectory where s_i^n is infinite, $J_1(n) = \min\{i : s_i^n = \infty\}$. For technical reasons we introduce another three bad events. Let

$$J_2(n) = \min\{i : x_{i+1}^n \notin \mathcal{E}(n)\}, \tag{21}$$

that means that the end of the J_2-th part of the trajectory is too close to some deep trap. The reason why we introduce this time is that when a part of the trajectory starts too close to some deep trap, the chance of hitting this trap is large, and thus the value of the score is strongly influenced by the mean waiting time of this trap.

For similar reasons we introduce

$$J_3(n) = \min\left\{i : \operatorname{dist}(x_i^n, \mathbb{D}(n)^c) \le \sqrt{\pi^{-1} 2^n n^\gamma}\right\}, \tag{22}$$

i.e. the J_3-th part is the first part that starts too close to the border of $\mathbb{D}(n)$ and X can therefore exit from the large disk during it.

Further, let

$$J_4(n) = \min\{i : X_d[0, j_i^n) \cap T_\varepsilon^M \cap X_d[j_i^n, j_{i+1}^n) \neq \emptyset\}, \tag{23}$$

which means that X_d returns during part J_4 to some deep trap visited in previous parts of the trajectory. Let $J(n) = \min\{J_1(n), \dots, J_4(n)\}$. The value of J is the index of the first part of the trajectory where at least one of the following *bad events* happens

(i) X_d visits two different deep traps
(ii) X_d can exit $\mathbb{D}(n)$
(iii) X_d returns to some deep trap y (possibly visited in previous parts) after exiting $D_y(2^n n^{-\kappa})$
(iv) the end of this part of trajectory is too close (in the sense of (21)) to some deep trap.
(v) X_d hits a deep trap that is too close to the border of $D_{x_i^n}(2^n n^\gamma)$.

Note that (iii) includes (c) from the previous list, (ii) contains (b), and (i), (v) is same as (a), (d).

Let now $\tilde{s}_i^n$ be a suitably chosen collection of i.i.d. random variables whose distribution will be defined later (see proof of Proposition 7.1). We set

$$\bar{s}_i^n = \begin{cases} s_i^n & \text{if } i < J(n), \\ \tilde{s}_i^n & \text{otherwise.} \end{cases} \tag{24}$$

We redefine the process Y^n by

$$Y^n(t) = \frac{1}{2^{n/\alpha}} \sum_{i=0}^{\lfloor tn^{1-\alpha-\gamma} \rfloor} \bar{s}_i^n. \tag{25}$$

We want to compare this process with the well rescaled time change $S(n)$, namely with

$$\bar{S}^n(t) = \frac{1}{2^{n/\alpha}} S(j^n_{\lfloor tn^{1-\alpha-\gamma} \rfloor}). \tag{26}$$

To this end we should control several quantities. First, we should estimate the time spent in the shallow traps, that is in T^ε (Section 3). Second, we need to control the probability that X_d hits T_M before Λ, because we did not include the very deep traps into the definition of the score (Section 4). Finally, we need to be sure that the value of J is large enough, otherwise the process Y^n has no relevance for our model (Section 5).

If all these condition are satisfied, that means that Y^n is a good approximation of $\bar{S}_n$ at least at the start of the trajectory, we should study the behaviour of the sequence Y^n. We will show that it converges to a certain Lévy process (Section 6).

3. The shallow traps

As we already noted in the previous section, we want to show that the proportion of time that X spends in the shallow traps is negligible. It will be shown later that the time that X needs to leave disk $\mathbb{D}(n)$ is of order $2^{n/\alpha}$. We thus need to prove that the time spent in T^ε can be made arbitrarily small with respect to $2^{n/\alpha}$. This is the result of the following lemma, whose proof occupies the rest of this section.

Lemma 3.1. *There exists $K_1 = K_1(m)$ independent of ε such that for $\mathbb{P}$-a.e. random environment $\boldsymbol{\tau}$ and for n large enough*

$$\mathbb{E}\Big[\sum_{i=0}^{\Lambda_d - 1} e_i \tau_{X_d(i)} \chi\{X_d(i) \in T^\varepsilon\} \Big| \boldsymbol{\tau} \Big] \le K_1 \varepsilon^{1-\alpha} 2^{n/\alpha}. \tag{27}$$

Recall that Λ_d is the first time that the discrete time process X_d leaves the disk $\mathbb{D}(n)$. To prove this lemma we first describe the distribution of the shallow traps in the disk $\mathbb{D}(n)$. We divide the shallow traps into several groups. Let $i_0(n)$ be the integer satisfying $1 \le \varepsilon 2^{-i_0(n)} \frac{2^{n/\alpha}}{n} < 2$. For any $i \in \{1, \dots, i_0(n)\}$, recall that

$$T^{\varepsilon 2^{-i+1}}_{\varepsilon 2^{-i}} = \Big\{ x \in \mathbb{D}(n) : \varepsilon 2^{-i} \frac{2^{n/\alpha}}{n} \le \tau_x < \varepsilon 2^{-i+1} \frac{2^{n/\alpha}}{n} \Big\}. \tag{28}$$

Let C be a large positive constant. We use $H_1 = H_1(n, C, \varepsilon)$ to denote the event

$$H_1(n, C, \varepsilon) = \big\{ \boldsymbol{\tau} : \big| T^{\varepsilon 2^{-i+1}}_{\varepsilon 2^{-i}} \big| \le C n \varepsilon^{-\alpha} 2^{i\alpha}, \forall i \in \{1, \dots, i_0(n)\} \big\}. \tag{29}$$

We show that H_1 occurs with an overwhelming probability.

Lemma 3.2. *There exists K_2 independent of ε such that for n large enough and for some positive constants C and c.*

$$\mathbb{P}[H_1(n, K_2, \varepsilon)] \geq 1 - Cn\exp(-cn). \tag{30}$$

The proof is postponed.

Convention. At this place it is convenient to introduce one convention. Later in this paper we will need different properties of the environment that we will denote H_i, $i = 1, 2, \ldots$ For all these properties we will prove a result that allows an application of Borel-Cantelli lemma. When we prove such result we will suppose that these properties are verified. We thus may ignore a set of "unusual" environments whose probability is zero .

Proof of Lemma 3.1. The proof is divided into two parts. We first bound the time spent in "very" shallow traps: let ξ be large enough such that

$$(1-\xi)(1-\alpha) + 1 < 0. \tag{31}$$

We define the set $\mathcal{S}$ of very shallow traps by

$$\mathcal{S} = \big\{x \in \mathbb{D}(n) : \tau_x \leq 2^{n/\alpha} n^{-\xi} \ll \varepsilon 2^{n/\alpha}/n\big\}. \tag{32}$$

Let $G_{\mathbb{D}(n)}(\cdot,\cdot)$ denote the Green's function of the discrete time simple random walk in the disk $\mathbb{D}(n)$. Then we have

$$\mathbb{E}\Big[\sum_{j=0}^{\Lambda_d - 1} e_j \tau_{X_d(j)} \chi\{X_d(j) \in \mathcal{S}\}\Big|\boldsymbol{\tau}\Big] = \sum_{x \in \mathbb{D}(n)} G_{\mathbb{D}(n)}(0,x)\tau(x)\chi\{x \in \mathcal{S}\}, \tag{33}$$

The Green's function can be bounded by (see (224) in Appendix A)

$$G_{\mathbb{D}(n)}(0,x) \leq cn \quad \text{for all } x \in \mathbb{D}(n). \tag{34}$$

We thus have

$$\mathbb{E}\Big[\sum_{j=0}^{\Lambda_d - 1} e_j \tau_{X_d(j)} \chi\{X_d(j) \in \mathcal{S}\}\Big|\boldsymbol{\tau}\Big] \leq cn \sum_{x \in \mathbb{D}(n)} \tau(x)\chi\{x \in \mathcal{S}\}. \tag{35}$$

Let $i_1(n)$ be the integer satisfying

$$2^{-1+n/\alpha} n^{-\xi} \leq 2^{-i_1(n)} \varepsilon \frac{2^{n/\alpha}}{n} \leq 2^{n/\alpha} n^{-\xi}, \tag{36}$$

that is $i_1(n) \sim (\xi - 1)\log_2 n$. The expression (35) is bounded from above by

$$cn \sum_{x \in \mathbb{D}(n)} \tau(x)\chi\{\tau(x) \leq 2\} + cn \sum_{i=i_1(n)}^{i_0(n)} \sum_{x \in \mathbb{D}(n)} \tau(x)\chi\{x \in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\}. \tag{37}$$

By Lemma 3.2 and (31) this can be bounded by

$$\begin{aligned}
&\leq 2cnm2^n n^{1-\alpha} + Cn \sum_{i=i_1(n)}^{i_0(n)} \varepsilon 2^{-i+1} \frac{2^{n/\alpha}}{n} \cdot n\varepsilon^{-\alpha} 2^{i\alpha} \\
&\leq Cn\varepsilon^{1-\alpha} 2^{n/\alpha} \sum_{i=i_1(n)}^{i_0(n)} 2^{i(\alpha-1)} + o(2^{n/\alpha}) \\
&\leq C\varepsilon^{1-\alpha} 2^{n/\alpha} n^{1+(1-\xi)(1-\alpha)} + o(2^{n/\alpha}) = o(2^{n/\alpha}).
\end{aligned} \tag{38}$$

This finishes the first part.

In the second part we bound the time spent in $T^\varepsilon \setminus \mathcal{S}$. We treat separately the time spent in $T^{\varepsilon 2^{-i+1}}_{\varepsilon 2^{-i}}$ for $i \in 1, \ldots, i_1(n)$, where $i_1(n)$ is defined as above. Let K' be a large positive constant and let $A(n,i)$ be the event

$$A(n,i) = \Big\{ \sum_{x \in T^{\varepsilon 2^{-i+1}}_{\varepsilon 2^{-i}}} G_{\mathbb{D}(n)}(0,x)\tau(x) \geq K' 2^{n/\alpha} \varepsilon^{1-\alpha} 2^{-i(1-\alpha)} \Big\}. \tag{39}$$

From the definition of $T^{\varepsilon 2^{-i+1}}_{\varepsilon 2^{-i}}$ we have

$$\mathbb{P}[A(n,i)] \leq \mathbb{P}\Big[2 \sum_{x \in T^{\varepsilon 2^{-i+1}}_{\varepsilon 2^{-i}}} G_{\mathbb{D}(n)}(0,x) \geq K' n\varepsilon^{-\alpha} 2^{\alpha i}\Big]. \tag{40}$$

By Lemma 3.2, there are at most $K_2 n\varepsilon^{-\alpha} 2^{i\alpha}$ sites in $T^{\varepsilon 2^{-i+1}}_{\varepsilon 2^{-i}}$ $\mathbb{P}$-a.s. for large n. For $i = i_1(n)$ this number is of order $n^{1+\alpha(\xi-1)}$, for all others i's it is smaller.

Let y_i, $i = 1, \ldots, R_n$, be a collection of uniformly, independently chosen points in $\mathbb{D}(n)$. By an easy combinatorial argument it is possible to prove that if R_n is $o(2^{n/2} n^{(1-\alpha)/2})$, then the probability that two of them are at the same place tends to zero. Since this is evidently satisfied for the number of sites in any of $T^{\varepsilon 2^{-i+1}}_{\varepsilon 2^{-i}}$, we can bound the sum in (40) by the sum over the random collection y_i, $i = 1, \ldots, K_2 n\varepsilon^{-\alpha} 2^{i\alpha}$. For any small, positive c and for n large enough we thus have

$$\mathbb{P}[A(n,i)] \leq (1+c)\mathbb{P}\Big[2 \sum_{i=1}^{K_2 n\varepsilon^{-\alpha} 2^{i\alpha}} G_{\mathbb{D}(n)}(0,y_i) \geq K' n\varepsilon^{-\alpha} 2^{\alpha i}\Big]. \tag{41}$$

It is known that there exist constants λ and C not depending on n such that (see Lemma A.2 for proof of this claim)

$$\mathbb{E}\big[\exp\big(\lambda G_{\mathbb{D}(n)}(0,y_1)\big)\big] \leq C. \tag{42}$$

By standard argument we can thus choose K' not depending on i such that

$$\mathbb{P}[A(n,i)] \leq c\exp(-c' n\varepsilon^{-\alpha} 2^{i\alpha}). \tag{43}$$

Since $i_1(n) \ll n$, we get by summation

$$\mathbb{P}\Big[\bigcup_{i=1}^{i_1(n)} A(n,i)\Big] \le cn\exp(-c'n\varepsilon^{-\alpha}), \tag{44}$$

and thus for n large enough none of $A(n,i)$ occurs $\mathbb{P}$-a.s. However, if it is the case, we have (using also the result of the first part of the proof)

$$\begin{aligned}\mathbb{E}\Big[\sum_{j=0}^{\Lambda_d-1} e_j\tau_{X_d(j)}\chi(X_d(j)\in T^\varepsilon)\Big|\boldsymbol{\tau}\Big]\\ \le \sum_{i=0}^{i_1(n)} K'2^{n/\alpha}\varepsilon^{1-\alpha}2^{-i(1-\alpha)} + o(2^{n/\alpha}) \le K_1 2^{n/\alpha}\varepsilon^{1-\alpha}.\end{aligned} \tag{45}$$

This finishes the proof. □

It remains to show Lemma 3.2.

Proof of Lemma 3.2. We first study the size of $T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}$ for some fixed index i. The probability $p_{n,i}$ that a site in $\mathbb{D}$ is in $T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}$ is

$$p_{n,i} = \varepsilon^{-\alpha}\frac{n^\alpha}{2^n}2^{i\alpha}\Big[L\Big(\varepsilon 2^{-i}\frac{2^{n/\alpha}}{n}\Big) - \Big(\frac{1}{2}\Big)^\alpha L\Big(\varepsilon 2^{-i+1}\frac{2^{n/\alpha}}{n}\Big)\Big]. \tag{46}$$

Recall that L defined in (14) is bounded, so the expression in the brackets can be bounded from above uniformly in i by some constant depending only on the function L. Hence,

$$p_{n,i} \le c\varepsilon^{-\alpha}\frac{n^\alpha}{2^n}2^{i\alpha}. \tag{47}$$

Applying exponential Markov bound we get for $\lambda > 0$, using (47) and the fact that $(1+1/n)^n \le e$,

$$\begin{aligned}\mathbb{P}\big[\big|T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big| \ge K_2 n\varepsilon^{-\alpha}2^{i\alpha}\big] &\le \exp(-\lambda K_2 n\varepsilon^{-\alpha}2^{i\alpha})\mathbb{E}\big[\exp\big(\lambda\big|T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big|\big)\big]\\ &= \exp(-\lambda K_2 n\varepsilon^{-\alpha}2^{i\alpha})\big[(1-p_{n,i}) + p_{n,i}e^\lambda\big]^{m2^n n^{1-\alpha}}\\ &\le \exp\big[n\varepsilon^{-\alpha}2^{i\alpha}(-K_2\lambda + mce^\lambda)\big].\end{aligned} \tag{48}$$

If K_2 is chosen large enough, the expression in the parentheses is negative and thus the required probability decreases exponentially. The probability of H_1^c satisfies

$$\begin{aligned}\mathbb{P}[H_1^c] = \mathbb{P}\Big[\bigcup_{i=1}^{i_0(n)}\big(\big|T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}\big| \ge K_2 n\varepsilon^{-\alpha}2^{i\alpha}\big)\Big]\\ \le \sum_{i=1}^{i_0(n)}\exp\big\{n\varepsilon^{-\alpha}2^{i\alpha}(-K_2\lambda + mce^\lambda)\big\}\\ \le i_0(n)\exp\big\{n\varepsilon^{-\alpha}(-K_2\lambda + mce^\lambda)\big\}.\end{aligned} \tag{49}$$

Since $i_0(n) \le n/\alpha$, the proof is finished. $\square$

4. Very deep traps

In this section we estimate the probability of hitting a very deep trap. The aim is to show that these sites may be neglected from the analysis.

Lemma 4.1. *For every $\delta > 0$ and m there exists M such that for n large enough and for $\mathbb{P}$-a.e. environment $\boldsymbol{\tau}$*

$$\mathbb{P}\big[X(t) \text{ hits } T_M(n) \text{ before } \Lambda(n)|\boldsymbol{\tau}\big] \le \delta. \tag{50}$$

Proof. The standard large deviation argument gives

$$\mathbb{P}[|T_M(n)| > Cnm/M^\alpha] \le C' \exp(-cnm/M^\alpha) \tag{51}$$

for some constants C, C' and c. We can thus take $\mathbb{P}$-a.s. n large enough such that $|T_M(n)| \le Cnm/M^\alpha$. Let A be an uniformly chosen random subset of $\mathbb{D}(n)$ with Cnm/M^α elements. Then

$$\begin{aligned}\mathbb{P}\big[\mathbb{P}[X \text{ hits } T_M \text{ before } \Lambda|\boldsymbol{\tau}] > \delta\big] \\ \le \mathbb{P}\big[\mathbb{P}[X \text{ hits } A \text{ before } \Lambda|A] > \delta\big].\end{aligned} \tag{52}$$

Further, let $\{y_i\}$, $i = 1, \dots, Cnm/M^\alpha$ be a collection of independently, uniformly chosen random points in $\mathbb{D}(n)$. As in the previous section we can replace A by this collection. The expression (52) is then bounded by

$$\le (1+c)\mathbb{P}\Big[\sum_{i=1}^{Cnm/M^\alpha} \mathbb{P}[X \text{ hits } y_i \text{ before } \Lambda|y_i] \ge \delta\Big] \tag{53}$$

for some small positive c. Since the terms in the sumation are independent, we can bound the last expression, using again the exponential Markov inequality, by

$$\le (1+c)\exp(-\delta\lambda_n)\mathbb{E}\big[\exp\big(\lambda_n \mathbb{P}[X \text{ hits } y_i \text{ before } \Lambda|y_i]\big)\big]^{Cnm/M^\alpha}. \tag{54}$$

The inequality (226) from Appendix A applied on the disk $\mathbb{D}(n)$ gives

$$\mathbb{E}\big[\exp\big(-(n\log 2/2 + o(n))\mathbb{P}[X \text{ hits } y_1 \text{ before } \Lambda]\big)\big] \le C. \tag{55}$$

Therefore, taking $\lambda_n = b\log\sqrt{\pi^{-1}m2^n n^{1-\alpha}}$, $b < 1$

$$\mathbb{P}\big[\mathbb{P}[X \text{ hits } T_M \text{ before } \Lambda|\boldsymbol{\tau}] > \delta\big] \le \exp\big\{-\delta cn + c'mn/M^\alpha + o(n)\big\}. \tag{56}$$

The lemma then follows by taking M large enough and applying Borel-Cantelli argument. $\square$

5. J IS LARGE ENOUGH

To justify the approximation of $\bar{S}^n$ by Y^n we should now prove that the index of the first bad part, J, is large enough. More precisely, we should show that one can choose κ and m such that, with large probability, the index J of the first bad part of the trajectory of X is sufficiently large for our purposes.

Lemma 5.1. *For any δ, k, and $\mathbb{P}$-a.e. $\boldsymbol{\tau}$ there exist m and κ not depending on ε and M such that for n large enough*

$$\mathbb{P}\big[J(n)n^{\alpha+\gamma-1} \geq k|\boldsymbol{\tau}\big] \geq 1-\delta. \tag{57}$$

To prove this lemma we should verify that all events described in Section 2 happen with low probability. This is the goal of all following technical lemmas. The proof of Lemma 5.1 can be found at the end of this section.

Event (i). The most complicated part of the proof is to show that X does not hit two deep traps during one part of the trajectory. The following lemma is a little bit more precise than is needed to bound J, however, we will need this more precise result later. We use p_ε^M to denote the factor $\varepsilon^{-\alpha} - M^{-\alpha}$.

Lemma 5.2. *Let*

$$V_{x_0}(n) = \sum_{y\in T_\varepsilon^M} \mathbb{P}_{x_0}\big[X_d \text{ hits } y \text{ before exiting } D_{x_0}(2^n n^\gamma)|\boldsymbol{\tau}\big], \tag{58}$$

where $\mathbb{P}_{x_0}$ denotes the law of the simple random walk X_d started at x_0. Then for any δ and $\mathbb{P}$-a.e. $\boldsymbol{\tau}$ there exists n_0 such that for all $n > n_0$ and $x_0 \in \mathcal{E}(n)$ (see (13) for definition of $\mathcal{E}(n)$),

$$\frac{\mathcal{K}(1-\delta)p_\varepsilon^M}{n^{1-\alpha-\gamma}} \leq V_{x_0}(n) \leq \frac{\mathcal{K}(1+\delta)p_\varepsilon^M}{n^{1-\alpha-\gamma}} \tag{59}$$

with $\mathcal{K} = (\log 2)^{-1}$.

To prove this lemma we should describe the distribution of the deep traps inside $\mathbb{D}(n)$. This description is contained in Lemmas 5.3 and 5.4.

First, we will show that the deep traps are distributed almost homogeneously around the disk. Let $\nu < \gamma < 1-\alpha$ and let $H_2 = H_2(n,\delta,\varepsilon,M)$ be the set of configurations of the environment satisfying the "homogeneity" condition:

$$\begin{aligned} H_2 = \big\{\boldsymbol{\tau}\colon \big|T_\varepsilon^M \cap B_x(2^n n^\nu)\big| \in \big[(1-\delta)p_\varepsilon^M n^{\nu+\alpha}, (1+\delta)p_\varepsilon^M n^{\nu+\alpha}\big] \\ \text{for all } x \text{ such that } B_x(2^n n^\nu) \subset \mathbb{D}(n).\big\} \end{aligned} \tag{60}$$

Lemma 5.3. *For any ε, M, and δ there exist positive constants C and c such that for n large enough*

$$\mathbb{P}[H_2] \geq 1 - Cn^{1-\alpha-\nu}\delta^{-2}\exp(-cn^{\nu+\alpha}). \tag{61}$$

Proof. We divide the complement of H_2 into two parts. First, we treat the case when there is a region in $\mathbb{D}$ where there are not enough deep traps. Let A be the event that there is a square of area $2^n n^\nu$ in $\mathbb{D}(n)$ where there are less than $(1-\delta)p_\varepsilon^M n^{\nu+\alpha}$ sites from $T_\varepsilon^M(n)$,

$$A = \{\exists x \in \mathbb{D} : \left|T_\varepsilon^M \cup B_x(2^n n^\nu)\right| < (1-\delta)p_\varepsilon^M n^{\nu+\alpha}, D_x(2^n n^\nu) \subset \mathbb{D}\}. \tag{62}$$

We use G to denote the grid $\lfloor 2^{n/2} n^{\nu/2}\delta/5\rfloor \mathbb{Z}^2$. Every square of area $2^n n^\nu$ contains at least one square of area $2^n n^\nu(1-\delta/2)$ with the centre in G for n sufficiently large. Hence, if A is true, then there is a square of area $2^n n^\nu(1-\delta/2)$ which has centre $x \in G$, and which contains less than $(1-\delta)p_\varepsilon^M n^{\nu+\alpha}$ sites. We use A_x to denote the last event. We have

$$\mathbb{P}[A] \le \sum_x \mathbb{P}[A_x] = C'\delta^{-2} n^{1-\alpha-\nu}\mathbb{P}[A_x], \tag{63}$$

where the sum runs over all $x \in G$ such that $B_x((1-\delta/2)2^n n^\nu) \subset \mathbb{D}$. We used the obvious fact that $\mathbb{P}[A_x]$ does not depend on x. The probability of A_x can be bounded using standard methods. Take $\eta > 0$. For n large enough, the probability p that a site is in $T_\varepsilon^M(n)$ is larger than $(1-\eta)p_\varepsilon^M 2^{-n} n^\alpha$. For $\lambda > 0$ we have

$$\begin{aligned}\mathbb{P}[A_x] &\le \exp(\lambda(1-\delta)n^{\nu+\alpha}p_\varepsilon^M)\big[(1-p)+e^{-\lambda}p\big]^{2^n n^\nu(1-\delta/2)} \\ &\le \exp(\lambda(1-\delta)n^{\nu+\alpha}p_\varepsilon^M)\Big[1+(e^{-\lambda}-1)\frac{(1-\eta)n^\alpha p_\varepsilon^M}{2^n}\Big]^{2^n n^\nu(1-\frac{\delta}{2})}.\end{aligned} \tag{64}$$

If n is large enough, the last expression is bounded by

$$\mathbb{P}[A_x] \le \exp\left[n^{\nu+\alpha}p_\varepsilon^M\big(\lambda(1-\delta)+(e^{-\lambda}-1)(1-\eta)^2(1-\delta/2)\big)\right]. \tag{65}$$

It is not difficult to show that for any δ there exist η and λ such that the exponent is negative. Hence, we have

$$\mathbb{P}[A] \le C' n^{1-\alpha-\nu}\delta^{-2}\exp(-c' n^{\nu+\alpha}). \tag{66}$$

In the second part of the proof we exclude the possibility that there are places in $\mathbb{D}$ where the deep traps are too dense. Let B be the event that there is a square of area $2^n n^\nu$ intersecting $\mathbb{D}(n)$ where is more than $(1+\delta)\varepsilon^{-\alpha}n^{\nu+\alpha}$ sites from $T_\varepsilon^M(n)$. The probability of B can be bounded exactly in the same way as the probability of A, one should only consider the squares with area $2^n n^\nu(1+\delta/2)$ and centres in G. We thus have

$$\mathbb{P}[H_2(n)^c] \le \mathbb{P}[A \cup B] \le C n^{1-\alpha-\nu}\delta^{-2}\exp(-c n^{\nu+\alpha}). \tag{67}$$

This finishes the proof. □

The lemma we have just proved is not precise enough to bound the probability of hitting traps that are closer than $\sqrt{2^n n^\nu}$ to the starting point. The following lemma will serve us for that bound. Again, it describes some sort of homogeneity of the environment

We consider the events $H_3(i) = H_3(i, n, \varepsilon, M)$,

$$H_3(i) = \big\{\exists x \in \mathbb{D}(n) : \big|B_x(2^{n+i}n^{-\kappa}) \cap T_\varepsilon^M\big| \le 4\log^2 n(1 \vee 2^i n^{\alpha-\kappa}\varepsilon^{-\alpha})\big\}, \tag{68}$$

where $a \vee b$ denotes the maximum of a, b. We define H_3 by

$$H_3 = \bigcap_{i=-1}^{\infty} H_3(i). \tag{69}$$

Observe that $2^{n+i}n^{-\kappa} \ll 2^n n^\nu$ for fixed i and n large enough. So, we study here much smaller squares than in the previous lemma. Hence, the description of the homogeneity is more precise in this direction. On the other hand, we prove only the upper bound on the number of the deep traps in these squares and this bound is also "weaker" than the previous bound.

Lemma 5.4. *There exists a constant C such that*

$$\mathbb{P}[H_3] \ge 1 - Cn^{-3}. \tag{70}$$

Proof. Fix some i and consider the lattice $G_i = \mathbb{Z}^2\sqrt{2^{n+i}n^{-\kappa}}$. If there is x such that $|B_x(2^{n+i}n^{-\kappa}) \cap T_\varepsilon^M| \ge 4\log^2 n(1 \vee 2^i n^{\alpha-\kappa}\varepsilon^{-\alpha})$, then there is a point $y \in G_i$ such that $B_y(4 \cdot 2^{n+i}n^{-\kappa})$ contains more than $4\log^2 n(1 \vee 2^i n^{\alpha-\kappa}\varepsilon^{-\alpha})$ sites from T_ε^M. The number of squares with area $4 \cdot 2^{n+i}n^{-\kappa}$ and centres in G_i that intersect $\mathbb{D}(n)$ is bounded by $Cn^{1-\alpha+\kappa}2^{-i}$.

Consider now one such square. The probability that it contains too many sites from T_ε^M can be bounded by standard arguments

$$\begin{aligned}&\mathbb{P}\big[|B(4 \cdot 2^{n+i}n^{-\kappa}) \cap T_\varepsilon^M| \ge 4\log^2 n(1 \vee 2^i n^{\alpha-\kappa}\varepsilon^{-\alpha})\big] \le \\ &\quad c\exp\big(-\lambda 4\log^2 n(1 \vee 2^i n^{\alpha-\kappa}\varepsilon^{-\alpha}) + 4p_\varepsilon^M(e^\lambda - 1)2^i n^{\alpha-\kappa}\varepsilon^{-\alpha}\big).\end{aligned} \tag{71}$$

Since $\alpha - \kappa < 0$, we can choose λ such that for n large enough the last expression is bounded by $(1/2)^{\log^2 n}$. Summation over i and over all squares that intersect $\mathbb{D}(n)$ gives us

$$\mathbb{P}[H_6^c] \le \sum_{i=-1}^{\infty} C2^{-i}n^{1-\alpha+\kappa}(1/2)^{\log^2 n} \le Cn^{-3}. \tag{72}$$

□

We now have all ingredients to prove Lemma 5.2.

Proof of Lemma 5.2. We can suppose that x_0 is the origin. We use ξ to denote the exit time from $D(2^n n^\gamma)$. Let γ' be a constant satisfying $\nu < \gamma' < \gamma$. We divide the sum $V_0(n)$ into two parts. First, we sum over all deep traps that are far enough from the origin. Precisely, we consider the deep traps that are in $D(2^n n^\gamma) \setminus D(2^n n^{\gamma'})$. Let I_1 denotes the sum over such traps. We use I_2 to denote the sum over the remaining deep traps.

To show the upper bound on I_1, we cover the set $D(2^n n^\gamma) \setminus D(2^n n^{\gamma'})$ by squares of area $2^n n^\nu$ and centres in $\sqrt{2^n n^\nu}\mathbb{Z}^2$. Let $x_1, \dots, x_R$ denote the set of centres of such squares that intersect $D(2^n n^\gamma) \setminus D(2^n n^{\gamma'})$. Since $\nu < \gamma'$, the size of each such square is negligible with respect to its distance to the origin. All deep traps in such squares have thus almost the same chance to be hit. We use expression (223) from Appendix A to estimate probability that X hits some point before exiting from $D(2^n n^\gamma)$. Let r_n be the radius of this disk, $r_n = \sqrt{\pi^{-1} 2^n n^\gamma}$.

$$I_1 \le \sum_{i=1}^{R} \sum_{\substack{y_j \in B_{x_i}(2^n n^\nu) \\ y_j \in T_\varepsilon^M}} \Big(1 - \frac{\log |y_j|}{\log r_n} + O(n^{-2})\Big)$$

$$= \sum_{i=1}^{R} \big|B_{x_i}(2^n n^\nu) \cap T_\varepsilon^M\big| \Big(1 - \frac{\log |x_i|}{\log r_n} + O(n^{-1+(\nu-\gamma')/2})\Big), \quad (73)$$

where we use the estimate

$$\frac{\log |y_j|}{\log r_n} - \frac{\log |x_i|}{\log r_n} = O(n^{-1+(\nu-\gamma')/2}) \quad (74)$$

that is valid for any $y_j \in B_{x_i}(2^n n^\nu)$.

From Lemma 5.3 we know that for n large enough $|B_{x_i}(2^n n^\nu) \cap T_\varepsilon^M| \le n^{\nu+\alpha} p_\varepsilon^M (1 + \delta/2)$ and thus

$$I_1 \le \sum_{i=1}^{R} n^{\nu+\alpha} p_\varepsilon^M (1 + \delta/2) \Big(1 - \frac{\log |x_i|}{\log r_n} + O(n^{-1+(\nu-\gamma')/2})\Big). \quad (75)$$

We now replace the summation by integration making again an error of order $O(n^{-1+(\nu-\gamma')/2})$. I_1 is thus bounded from above by

$$\int_{D(2^n n^\gamma) \setminus D(2^n n^{\gamma'})} \frac{n^{\nu+\alpha} p_\varepsilon^M}{2^n n^\nu} \Big(1 + \frac{\delta}{2}\Big) \Big(1 - \frac{\log |x|}{\log r_n} + O(n^{-1+(\nu-\gamma')/2})\Big)\, dx. \quad (76)$$

The integration gives

$$I_1 \le \frac{n^{\alpha+\gamma-1} p_\varepsilon^M}{\log 2} \Big(1 + \frac{\delta}{2}\Big)(1 + o(1)) \le \frac{n^{\alpha+\gamma-1} p_\varepsilon^M}{\log 2} \Big(1 + \frac{3\delta}{4}\Big) \quad (77)$$

for n large enough. This finishes the proof of the upper bound for I_1. The proof of the lower bound is analogous. After a very similar calculation we get

$$I_1 \ge \frac{n^{\alpha+\gamma-1} p_\varepsilon^M}{\log 2} \Big(1 - \frac{3\delta}{4}\Big). \quad (78)$$

We should now estimate the sum I_2 over all sites $x \in T_\varepsilon^M \cap (D(2^n n^{\gamma'}) \setminus D(2^n n^{-\kappa}))$. The disk $D(2^n n^{-\kappa})$ can be excluded since by the assumptions of the lemma $x \in \mathcal{E}(n)$ and so there are no deep traps in this disk. We cover the domain by objects comprising eight squares of area $2^{n+i} n^{-\kappa}$ whose union is the square, centred at the origin, of nine times

larger area with the middle square cut off. The parameter i takes values in the set $\{-1, 0, 1, \dots, (\gamma' + \kappa)\log_2 n\}$. We use this covering because if the trap is too close to origin, we should know more precisely its position to estimate its hitting probability. Our covering becomes clearly finer when the origin is approached.

Any point inside the i-th object from the previous paragraph has distance from origin at least $\sqrt{2^{n+i}n^{-\kappa}}/2$. In each of the eight squares there is, by Lemma 5.4, at most $4\log^2 n(1 \vee 2^i n^{\alpha-\kappa}\varepsilon^{-\alpha})$ sites from T_ε^M. By formula (223) for the hitting probability of a point in $\mathbb{D}(2^n n^\gamma)$ we have

$$I_2 \le 8 \sum_{i=-1}^{(\gamma'+\kappa)\log_2 n} \left[1 - \frac{\log(\sqrt{2^{n+i}n^{-\kappa}}/2)}{\log r_n} + O\Big(\frac{2^{-n-i}n^\kappa}{\log r_n}\Big) + O(\log^{-2} r_n)\right] \cdot 4\log^2 n(1 \vee 2^i n^{\alpha-\kappa}\varepsilon^{-\alpha}). \tag{79}$$

The expression in the brackets can be easily bounded by $Cn^{-1}\log n$ with some large constant C. Hence,

$$I_2 \le C \sum_{i=-1}^{(\gamma'+\kappa)\log_2 n} \frac{\log n}{n} \log^2 n(1 \vee 2^i n^{\alpha-\kappa}\varepsilon^{-\alpha}). \tag{80}$$

Since the expression inside of the summation is increasing in i, the last display can be trivially estimated by $(\gamma' + \kappa)\log_2 n$ times the last term. This gives

$$I_2 \le Cn^{\alpha+\gamma'-1}\log^4 n \ll \frac{n^{\alpha+\gamma-1}p_\varepsilon^M}{\log 2}\Big(1 + \frac{\delta}{4}\Big). \tag{81}$$

Putting together (77), (78), and (81) we get

$$\frac{n^{\alpha+\gamma-1}p_\varepsilon^M}{\log 2}(1-\delta) \le I_1 \le V_0(n) = I_1 + I_2 \le \frac{n^{\alpha+\gamma-1}p_\varepsilon^M}{\log 2}(1+\delta). \tag{82}$$

This finishes the proof of Lemma 5.2. □

Using exactly the same approach as above and Lemma 5.6 below we show

Lemma 5.5. *For $x \in T_\varepsilon^M$, let us redefine*

$$V_x(n) = \sum_{y \in T_\varepsilon^M \setminus \{x\}} \mathbb{P}_x\big[X_d \text{ hits } y \text{ before exiting } D_x(2^{n+1}n^\gamma) | \boldsymbol{\tau}\big], \tag{83}$$

where $\mathbb{P}_x$ denotes the law of the simple random walk X_d started at x. Then for any δ and $\mathbb{P}$-a.e. $\boldsymbol{\tau}$ there exists n_0 such that for all $n > n_0$ and all $x \in T_\varepsilon^M$,

$$V_x(n) \le \frac{Cp_\varepsilon^M}{n^{1-\alpha-\gamma}}. \tag{84}$$

Let $H_4 = H_4(n,\varepsilon)$ be the event

$$H_4(n,\varepsilon) = \left\{\boldsymbol{\tau} : \min\{|x-y| : x,y \in T_\varepsilon(n)\} \geq 2\sqrt{\pi^{-1}2^n n^{-\kappa}}\right\}. \tag{85}$$

The constant 2 before the square root is not necessary for the current application, but it will be used later.

Lemma 5.6. *There exists constant $C = C(\varepsilon, m)$ such that*

$$\mathbb{P}[H_4] \geq 1 - Cn^{1+\alpha-\kappa}. \tag{86}$$

Proof. Let $B(x)$ be the event

$$B(x) = \left\{x \in T_\varepsilon(n)\right\} \cap \left\{\exists y \in T_\varepsilon(n), |y-x| \leq 2\sqrt{\pi^{-1}2^n n^{-\kappa}}\right\}. \tag{87}$$

Then

$$\mathbb{P}[B(x)] \leq C\frac{n^{2\alpha-\kappa}}{2^n}\varepsilon^{-2\alpha}. \tag{88}$$

and the result follows by summation over all $x \in \mathbb{D}(n)$. □

The following lemma is an easy consequence of Lemma 5.2. It is the actual estimate of the probability of hitting a deep trap.

Lemma 5.7. *For any $\delta > 0$ and $\mathbb{P}$-a.e. $\boldsymbol{\tau}$, there exists n_0 such that for $n > n_0$ and for all $x \in \mathcal{E}(n)$, the probability that the simple random walk started at x hits exactly one site from $T_\varepsilon^M(n)$ before exiting $D_x(2^n n^\gamma)$ is in interval*

$$\left(\mathcal{K}(1-\delta)p_\varepsilon^M n^{\alpha+\gamma-1}, \mathcal{K}(1+\delta)p_\varepsilon^M n^{\alpha+\gamma-1}\right). \tag{89}$$

The probability that it hits more than one deep trap is bounded by

$$\mathbb{P}[X \textit{ hits at least two sites from } T_\varepsilon^M] \leq Cn^{2(\alpha+\gamma-1)}(p_\varepsilon^M)^2 \tag{90}$$

for some positive constant C.

Proof. Let $T_\varepsilon^M \cap D_x(2^n n^\gamma) = \{x_1, \ldots, x_L\}$. Assume that some point x_i was hit by X before the exit from $D(2^n n^\gamma)$.

We apply now Lemma 5.5 and the Strong Markov property. We thus have

$$\sum_{j\neq i} \mathbb{P}[X \text{ hits } x_j | X \text{ hit } x_i] \leq Cn^{\alpha+\gamma-1}p_\varepsilon^M. \tag{91}$$

The Bonferroni inequalities give

$$\begin{aligned}
\mathbb{P}[X \text{ hits } T_\varepsilon^M] &\leq \sum_i \mathbb{P}[X \text{ hits } x_i] \leq \mathcal{K}(1+\delta)p_\varepsilon^M n^{\alpha+\gamma-1} \\
\mathbb{P}[X \text{ hits } T_\varepsilon^M] &\geq \sum_i \mathbb{P}[X \text{ hits } x_i] - \frac{1}{2}\sum_i\sum_{j\neq i}\mathbb{P}[X \text{ hits } x_i \text{ and } x_j] \\
&\geq \mathcal{K}(1-\delta)p_\varepsilon^M n^{\alpha+\gamma-1} - C(p_\varepsilon^M)^2 n^{2(\alpha+\gamma-1)} \geq \mathcal{K}(1-2\delta)p_\varepsilon^M n^{\alpha+\gamma-1}
\end{aligned} \tag{92}$$

for n large enough. Similarly we get from the Strong Markov property and Lemma 5.5

$$\mathbb{P}[X \text{ hits at least two points from } T_\varepsilon^M] \leq C(p_\varepsilon^M)^2 n^{2(\alpha+\gamma-1)}. \tag{93}$$

This finishes the proof of Lemma 5.7. □

Event (iv). To find a lower bound for J, we should further verify that the probability that a part of the trajectory ends too close to some deep trap is small.

Lemma 5.8. *For $\mathbb{P}$-a.e. $\boldsymbol{\tau}$, the probability that the simple random walk started at arbitrary $x \in \mathbb{D}(n)$ exits $D_x(2^n n^\gamma)$ at some point that is in $\mathbb{D}(n) \setminus \mathcal{E}(n)$ is smaller than $Cn^{2-\kappa/2-\gamma/2}$.*

Proof. We start again with the description of the properties of the environment. Let r_n be the radius of the disk $D(2^n n^\gamma)$. We use $A_x(2^n n^\gamma)$ to denote the annular ring with the centre x, the inner radius $r_n - \sqrt{\pi^{-1} 2^n n^{-\kappa}}$, and the outer radius $r_n + \sqrt{\pi^{-1} 2^n n^{-\kappa}}$. Let $H_5 = H_5(n, \varepsilon, M)$ be the event

$$H_5 = \left\{ \boldsymbol{\tau} : |T_\varepsilon^M(n) \cap A_x(2^n n^\gamma)| \leq n^2 \text{ for all } x \in \mathbb{D}(n) \right\}. \tag{94}$$

Lemma 5.9. *For n large there exist constants C and c such that*

$$\mathbb{P}[H_5] \geq 1 - C 2^n n^{1-\alpha} \exp(-cn^2). \tag{95}$$

Proof. There are less than $C 2^n n^{\gamma/2-\kappa/2}$ points in the annulus $A_x(2^n n^\gamma)$. The probability that a trap is in $T_\varepsilon^M(n)$ is of order $p_\varepsilon^M 2^n n^{-\alpha}$. The standard application of Markov inequality gives

$$\mathbb{P}\left[|A_x(2^n n^\nu) \cap T_\varepsilon^M(n)| > n^2 \right] \leq \exp(-c(\varepsilon, M) n^2). \tag{96}$$

The result follows by summation over all $x \in \mathbb{D}(n)$. □

We can now finish the proof of Lemma 5.8. We use the fact that probability of exiting the disk of radius R in a particular point at its border is $O(1/R)$ (see [Law91] Lemma 1.7.4). From Lemma 5.9 we know that there are less than n^2 deep traps in annulus $A_x(2^n n^\gamma)$. This implies that there are at most $cn^2 \sqrt{2^n n^{-\kappa}}$ points on the border of $D_x(2^n n^\gamma)$ that are close to some deep trap. The required probability is thus bounded from above by

$$C \sqrt{2^{-n} n^{-\gamma}} n^2 \sqrt{2^n n^{-\kappa}} = C n^{2-\kappa/2-\gamma/2}. \tag{97}$$

□

Event (v). The next lemma excludes the possibility of hitting a deep trap that is too close to the border of the disk with area $2^n n^\gamma$ around the starting point.

Lemma 5.10. *For any $x \in \mathbb{D}$, the probability that the random walk started at x hits a deep trap in $A_x(2^n n^\gamma)$ before the exit from $D_x(2^n n^\gamma)$ is smaller than $Cn^{2-\gamma/2-\kappa/2}$.*

Proof. We need to estimate the probability that we hit some point y that is in the distance smaller than $\sqrt{\pi^{-1} 2^n n^{-\kappa}}$ from the border of $D_x(2^n n^\gamma)$. We use (224) to estimate this probability. The advantage

of (224) against (223) is that the error terms are much smaller. Since for any disk D centred at x

$$G_D(x,y) = \mathbb{P}_x[X \text{ hits } y \text{ before exit from } D]G_D(y,y) \tag{98}$$

and $G_D(y,y) \geq 1$, we know that $\mathbb{P}_x(X \text{ hits } y) \leq G_D(x,y)$. According to Lemma 5.9 there are at most n^2 deep traps in $A_x(2^n n^\gamma)$. We thus have

$$\begin{aligned}
&\mathbb{P}_x[X \text{ hits } T_\varepsilon^M \cap A_x(2^n n^\gamma) \text{ before exiting } D_x(2^n n^\gamma)] \\
&\leq \frac{2n^2}{\pi}\Big[\log\sqrt{\pi^{-1}2^n n^\gamma} - \log\big(\sqrt{\pi^{-1}2^n n^\gamma}(1-n^{-\gamma/2-\kappa/2})\big) + O(2^{-n/2})\Big] \\
&\leq -cn^2\log(1-n^{-\gamma/2-\kappa/2}) \leq Cn^{2-\gamma/2-\kappa/2}.
\end{aligned} \tag{99}$$

This finishes the proof. □

Event (iii). Finally, we need to show that X almost never returns to a deep trap after exiting a disk of area $2^n n^{-\kappa}$ around it. We do not need to consider the traps that are closer than $\sqrt{\pi^{-1}2^n n^{-\kappa}}$ to the border of $\mathbb{D}$ because hitting such traps has already been dealt with when considering (ii) and (v) defining the "bad" event.

Lemma 5.11. *There exists a constant C such that for any x satisfying $D_x(2^n n^{-\kappa}) \cap \mathbb{D}(n)^c = \emptyset$, the probability that X returns to x before Λ after exiting disk $D_x(2^n n^{-\kappa})$ is smaller than $Cn^{-1}\log n$.*

Proof. Let p_{ret} denotes the required probability and let ξ be the first time when X exits $D_x(2^n n^{-\kappa})$. Obviously, $\xi < \Lambda$. By the Markov property

$$\begin{aligned}
G_{\mathbb{D}}(x,x) &= \sum_{i=0}^{\Lambda}\mathbb{P}_x[X_d(i)=x] = \sum_{i=0}^{\xi}\mathbb{P}_x[X_d(i)=x] + \sum_{i=\xi+1}^{\Lambda}\mathbb{P}_x[X_d(i)=x] \\
&= G_{D(2^n n^{-\kappa})}(0,0) + p_{\text{ret}}G_{\mathbb{D}}(x,x).
\end{aligned} \tag{100}$$

Hence,

$$p_{\text{ret}} = 1 - \frac{G_{D(2^n n^{-\kappa})}(0,0)}{G_{\mathbb{D}}(x,x)} \leq 1 - \frac{G_{D(2^n n^{-\kappa})}(0,0)}{G_{2\mathbb{D}}(0,0)}, \tag{101}$$

where $2\mathbb{D}$ denotes the disk with centre the origin and twice the radius of $\mathbb{D}$. Using the expression (225) we get

$$p_{\text{ret}} \leq 1 - \frac{\log(2^n n^{-\kappa}) + O(1)}{\log(2\cdot 2^n n^{1-\alpha}) + O(1)} \leq Cn^{-1}\log n. \tag{102}$$

This finishes the proof. □

Proof of Lemma 5.1. We have now all ingredients to prove Lemma 5.1. We should prove that the probability that some of the events (i)–(v) from Section 2 happen during first $Cn^{1-\alpha-\gamma}$ parts can be made very

small. We will use $J_{(i)}, \ldots, J_{(v)}$ to denote the first part where (i), ..., resp. (v) occurs.

The simplest condition is (ii). This condition requires that X cannot exit $\mathbb{D}$ during the good part of the trajectory. That means that starting point of a part of the trajectory satisfying (ii) should be in the annular ring with the outer radius $\sqrt{\pi^{-1}m2^n n^{1-\alpha}}$ (which is the radius of $\mathbb{D}$) and the inner radius $\sqrt{\pi^{-1}m2^n n^{1-\alpha}} - \sqrt{\pi^{-1}2^n n^{\gamma}}$. The sequence of starting points x_i^n is a random walk on $\mathbb{Z}^2$. It follows from the invariance principle for random walks that the law of $J_{(ii)}n^{\alpha+\gamma-1}m^{-1/2}$ converges as $n \to \infty$ to the exit time for a standard two dimensional Brownian motion from the unit disk, having started at the origin. In particular this distribution does not put mass at the value 0 and does not depend on m.

It is thus possible to fix m large enough such that

$$\mathbb{P}[J_{(ii)}n^{\alpha+\gamma-1} \geq k|\boldsymbol{\tau}] \geq 1 - \delta/4. \tag{103}$$

From the same reason we can choose $K > k$ such that

$$\mathbb{P}[J_{(ii)}n^{\alpha+\gamma-1} \leq K|\boldsymbol{\tau}] \geq 1 - \delta/4. \tag{104}$$

Hence, outside a set of probability $\delta/2$ the number of parts before $J_{(ii)}$ is in interval $(kn^{1-\alpha-\gamma}, Kn^{1-\alpha-\gamma})$. We use A to denote this event.

Conditionally on A, we will show that

$$\mathbb{P}\big[\min(J_{(i)}, J_{(iii)}, J_{(iv)}, J_{(v)}) \leq J_{(ii)}\big|\boldsymbol{\tau}, A\big] \to 0 \quad \text{as} \quad n \to \infty. \tag{105}$$

The claim of the lemma is then an easy consequence of this fact and the previous paragraph. Observe that (105) means that in the majority of cases the first bad event that happens is the possibility of exit from $\mathbb{D}$. The probability of all other events is negligible.

We start with condition (iv). According to it, the part is bad if its end is not in $\mathcal{E}(n)$. Lemma 5.8 states that the probability that this happens during a particular part of trajectory is of order $n^{2-\kappa/2-\gamma/2}$. Since the number of parts before $J_{(ii)}$ is bounded by $Kn^{1-\alpha-\gamma}$, the probability that (iv) happens is bounded by $Kn^{3-\alpha-\gamma/2-\kappa/2}$. However, κ can be chosen large enough to assure that this bound converges to 0. We thus have

$$\mathbb{P}[J_{(iv)} < J_{(ii)}|\boldsymbol{\tau}, A] \to 0. \tag{106}$$

Using a very similar reasoning and Lemma 5.10 we get exactly the same estimate for condition (v). Hence,

$$\mathbb{P}[J_{(v)} < J_{(ii)}|\boldsymbol{\tau}, A] \to 0. \tag{107}$$

Condition (i) requires that X does not visit two deep traps during one part of the trajectory. We use B to denote the event $A \cap \{J_{(iv)} > J_{(ii)}\}$. We show

$$\mathbb{P}[J_{(i)} < J_{(ii)}|B, \boldsymbol{\tau}] \to 0. \tag{108}$$

Since we assume that $J_{(iv)} \geq J_{(ii)}$, we can apply Lemma 5.7. It claims that probability of hitting two deep traps during one part is of order $n^{2(\alpha+\gamma-1)}$. By the same argument as before we can bound the probability in (108) by $Kn^{\alpha+\gamma-1}$ and it tends to 0 as $n \to \infty$.

The last condition (iii) demands that X does not return to a deep trap after exiting the disk of area $2^n n^{-\kappa}$ around it. For one particular trap probability of such event can be bounded by $cn^{-1}\log n$ by Lemma 5.11. According to Lemma 5.2, the probability of visiting a deep trap during one part of the trajectory is of order $n^{\alpha+\gamma-1}$. Let N denotes the number of visited deep traps before Λ. Conditionally on B, it is not difficult to show using Markov inequality that

$$\mathbb{P}[N \geq n^{1/2}|B, \boldsymbol{\tau}] \leq Cn^{-1/2}. \tag{109}$$

We have thus

$$\begin{aligned}
&\mathbb{P}[J_{(iii)} < J_{(ii)}|B, \boldsymbol{\tau}] \\
&\quad \leq \mathbb{P}[J_{(iii)} < J_{(ii)}|B, \boldsymbol{\tau}, N \leq n^{1/2}]\mathbb{P}[N \leq n^{1/2}|B, \boldsymbol{\tau}] + \mathbb{P}[N \geq n^{1/2}|B, \boldsymbol{\tau}] \\
&\quad \leq cn^{-1/2}\log n + Cn^{-1/2} \to 0 \qquad \text{as } n \to \infty.
\end{aligned} \tag{110}$$

The claim (105) that follows easily from (106)–(110). This finishes the proof of Lemma 5.1. □

6. Properties of the score

In this section we will prove the convergence of the sequence of processes Y^n to a Lévy process. This result is contained in Proposition 6.5. Recall that Y^n was defined in (25) as a well rescaled sum of scores. Hence, we should first study the properties of the score.

The score of the i-th part of the trajectory depends on the history only through its starting point x_i^n. We thus associate to every point $x \in \mathcal{E}(n)$ the random variable s_x, which has the same distribution as the score of a part of the trajectory of X that is started at x. We can ignore the points in $\mathbb{D}(n) \setminus \mathcal{E}(n)$ because we do not consider the parts of trajectory started in this set (see definition of J). We have got already some information which can help us to describe the distribution of the random variables s_x. According to Lemma 5.7, the probability of hitting two deep traps in the disk $D_x(2^n n^\gamma)$ is of order $n^{2(\alpha+\gamma-1)}$, and the probability of hitting one deep trap is with high precision $\mathcal{K}p_\varepsilon^M n^{\alpha+\gamma-1}$. Otherwise X does not hit any deep trap. In the last case $s_x = 0$ (if none of (i)–(v) of Section 2 happen).

We want now to study more precisely the distribution of s_x conditionally on $s_x < \infty$. To achieve it we should gain more information about the depth of the trap that X hits as the first. The idea behind the proof is that as n increases the density of deep traps becomes lower, and the hitting measure of T_ε^M charges more and more sites. The distribution of the depth of the first visited trap should be thus close to

the original distribution of the depth of the trap conditioned on being between $\varepsilon 2^{n/\alpha}/n$ and $M2^{n/\alpha}/n$.

To prove this heuristics we divide the set of deep traps into several parts and we estimate the probability of hitting each of them. Let $h(x)$ be a function satisfying

$$h(x) \geq (\log x)^{-1}, \qquad \lim_{x\to\infty} h(x) = 0, \tag{111}$$

and (with L defined in (14))

$$L(2^{n/\alpha}n^{-1}x) - 1 = o(h(n)) \quad \text{for all } x \geq \varepsilon. \tag{112}$$

Such function exists because $\lim_{x\to\infty} L(x) = 1$. Let $z_n(i)$ satisfy $\varepsilon = z_n(0) < z_n(1) < \cdots < z_n(R) = M$ and $z_n(i+1) - z_n(i) \in (h(n), 2h(n))$ for all $i \in \{0, \dots R-1\}$.

We now estimate the probability of hitting a trap in $T_{z_n(i)}^{z_n(i+1)}$. We use p_i^n to denote

$$p_i^n = z_n(i)^{-\alpha} - z_n(i+1)^{-\alpha}. \tag{113}$$

Lemma 6.1. *For any $\delta > 0$ and $\mathbb{P}$-a.e. $\boldsymbol{\tau}$ there exists n_0 such that for all $n > n_0$, for all $x \in \mathcal{E}(n)$, and for all $i = \{0, \dots, R-1\}$ the probability that the simple random walk started at x hits a trap in $T_{z_n(i)}^{z_n(i+1)}$ before the exit from $D_x(2^n n^\gamma)$ is in the interval*

$$\big[\mathcal{K}(1-\delta)n^{\alpha+\gamma-1}p_i^n, \mathcal{K}(1+\delta)n^{\alpha+\gamma-1}p_i^n\big]. \tag{114}$$

Proof. The proof is very similar to the proof of Lemma 5.2. We should first improve the bounds on the homogeneity of the environment that we have proved in Lemma 5.3.

Let $H_6 = H_6(n, \delta, \varepsilon, M)$ be the event that for every square $B_x(2^n n^\nu)$ in $\mathbb{D}(n)$ and for every $i \in \{0, \dots R-1\}$ the number of sites in $T_{z_n(i)}^{z_n(i+1)} \cap B_x(2^n n^\nu)$ is in the interval

$$\big[(1-\delta)n^{\alpha+\nu}p_i^n, (1+\delta)n^{\alpha+\nu}p_i^n\big]. \tag{115}$$

We prove that H_6 occurs $\mathbb{P}$-a.s. for n large enough.

Lemma 6.2. *For any δ there exist constants c and C such that for n large enough*

$$\mathbb{P}[H_6] \geq 1 - C\log(n)n^{1-\alpha-\nu}\delta^{-2}\exp\big(-cn^{\nu+\alpha}h(n)\big). \tag{116}$$

Using this lemma it is not difficult to finish the proof of Lemma 6.1. We will not give the detailed reasoning, because the proof follows the same line as the proof of Lemma 5.2. The only change is that Lemma 6.1 should be used instead of Lemma 5.3. □

Proof of Lemma 6.2. To show that H_6 occurs $\mathbb{P}$-a.s. for n large enough we will need the following technical lemma that estimates the probability that a trap is in $T_{z_n(i)}^{z_n(i+1)}$.

Lemma 6.3. *For any $\eta > 0$ there exist n_0 such that for all $n \geq n_0$ and all $i = 0, \dots, R-1$*

$$\mathbb{P}\big[0 \in T_{z_n(i)}^{z_n(i+1)}\big] \in \Big((1-\eta)\frac{n^\alpha}{2^n}p_i^n, (1+\eta)\frac{n^\alpha}{2^n}p_i^n\Big). \tag{117}$$

Proof. Let $g(x) = L(x) - 1$. Then by (14) we have

$$\begin{aligned}\mathbb{P}\big[0 \in T_{z_n(i)}^{z_n(i+1)}\big] &= \mathbb{P}\Big[\tau_0 \in \Big[z_n(i)\frac{2^{n/\alpha}}{n}, z_n(i+1)\frac{2^{n/\alpha}}{n}\Big)\Big] \\ &= \frac{n^\alpha}{2^n}\Big[p_i^n + \frac{g(2^{n/\alpha}n^{-1}z_n(i))}{z_n(i)^\alpha} - \frac{g(2^{n/\alpha}n^{-1}z_n(i+1))}{z_n(i+1)^\alpha}\Big].\end{aligned} \tag{118}$$

We should thus show that

$$\frac{g(2^{n/\alpha}n^{-1}z_n(i))}{z_n(i)^\alpha} - \frac{g(2^{n/\alpha}n^{-1}z_n(i+1))}{z_n(i+1)^\alpha} = o(p_i^n). \tag{119}$$

However, this is obviously true since

$$p_i^n = (z_n(i))^{-\alpha} - (z_n(i+1))^{-\alpha} \geq ch(n) \tag{120}$$

for some c depending only on M, and $g(2^{n/\alpha}n^{-1}z_j^n) = o(h(n))$ by (112). □

The remaining part of the proof of Lemma 6.2 is analogous to the proof of Lemma 5.3. We only explain the appearance of the additional factors $\log(n)$ and $h(n)$ that are in (116) but not in (61). The logarithm before the exponential is due to the summation over all possible values of i and (111). The factor $h(n)$ inside the exponent comes from Lemma 6.3 which replaces the bound on p before (64), and from the existence of constants $c(\varepsilon, M)$, $C(\varepsilon, M)$ such that

$$ch(n) \leq \frac{1}{z_n(i)^\alpha} - \frac{1}{z_n(i+1)^\alpha} \leq Ch(n). \tag{121}$$

This finishes the proof. □

Using Lemma 6.1 we can now describe the behaviour of random variables s_x. Due to condition (ii) from Section 2, all good parts of the trajectory starts at sites that are in the distance larger than $\sqrt{\pi^{-1}2^n n^\gamma}$ from the border of $\mathbb{D}(n)$. That is why we introduce $\mathcal{E}_0(n) = \{x \in \mathcal{E}(n) : D_x(2^n n^\gamma) \cap \mathbb{D}(n)^c = \emptyset\}$. The random variables s_x then satisfy

Lemma 6.4. *For $\mathbb{P}$-a.e. random environment $\boldsymbol{\tau}$*

$$\begin{aligned}\lim_{n\to\infty} \max_{x\in\mathcal{E}_0(n)} \frac{1 - \mathbb{E}[\exp(-\frac{\lambda s_x}{2^{n/\alpha}})|s_x < \infty, \boldsymbol{\tau}]}{n^{\alpha+\gamma-1}} &= F(\lambda), \\ \lim_{n\to\infty} \min_{x\in\mathcal{E}_0(n)} \frac{1 - \mathbb{E}[\exp(-\frac{\lambda s_x}{2^{n/\alpha}})|s_x < \infty, \boldsymbol{\tau}]}{n^{\alpha+\gamma-1}} &= F(\lambda),\end{aligned} \tag{122}$$

with

$$F(\lambda) = F(\lambda; \varepsilon, M, \alpha) = \mathcal{K}\Big(p_\varepsilon^M - \int_\varepsilon^M \frac{\alpha}{1 + \mathcal{K}'\lambda z} \cdot \frac{1}{z^{\alpha+1}} dz\Big) \tag{123}$$

and $\mathcal{K}' = \pi^{-1}\log 2$.

Proof. By Lemmas 5.2, 5.8, and 5.10 we know that if κ is large enough, $\mathbb{P}[s_x = \infty] = O(n^{2(\alpha+\gamma-1)})$. Since this probability is much smaller than any other probability that will be used in the following computation, the conditioning on $s_x < \infty$ has almost no effect. Actually,

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\Big|s_x<\infty,\boldsymbol{\tau}\Big]\\ &= \mathbb{P}[s_x<\infty|\boldsymbol{\tau}]^{-1}\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\chi\{s_x<\infty\}\Big|\boldsymbol{\tau}\Big] \qquad (124)\\ &= \mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\Big|\boldsymbol{\tau}\Big]\big(1+O(n^{2(\alpha+\gamma-1)})\big).\end{aligned}$$

If the process X hits deep trap y in $D_x(2^n n^\gamma)$ and nothing unusual happens, then the random variable s_x is a sum of a geometrically distributed number of exponential random variables with mean τ_y. The mean of the geometrically distributed number of visits of y is equal to $G_{D(2^n n^{-\kappa})}(0,0)$, where by (225)

$$G_{D(2^n n^{-\kappa})}(0,0) = \frac{2}{\pi}\log\sqrt{\pi^{-1}2^n n^{-\kappa}} + O(1) = \frac{n}{\pi}\log 2 + O(\log n). \quad (125)$$

Since the geometrically long sum of exponential random variables is again exponentially distributed, the score s_x is in this case an exponential random variable with mean $\tau_y(n\log 2/\pi + O(\log n))$. This implies that conditionally on hitting a trap with the depth τ_y the Laplace transform of $s_x/2^{n/\alpha}$ equals

$$\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\Big|\tau_y\Big] = \frac{1}{1+\lambda\tau_y 2^{-n/\alpha}(n\log 2/\pi + O(\log n))}. \quad (126)$$

We now estimate the Laplace transform $\mathbb{E}\big[\exp(-\lambda s_x 2^{-n/\alpha})|\boldsymbol{\tau}\big]$. We start with a lower bound. Choose $\delta > 0$. By Lemmas 5.7, 6.1, and expression (126) we have for n large enough

$$\begin{aligned}\mathbb{E}\Big[\exp\Big(-\frac{\lambda s_x}{2^{n/\alpha}}\Big)\Big|\boldsymbol{\tau}\Big] &\geq \Big(1-(1+\delta)\mathcal{K}p_\varepsilon^M n^{\alpha+\gamma-1}\Big)\\ &+ \mathcal{K}n^{\alpha+\gamma-1}\sum_{i=1}^{R}\frac{1-\delta}{1+\lambda\frac{z_n(i)}{2^{n/\alpha}}\frac{2^{n/\alpha}}{n}\frac{n}{\pi}\log 2+o(1)}\Big(\frac{1}{(z_{i-1}^n)^\alpha}-\frac{1}{(z_n(i))^\alpha}\Big).\end{aligned} \quad (127)$$

The last expression can be bounded from bellow by

$$1-\mathcal{K}n^{\alpha+\gamma-1}\Big(p_\varepsilon^M - \int_\varepsilon^M \frac{\alpha}{1+\mathcal{K}'\lambda z}\frac{1}{z^{\alpha+1}}dz\Big) - \delta C n^{\alpha+\gamma-1}p_\varepsilon^M, \quad (128)$$

with C being a constant not depending on δ. The last expression together with (124) give

$$\limsup_{n\to\infty} \max_{x\in\mathcal{E}_0(n)} \frac{1-\mathbb{E}[\exp(-\frac{\lambda s_x}{2^{n/\alpha}})|s_x<\infty,\boldsymbol{\tau}]}{n^{\alpha+\gamma-1}} \leq \mathcal{K}\Big(p_\varepsilon^M - \int_\varepsilon^M \frac{\alpha}{1+\mathcal{K}'\lambda z}\frac{1}{z^{\alpha+1}}dz\Big) + C\delta p_\varepsilon^M. \quad (129)$$

Since δ can be taken arbitrarily small, the proof of the upper bound for the first expression in (122) is finished. The proof of the lower bound for the second expression in (122) is completely similar. □

We can finally show the convergence of the sequence Y^n to a Lévy process (see [Ber96] for complete treatment of Lévy processes). The following proposition will be used later to prove aging.

Proposition 6.5. *For $\mathbb{P}$-a.e. realisation of the environment, the sequence of processes $Y^n(t)$ converges weakly in the Skorokhod topology on $D([0,\infty))$ to the Lévy process $Y(t)$ with the Lévy measure*

$$\rho(dx) = \frac{\alpha\mathcal{K}}{\mathcal{K}'}\int_\varepsilon^M \frac{1}{z^{\alpha+2}}\exp\Big(-\frac{x}{\mathcal{K}'z}\Big)\,dz\,dx. \quad (130)$$

Proof. We first prove the weak convergence of finite dimensional distributions. Let $0=t_0<t_1<\cdots<t_\ell$. We will show the convergence of Laplace transforms. By definition of Y^n

$$\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{\ell}\lambda_i\big(Y^n(t_i)-Y^n(t_{i-1})\big)\Big)\Big] = \mathbb{E}\Big[\prod_{i=1}^{\ell}\prod_{j\in B(n,i)}\exp\Big(-\frac{\lambda_i}{2^{n/\alpha}}s_j^n\Big)\Big], \quad (131)$$

where $B(n,i)=\{\lfloor n^{1-\alpha-\gamma}t_{i-1}\rfloor+1,\ldots,\lfloor n^{1-\alpha-\gamma}t_i\rfloor\}$.

If $j<J$, then the random variables s_j^n are determined by behaviour of X, otherwise they are equal to $\tilde{s}_j^n$. Since $\tilde{s}_j^n$'s are independent of all other randomness, we can write

$$= \sum_{k=0}^{\infty}\mathbb{P}[J=k]\mathbb{E}\Big[\prod_{i=1}^{\ell}\prod_{\substack{j\in B(n,i)\\ j<J}}\exp\Big(-\frac{\lambda_i}{2^{n/\alpha}}s_j^n\Big)\Big|J=k\Big] \mathbb{E}\Big[\prod_{i=1}^{\ell}\prod_{\substack{j\in B(n,i)\\ j\geq J}}\exp\Big(-\frac{\lambda_i}{2^{n/\alpha}}\tilde{s}_j^n\Big)\Big|J=k\Big]. \quad (132)$$

At this place it is necessary to define the distribution of $\tilde{s}_j^n$. We require that $\tilde{s}_i^n$'s satisfy the same relation as s_x in the limit, i.e.

$$\mathbb{E}\Big[\exp\Big(-\frac{\lambda}{2^{n/\alpha}}\tilde{s}_j^n\Big)\Big] = 1-F(\lambda)n^{\alpha+\gamma-1}. \quad (133)$$

We have obviously chosen the $\tilde{s}_j^n$'s in the way that the second expectation in (132) does not pose any problems. We should thus control only the first one.

Let $\boldsymbol{y} = \{y_0, \dots, y_k\} \in \mathcal{E}(n)^{k+1}$. We use $\boldsymbol{x}_n$ to denote the sequence $x_0^n, \dots, x_k^n$ of starting points of the parts of the trajectory. We have

$$\mathbb{E}\Big[\prod_{i=1}^{\ell} \prod_{\substack{j\in B(n,i)\\ j<k}} \exp\Big(-\frac{\lambda_i}{2^{n/\alpha}} s_j^n\Big)\Big]$$
$$= \sum_{\boldsymbol{y}} \mathbb{P}[\boldsymbol{x}_n = \boldsymbol{y}]\mathbb{E}\Big[\prod_{i=1}^{\ell} \prod_{\substack{j\in B(n,i)\\ j<k}} \exp\Big(-\frac{\lambda_i}{2^{n/\alpha}} s_j^n\Big)\Big|\boldsymbol{x}_n = \boldsymbol{y}\Big]. \quad (134)$$

Only the last term of the product depends on y_k. We can thus sum over all possible values of the endpoint of the last part. Let $\boldsymbol{x}_n'$, resp. $\boldsymbol{y}'$, denote the sequences $\boldsymbol{x}_n$ and $\boldsymbol{y}$ without the last element. We get

$$= \sum_{\boldsymbol{y}'} \mathbb{P}[\boldsymbol{x}_n' = \boldsymbol{y}']\mathbb{E}\Big[\prod_{i=1}^{\ell} \prod_{\substack{j\in B(n,i)\\ j<k}} \exp\Big(-\frac{\lambda_i}{2^{n/\alpha}} s_j^n\Big)\Big|\boldsymbol{x}_n' = \boldsymbol{y}'\Big]. \quad (135)$$

Conditionally on the value x_{k-1}^n, the random variable s_{k-1}^n is independent of the rest. The expectation in the last formula can be thus written as

$$\mathbb{E}\Big[\prod_{i=1}^{\ell} \prod_{\substack{j\in B(n,i)\\ j<k-1}} \exp\Big(-\frac{\lambda_i}{2^{n/\alpha}} s_j^n\Big)\Big|\boldsymbol{x}_n' = \boldsymbol{y}'\Big]\mathbb{E}\Big[\exp\Big(-\frac{\lambda_r}{2^{n/\alpha}} s_{x_{k-1}^n}\Big)\Big|s_{x_{k-1}^n} < \infty\Big], \quad (136)$$

where the index r satisfies $k-1 \in B(n,r)$. According to Lemma 6.4, the second expectation can be bounded from above by

$$1 - (1-\delta)F(\lambda_r)n^{\alpha+\gamma-1} \quad (137)$$

if n is large enough.

We can now repeat the same manipulation with the last but one value of j, etc. At the end, putting the result of this iteration into (132), we get

$$\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{\ell} \lambda_i\big(Y^n(t_i) - Y^n(t_{i-1})\big)\Big)\Big]$$
$$\le \prod_{i=1}^{\ell} \big(1 - (1-\delta)F(\lambda_i)n^{\alpha+\gamma-1}\big)^{\lfloor n^{1-\alpha-\gamma}(t_i - t_{i-1})\rfloor}. \quad (138)$$

Taking the limits we obtain

$$\limsup_{n\to\infty} \mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{\ell}\lambda_i\big(Y^n(t_i)-Y^n(t_{i-1})\big)\Big)\Big] \le \exp\Big[-\sum_{i=1}^{\ell}(1-\delta)F(\lambda_i)(t_i-t_{i-1})\Big]. \quad (139)$$

In the same way we obtain a lower bound. Since δ was arbitrary we have

$$\lim_{n\to\infty} \mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{\ell}\lambda_i\big(Y^n(t_i)-Y^n(t_{i-1})\big)\Big)\Big] = \exp\Big[-\sum_{i=1}^{\ell}F(\lambda_i)(t_i-t_{i-1})\Big]. \quad (140)$$

The corresponding Laplace transform of $Y(t)$ is easy to calculate. We have

$$\mathbb{E}\Big[\exp\Big(-\sum_{i=1}^{\ell}\lambda_i\big(Y(t_i)-Y(t_{i-1})\big)\Big)\Big] = \exp\Big[-\sum_{i=1}^{\ell}\Psi(\lambda_i)(t_i-t_{i-1})\Big], \quad (141)$$

where $\Psi(\lambda)$ is the Laplace exponent of Y. By Lévy-Khintchine formula it is equal to

$$\Psi(\lambda) = \int_0^\infty (1-e^{-\lambda x})\rho(dx). \quad (142)$$

An easy integration gives the same result as (140).

To prove the weak convergence it remains to verify that the sequence Y_n is tight. We use Theorem 16.8 from [Bil99]. We should show that for any N and δ_1, δ_2 there exist a, n_0, and η such that

(i) $\mathbb{P}[\sup_{t\in[0,N]}|Y_n(t)|\ge a] < \delta_1$ for all $n > n_0$

(ii) $\mathbb{P}[w(Y^n,\eta,N)\ge\delta_2] < \delta_1$ for all $n > n_0$,

where

$$w(f,\eta,N) = \inf_{\{t_i\}}\max_{0<i\le r}\sup\{|f(s)-f(t)| : s,t\in[t_{i-1},t_i)\} \quad (143)$$

and the infimum runs over all finite collections $\{t_i\}$ such that $0 < t_i - t_{i-1} < \eta$, $t_0 = 0$, and $t_r = N$.

Proof of (i) Since Y^n are increasing, (i) is equivalent to the tightness of the sequence $Y^n(N)$. From convergence of finite dimensional distribution we know that the Laplace transforms of $Y^n(N)$ converge to $\mathcal{L}_{Y(N)}(\lambda) = \mathbb{E}[\exp(-\lambda Y(N))]$. It is sufficient to verify that this Laplace transform satisfies $\lim_{\lambda\to 0}\mathcal{L}_{Y(N)}(\lambda) = 1$. However, $\mathcal{L}_{Y(N)}$ is continuous

and

$$\mathcal{L}_{Y(N)}(0) = \exp(-NF(0)) = \exp\Big[-N\mathcal{K}\Big(p_\varepsilon^M - \int_\varepsilon^M \frac{\alpha}{z^{\alpha+1}}dz\Big)\Big] = 1. \tag{144}$$

Proof of (ii) According to Lemma 5.7, the expected number of jumps of Y^n in the interval $[0, N]$ can be bounded by some constant C not depending on n. Markov inequality then gives the existence of some C' such that the probability that the number of jumps of Y^n exceeds C' is smaller than $\delta_1/2$ for all n large enough. If the number of jumps is finite, we can take $\{t_i\}$ being the superset of the set of all jumps. The process Y^n is then constant on any interval $[t_{i-1}, t_i)$ and thus $w(Y^n, \eta, N) = 0$. This completes the proof of Proposition 6.5. □

7. Proof of aging

We prove here the following proposition that is a more precise version of Theorem 1.1.

Proposition 7.1. *For $\mathbb{P}$-a.e. realisation of the environment $\boldsymbol{\tau}$ and for every $0 < \theta < \infty$*

$$\lim_{t\to\infty} R(t, t+\theta t) = \int_0^{1/1+\theta} \frac{\sin\alpha\pi}{\pi} u^{\alpha-1}(1-u)^{-\alpha}\, du \equiv R(\theta). \tag{145}$$

An easy calculation gives

Corollary 7.2. *The function $R(\theta)$ satisfies*

$$\lim_{\theta\to 0} R(\theta) = 1 \qquad \textit{and} \qquad \lim_{\theta\to\infty} R(\theta) = 0. \tag{146}$$

Proof. **I.** We introduce some additional notation. Let $Z(t) = Z(t; \varepsilon, M)$ be a Lévy process with the Lévy measure

$$\rho'(dx) = \frac{\alpha\mathcal{K}}{\mathcal{K}'}\Big(\int_0^\varepsilon + \int_M^\infty\Big)\frac{1}{z^{\alpha+2}}\exp\Big(-\frac{x}{\mathcal{K}'z}\Big)\, dz\, dx, \tag{147}$$

independent of the processes Y, Y^n. We define the new family of processes,

$$\tilde{Y}^n(t) = Y^n(t) + Z(t) \qquad \text{and} \qquad \tilde{Y}(t) = Y(t) + Z(t). \tag{148}$$

The advantage of this new class is that the Lévy measure of $\tilde{Y}$ is given by

$$\begin{aligned}\rho(dx) + \rho'(dx) &= \frac{\alpha\mathcal{K}}{\mathcal{K}'}\int_0^\infty \frac{1}{z^{\alpha+2}}\exp\Big(-\frac{x}{\mathcal{K}'z}\Big)\, dz\, dx \\ &= \frac{\alpha^2\Gamma(\alpha)\mathcal{K}(\mathcal{K}')^\alpha}{x^{\alpha+1}}\, dx,\end{aligned} \tag{149}$$

and thus $\tilde{Y}$ is an α-stable subordinator. As an easy consequence of the previous section we know that the sequence $\tilde{Y}^n$ converges weakly to $\tilde{Y}$

whatever the values chosen for ε and M. Let $\mathcal{R}_n = \mathcal{R}(\tilde{Y}^n)$, $\mathcal{R} = \mathcal{R}(\tilde{Y})$ denote the range of $\tilde{Y}^n$, resp. of $\tilde{Y}$.

Fix $\theta > 0$. Let δ_1, $\delta_2 > 0$ be arbitrarily small but fixed. We will now fix the values of M, m, ε as functions of δ_1, δ_2 and n as a function of δ_1, δ_2 and t . First, let $n(t)$ be the integer satisfying

$$1 \leq \frac{t}{2^{n(t)/\alpha}} < 2^{1/\alpha}. \tag{150}$$

Obviously, $n(t) \to \infty$ as $t \to \infty$. In this section $n = n(t)$ is always connected with t via (150). We use $s = s(t)$ to denote the rescaled value of t, $s = t2^{-n(t)/\alpha}$. By (150) s satisfies $1 \leq s < 2^{1/\alpha}$. In the same way we rescale the value $(1+\theta)t$. The process $\tilde{Y}^n$ that we will use to approximate the time change $\bar{S}^n$ should be thus relevant until the level $(1+\theta)s < (1+\theta)2^{1/\alpha}$. Let t_0 be such that

$$\mathbb{P}[\tilde{Y}(t_0) < (1+\theta)2^{1/\alpha}] < \delta_1. \tag{151}$$

By the weak convergence of $\tilde{Y}^n$ to $\tilde{Y}$ we can take t (and so $n = n(t)$) large enough such that

$$\mathbb{P}[\tilde{Y}^{n(t)}(t_0) \geq (1+\theta)2^{1/\alpha}] > 1 - 2\delta_1. \tag{152}$$

There are $J(n)$ relevant parts of the trajectory of the process X. For every time unit we need $n^{1-\alpha-\gamma}$ parts. So, we should choose m in such a way that

$$\mathbb{P}[J(n)n^{\alpha+\gamma-1} \geq t_0] > 1 - \delta_1. \tag{153}$$

By Lemma 5.1, this can be done independently of ε and M. Let A_1 be the event $\left\{\tilde{Y}^n(t_0) \geq (1+\theta)s \text{ and } J(n) \geq t_0 n^{1-\alpha-\gamma}\right\}$. Then, by (152) and (153),

$$\mathbb{P}[A_1] \geq 1 - 3\delta_1. \tag{154}$$

We can now fix the values of ε and M. Later, we want to work with the processes $\tilde{Y}^n$ instead of Y^n. We should thus guarantee that the artificial addition of process Z is not relevant. We take ε_1 and M_1, such that

$$\mathbb{P}[Z(t_0; \varepsilon_1, M_1) > \delta_2] < \delta_1. \tag{155}$$

We want also safely ignore the error introduced by the very deep and the shallow traps. By Lemma 4.1, we can take M_2 such that

$$\mathbb{P}[X \text{ hits } T_{M_2} \text{ before } \Lambda_d(n)] < \delta_1. \tag{156}$$

Further, by Lemma 3.1, we know that there is a constant K_1 (not depending on ε, M or n), such that $\boldsymbol{\tau}$-a.s. for n (or equivalently t) large enough

$$\mathbb{E}\Big[\frac{1}{2^{n/\alpha}} \cdot \text{time spent in } T^\varepsilon \text{ before } \Lambda_d(n) \,\Big|\, \boldsymbol{\tau}\Big] \leq K_1 \varepsilon^{1-\alpha}, \tag{157}$$

and thus

$$\mathbb{P}\Big[\frac{1}{2^{n/\alpha}} \cdot \text{time spent in } T^\varepsilon \text{ before } \Lambda_d(n) \;> \delta_2 \,\Big|\, \boldsymbol{\tau}\Big] \leq \delta_2^{-1} \varepsilon^{1-\alpha} K_1. \tag{158}$$

Let us take ε_2 such that $\delta_2^{-1}\varepsilon_2^{1-\alpha}K_1 < \delta_1$. The constants ε and M are then defined by

$$\varepsilon = \min(\varepsilon_1, \varepsilon_2) \qquad \text{and} \qquad M = \max(M_1, M_2). \tag{159}$$

This choice of constants ensures that the distance between the rescaled time change $\bar{S}^n$ and the process $\tilde{Y}^n$ is small. Precisely, let

$$A_2 = \big\{ |\bar{S}^n(t) - \tilde{Y}^n(t)| \le 2\delta_2 \, \forall t \le t_0 \big\}. \tag{160}$$

Then our choice of constants gives

$$\mathbb{P}\big[A_2 | A_1\big] \ge 1 - 3\delta_1. \tag{161}$$

Let $A = A_1 \cap A_2$. Then from (154) and (161) follows that for t large enough

$$\mathbb{P}[A] \ge 1 - 6\delta_1. \tag{162}$$

II. Later we will take the limit $n \to \infty$ for fixed value of $s \in [1, 2^{1/\alpha}]$ instead of taking limit $t \to \infty$. We will show that this limit does not depend on s. To be able to show the existence of the limit $t \to \infty$ we will need uniformity of convergence in s. The proof of the following auxiliary lemma is left to the reader.

Lemma 7.3. *Let $\mathcal{P}_u(s, Y) = \mathbb{P}\big[[s, s+u] \cap \mathcal{R}(Y) \ne \emptyset\big]$ for Y being $\tilde{Y}^n$ or $\tilde{Y}$. Then for any $u < \theta 2^{1/\alpha}$*

$$\lim_{n\to\infty} \mathcal{P}_u(s, \tilde{Y}^n) = \mathcal{P}_u(s, \tilde{Y}) \tag{163}$$

uniformly for $s \in [1, \theta 2^{1/\alpha}]$.

III. We now study the event $G(t) = \big\{X(t) = X((1+\theta)t)\big\}$ for t large. We divide the probability space into three disjoint parts,

$$\begin{aligned} E_1(n,s) &= \big\{ \operatorname{dist}(s, \mathcal{R}_n) \le 2\delta_2 \text{ or } \operatorname{dist}((1+\theta)s, \mathcal{R}_n) \le 2\delta_2 \big\} \\ E_2(n,s) &= \big\{ \operatorname{dist}(s, \mathcal{R}_n) > 2\delta_2, \operatorname{dist}((1+\theta)s, \mathcal{R}_n) > 2\delta_2 \text{ and} \\ &\qquad \big(s, (1+\theta)s\big) \cap \mathcal{R}_n \ne \emptyset \big\} \\ E_3(n,s) &= \big\{ [s - 2\delta_2, (1+\theta)s + 2\delta_2] \cap \mathcal{R}_n = \emptyset \big\}. \end{aligned} \tag{164}$$

This division has the following reason. On event A_2 and therefore on event A, to precision $2\delta_2$, any interval that does not intersect $\mathcal{R}_n$ corresponds to a time period that X spent in $D_y(2^n n^{-\kappa})$ around some deep trap y. Heuristically the points of the range correspond to times when the walk did not meet any deep trap for a long time.

We wish to show that essentially event $G(t)$ is the same as event $E_3(n,s)$. Obviously

$$\begin{aligned} \mathbb{P}[G(t) \cap E_3(n,s)] &\le \mathbb{P}[G(t)] \\ &\le \mathbb{P}[E_3(n,s)] + \mathbb{P}[E_1(n,s)] + \mathbb{P}[G(t) \cap E_2(n,s)] \end{aligned} \tag{165}$$

We should thus estimate all quantities in the last display. When E_1 occurs, at least one of the values s, $(1+\theta)s$ is too close to $\mathcal{R}_n$. Hence,

we cannot know precisely what happens with the process X in this situation. However, the probability of E_1 is small. Indeed,

$$\mathbb{P}[E_1] \leq \mathbb{P}[\mathrm{dist}(s, \mathcal{R}_n) \leq 2\delta_2] + \mathbb{P}[\mathrm{dist}((1+\theta)s, \mathcal{R}_n) \leq 2\delta_2]. \quad (166)$$

If n is large, we can bound the first term in the last expression by

$$\mathbb{P}[\mathrm{dist}(s, \mathcal{R}_n) \leq 2\delta_2] \leq \delta_1 + 1 - \mathbb{P}[\mathcal{R} \cap [s - 2\delta_2, s + 2\delta_2] = \emptyset]. \quad (167)$$

The constant δ_1 comes from the approximation of $\mathcal{R}_n$ by $\mathcal{R}$ and by Lemma 7.3 can be chosen independent of s. Since $\tilde{Y}$ is a stable subordinator, the probability $\mathbb{P}[\mathcal{R} \cap [s - 2\delta_2, s + 2\delta_2] = \emptyset]$ can be evaluated using formulas from Lemma B.1,

$$\begin{aligned} \mathbb{P}[\mathrm{dist}(s, \mathcal{R}_n) \leq 2\delta_2] &\leq \delta_1 + 1 - \mathbb{P}[g(s + 2\delta_2) < s - 2\delta_2] \\ &= \delta_1 + 1 - \int_0^{\frac{s-2\delta_2}{s+2\delta_2}} \frac{\sin \alpha\pi}{\pi} u^{\alpha-1}(1-u)^{-\alpha}\, du \leq C\delta_1 + C'\delta_2^{1-\alpha} \quad (168) \end{aligned}$$

for some constants C, C' independent of s. In the same way we can estimate the second probability from (166). We have thus

$$\mathbb{P}[E_1] \leq C\delta_1 + C'\delta_2^{1-\alpha}. \quad (169)$$

If A occurs, then the realisation of E_2 means that $X(t)$ is in disk $D_{y_1}(2^n n^{-\kappa})$ and $X\big((1+\theta)t\big)$ is in $D_{y_2}(2^n n^{-\kappa})$ for some $y_1, y_2 \in T_\varepsilon^M$. By definition of J we have necessarily $y_1 \neq y_2$, and thus by Lemma 5.6

$$\mathbb{P}[G(t) \cap E_2(n,s) \cap A] = 0. \quad (170)$$

Hence,

$$\mathbb{P}[G(t) \cap E_2(n,s)] \leq 1 - \mathbb{P}[A] \ \leq C\delta_1 \quad (171)$$

The most interesting event is E_3. The probability of E_3 can be calculated in a similar manner to the probability of E_1. For n large enough

$$\big|\mathbb{P}[E_3(n,s)] - \mathbb{P}\big[\mathcal{R} \cap [s - 2\delta_2, (1+\theta)s + 2\delta_2] = \emptyset\big]\big| \leq \delta_1, \quad (172)$$

which implies

$$\left|\mathbb{P}[E_3(n,s)] - \int_0^{1/1+\theta} \frac{\sin \alpha\pi}{\pi} u^{\alpha-1}(1-u)^{-\alpha}\, du\right| \leq (C\delta_1 + C'\delta_2). \quad (173)$$

The constants C and C' can be chosen again independent of s. Note also that the main term does not depend on s.

We will now show that $\mathbb{P}[G(t) \cap E_3(n,s)]$ is close to $\mathbb{P}[E_3(n,s)]$. Let $D = D(t,\theta)$ be the event that for some $y \in T_\varepsilon^M$,

$$t^n \equiv \inf\{u : X(u) = y\} < t \quad (174)$$

and

$$s^n \equiv \sup\{u < \inf\{v > t^n : X(v) \notin D_y(2^n n^{-\kappa})\} : X(u) = y\} > t(1+\theta). \quad (175)$$

Obviously we have that the event $\{E_3(n,s)\} \setminus \{G(t) \cap E_3(n,s)\}$ is contained in the event $A^c \cup (D \cap (\{X(t) \neq y\} \cup \{X(t(1+\theta)) \neq y\})$.

Lemma 7.4. *The probability of the event D intersected with $\{X(t) \neq y\} \cup \{X(t(1+\theta)) \neq y\}$ tends to zero as t tends to infinity.*

We use this lemma to finish the proof of Proposition 7.1. For t large enough we have putting (169) and (171), into (165) we get

$$\mathbb{P}[G(t)] \leq C\delta_1 + C'\delta_2^{1-\alpha} + \mathbb{P}[E_3(n,s)]. \tag{176}$$

Similarly, we obtain the lower bound (for t sufficiently large)

$$\mathbb{P}[G(t)] \geq \mathbb{P}[E_3(n,s)] - C\delta_1. \tag{177}$$

Since the expression (173) for E_3 and also the constants in error terms do not depend on s, and since δ_1 and δ_2 can be taken arbitrarily small, we have

$$\lim_{t\to\infty} \mathbb{P}[G(t)] = \int_0^{1/1+\theta} \frac{\sin \alpha\pi}{\pi} u^{\alpha-1}(1-u)^{-\alpha}\, du. \tag{178}$$

This finishes the proof. □

IV. It remains to show Lemma 7.4

Proof of Lemma 7.4. It will suffice to show that

$$\mathbb{P}[X(t) = y | D, t^n, y, \boldsymbol{\tau}], \mathbb{P}[X(t(1+\theta)) = y | D, t^n, y, \boldsymbol{\tau}] \tag{179}$$

tend to one as t tends to infinity. We will only treat the first probability, the proof of the second convergence being entirely similar.

The Markov process $(X(t^n + s) : s^n - t^n \geq s \geq 0)$, given $D, t^n, y, \boldsymbol{\tau}$ is equal in law to the process $(U(s) : s \in [0, s^n - t^n])$ conditioned on the event $\{S > s^n - t^n\}$ where U and S are constructed as follows:

(i) U stays at site y for an exponential, mean τ_y, amount of time, then

(ii) with probability $p(n)$, the probability that a random walk starting at y escapes $\mathcal{D} \equiv D_y(2^n n^{-\kappa})$ before returning to site y, the process terminates and S is the termination time. With probability $1 - p(n)$ the process U performs an excursion away from y conditioned not to leave $\mathcal{D}$. At the end of the excursion it returns to y and step (i) resumes and so on.

The important point is that the number $p(n)$ is of order $1/n$ while (recall $y \in T_\varepsilon^M$) the mean time spent at y per visit exceeds $\varepsilon 2^{n/\alpha}/n$. Thus the conditioning event has probability bounded below by $C(\varepsilon, \theta)$. Hence it will suffice to show that $\mathbb{P}[\bar{U}(t - t^n) \neq y | \boldsymbol{\tau}]$ tends to zero as t tends to infinity $\boldsymbol{\tau}$-a.s. where process $(\bar{U}(u) : u \geq 0)$ is a Markov process that alternates staying at site y an exponential amount of time with mean τ_y and performing excursions away from y conditioned to stay within $\mathcal{D}$ (again staying at each site a time according to $\boldsymbol{\tau}$).

We first show that $\boldsymbol{\tau}$-a.s. for t (and therefore n) sufficiently large, the expected duration of a conditioned excursion from y is very small

compared to τ_y uniformly over possible $y \in T_\varepsilon^M$. It is easy to prove that in the neighbourhood of y there are only traps shallower than $\varepsilon n^{-5/(1-\alpha)} 2^{n/\alpha}/n$. Indeed, as in the proof of Lemma 5.6, let

$$B(y) = \Big\{ y \in T_\varepsilon^M, \exists x \in \mathcal{D}, \tau_x \geq \varepsilon n^{-\frac{5}{1-\alpha}} \frac{2^{n/\alpha}}{n} \Big\}. \tag{180}$$

Then,

$$\mathbb{P}[B(y)] \leq C 2^n n^{-\kappa} \frac{n^{2\alpha} n^{\frac{5\alpha}{1-\alpha}}}{2^{2n}}. \tag{181}$$

The summation over all sites in $\mathbb{D}(n)$ gives

$$\mathbb{P}\Big[\bigcup_{y \in \mathbb{D}(n)} B(y) \Big] \leq C n^{1+\alpha-\kappa} n^{\frac{5\alpha}{1-\alpha}} \tag{182}$$

and the claim follows easily by the Borel-Cantelli lemma taking κ large enough.

Next, we estimate the expected number of visits to $z \in \mathcal{D} \setminus \{y\}$ during an excursion that does not leave the disk. It is a well known fact that the expected number of visits of $z \in \mathbb{Z}^2$ by the simple random walk during one excursion from the origin is equal to one. So,

$$\begin{aligned} 1 &= \mathbb{E}[\# \text{ visits of } z] \\ &= \mathbb{E}[\# \text{ visits of } z | X_d \text{ does not leave } \mathcal{D}] \mathbb{P}[X_d \text{ does not leave } \mathcal{D}] \\ &+ \mathbb{E}[\# \text{ visits of } z | X_d \text{ leaves } \mathcal{D}] \mathbb{P}[X_d \text{ leaves } \mathcal{D}]. \end{aligned} \tag{183}$$

It follows that for n large enough

$$\begin{aligned} &\mathbb{E}[\# \text{ of visits of } z | X_d \text{ does not leave } \mathcal{D}] \\ &\qquad \leq \big(\mathbb{P}[\text{excursion does not leave } \mathcal{D}] \big)^{-1} \\ &\qquad \leq (1 - G_{D(2^n n^{-\kappa})}(0,0)^{-1})^{-1} \leq 1 + C/n \leq 2. \end{aligned} \tag{184}$$

The expected duration of the i-th excursion, V_i, thus satisfies

$$\mathbb{E}[V_i] \leq 2 \sum_{z \in \mathcal{D} \setminus \{y\}} \tau_z \leq 2 \sum_{z \in \mathbb{D}(n)} \tau_z \chi\{\tau_z \leq n^{-5/(1-\alpha)} \varepsilon 2^{n/\alpha}/n\}. \tag{185}$$

The last sum can be bounded using Lemma 3.2. Let $i_2(n)$ be such that $2^{-i_2(n)} \leq n^{-5/(1-\alpha)} \leq 2^{-i_2(n)+1}$. Then, a.s. for n large

$$\begin{aligned} \mathbb{E}[V_i] &\leq 2 \sum_{z \in \mathbb{D}} \tau_z \chi\{\tau_z \leq 2\} + 2 \sum_{i=i_2(n)}^{i_0(n)} \sum_{z \in T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}}} \tau_z \\ &\leq 4 \cdot 2^n n^{1-\alpha} + 2 \sum_{i=i_2(n)}^{i_0(n)} \varepsilon \frac{2^{n/\alpha}}{n} 2^{-i+1} \big| T_{\varepsilon 2^{-i}}^{\varepsilon 2^{-i+1}} \big| \\ &\leq 4 \cdot 2^n n^{1-\alpha} + C 2^{n/\alpha} \sum_{i=i_2(n)}^{i_0(n)} 2^{-(1-\alpha)i} \leq C 2^{n/\alpha} n^{-5}. \end{aligned} \tag{186}$$

Since the expected number of excursions of $\bar{U}$ before time $(1+\theta)t$ is bounded by a multiple of n, the mean of the total time spent by $\bar{U}$ during the interval $[0,(1+\theta)t+2^{n/\alpha}/n^2]$ away from y is easily bounded by $C2^{n/\alpha}n^{-4}$ for C depending on ε but not on t.

We claim that (for n sufficiently large) for any $u \in [0,(1+\theta)t]$, $\mathbb{P}[\bar{U}(u) \neq y] \leq 2C/n^2$. Suppose not. Then for some u_0, $\mathbb{P}[Y(u_0) \neq y] \geq 2C/n^2$. We have that the expected total time spent by $\bar{U}$ away from y in interval $[u_0, u_0 + 2^{n/\alpha}/n^2]$ is bounded by $C2^{n/\alpha}n^{-4}$, so there exists $v_0 \in [u_0, u_0 + 2^{n/\alpha}/n^2]$ so that $\mathbb{P}[\bar{U}(v_0) \neq y] \leq C/n^2$. On the other hand, by the Markov property for $\bar{U}$ if λ is the time of the first jump from y

$$\mathbb{P}[\bar{U}(v_0) \neq y] \geq \mathbb{P}[\bar{U}(v_0) \neq y \cap \{\lambda > v_0 - u_0\}] \\ > \frac{1}{2}\mathbb{P}[\bar{U}(u_0) \neq y] \geq C/n^2. \quad (187)$$

for n sufficiently large. This contradiction gives the desired result and with it the lemma is proven.

□

8. Proof of subaging

In this section we prove the subaging behaviour of the function $\Pi(t_w, t_w + t)$. Recall that this function has been defined as the probability that X does not jump between t_w and $t_w + t$. If we know that at time t_w the process X is in a trap y with depth τ_y, then this probability is easy to obtain, by the Markov property

$$\mathbb{P}\big[X(t') = X(t)\,\forall t' \in [t_w, t_w + t]\big|\tau_{X(t_w)}\big] = \exp\Big(-\frac{t}{\tau_{X(t_w)}}\Big). \quad (188)$$

We should thus gain an information about the depth $\tau_{X(t_w)}$. We would like to deduce its distribution from the behaviour of processes $\tilde{Y}^n$ and $\tilde{Y}$, because these are the only objects we really control. It should be obvious that the depth of the trap where X is at time t_w depends on the size of the jump of $\tilde{Y}^n$ that intersects the level $t_w/2^{n/\alpha}$. Hence, to find an expression for the function $\Pi(t_w, t_w + t)$ we should control two basic objects. First, the distribution of the size of the jump of $\tilde{Y}^n$ that intersect certain level, and second, the conditional distribution of $\tau_{X(t_w)}$ knowing the size of this jump.

We start by controlling the size of the jump. Let $\ell_n = \ell_n(s)$ be the size of the jump of $\tilde{Y}^n$ that intersect the level s,

$$\ell_n(s) = \inf\{x \in \mathcal{R}_n : x > s\} - \sup\{x \in \mathcal{R}_n : x \leq s\}, \quad (189)$$

and let $\ell = \ell(s)$ be the same size for the limiting process $\tilde{Y}$. We use μ_s^n, resp. μ_s to denote the distributions of $\ell_n(s)$ and $\ell(s)$.

The following lemma is a consequence of Proposition 6.5 and the $\mathbb{P}$-a.s. continuity of the functional $Y \to \inf\{x \in \mathcal{R}(Y) : x > s\} - \sup\{x \in \mathcal{R}(Y) : x \leq s\}$ in the Skorokhod topology on $D([0,\infty))$.

Lemma 8.1. *The sequence μ_s^n converges weakly to μ_s uniformly in $s \in [1, 2^{1/\alpha}]$, that is for every bounded continuous function g*

$$\int g(\ell)\mu_s^n(d\ell) \xrightarrow{n\to\infty} \int g(\ell)\mu_s(d\ell) \qquad \textit{uniformly in } s \in [1, 2^{1/\alpha}]. \tag{190}$$

As a consequence of the scaling invariance of $\tilde{Y}$ (recall that $\tilde{Y}$ is a stable subordinator) we get the following relation between the measures μ_s,

$$\mu_s([a,b]) = \mu_1([a/s, b/s]) \tag{191}$$

for any interval $[a,b] \subset (0,\infty)$.

The control of $\tau_{X(t_w)}$ knowing the size of the jump is more complicated. It occupies the majority of the proof of the following proposition that is a refined version of Theorem 1.2.

Proposition 8.2. *For $\mathbb{P}$-a.e. realisation of the environment $\boldsymbol{\tau}$,*

$$\lim_{t\to\infty} \Pi\Big(t, t + \frac{\theta t}{\log t}\Big) = \int_0^\infty \Big(\frac{\ell\pi}{\ell\pi + \theta\alpha}\Big)^{1+\alpha} \mu_1(d\ell) \equiv \Pi(\theta). \tag{192}$$

By an easy application of dominated convergence theorem we get

Corollary 8.3. *The function $\Pi(\theta)$ satisfies*

$$\lim_{\theta\to 0} \Pi(\theta) = 1 \qquad \textit{and} \qquad \lim_{\theta\to\infty} \Pi(\theta) = 0. \tag{193}$$

Proof of Proposition 8.2. We proceed similarly as in the proof of aging. We take $n(t)$ as in (150) and we define $s = s(t) = t/2^{n(t)/\alpha}$. Next, we choose δ_1 and δ_2, and we set the constants ε, M and m in the same manner as before. We thus know that the process $\tilde{Y}^n$ is a good approximation of the rescaled time change $\bar{S}^n$. That means that $\mathbb{P}[A] = \mathbb{P}[A_1 \cap A_2] \geq 1 - C\delta_1$ with A_1, A_2 defined as in the previous section. For the following discussion we will suppose that A occurs and we take account of the remaining part of the probability space at the end of the proof.

As we have already discussed, it is necessary to obtain the conditional distribution of $\tau_{X(t)}$ knowing $\ell_n(s)$. Similarly as in the proof of aging not much can be done if the distance between s and $\mathcal{R}_n$ is smaller than $2\delta_2$, because the approximation is not sufficiently precise. However, the probability of this bad case can be bounded by $C\delta_1 + C'\delta_2^{1-\alpha}$ uniformly in s in the same way as in (169).

Let $E = E(n,s)$ denote the event $\mathrm{dist}(s, \mathcal{R}_n) > 2\delta_2$. If E occurs, then the situation is more favourable. We know that X was at time t

inside a disk $D_y(2^n n^{-\kappa})$ around some deep trap $y = y(n,s)$. Moreover, similarly as in Lemma 7.4, we can show

$$\mathbb{P}[X(t) = y(n,s)|E(n,s)] \to 1 \quad \text{as} \quad t \to \infty. \tag{194}$$

We will thus compute the conditional distribution of $\tau_{y(n,s)}$ knowing $\ell_n(s)$ instead of the distribution of $\tau_{X(t)}$. As we have already discussed in the proof of Lemma 6.4, the size ℓ of the jump that is the result of the visit of y satisfies

$$2^{n/\alpha}\ell = \tau_y \sum_{i=1}^{\xi} e_i', \tag{195}$$

where ξ is a geometrically distributed random variable with mean

$$G_{D(2^n n^{-\kappa})}(0,0) = n \log 2/\pi + o(n) = \mathcal{K}' n + o(n), \tag{196}$$

and e_i' are i.i.d., exponential random variables with mean one. It is convenient to introduce the rescaled depth of trap, $\sigma_x = \tau_x n/2^{n/\alpha}$. Equation (195) then becomes

$$\ell = \frac{\sigma_y}{n} \sum_{i=1}^{\xi} e_i'. \tag{197}$$

As can be seen from Lemma 6.1, the distribution ν_n of σ_y converges weakly to the distribution ν given by

$$\nu(dx) = \frac{\alpha}{\varepsilon^{-\alpha} - M^{-\alpha}} \cdot \frac{1}{x^{\alpha+1}}\, dx \quad \text{for} \quad \varepsilon \le x \le M. \tag{198}$$

The random variable $n^{-1}\sum_{i=1}^{\xi} e_i'$ is an exponential random variable with mean $\mathcal{K}' + o(1)$. Let f_n denote its density, and let f denote the density of the limiting distribution,

$$f(x) = \exp\big(-x/\mathcal{K}'\big)/\mathcal{K}'. \tag{199}$$

We use F_ℓ^n to denote the distribution function of $\sigma_{y(n,s)}$ conditionally on $\ell_n(s) = \ell$,

$$F_\ell^n(a) = \mathbb{P}[\sigma_{y(n,s)} \le a | \ell_n(s) = \ell]. \tag{200}$$

Lemma 8.4. *The function F_ℓ^n can be written as*

$$F_\ell^n(a) = \frac{\int_\varepsilon^a \frac{1}{x} f_n(\frac{\ell}{x}) \nu_n(dx)}{\int_\varepsilon^M \frac{1}{x} f_n(\frac{\ell}{x}) \nu_n(dx)} \tag{201}$$

Proof. We should verify that for any event B that is measurable with respect to the σ-algebra generated by the random variable $\ell_n(s)$

$$\int_B \chi\{\sigma_y \le a\} d\mathbb{P} = \int_B F_\ell^n(a)\, d\mathbb{P}. \tag{202}$$

It is sufficient to verify the last expression for an event B that has the form $\{\ell_n(s) \in I\}$ for some interval $I \subset [0,\infty)$. The left hand side of (202) can be then written as

$$\int_B \chi\{\sigma_y \le a\} d\mathbb{P} = \int_\varepsilon^a \int_{I/x} f_n(z)\, dz\, \nu_n(dx). \tag{203}$$

To compute the right hand side we should first find the distribution of $\ell_n(s)$

$$\mathbb{P}[\ell_n(s) \le u] = \int_\varepsilon^M \int_0^{u/x} f_n(z)\, dz\, \nu_n(dx). \tag{204}$$

The right hand side of (202) then equals

$$\int_I \frac{\int_\varepsilon^a \frac{1}{x} f_n(\frac{\ell}{x})\nu_n(dx)}{\int_\varepsilon^M \frac{1}{x} f_n(\frac{\ell}{x})\nu_n(dx)} d\Big(\int_\varepsilon^M \int_0^{\ell/x} f_n(z) dz\, \nu_n(dx)\Big)$$
$$= \int_I \frac{\int_\varepsilon^a \frac{1}{x} f_n(\frac{\ell}{x})\nu_n(dx)}{\int_\varepsilon^M \frac{1}{x} f_n(\frac{\ell}{x})\nu_n(dx)} \Big(\int_\varepsilon^M \frac{1}{x} f_n(\ell/x)\, \nu_n(dx)\Big)\, d\ell. \tag{205}$$

Making the substitution $z = \ell/x$ and changing the order of integration it is easy to get the same expression as in (203). This finishes the proof. □

As an consequence of the previous lemma we get

Lemma 8.5. *For any bounded continuous function g*

$$\int g(a) dF_\ell^n(a) \xrightarrow{n\to\infty} \int g(a) dF_\ell(a), \tag{206}$$

where

$$F_\ell(a) = \frac{\int_\varepsilon^a z^{-\alpha-2} \exp(\ell/\mathcal{K}' z) dz}{\int_\varepsilon^M z^{-\alpha-2} \exp(\ell/\mathcal{K}' z) dz}. \tag{207}$$

Moreover, if $K \subset (0,\infty)$ compact and g has bounded first derivative, then the convergence is uniform in $\ell \in K$.

Proof. It is easy to show using the weak convergence of ν_n and properties of f_n that the nominator, resp. the denominator of (201), converge to

$$\int_\varepsilon^b \frac{1}{x} f\Big(\frac{\ell}{x}\Big)\, \nu(dx), \tag{208}$$

with $b = a$ resp. $b = M$. Inserting (198) and (199) into the last expression we get

$$\int_\varepsilon^b \frac{1}{x} f\Big(\frac{\ell}{x}\Big)\, \nu(dx) = \int_\varepsilon^b z^{-\alpha-2} \exp(\ell/\mathcal{K}' z)\, dz, \tag{209}$$

which proves the pointwise convergence. The uniform convergence can be then proved using standard methods. □

We have now all ingredients to finish the proof of Proposition 8.2. Let $G = G(t)$ denote the event

$$G = \big\{X(t') = X(t)\,\forall t' \in [t, t + \theta t/\log t]\big\}. \tag{210}$$

Then,

$$\begin{aligned}\mathbb{P}[G] &= \int_0^\infty \mathbb{P}[G|\ell_n(s) = \ell]\mu_s^n(d\ell)\\ &= \int_0^\infty \mathbb{P}[G|\ell \cap (A\cap E)]\big(1 - \mathbb{P}\big[(A\cap E)^c|\ell\big]\big)\mu_s^n(d\ell) \qquad (211)\\ &+ \int_0^\infty \mathbb{P}[G|\ell\cap(A\cap E)^c]\mathbb{P}[(A\cap E)^c|\ell]\mu_s^n(d\ell) =\end{aligned}$$

The second integral can be bounded by $\mathbb{P}[(A\cap E)^c] \le C\delta_1 + C'\delta_2^{1-\alpha}$. The first one can be bounded from above by

$$\int_0^\infty \mathbb{P}[G|\ell\cap(A\cap E)]\mu_s^n(d\ell) \equiv I(t) \tag{212}$$

and from below by $I(t) - C\delta_1 - C'\delta_2^{1-\alpha}$. We should thus compute the value of $I(t)$. Using (188) we get

$$I(t) = \int_0^\infty \int_\varepsilon^M \exp\Big(-\frac{\theta t n}{a 2^{n/\alpha}\log t}\Big) dF_\ell^n(a)\,\mu_s^n(d\ell). \tag{213}$$

Taking $t = s2^{n/\alpha}$ we get

$$I(s2^{n/\alpha}) = \int_0^\infty \int_\varepsilon^M \exp\Big(-\frac{\theta s\alpha}{a\log 2 + cn^{-1}\log s}\Big) dF_\ell^n(a)\,\mu_s^n(d\ell). \tag{214}$$

It is not difficult to show using Lemmas 8.5 and 8.1, uniformly for $s \in [1, 2^{1/\alpha}]$,

$$\lim_{n\to\infty} I(s2^{n/\alpha}) = \int_0^\infty \int_\varepsilon^M \exp\Big(-\frac{\theta s\alpha}{a\log 2}\Big) dF_\ell(a)\,\mu_s(d\ell) \equiv I_\infty(s). \tag{215}$$

Inserting (207) into (215) we get

$$I_\infty(s) = \int_0^\infty \int_\varepsilon^M \exp\Big(-\frac{\theta s\alpha}{a\log 2}\Big)\frac{a^{-\alpha-2}\exp(-\ell/\mathcal{K}'a)}{\int_\varepsilon^M z^{-\alpha-2}\exp(\ell/\mathcal{K}'z)\,dz}\,da\,\mu_s(d\ell). \tag{216}$$

For any $c > 0$ the integral $\int_0^\infty \exp(-c/z)z^{-\alpha-2}\,dz = c^{-\alpha-1}\Gamma(\alpha+1)$. We introduce the following notation. Let

$$g_c(\varepsilon, M) = \frac{1}{\Gamma(\alpha+1)}\Big(\int_0^\varepsilon + \int_M^\infty\Big) e^{-c/z} z^{-\alpha-2}\,dz, \tag{217}$$

and

$$d_1 = \frac{\theta s\alpha}{\log 2} + \frac{\ell}{\mathcal{K}'} \qquad \text{and} \qquad d_2 = \frac{\ell}{\mathcal{K}'}. \tag{218}$$

Then

$$I_\infty(s) = \int_0^\infty \frac{d_1^{-\alpha-1} - g_{d_1}(\varepsilon, M)}{d_2^{-\alpha-1} - g_{d_2}(\varepsilon, M)}\mu_s(d\ell). \tag{219}$$

The difference between $I_\infty(s)$ and $J(s) \equiv \int_0^\infty (d_2/d_1)^{1+\alpha}\mu_s(d\ell)$ is small for ε small and M large. To see this consider

$$\lim_{\substack{\varepsilon\to 0\\ M\to\infty}} I_\infty(s)$$
$$= \lim_{\substack{\varepsilon\to 0\\ M\to\infty}} \left[\int \frac{d_1^{-\alpha-1}}{d_2^{-\alpha-1} - g_{d_2}(\varepsilon, M)}\mu_s(d\ell) - \int \frac{g_{d_1}(\varepsilon, M)}{d_2^{-\alpha-1} - g_{d_2}(\varepsilon, M)}\mu_s(d\ell)\right]. \tag{220}$$

Both terms converge due to the monotone convergence theorem, first one to $J(s)$ and second one to 0 uniformly in s. From the scaling relation (191) we get that $J(s)$ actually does not depend on s,

$$J(1) = \int_0^\infty \left(\frac{\ell\pi}{\ell\pi + \theta\alpha}\right)^{1+\alpha}\mu_1(d\ell). \tag{221}$$

Since $\varepsilon \to 0$ and $M \to \infty$ when $\delta_1, \delta_2 \to 0$, there exists a function $h(\delta_1, \delta_2)$ such that $h(\delta_1, \delta_2) \to 0$ as $\delta_1, \delta_2 \to 0$ satisfying $|I_\infty(s) - J(1)| \le h(\delta_1, \delta_2)$ for all s. Using this, (215), (220), and the bounds in the paragraph after (211) we get that for n larger than some $n(\delta_1, \delta_2)$ and for any $s \in [1, 2^{1/\alpha}]$

$$\left|\mathbb{P}[G(s2^{n/\alpha})] - J(1)\right| \le \left(C\delta_1 + C'\delta_2^{1-\alpha} + h(\delta_1, \delta_2)\right). \tag{222}$$

Since δ_1 and δ_2 can be taken arbitrarily small, the proof is finished. □

Appendix A. Some properties of the simple random walk

We summarise here some known properties of Green's function and hitting probabilities of the simple random walk on $\mathbb{Z}^2$ that is killed when it exits the disk D with radius r. Let ξ denote the exit time from this disk.

The most important formula that we use repeatedly is

$$\mathbb{P}[X \text{ hits } x \text{ before } \xi] = 1 - \frac{\log|x|}{\log r} + O\left(\frac{|x|^{-2}}{\log r}\right) + O(\log^{-2} r). \tag{223}$$

The proof of it can be found for example in Lawler [Law91], Proposition 1.6.7. We use also a similar expansion for the Green's function,

$$G_D(0, x) = \frac{2}{\pi}(\log r - \log|x|) + O(|x|^{-2}) + O(r^{-1}). \tag{224}$$

For $G_D(0,0)$ there is the following formula ([Law91], Theorem 1.6.6)

$$G_D(0,0) = \frac{2}{\pi}\log r + k + O(r^{-1}). \tag{225}$$

As an easy consequence of formula (223) we get following lemma:

Lemma A.1. *Let y be an uniformly chosen point in D. Then there exists constant C and $b < 1$ independent of r such that*

$$\mathbb{E}\left[\exp\left(b\log r\,\mathbb{P}[X \textit{ hits } y \textit{ before } \xi]\right)\right] \le C. \tag{226}$$

Proof. Let a be a positive constant and let D_a denotes the disk with radius a. Then by (223) we have

$$\begin{aligned}
&\mathbb{E}\big[\exp\big(b\log r\mathbb{P}[X \text{ hits } y \text{ before } \xi]\big)\big] \\
&\quad\le \frac{1}{\pi r^2}\sum_{y\in D_a}\exp(b\log r)+\frac{1}{\pi r^2}\sum_{D\setminus D_a}\exp\big(b\log r\mathbb{P}[X \text{ hits } y \text{ before } \xi]\big) \\
&\quad\le \frac{C}{r^{2-b}}+\frac{1}{\pi r^2}\sum_{y\in D\setminus D_a}\exp\{b\log r-b\log|y|+O(|y|^{-2})+O(\log^{-1}r)\} \\
&\quad\le C+\frac{1}{\pi r^{2-b}}\sum_{y\in D\setminus D_a}\frac{C}{|y|^b}\le C+Cr^{b-1}\int_a^r y^{-b}\,dy\le C.
\end{aligned} \tag{227}$$

This finishes the proof. □

Similarly we get

Lemma A.2. *There exist $\lambda>0$ and C independent of r such that*

$$\mathbb{E}[\exp(\lambda G_D(0,y))]\le C. \tag{228}$$

Appendix B. Some properties of stable subordinators

Let Y be a stable subordinator with the Lévy measure

$$\pi(dx)=kx^{-\alpha-1}\chi\{x\ge 0\}\,dx,\qquad k>0. \tag{229}$$

We use $\mathcal{R}=\mathcal{R}(Y)$ to denote the range of this process. Let $U(dx)$ denote its potential measure that is defined by

$$U(A)=\int_0^\infty \mathbb{P}(Y(t)\in A)\,dt\quad\text{for any } A\in\mathcal{B}(\mathbb{R}). \tag{230}$$

For every $x>0$, let

$$g(x)=\sup\{y\in\mathcal{R}: y\le x\}, \tag{231}$$

and let

$$d(x)=\inf\{y\in\mathcal{R}: y\ge x\}. \tag{232}$$

Then it follows from Bertoin [Ber96], Theorems III.2, III.6, and the discussion following the second theorem that

Lemma B.1. *(i) For each fixed $x\ge 0$ and every $0\le y\le x<z$, we have*

$$\mathbb{P}(g(x)\in dy, d(x)\in dz)=U(dy)\pi(dz-y). \tag{233}$$

(ii) For every $x>0$ the random variable $x^{-1}g(x)$ has the distribution

$$\frac{s^{\alpha-1}(1-s)^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}ds=\frac{\sin\alpha\pi}{\pi}s^{\alpha-1}(1-s)^{-\alpha}ds\qquad(0<s<1). \tag{234}$$

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