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Qualitative theory and identification of a class of mechanical systems

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1. Introduction

We consider mechanical systems which can be described by the scalar differential equation

$$m\ddot{y} + h(y, \dot{y}) + r(y) = 0\,, \tag{1.1}$$

where m is the mass, h describes the damping force and r is the restoring force.
We suppose that we have some preliminary knowledge about h and r, e.g. h has the form $h(y, \dot{y}) = \dot{y}\, g(y)$. Our goal is to get a more detailed description of h and r by applying some periodic force (excitation) to (1.1), that means, by studying the system

$$m\ddot{y} + h(y, \dot{y}) + r(y) = P(t), \quad P(t+\omega) = P(t). \tag{1.2}$$

The qualitative behaviour of the autonomous system (1.1) can be determined by investigating the singular trajectories (equilibria, limit cycles, separatrices) and their stability in the $(y, \dot{y})$ - plane, the so called phase plane [1]. The acceleration $\ddot{y}$ is uniquely determined by the state y and the velocity $\dot{y}$ according to equation (1.1). The qualitative study of the ω - periodic system (1.2) is based on the investigation of the Poincaré map in the phase plane [3].
As mentioned above, our goal is to improve our knowledge of the functions h and r by applying a periodic force to the equation under consideration.
In practice, we have to do measurements in order to identify unknown parameters and functions. Usually, in this process the method of least squares plays an imported role. The aim of this note is to show that there is another approach in order to get more information about the unknown or only partly known system characteristics, provided we are able to measure not only state and velocity of the mechanical system under periodic excitation but also the acceleration. The role of acceleration in studying mechanical systems (1.1) has been demonstrated in [2]. In what follows we emphasize its importance also in investigating inverse problems.

2. Test of non-linearity

If we do not know the function h and r, the first question is the following: Can we conclude from measurements whether the system is linear or non-linear?
We denote by $\{\Pi_k\}=\{\bar{y}_k, \dot{\bar{y}}_k, \ddot{\bar{y}}_k\}$, $k=1,\ldots,n$, a sequence of points describing the measured state, velocity and acceleration of system (1.2) at the moment $t=t_k=t_0+k\,\omega$. If we represent these points in the extended phase space $(y, \dot{y}, \ddot{y})$ we get a set of points parameterised by the time t_k (see Fig.1).

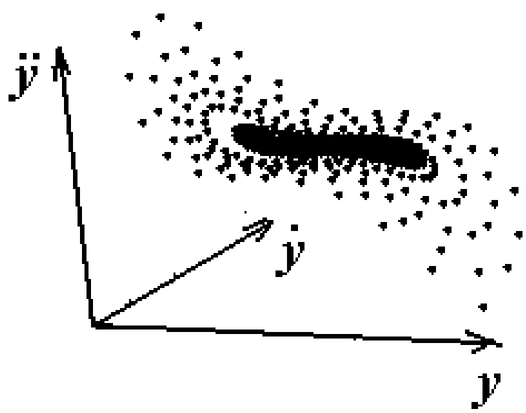

Fig. 1: Set of measured points $\prod_k$ in the extended phase space

In the ideal case, when we have no measuring error, it holds

$$m\,\ddot{\bar{y}}_k + h(\bar{y}_k, \dot{\bar{y}}_k) + r(\bar{y}_k) = C \text{ for } \quad k=1,\ldots,n, \tag{2.1}$$

where $C=P(t_o)=P(t_k)$ for all k. That mean, all points $\prod_k$ are located on the surface defined by $m\,w+h(u,v)+r(u)=C$ in the (u,v,w) - space. If h and r are linear functions, then all points $\prod_k$ must be located in a plane E. Therefore, if there are two numbers α_1 and α_2 such that all points $\prod_k$ satisfy

$$m\,\ddot{\bar{y}}_k + \alpha_1\,\dot{\bar{y}}_k + \alpha_2\,\bar{y}_k = C \text{ for } k=1,\ldots,n, \tag{2.2}$$

then this is an indicator that (1.2) represents a linear system. If we replace the force $P(t)$ by $\beta P(t)$, $\beta > 0$, and the corresponding set $\Pi_k^{(\beta)}$ satisfies $\Pi_k^{(\beta)} = \beta \Pi_k$, then this is another indicator of linearity.

Of course, in practice we have some measure error. If we can find constants α_1 and α_2 so that all measured points are located near the plane defined by α_1, α_2 and C, then we can conclude that system (1.2) is linear or weakly non-linear.

If we project the sequence $\{\Pi_k\}$ along the plane E into the planes $(\bar{y}, \dot{\bar{y}})$, $(\ddot{\bar{y}}, \bar{y})$ and $(\ddot{\bar{y}}, \dot{\bar{y}})$, then all points are located on a straight line.

As an example, we consider the linear equation

$$\ddot{y} + 0.1\dot{y} + y = 0.$$

We assume that the behaviour of the system until the time $t = 0$ is characterized by the stable equilibrium state $y = 0$. At the moment $t = 0$ we apply the external force $\cos(t)$ and investigate numerically the corresponding initial value problem

$$\ddot{y} + 0.1\dot{y} + y = \cos, \ y(0) = \dot{y}(0) = 0. \tag{2.3}$$

As result we obtain the sequence of points $\{\Pi_k\} = \{\bar{y}_k, \dot{\bar{y}}_k, \ddot{\bar{y}}_k\}$ describing state, velocity and acceleration at the moment $t = t_k = 2\pi k$, $k = 0, 1, 2\dots$. It is obvious that all these points satisfy the relationship

$$\ddot{\bar{y}}_k + 0.1\dot{\bar{y}}_k + \bar{y}_k = 1.$$

If we introduce the notation $F(k) := \ddot{\bar{y}}_k + 0.1\dot{\bar{y}}_k + \bar{y}_k$, then Fig.2. shows that the points $\{\Pi_k\}$ define a straight line in the (k, F) - plane.

Next we investigate a system described by the equation

$$\ddot{y} + 0.1\dot{y} + y + \beta y^3 = 0. \tag{2.4}$$

To test this system we apply the force $\cos t$, investigate the initial value problem

$$\ddot{y} + 0.1\dot{y} + y + \beta y^3 = \cos, \ y(0) = \dot{y}(0) = 0 \tag{2.3}$$

and compute the corresponding sequence $\{\Pi^{\beta}{}_k\}$ satisfying

$$\ddot{\bar{y}}_k + 0.1\dot{\bar{y}}_k + \bar{y}_k = 1 - \beta \bar{y}_k^3.$$

Fig.2. shows that in the cases $\beta = 0.2; 0.3; 0.5$ these points do not define a straight line, and that the deviation from a straight line increases with increasing β.

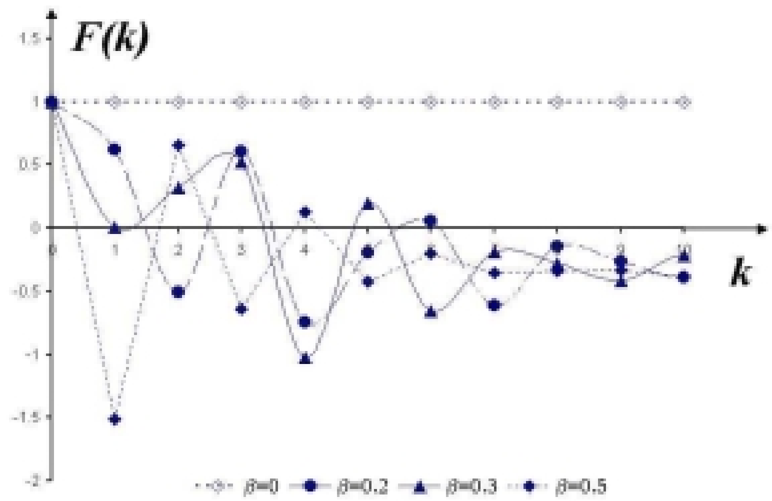

Fig. 2: Representation of the sequences Π_k^β in the (k,F) - plane.

3. System with unknown damping force or unknown restoring force

We consider the differential equation

$$m\,\ddot{y} + h(y,\dot{y}) + r(y) = 0\,, \tag{3.1}$$

where one of the functions r and h is known. In order to determine the unknown function we suppose that the system is in the stable equilibrium state $y = 0$ and apply at the moment $t = 0$ the ω - periodic force $P(t)$, that is, we consider the initial value problem

$$m\,\ddot{y} + h(y,\dot{y}) + r(y) = P(t),\quad y(0) = 0\,,\ \dot{y}(0) = 0 \tag{3.2}$$

and measure the acceleration $\ddot{y}$, the velocity $\dot{y}$ and the state y at the moments $t = t_k = k\omega$, $k = 1, 2, \ldots n$. The obtained result is denoted by $\{\Pi_k\} = \{\bar{y}_k, \dot{\bar{y}}_k, \ddot{\bar{y}}_k\}$. Since $P(t_k) = P(0) = c$, all points Π_k satisfy

$$h(\bar{y}_k, \dot{\bar{y}}_k) + r(\bar{y}_k) = c - m\ddot{\bar{y}}_k ,$$

provided we have no measure error. In case that h is given we plot the result in the $(y, r(y))$- plane and get a parameterised representation of $r(y)$. If the obtained points are not enough to get a satisfactory representation of $r(y)$ we can also consider the initial value problem

$$m\ddot{y} + h(y, \dot{y}) + r(y) = \alpha P(t) + \beta , \quad y(0) = 0 , \; \dot{y}(0) = 0 , \tag{3.3}$$

where α and β are any numbers. Analogously we can determine h parametrically in the $(y, \dot{y}, h(y, \dot{y}))$ - space.

As an example, we consider the non-linear equation

$$\ddot{y} + 0.015\dot{y} + r(y) = 0$$

with $r(y) = y + y^3$.

If we apply the force $\alpha \cos t + \beta$ and study the initial value problem

$$\ddot{y} + 0.015\dot{y} + r(y) = \alpha \cos t + \beta, \; y(0) = 0 , \; \dot{y}(0) = 0$$

we can compute the corresponding sequence $\{\Pi_k\}$ representing the function $r(y) = y + y^3$ (see Fig. 3 and Fig. 4).

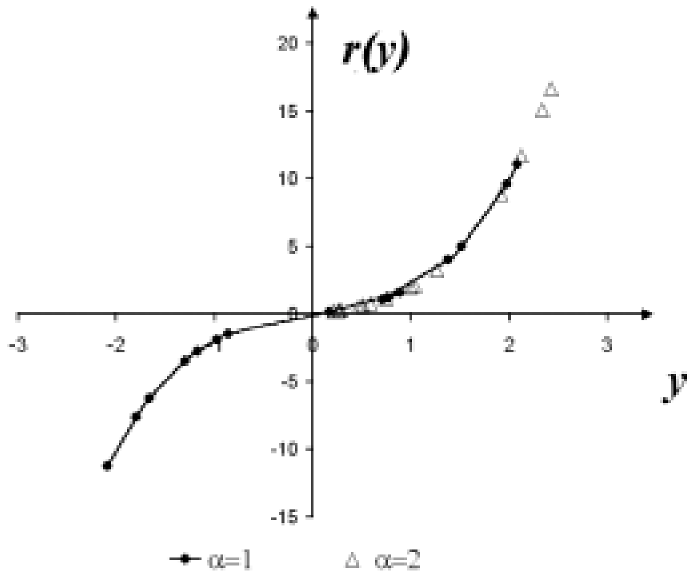

Fig. 3: Parametrised representation of $r(y)$ by Π_k^{α} with $\alpha = 1; 2$ and $\beta = 0$.

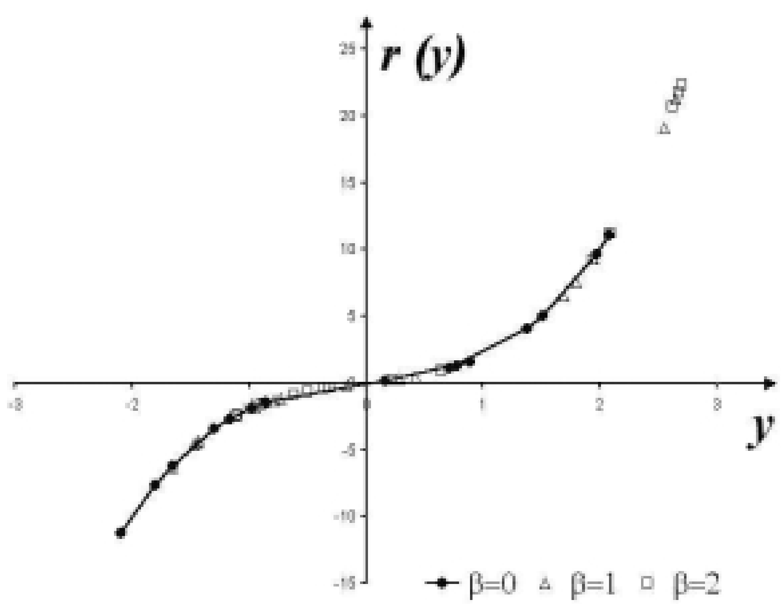

Fig. 4: Parametrised representation of $r(y)$ by Π_k^{α} with $\beta = -2;\ 2$ and $\alpha = 1$.

4. Systems with large mass

We consider the differential equation

$$m\,\ddot{y} + h(y, \dot{y}) + r(y) = 0\,, \tag{4.1}$$

where we assume that m is given and that the damping force h has the form $h(y, \dot{y}) = \dot{y}^k\, g(y)$, $k \geq 1$. We suppose that neither g nor r are known. To determine the unknown function r we consider the initial value problem

$$m\,\ddot{y} + \dot{y}^k\, g(y) + r(y) = P(t)\ , \quad y(0) = 0\,,\ \dot{y}(0) = 0\,, \tag{4.2}$$

where P is ω - periodic with ω >> 1. Introducing the slow time τ by $t \;= \omega\tau$, and using the notation $z(\omega\,\tau) = \;\tilde{z}(\tau)$, we get from (4.2)

$$\frac{m}{\omega^{\,2}}\tilde{y}'' + \frac{\tilde{y}'^k}{\omega^{\,k}} g(\tilde{y}) + r(\tilde{y}) = \tilde{P}\,(\tau)\ ,\ \tilde{y}(0) = 0\,,\ \tilde{y}'(0) = 0\,. \tag{4.3}$$

Setting $\omega = \sqrt{m}, \varepsilon = m^{-k/2}$ we obtain

$$\tilde{y}'' + \varepsilon\,\tilde{y}'^k\, g(\tilde{y}) + r(\tilde{y}) = \tilde{P}\,(\tau)\ . \tag{4.4}$$

If we suppose that ε is small (that is m is large), then we can study the initial value problem

$$\tilde{y}'' + r(\tilde{y}) = \tilde{P}\,(\tau)\ ,\ \tilde{y}(0) = 0\,,\ \tilde{y}'(0) = 0\,.$$

We denote the solution of this problem for $t = t_k = k\,\omega$ by $\bar{y}_k$. Thus, we have

$$\bar{y}_k'' + r(\bar{y}_k) = \ \bar{P}\,(t_k)\ = c\ \text{ for } k = 0, 1, \ldots n\,,$$

and we can determine $r\,(y)$ parametrically from the relations

$$r(\bar{y}_k)\ \ = c - \ \bar{y}_k''\ \ ,\ k = 0, 1, \ldots n\,. \tag{4.5}$$

As mentioned above, if we need more points to determine $r\,(y)$, then we can consider the modified problem

$$\tilde{y}'' + r(\tilde{y}) = \alpha\ \tilde{P}(\tau) + \beta,\ \tilde{y}(0) = 0,\ \tilde{y}'(0) = 0,$$

where α and β are real numbers.
If we have determined r, we can use (4.4) to obtain a parameterised representation of $g(y)$.
As an example, we consider the non-linear equation

$$100\,\ddot{y} + 0.015\dot{y} + y + y^3 = 0$$

According to our investigation above we arrive at the initial value problem

$$100\ddot{\tilde{y}} + 0.015\dot{\tilde{y}} + \tilde{y} + \tilde{y}^3 = \tilde{P}(\tau), \tilde{y}(0) = 0,\ \tilde{y}'(0) = 0, \tag{4.6}$$

as $P(\tau)$ we choose $\alpha\cos\tau + \beta$.
The investigation of (4.6) yields the sequence of points $\{\Pi_k^{\alpha,\beta}\}$. Representing $\{\Pi_k^{\alpha,\beta}\}$ in the $(y, r(\tilde{y}))$ - plane for $\beta = 0$ and $\alpha = 10, 20$ we get the following picture.

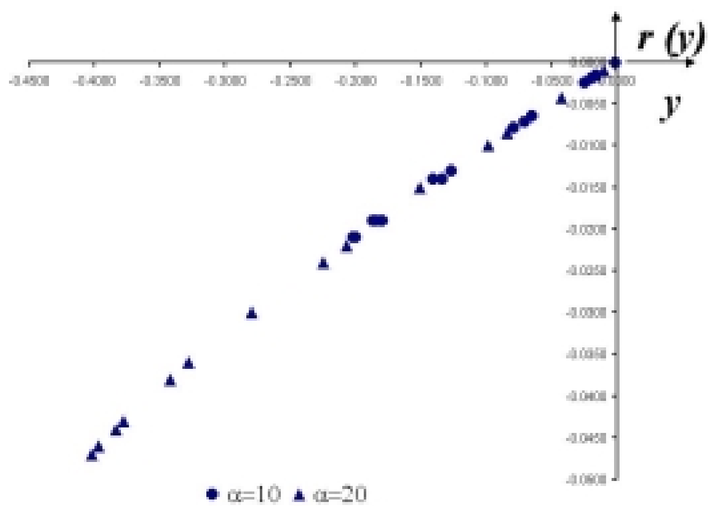

Fig. 5: Influence of the parameter α to the dependency $r(y)$ for the fixed value of $\beta = 0$.

This picture shows that the sequence $\{\Pi_k^{\alpha,\beta}\}$ determines the behaviour of r only for $-0.45 \le \tilde{y} \le 0$. To obtain more points we apply the force $20\cos\tau + \beta$. Fig.6 shows the corresponding parametrical representation of r which sufficiently accurate.

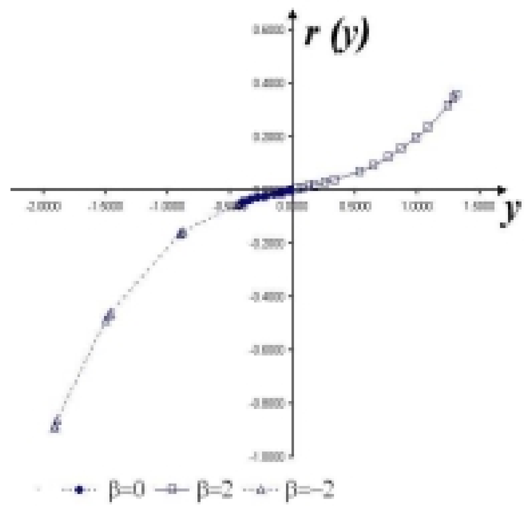

Fig. 6: Influence of the parameter β on the dependency $r(y)$ for the fixed value of $\alpha = 20$.

As it seen from Fig. 7, the results obtained by the (4.5) are close to exact soluton. All points lay in one curve.

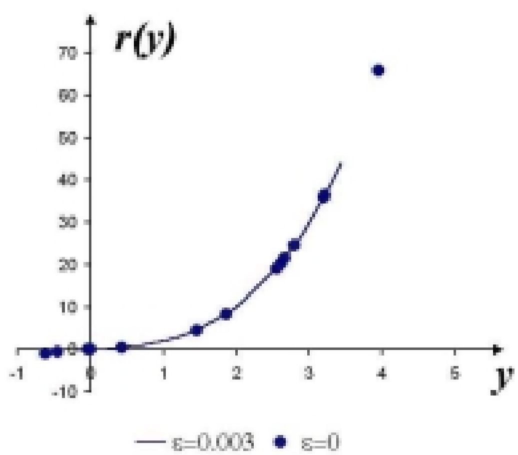

Fig. 7: Comparisson of the results obtained by exact formula and dependence (4.5).

5. System with a special damping force

We consider a mechanical system described by the differential equation

$$m\,\ddot{y} + \dot{y}^2\,g(y) + r(y) = 0 \tag{5.1}$$

with $g(0) > 0;\ r(0) = 0$. We assume that m is given, and that $y = 0$ is a stable equilibrium state of (5.1). In order to determine $g(y)$ and $r(y)$ we apply an ω - periodic force to (5.1), that is, we investigate the initial value problem

$$m\,\ddot{y} + \dot{y}^2\,g(y) + r(y) = P(t)\ ,\ \ y(0) = 0\,,\ \dot{y}(0) = 0\,. \tag{5.2}$$

Using the transformation $t\ \ = \omega\tau$, we get from (5.2)

$$\frac{m}{\omega^{\ 2}}\,\tilde{y}'' + \frac{\tilde{y}'^2}{\omega^{\ 2}}\,g(\tilde{y}) + r(\tilde{y}) = \tilde{P}\ (\tau)\ ,\ \tilde{y}(0) = 0\,,\ \tilde{y}'(0) = 0\,. \tag{5.3}$$

If we assume that $\omega = m$, then we obtain from (5.3)

$$\tilde{y}'' + \tilde{y}'^2\,g(\tilde{y}) + \omega\,r(\tilde{y}) = \omega\tilde{P}\ (\tau)\ \ \tilde{y}(0) = 0\,,\ \tilde{y}'(0) = 0\,. \tag{5.4}$$

Now we suppose $\omega << 1$. In that case the initial value problem

$$\tilde{y}'' + \tilde{y}'^{\,2}\,g\,(\tilde{y}) = 0\ \ ,\ \tilde{y}(0) = 0\,,\ \tilde{y}'(0) = 0$$

is an approximation of the initial value problem (5.3). If we suppose that the experimental investigation of (5.3) yields the sequence of measurements $\{\Pi_k\} = \{\overline{y}_k, \dot{\overline{y}}_k, \ddot{\overline{y}}_k\}$, then we can get a parameterised representation of $g(\tilde{y})$ by mean of the relation

$$g(\tilde{y}) = -\frac{\tilde{y}''_k}{\tilde{y}'^{\,2}_k}\,,\ k = 0, 1, \ldots n\,.$$

To obtain more points for parameterised representation of $g(\tilde{y})$ we may proceed as in the section before, also in order to determine $r(y)$.
As an example we consider the dynamical system described by the equation

$$m\,\ddot{y} + \dot{y}\left(0.1 + 100\,y^2\right) + y = 0\,.$$

To test this system we apply the force $\alpha \cos \omega t + \beta$ where $\omega = 0.00001; 0.00005; 0.0001$ and $\alpha = 1$; $\beta = -1$. According to our investigation above we arrive at the initial value problem

$$\ddot{\tilde{y}} + \dot{\tilde{y}}^2\left(0.1 + 100\,\tilde{y}^2\right) + \omega\,\tilde{y} = \alpha\cos\tau + \beta\,,\quad \tilde{y}(0) = 0\,,\ \tilde{y}'(0) = 0\,, \tag{5.4}$$

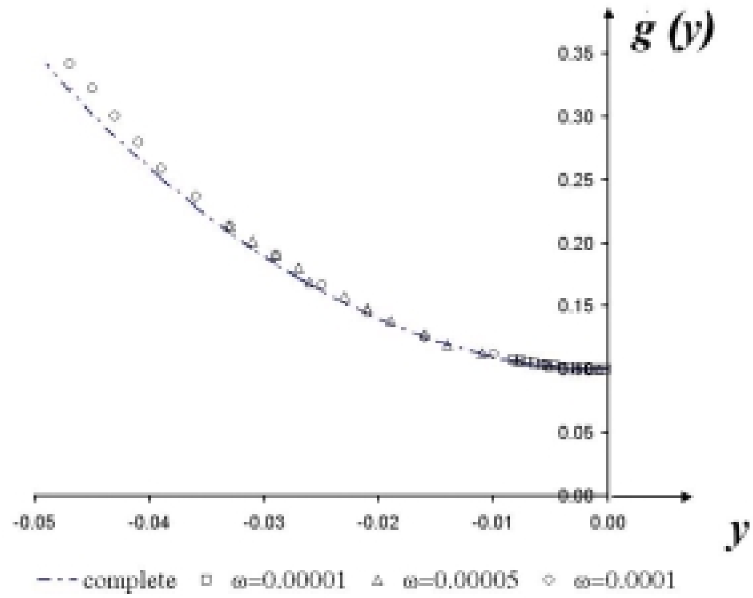

Fig. 8: Influence of the parameter ω to the dependency $g(y)$ for the fixed value of $\beta = -1$ and $\alpha = 1$.

Summary

We have considered mechanical systems which can be described by the nonlinear differential equation $m\,\ddot{y} + h(y, \dot{y}) + r(y) = 0$ and which are characterized by the property that there exists no information or only partial information on the damping force *h* or the restoring force *r*. We have charaterized several classes of such systems where by applying a periodic force and by measuring acceleration, velocity and displacement, the functions *h* or *r* can be easily identified.

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