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Nonlocal phase-field models for non-isothermal phase transitions and hysteresis

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Abstract

In this paper a nonlocal phase-field model for non-isothermal phase transitions with a non-conserved order parameter is studied. The paper complements recent investigations by S. Zheng and the second author and treats the case when the part of the free energy density forcing the order parameter to attain values within the physically meaningful range $[0,1]$ is not given by a logarithmic expression but by the indicator function of $[0,1]$. The resulting field equations form a system of integro-partial differential inclusions that are highly nonlinearly coupled. For this system, results concerning global existence, uniqueness and large-time asymptotic behaviour are derived. The main results are proved by first transforming the system of inclusions into an equivalent system of equations in which hysteresis operators occur, and then employing techniques similar to those recently developed by the authors for phase-field systems involving hysteresis operators.

1 Introduction

In a number of recent papers (see, for instance, [1], [3], [6] and the references given therein), integrodifferential (nonlocal) models for isothermal phase transitions with either conserved or non-conserved order parameters have been studied, leading to a number of results concerning existence, uniqueness, and asymptotic behaviour of solutions. In the recent papers [5], [12] the more difficult *non-isothermal* case has been treated, where [5] studied conserved order parameters in phase separation phenomena and [12] non-conserved order parameters. In both [5], [12] the corresponding free energy density was assumed to contain a logarithmic part that forces the order parameter to attain values within the physically meaningful range $[0,1]$. In this paper, we complement the results of [12] by investigating the case when the logarithmic part is replaced by the indicator function $I_{[0,1]}$ of the interval $[0,1]$. Results concerning existence, uniqueness and asymptotic behaviour for $t \to +\infty$ resembling those established in [12] for the smooth case will also be proved for this non-smooth case. As it turns out, the results are even more complete than those of [12] since a certain crucial assumption is not needed in our setting.

To give a complete description of the corresponding mathematical problem, consider non-isothermal phase transitions occurring in a thermally insulated container $\Omega \subset \mathbb{R}^N$ that forms an open and bounded domain with Lipschitzian boundary $\partial\Omega$. The physical process is described by the time evolution of a non-conserved order parameter $\chi \in [0,1]$ and of the absolute temperature $\theta \geq 0$. If we denote

$\Omega_T := \Omega \times (0,T)$, where $T > 0$ is some final time, and if $\mathbf{n}$ is the outward unit normal to $\partial\Omega$, then the resulting model equations have the form

$$\mu(\theta)\,\chi_t + \theta\,F_1'(\chi) + F_2'(\chi) + Q[\chi] \in -\,\partial I_{[0,1]}(\chi)\,, \quad \text{in } \Omega_T\,, \tag{1.1}$$

$$Q[\chi](x,t) = \int_\Omega K(x-y)\,(1-2\,\chi(y,t))\,dy\,, \quad \text{in } \Omega_T\,, \tag{1.2}$$

$$C_V\,\theta_t + (F_2'(\chi) + Q[\chi])\,\chi_t - \kappa\,\Delta\theta = 0\,, \quad \text{in } \Omega_T\,, \tag{1.3}$$

$$\frac{\partial\theta}{\partial\mathbf{n}} = 0\,, \quad \text{on } \partial\Omega \times (0,T)\,, \tag{1.4}$$

$$\chi(\cdot,0) = \chi_0\,, \quad \theta(\cdot,0) = \theta_0\,, \quad \text{in } \Omega \tag{1.5}$$

with a given kernel function $K : \mathbb{R}^N \to [0,\infty)$ such that $K(x) = K(-x)$ for all $x \in \mathbb{R}^N$. A canonical choice for K consists in considering a function $K_1 : [0,\infty) \to [0,\infty)$ and putting $K(x) = K_1(|x|)$ for $x \in \mathbb{R}^N$. Indeed, we may always assume that K is defined only on the set $\Omega - \Omega$ and extend it by 0 outside.

System (1.1)–(1.5) forms an initial-boundary value problem for a system in which an integrodifferential inclusion is coupled to a parabolic differential equation. It is the aim of this work to prove results concerning its well-posedness and large-time asymptotic behaviour (see Theorems 2.2 and 5.1 below).

Before going into mathematical details, we give a brief derivation of system (1.1)–(1.5). To this end, suppose that the order parameter χ represents the local volume fraction (concentration) of one of the phases, say, of the high temperature phase. For instance, if a solid-liquid transition is considered, the sets $\{\chi = 0\}$, $\{\chi = 1\}$, and $\{0 < \chi < 1\}$, correspond to solid, liquid, and mushy region, in that order. We start from the non-local free energy density

$$\begin{aligned} F(\chi,\theta) &= C_V\,\theta\,(1-\ln(\theta)) + \theta\,F_1(\chi) + F_2(\chi) + \theta\,I_{[0,1]}(\chi) \\ &\quad + \chi\int_\Omega K(x-y)\,(1-\chi(y))\,dy\,. \end{aligned} \tag{1.6}$$

Here, $C_V > 0$ is the specific heat. The functions F_1, F_2 are smooth where F_2 is usually concave (often a linear function or a quadratic function having a negative leading term). Typical choices are $F_1(\chi) = -L\,\chi/\theta_c$, $F_2(\chi) = L\,\chi$, where $L > 0$ and $\theta_c > 0$ represent latent heat of phase transition and phase transition temperature, respectively. Moreover, $I_{[0,1]}(\chi) = \left\{ \begin{array}{ll} 0 & \text{if } \chi \in [0,1] \\ +\infty & \text{otherwise} \end{array} \right\}$ is the indicator function of $[0,1]$, and

$$\partial I_{[0,1]}(\chi) = \begin{cases} (-\infty,0] & \text{if } \chi = 0 \\ \{0\} & \text{if } 0 < \chi < 1 \\ [0,+\infty) & \text{if } \chi = 1 \end{cases}$$

denotes its subdifferential. Note that the system corresponding to (1.1)–(1.5) can be viewed as a non-local version of a relaxed Stefan problem of Penrose-Fife type (cf. [4], [10]).

Following the rules of thermodynamics, we introduce the densities of entropy S and internal energy E by

$$\begin{aligned} S(\chi,\theta) &= -\delta_\theta F(\chi,\theta) = C_V \ln(\theta) - F_1(\chi) - I_{[0,1]}(\chi)\,, \\ E(\chi,\theta) &= F(\chi,\theta) + \theta\, S(\chi,\theta) \\ &= C_V\,\theta + F_2(\chi) + \chi \textstyle\int_\Omega K(x-y)\,(1-\chi(y))\,dy\,, \end{aligned} \tag{1.7}$$

where δ_θ denotes the variation with respect to θ. To find equilibrium values for χ and θ, we maximize the total entropy functional

$$\mathcal{S}[\chi,\theta] := \int_\Omega S(\chi,\theta)\,dx = \int_\Omega \Big(C_V \ln(\theta) - F_1(\chi) - I_{[0,1]}(\chi) \Big)\,dx \tag{1.8}$$

under the constraint that total internal energy be conserved, i. e. that

$$\begin{aligned} \mathcal{E}[\chi,\theta] &:= \int_\Omega E(\chi,\theta)\,dx \\ &= \int_\Omega \Big(C_V\,\theta + F_2(\chi) + \chi \int_\Omega K(x-y)\,(1-\chi(y))\,dy \Big)\,dx = \text{const.} \end{aligned} \tag{1.9}$$

Applying Lagrange's method, we maximize the augmented entropy

$$\mathcal{S}_\lambda[\chi,\theta] := \mathcal{S}[\chi,\theta] + \lambda\,\mathcal{E}[\chi,\theta]\,. \tag{1.10}$$

The search for critical points leads to the Euler-Lagrange equations

$$\begin{aligned} \delta_\chi \mathcal{S}_\lambda &= -F_1'(\chi) - \partial I_{[0,1]}(\chi) + \lambda\,F_2'(\chi) + \lambda\,Q[\chi] \ni 0\,, \\ \delta_\theta \mathcal{S}_\lambda &= \tfrac{C_V}{\theta} + \lambda\,C_V = 0\,, \end{aligned} \tag{1.11}$$

with $Q[\chi]$ given by (1.2). From the second identity in (1.11) the Lagrange multiplier is easily identified as $\lambda = -1/\theta$.

We now postulate that the evolution of χ runs in the direction of $\delta_\chi \mathcal{S}_\lambda$ at a rate which is proportional to it. More precisely, we assume that the evolution of χ is governed by the equation $\hat\mu(\theta)\,\chi_t = \delta_\chi \mathcal{S}_\lambda[\chi,\theta]$ which is identical to (1.1) with $\mu(\theta) = \theta\,\hat\mu(\theta)$.

To derive an evolution equation for the temperature, we have to keep in mind the energy conservation law (1.9), that is,

$$\frac{d}{dt}\mathcal{E}[\chi,\theta] = 0\,, \tag{1.12}$$

or equivalently

$$\int_\Omega (C_V\,\theta_t + (F_2'(\chi) + Q[\chi])\,\chi_t)\,dx = 0\,. \tag{1.13}$$

Formally, by (1.13), there exists a vector function $\mathbf{q}$ (the *heat flux*) such that $\mathbf{q}\cdot\mathbf{n} = 0$ on $\partial\Omega$ and

$$C_V\,\theta_t + (F_2'(\chi) + Q[\chi])\,\chi_t + \nabla\cdot\mathbf{q} = 0\,. \tag{1.14}$$

Assuming the Fourier law $\mathbf{q} = -\kappa\nabla\theta$, where $\kappa > 0$ denotes the constant heat conductivity, we obtain (1.3), (1.4) as energy balance.

Next, we study the thermodynamic consistency of the model. Assuming that $\theta > 0$ (which will have to be verified below), we obtain from a straightforward calculation, using (1.1), (1.14), and the boundary condition (1.4), that

$$\int_\Omega \left[\frac{dS}{dt}(\chi,\theta) + \nabla\cdot\left(\frac{\mathbf{q}}{\theta}\right)\right] dx = \int_\Omega \left[\frac{dS}{dt}(\chi,\theta) - \frac{\nabla\cdot\mathbf{q}}{\theta} + \frac{\kappa}{\theta^2}|\nabla\theta|^2\right] dx$$
$$= \int_\Omega \left[\frac{\kappa}{\theta^2}|\nabla\theta|^2 + \mu(\theta)\,\chi_t^2\right] dx \geq 0. \quad (1.15)$$

Therefore, the Clausius-Duhem inequality (i. e. the Second Principle of Thermodynamics) is satisfied in integrated form.

The main mathematical novelties of the results stated below in comparison to other non-isothermal phase-field models for non-conserved order parameters lie in the occurrence of the integral expression $Q[\chi]$ in the equations and in the fact that in (1.1) the indicator function $I_{[0,1]}$ occurs while no diffusive term is present. This entails a loss of spatial smoothness of the unknown χ. In comparison with the recent paper [12] the main difference is that in [12] the expression $\theta\, I_{[0,1]}$ in the free energy density (1.6) was replaced by a smooth expression of the form

$$(\beta_1 + \beta_2\,\theta)\,F_3(\chi)\,, \quad (1.16)$$

with a nonlinearity $F_3 : (0,1) \to \mathbb{R}$ which is typically of the form

$$F_3(\chi) = \chi\ln(\chi) + (1-\chi)\ln(1-\chi)\,. \quad (1.17)$$

Note that in [12] it has been necessary to assume that both β_1 and β_2 are positive, while in this paper we only need the positivity of β_2 (which is here normalized to unity).

The plan of this paper is as follows: In Section 2, we transform the system (1.1)–(1.5) into an equivalent system without inclusions by introducing the so-called *generalized freezing index* as a new variable replacing χ. The resulting system of equations contains nonlinearities of *hysteresis* type at several places and has exactly the same form as the one considered in [8]. However, due to the presence of the nonlocal term (1.2), one of the involved hysteresis operators does not meet the conditions imposed in [8]. Therefore, the line of argumentation employed there to prove existence and uniqueness needs to be modified substantially.

We prepare the existence and uniqueness proof, which will be carried out in Section 4, by studying a related "ordinary" differential equation in Section 3. In the final Section 5, we modify techniques developed in [9] in order to prove a result concerning the asymptotic behaviour as $t \to +\infty$.

In what follows, the norms of the standard Lebesgue spaces $L^p(\Omega)$, for $1 \leq p \leq \infty$, will be denoted by $|\cdot|_p$. Finally, we shall use the usual denotations $W^{m,p}(\Omega)$ and $H^m(\Omega)$, $m \in \mathbb{N}$, $1 \leq p \leq \infty$, for the standard Sobolev spaces.

2 The associated hysteresis system

In this section, we construct and study a system of differential equations involving hysteresis operators which is equivalent to system (1.1)–(1.5). We recall (cf. [2], [7]) that a mapping $\mathcal{H} : C[0,T] \to C[0,T]$ is called a *hysteresis operator* if it is

(i) *causal*, that is, the implication $u(t) = v(t) \quad \forall\, t \in [0,t_0] \ \Rightarrow\ \mathcal{H}[u](t_0) = \mathcal{H}[v](t_0)$ holds for every $u, v \in C[0,T]$ and $t_0 \in [0,T]$,

and

(ii) *rate-independent*, that is, for every $u \in C[0,T]$ and every continuous increasing function α mapping $[0,T]$ onto $[0,T]$ we have $\mathcal{H}[u \circ \alpha](t) = \mathcal{H}[u](\alpha(t))$ for all $t \in [0,T]$.

Let us note (cf. [7]) that hysteresis operators are exactly those that admit a local representation by means of superposition operators in each interval of monotonicity of the input, with a possible branching when the input changes direction. We also note that hysteresis operators have a natural extension to input functions u depending on both time and space variables: Given a hysteresis operators $\mathcal{H}$, we define the operator $\hat{\mathcal{H}}$ by simply putting $\hat{\mathcal{H}}[u](x,t) := \mathcal{H}[u(x,\cdot)](t)$, for $(x,t) \in \Omega \times [0,T]$. Usually, one does not distinguish between the operators $\hat{\mathcal{H}}$ and $\mathcal{H}$ and denotes them both by $\mathcal{H}$. We adopt this convention in this paper.

We now introduce the *generalized freezing index*

$$w(x,t) = w_0(x) \ - \ \int_0^t \left[\frac{1}{\mu(\theta)} \left(\theta\, F_1'(\chi) + F_2'(\chi) + Q[\chi] \right) \right] (x,\tau)\, d\tau \,, \qquad (2.1)$$

with some given initial condition w_0. Using (2.1), we obtain from (1.1) that for almost every $x \in \Omega$ it holds

$$\chi_t(x,t) - w_t(x,t) \ \in \ -\,\partial I_{[0,1]}\,(\chi(x,t))\,, \quad \text{for almost every } \ t \in (0,T)\,, \qquad (2.2)$$

or, equivalently,

$$\begin{aligned} &\chi(x,t) \in [0,1]\,, \quad (\chi_t(x,t) - w_t(x,t))(\chi(x,t) - \varphi) \le 0 \quad \forall\, \varphi \in [0,1]\,, \\ &\text{for almost every } \ t \in (0,T)\,. \end{aligned} \qquad (2.3)$$

The variational inequality (2.3) enables us to apply the theory of hysteresis operators and to simplify the problem stated by the equations (1.1)–(1.5). To this end, we recall the following result (cf. [7]).

Proposition 2.1 *Let $Z \subset \mathbb{R}$ denote a closed and nonempty interval, and let $\chi^0 \in Z$ and $w \in W^{1,1}(0,T)$ be given. Then there exists a unique $\chi \in W^{1,1}(0,T)$ with $\chi(0) = \chi^0$ such that it holds*

$$\chi(t) \in [0,1]\,, \quad (\chi_t(t) - w_t(t))(\chi(t) - \varphi) \le 0 \quad \forall\, \varphi \in [0,1]\,, \quad \textit{for a. e. } \ t \in (0,T)\,. \qquad (2.4)$$

The associated solution operator $\mathfrak{s}_Z : Z \times W^{1,1}(0,T) \to W^{1,1}(0,T) : (\chi^0, w) \mapsto \mathfrak{s}_Z[\chi^0, w] = \chi$, *is Lipschitz continuous and admits a Lipschitz continuous extension as mapping from* $Z \times C[0,T]$ *into* $C[0,T]$. *Moreover, we have:*

(i) *If* $(\chi^{0,i}, w_i) \in Z \times C[0,T]$, $\chi_i = \mathfrak{s}_Z[\chi^{0,i}, w]$, $i = 1,2$, *and* $t \in [0,T]$, *then*

$$\big|\chi_1(t) - \chi_2(t)\big| \le \big|\chi^{0,1} - \chi^{0,2}\big| + 2 \max_{0\le\tau\le t} |w_1(\tau) - w_2(\tau)|. \tag{2.5}$$

(ii) *If* $(\chi^0, w) \in Z \times W^{1,1}(0,T)$, $\chi = \mathfrak{s}_Z[\chi^0, w]$, *and* $t \in [0,T]$ *then*

$$\chi_t^2(t) = \chi_t(t)\, w_t(t) \le w_t^2(t). \tag{2.6}$$

(iii) *If* $(\chi^0, w) \in Z \times W^{2,1}(0,T)$, *then* $\chi = \mathfrak{s}_Z\big[\chi^0, w\big] \in W^{1,\infty}(0,T)$ *and the function*

$$t \mapsto \int_0^t w_{tt}(\tau)\, \chi_t(\tau)\, d\tau - \frac{1}{2}\chi_t(t)\, w_t(t)$$

is a. e. equal to a non-decreasing function.

The operator $\mathfrak{s}_Z$ is called *stop operator*. The hysteretic input-output behaviour of $\mathfrak{s}_{[0,1]}$ is illustrated in Fig. 1. Along the upper (lower) threshold line $\{\chi = 1\}$, $(\{\chi = 0\})$, the process is irreversible and can only move to the right (to the left, respectively), while in between, motions in both directions are admissible. Property (iii) is related to the so-called *clockwise convexity* of the stop, see also [9, Section 5]. This is similar to Prandtl's model of perfect elastoplasticity, where the horizontal parts of the diagram correspond to plastic yielding and the intermediate lines can be interpreted as linearly elastic trajectories.

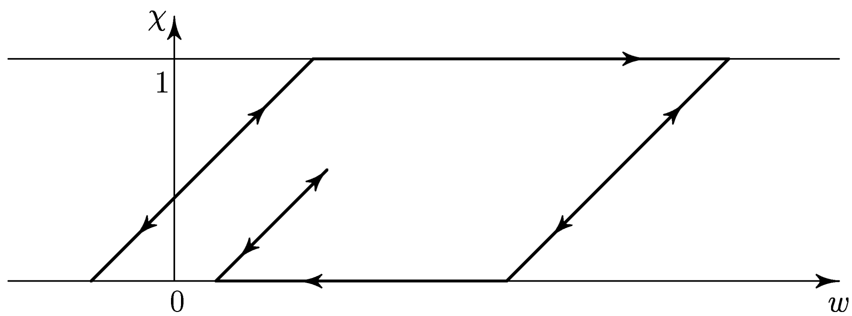

Figure 1: A diagram of $\chi = \mathfrak{s}_{[0,1]}[0, w]$.

Using Proposition 2.1, the notational convention for the extension of hysteresis operators to input functions acting on both space and time, as well as the abbreviation $\mathfrak{s}[w](x,t) := \mathfrak{s}_{[0,1]}[\chi_0(x), w(x,\cdot)](t)$, we can eliminate $\chi = \mathfrak{s}[w]$ from (1.1)–(1.3) to arrive at the system

$$\mu(\theta)\, w_t + \theta\, F_1'(\mathfrak{s}[w]) + F_2'(\mathfrak{s}[w]) + \mathcal{Q}[w] = 0\,, \tag{2.7}$$

$$\mathcal{Q}[w](x,t) := (Q \circ \mathfrak{s})[w](x,t) = \int_\Omega K(x-y)(1 - 2\,\mathfrak{s}[w](y,t))\,dy\,, \tag{2.8}$$

$$C_V\theta_t + (F_2'(\mathfrak{s}[w]) + \mathcal{Q}[w])(\mathfrak{s}[w])_t - \kappa\,\Delta\theta = 0\,, \tag{2.9}$$

with the initial conditions

$$w(x,0) = w_0(x), \quad \theta(x,0) = \theta_0(x)\,, \tag{2.10}$$

and the boundary condition (1.4). More generally, we are in the situation of a system of differential equations with hysteresis of the form

$$\mu(\theta)\,w_t + \theta\,\mathcal{H}_1[w] + \mathcal{H}_2[w] = 0\,, \tag{2.11}$$

$$C_V\theta_t + \mathcal{H}_2[w]\,\mathcal{G}[w]_t - \kappa\,\Delta\theta = 0\,. \tag{2.12}$$

It differs from the class of phase-field systems with hysteresis which has been studied in [8] in two respects. First, the term $(F_2'(\mathfrak{s}[w]) + \mathcal{Q}[w])(\mathfrak{s}[w])_t$ in (2.9) is not simply the time derivative of a hysteresis potential, and second, the nonlocality of $\mathcal{Q}$ renders a direct application of the techniques developed there impossible.

We are now in the position to formulate the main result of this paper.

Theorem 2.2 *Let $F_1, F_2 \in C^2[0,1]$, let $K \in L^1(\mathbb{R}^N)$ be a given even non-negative function, and let the function $\mu : (0,\infty) \to (0,\infty)$ be Lipschitz continuous on compact subsets of $(0,\infty)$ and satisfy the condition*

$$\exists\,\mu_0 > 0 : \mu(\theta) \geq \mu_0 \,\min\{\theta, 1\} \quad \forall\,\theta > 0. \tag{2.13}$$

Moreover, let the initial data satisfy $w_0 \in L^\infty(\Omega)$, $\theta_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ and $\theta_0(x) \geq \delta > 0$ a. e. in Ω. Then the initial-boundary value problem (1.4), (2.7)–(2.10) has a unique solution pair $(w,\theta) \in L^\infty(\Omega_T) \times L^\infty(\Omega_T)$ satisfying $w_t \in L^\infty(\Omega_T)$ and θ_t, $\Delta\theta \in L^2(\Omega_T)$, such that Eqs. (2.7)–(2.9) are satisfied a. e. in Ω_T and such that θ is positive a. e. in Ω_T. Moreover, there is some constant $\beta > 0$, independent of $T > 0$, such that

$$\theta(x,t) \geq \delta\,\mathrm{e}^{-\beta\,t} \quad a.\,e.\ in\ \Omega_T. \tag{2.14}$$

The proof of this result will be given in the following Sections 3 and 4.

Remark 2.3 Hypothesis (2.13) is satisfied if $\mu(\theta) = \hat{\mu}\,\theta^\alpha$ for some $\hat{\mu} > 0$ and $\alpha \in [0,1]$. Note that for $\alpha = 1$ a nonlocal analogue to a relaxed Stefan problem of Penrose-Fife type results, while for $\alpha = 0$ we obtain a nonlocal analogue to the relaxed Stefan problem of Caginalp type.

3 An auxiliary problem

In this section, we study an auxiliary problem as preparation to the proof of Theorem 2.2. To this end, assume that a function $\theta \in L^\infty(\Omega_T)$ is given. We then consider the "ordinary" differential equation

$$w_t(x,t) = \gamma[w,\theta](x,t)\,, \quad w(x,0) = w_0(x)\,, \tag{3.1}$$

where $w_0 \in L^\infty(\Omega)$ is given and $\gamma : L^2(\Omega; C[0,T]) \times L^\infty(\Omega_T) \to L^\infty(\Omega_T)$ is a mapping satisfying the following hypothesis.

Hypothesis 3.1

(i) There exists $\Gamma_0 > 0$ such that for every $w \in L^2(\Omega; C[0,T])$ and $\theta \in L^\infty(\Omega_T)$ we have

$$|\gamma[w,\theta](x,t)| \;\le\; \Gamma_0\,(1 + |\theta(x,t)|) \quad \text{a.e. in } \Omega_T. \tag{3.2}$$

(ii) There exist an even non-negative function $K \in L^1(\mathbb{R}^N)$, and a non-decreasing function $\Gamma : [0,+\infty) \to [0,+\infty)$ such that for every $w_1, w_2 \in L^2(\Omega; C[0,T])$, $R > 0$, and $\theta_1, \theta_2 \in L^\infty(\Omega_T)$, with $\max\{|\theta_1|_\infty, |\theta_2|_\infty\} \le R$, we have

$$\begin{aligned}|\gamma[w_1,\theta_1](x,t) - \gamma[w_2,\theta_2](x,t)| \;&\le\; \Gamma(R)\Big(|\theta_1(x,t) - \theta_2(x,t)| \\ &+ |w_1(x,\cdot) - w_2(x,\cdot)|_{[0,t]} + \int_\Omega K(x-y)\,|w_1(y,\cdot) - w_2(y,\cdot)|_{[0,t]} dy\Big)\,.\end{aligned} \tag{3.3}$$

Here we have used the abbreviation $|w|_{[0,t]} = \max_{\tau\in[0,t]} |w(\tau)|$ for $t \in [0,T]$.

For the existence and uniqueness of solutions to Eq. (3.1) we cannot simply refer to [8], since there the non-local convolution term was not present. We nevertheless show here that the contraction argument works in appropriate function spaces.

Lemma 3.2 *Let Hypothesis 3.1 hold, and let $w_0 \in L^\infty(\Omega)$ be given. Then (3.1) admits for every $\theta \in L^\infty(\Omega_T)$ a unique solution $w \in L^\infty(\Omega; C[0,T])$ such that $w_t \in L^\infty(\Omega_T)$. Moreover, there exists a non-decreasing function $\hat\Gamma : [0,+\infty) \to [0,+\infty)$ such that for every $R > 0$ and every $\theta_1, \theta_2 \in L^\infty(\Omega_T)$ satisfying $\max\{|\theta_1|_\infty, |\theta_2|_\infty\} \le R$, the corresponding solutions w_1, w_2 satisfy the inequality*

$$|((w_1)_t - (w_2)_t)(\cdot,t)|_2^2 \;\le\; \hat\Gamma(R)\left(|(\theta_1 - \theta_2)(\cdot,t)|_2^2 + \int_0^t |(\theta_1 - \theta_2)(\cdot,\tau)|_2^2\, d\tau\right)\,. \tag{3.4}$$

Proof. We first derive some useful estimates. We will see that the L^2-framework will play a central role when Eq. (3.1) is later coupled with the energy balance equation. For any function $v \in L^2(\Omega_T)$ and $(x,t) \in \Omega_T$ set

$$G_0[v](x,t) \;=\; w^0(x) + \int_0^t v(x,\tau)\, d\tau\,. \tag{3.5}$$

Let $v_1, v_2 \in L^2(\Omega_T)$, $\theta_1, \theta_2 \in L^\infty(\Omega_T)$, and $R \geq \max\{|\theta_1|_\infty, |\theta_2|_\infty\}$ be arbitrarily given. Put

$$w_i = G_0[v_i], \quad V_i = \gamma[w_i, \theta_i], \qquad i = 1, 2, \tag{3.6}$$

$$\bar{v} = v_1 - v_2, \quad \bar{V} = V_1 - V_2, \quad \bar{\theta} = \theta_1 - \theta_2. \tag{3.7}$$

From (3.3) it follows for a. e. $(x,t) \in \Omega_T$ that

$$|\bar{V}(x,t)| \leq \Gamma(R) \left(|\bar{\theta}(x,t)| + \int_0^t |\bar{v}(x,\tau)| \, d\tau + \int_\Omega K(x-y) \left(\int_0^t |\bar{v}(y,\tau)| \, d\tau \right) dy \right). \tag{3.8}$$

Using Young's inequality for convolutions, and putting $|K|_1 = |K|_{L^1(\mathbb{R}^N)}$, we obtain from (3.8) for a. e. $t \in (0,T)$ that

$$\begin{aligned} |\bar{V}(\cdot,t)|_2 &\leq \Gamma(R) \left(|\bar{\theta}(\cdot,t)|_2 + (1 + |K|_1) \left| \int_0^t |\bar{v}(\cdot,\tau)| \, d\tau \right|_2 \right) \\ &\leq \Gamma(R) \left(|\bar{\theta}(\cdot,t)|_2 + (1 + |K|_1) T^{1/2} \left(\int_0^t |\bar{v}(\cdot,\tau)|_2^2 \, d\tau \right)^{1/2} \right), \end{aligned} \tag{3.9}$$

whence

$$|\bar{V}(\cdot,t)|_2^2 \leq 2\Gamma^2(R) \left(|\bar{\theta}(\cdot,t)|_2^2 + (1 + |K|_1)^2 T \int_0^t |\bar{v}(\cdot,\tau)|_2^2 \, d\tau \right). \tag{3.10}$$

We now employ the above estimates for the existence and uniqueness proof. To this end, let $\theta \in L^\infty(\Omega_T)$ be given, and set $R = |\theta|_\infty$, $\theta_1 = \theta_2 = \theta$. We show that the mapping $v \mapsto G_1[v] := \gamma[G_0[v], \theta]$ is a contraction with respect to a suitable norm in $L^2(\Omega_T)$. Indeed, from (3.10) it follows that

$$|(|G_1[v_1] - G_1[v_2])(\cdot,t)|_2^2 \leq a \int_0^t |(v_1 - v_2)(\cdot,\tau)|_2^2 \, d\tau \quad \text{a. e.}, \tag{3.11}$$

with

$$a = 2\Gamma^2(R) \, (1 + |K|_1)^2 T. \tag{3.12}$$

We now fix some $b > a$, and we put for $v \in L^2(\Omega_T)$

$$\|v\|_b = \left(\int_0^T e^{-bt} |v(\cdot,t)|_2^2 \, dt \right)^{1/2}. \tag{3.13}$$

Then $\|\cdot\|_b$ is an equivalent norm in $L^2(\Omega_T)$, and from (3.11) it follows that

$$\begin{aligned} \|G_1[v_1] - G_1[v_2]\|_b^2 &\leq a \int_0^T e^{-bt} \int_0^t |(v_1 - v_2)(\cdot,\tau)|_2^2 \, d\tau \, dt \\ &= \frac{a}{b} \int_0^T (e^{-b\tau} - e^{-bT}) |(v_1 - v_2)(\cdot,\tau)|_2^2 \, d\tau \leq \frac{a}{b} \|v_1 - v_2\|_b^2. \end{aligned} \tag{3.14}$$

By the Banach Contraction Principle, there exists a unique $v \in L^2(\Omega_T)$ such that $G_1[v] = v$, hence $w = G_0[v]$ is the desired solution to Eq. (3.1).

Finally, let $\theta_1, \theta_2 \in L^\infty(\Omega_T)$ and $R \geq \max\{|\theta_1|_\infty, |\theta_2|_\infty\}$ be given, and let w_1, w_2 be the corresponding solutions to Eq. (3.1), and $v_i = (w_i)_t$ for $i = 1, 2$. With the notation of (3.10) and (3.12), we have

$$|\bar{v}(\cdot,t)|_2^2 \;\leq\; 2\Gamma^2(R)\,|\bar{\theta}(\cdot,t)|_2^2 + a\int_0^t |\bar{v}(\cdot,\tau)|_2^2\,d\tau\,, \tag{3.15}$$

whence (3.4) follows using a standard Gronwall argument. □

4 Proof of Theorem 2.2

We are now prepared to show the main result. Let $\varepsilon > 0$ be arbitrary but fixed (to be specified later). We construct a suitable "cutoff"-version of system (1.4), (2.7)–(2.10). To this end, we define the auxiliary functions

$$T_\varepsilon := \max\{\varepsilon, |s|\}, \quad \mu_\varepsilon(s) := \mu(T_\varepsilon(s)), \quad s \in \mathbb{R}, \tag{4.1}$$

and the operator

$$\gamma_\varepsilon[w,\theta] := -\frac{1}{\mu_\varepsilon(\theta)}\,\left(T_\varepsilon(\theta)\,F_1'(\mathfrak{s}[w]) \,+\, F_2'(\mathfrak{s}[w]) \,+\, \mathcal{Q}[w]\right)\,. \tag{4.2}$$

Recalling the assumptions on F_1, F_2, k, μ, as well as the boundedness and Lipschitz properties of the stop operator s stated in Proposition 2.1, we readily verify that γ_ε maps $L^2(\Omega; C[0,T]) \times L^\infty(\Omega_T)$ into $L^\infty(\Omega_T)$ and fulfils Hypothesis 3.1 with a suitable constant $\Gamma_0 > 0$ and a suitable function Γ which we need not specify. Next, we define the operator $\mathcal{H}[w] := -(F_2'(\mathfrak{s}[w]) + \mathcal{Q}[w])$ and consider the following system:

$$w_t = \gamma_\varepsilon[w,\theta]\,, \quad \text{in } \Omega_T\,, \tag{4.3}$$

$$\theta_t - \Delta\theta = \mathcal{H}[w](\mathfrak{s}[w])_t\,, \quad \text{in } \Omega_T\,, \tag{4.4}$$

together with the initial-boundary conditions (1.4), (2.10). We proceed in several steps.

<u>*Step 1:*</u> *The system (1.4), (2.10), (4.3), (4.4) has a unique solution* $(w^\varepsilon, \theta^\varepsilon) \in L^\infty(\Omega_T) \times L^\infty(\Omega_T)$ *such that* $w_t^\varepsilon \in L^\infty(\Omega_T)$ *and* $\theta_t^\varepsilon\,,\, \Delta\theta^\varepsilon \in L^2(\Omega_T)$.

To verify this claim, we proceed by successive approximation. We put $\theta^0(x,t) := \theta_0(x)$ and consider for $k \in \mathbb{N}$ the recursive scheme

$$w_t^k = \gamma_\varepsilon[w^k, \theta^{k-1}]\,, \quad \text{in } \Omega_T\,, \tag{4.5}$$

$$\theta_t^k - \Delta\theta^k + \theta^k = \theta^{k-1} + \mathcal{H}[w^k](\mathfrak{s}[w^k])_t\,, \quad \text{in } \Omega_T\,, \tag{4.6}$$

$$w^k(x,0) = w_0(x)\,, \quad \text{a.e. in } \Omega\,, \tag{4.7}$$

$$\theta^k(x,0) = \theta_0(x)\,, \quad \text{a.e. in } \Omega\,, \quad \frac{\partial \theta^k}{\partial \mathbf{n}} = 0\,, \quad \text{on } \partial\Omega\,. \tag{4.8}$$

Notice that if $\theta^{k-1} \in L^\infty(\Omega_T)$ is given, then Lemma 3.2 yields the existence of a unique solution $w^k \in L^\infty(\Omega_T)$ to (4.5), (4.7) such that $w_t^k \in L^\infty(\Omega_T)$ as well. But then standard linear parabolic theory shows that (4.6), (4.8) has a unique solution θ^k which belongs to $L^\infty(\Omega_T) \cap H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))$. Thus, the above recursive scheme is well-defined.

Next, observe that from (4.5), using the boundedness of the stop operator and the fact that $K \in L^1(\mathbb{R}^N)$, we can infer the existence of some $C_1 > 0$ (which is independent of $k \in \mathbb{N}$) such that

$$|(\mathfrak{s}[w^k])_t(x,t)| \,\le\, |w_t^k(x,t)| \,\le\, C_1 \left(1 \,+\, |\theta^{k-1}(x,t)|\right) \quad \text{a.e. in } \Omega_T\,. \tag{4.9}$$

Therefore, using the fact that also the operator $\mathcal{H}$ is globally bounded, we conclude that the term $\mathcal{H}[w^k](\mathfrak{s}[w^k])_t$ on the right-hand side of (4.6) is bounded by a constant which does not depend on $k \in \mathbb{N}$. Hence, it follows from estimates identical to those in the proof of [8, Theorem 3.1], for instance, that

$$|\theta^k|_\infty \,\le\, C_2\,, \tag{4.10}$$

with some $C_2 \,\ge\, |\theta_0|_\infty$ which is independent of $k \in \mathbb{N}$. Taking C_2 larger, if necessary, we then conclude that

$$|\theta^k|_{H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega))} \,\le\, C_2 \quad \forall\, k \in \mathbb{N}\,, \tag{4.11}$$

and then also

$$|w_t^k|_{L^\infty(\Omega_T)} \,\le\, C_2 \quad \forall\, k \in \mathbb{N}. \tag{4.12}$$

Hence, there is a subsequence of $\{(w^k,\theta^k)\}$, which is again indexed by $k \in \mathbb{N}$, and a pair (w,θ) such that we have the convergences

$$\begin{aligned} w_t^k \to w_t \quad &, \quad \text{weakly-star in } L^\infty(\Omega_T)\,, \\ \theta^k \to \theta \quad &, \quad \text{weakly-star in } L^\infty(\Omega_T)\,, \\ & \quad\ \ \text{weakly in } H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega)). \end{aligned} \tag{4.13}$$

Next, we will show that $\{w_t^k\}$ and $\{\theta^k\}$ are Cauchy sequences in $L^2(\Omega_T)$. This will imply, in particular, that the above convergences hold for the entire sequence $\{(w^k,\theta^k)\}$, and that $w^k \to w$ strongly in $L^2(\Omega; C[0,T])$.

To this end, let $k \in \mathbb{N}$ be fixed. We consider system (4.5)–(4.8) for k and $k+1$ and put $w := w^{k+1} - w^k$, $\theta := \theta^{k+1} - \theta^k$, $z := \theta^k - \theta^{k-1}$. We then have

$$w_t = \gamma_\varepsilon[w^{k+1},\theta^k] \,-\, \gamma_\varepsilon[w^k,\theta^{k-1}]\,, \tag{4.14}$$

$$\theta_t - \Delta\theta + \theta = z \,+\, \mathcal{H}[w^{k+1}](\mathfrak{s}[w^{k+1}])_t \,-\, \mathcal{H}[w^k](\mathfrak{s}[w^k])_t\,, \tag{4.15}$$

$$w(x,0) = \theta(x,0) = z(x,0) = 0 \quad \text{a.e. in } \Omega\,, \tag{4.16}$$

$$\frac{\partial\theta}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega\,. \tag{4.17}$$

In what follows, we denote by $C_i\,,\ i\in\mathbb{N}$, positive constants which may depend on the data but not on $k\in\mathbb{N}$.

Now, put $R := C_2$. Then $R \ge \max\{|\theta^{k+1}|_\infty\,,\ |\theta^k|_\infty\,,\ |\theta^{k-1}|_\infty\}$, and we can infer from Lemma 3.2 that for any $t\in[0,T]$ it holds

$$\int_0^t |w_t(\cdot,\tau)|_2^2\,d\tau \,\le\, C_3 \int_0^t |z(\cdot,\tau)|_2^2\,d\tau\,. \tag{4.18}$$

Next, we integrate (4.6) over $[0,\tau]$ for some $\tau\in[0,T]$. Integrating by parts, we obtain

$$\begin{aligned}
&\theta(x,\tau) \,-\, \int_0^\tau \Delta\theta(x,s)\,ds \,+\, \int_0^\tau \theta(x,s)\,ds \\
=\;& \int_0^\tau z(x,s)\,ds \,+\, \int_0^\tau \left[(\mathcal{H}[w^{k+1}]-\mathcal{H}[w^k])(\mathfrak{s}[w^k])_t\right](x,s)\,ds \\
&+\mathcal{H}[w^{k+1}](x,\tau)\,\left(\mathfrak{s}[w^{k+1}]-\mathfrak{s}[w^k]\right)(x,\tau) \\
&-\int_0^\tau \left[\left(\mathcal{H}[w^{k+1}]\right)_t\,\left(\mathfrak{s}[w^{k+1}]-\mathfrak{s}[w^k]\right)\right](x,s)\,ds\,.
\end{aligned} \tag{4.19}$$

Multiplying (4.19) by θ, and integrating over $\Omega\times[0,t]$ for $t\in[0,T]$, we find that

$$\begin{aligned}
&\int_0^t |\theta(\cdot,\tau)|_2^2\,d\tau \,+\, \frac{1}{2}\int_\Omega \left|\int_0^t \nabla\theta(x,\tau)\,d\tau\right|^2 dx \,+\, \frac{1}{2}\int_\Omega \left|\int_0^t \theta(x,\tau)\,d\tau\right|^2 dx \\
\le\;& \int_0^t \int_\Omega |\theta(x,\tau)| \left|\int_0^\tau z(x,s)\,ds\right| dx\,d\tau \\
&+ \int_0^t \int_\Omega |\theta(x,\tau)| \int_0^\tau \left[\left|\mathcal{H}[w^{k+1}]-\mathcal{H}[w^k]\right|\,\left|\left(\mathfrak{s}[w^k]\right)_t\right|\right](x,s)\,ds\,dx\,d\tau \\
&+ \int_0^t \int_\Omega \left[|\theta|\,\left|\mathcal{H}[w^{k+1}]\right|\,\left|\mathfrak{s}[w^{k+1}]-\mathfrak{s}[w^k]\right|\right](x,\tau)\,dx\,d\tau \\
&+ \int_0^t \int_\Omega |\theta(x,\tau)| \int_0^\tau \left[\left|\left(\mathcal{H}[w^{k+1}]\right)_t\right|\,\left|\mathfrak{s}[w^{k+1}]-\mathfrak{s}[w^k]\right|\right](x,s)\,ds\,dx\,d\tau \\
:=\;& I_1 \,+\, I_2 \,+\, I_3 \,+\, I_4\,.
\end{aligned} \tag{4.20}$$

Now let $\alpha>0$ be arbitrary (to be specified later). Using Young's inequality $|x\,y| \le \alpha\,|x|^2 \,+\, \frac{1}{4\alpha}\,|y|^2$ for all $x,y\in\mathbb{R}$, we readily see that

$$I_1 \,\le\, \alpha \int_0^t |\theta(\cdot,\tau)|_2^2\,d\tau \,+\, \frac{C_4}{\alpha}\int_0^t\int_0^\tau |z(\cdot,s)|_2^2\,ds\,d\tau\,. \tag{4.21}$$

Next, observe that for a. e. $(x,\tau)\in\Omega\times[0,T]$ it holds

$$\begin{aligned}
&|w(x,\cdot)|_{[0,\tau]} \;=\; \max_{0\le r\le\tau} |w(x,r)| \;\le\; \max_{0\le r\le\tau} \int_0^r |w_t(x,s)|\,ds \\
\le\;& \max_{0\le r\le\tau}\left(r\int_0^r |w_t(x,s)|^2\,ds\right)^{1/2} \;\le\; \sqrt{T}\,\left(\int_0^\tau |w_t(x,s)|^2\,ds\right)^{1/2}.
\end{aligned} \tag{4.22}$$

Hence, using Young's inequality, Eqs. (2.5) and (4.5), and the global boundedness of $\mathcal{H}$, we deduce that

$$
\begin{aligned}
I_3 \;&\leq\; \alpha \int_0^t |\theta(\cdot,\tau)|_2^2 \, d\tau \;+\; \frac{C_5}{\alpha} \int_0^t \int_\Omega \left| (\mathfrak{s}[w^{k+1}] - \mathfrak{s}[w^k])(x,\tau) \right|^2 \, dx \, d\tau \\
&\leq\; \alpha \int_0^t |\theta(\cdot,\tau)|_2^2 \, d\tau \;+\; \frac{2\, C_5}{\alpha} \int_0^t \int_\Omega |w(x,\cdot)|_{[0,\tau]}^2 \, dx \, d\tau \\
&\leq\; \alpha \int_0^t |\theta(\cdot,\tau)|_2^2 \, d\tau \;+\; \frac{C_6}{\alpha} \int_0^t \int_0^\tau |w_t(\cdot,s)|_2^2 \, ds \, d\tau \, . \qquad (4.23)
\end{aligned}
$$

Next, observe that Eqs. (4.9), (4.10) and the boundedness of $\mathfrak{s}$ and of F_2'' imply that for a. e. $(x,\tau) \in \Omega \times [0,T]$ it holds

$$
\begin{aligned}
\left| (\mathcal{H}[w^{k+1}])_t(x,\tau) \right| \;&\leq\; \left| F_2''(\mathfrak{s}[w^{k+1}](x,\tau)) \right| \, \left| (\mathfrak{s}[w^{k+1}])_t(x,\tau) \right| \\
&\quad + 2 \int_\Omega K(x-y) \, \left| (\mathfrak{s}[w^{k+1}])_t(y,\tau) \right| \, dy \;\leq\; C_7 \, . \qquad (4.24)
\end{aligned}
$$

Thus, by the same token as above, we can infer that

$$
I_4 \;\leq\; \alpha \int_0^t |\theta(\cdot,\tau)|_2^2 \, d\tau \;+\; \frac{C_8}{\alpha} \int_0^t \int_0^\tau |w_t(\cdot,s)|_2^2 \, ds \, d\tau \, . \qquad (4.25)
$$

Finally, Young's inequality, and Eqs. (4.9) and (4.10), yield that

$$
I_2 \;\leq\; \alpha \int_0^t |\theta(\cdot,\tau)|_2^2 \, d\tau \;+\; \frac{C_9}{\alpha} \int_0^t \int_\Omega \Big(\int_0^\tau |V(x,s)| \, ds \Big)^2 \, dx \, d\tau \, , \qquad (4.26)
$$

where $V = \mathcal{H}[w^{k+1}] - \mathcal{H}[w^k]$. Now, by our assumptions and by (2.5), we have

$$
\begin{aligned}
|V(x,s)| \;&\leq\; \max_{0 \leq \xi \leq 1} |F_2''(\xi)| \, \left| (\mathfrak{s}[w^{k+1}] - \mathfrak{s}[w^k])(x,s) \right| \\
&\quad + 2 \int_\Omega K(x-y) \, \left| (\mathfrak{s}[w^{k+1}] - \mathfrak{s}[w^k])(y,s) \right| \, dy \\
&\leq\; 2 \max_{0 \leq \xi \leq 1} |F_2''(\xi)| \, |w(x,\cdot)|_{[0,s]} \;+\; 4 \int_\Omega K(x-y) \, |w(y,\cdot)|_{[0,s]} \, dy \\
&\leq\; C_{10} \Big(\int_0^s |w_t(x,\sigma)| \, d\sigma \;+\; \int_\Omega K(x-y) \Big(\int_0^s |w_t(y,\sigma)| \, d\sigma \Big) \, dy \Big) \, . \qquad (4.27)
\end{aligned}
$$

From this it follows, as in the derivation of Eq. (3.10), that

$$
|V(\cdot,s)|_2^2 \;\leq\; C_{11} \int_0^s |w_t(\cdot,\sigma)|_2^2 \, d\sigma \, . \qquad (4.28)
$$

Summarizing the estimates (4.20)–(4.28), and chosing $\alpha < \frac{1}{4}$, we have shown the estimate

$$
\begin{aligned}
&\int_0^t \int_\Omega \left| \theta^{k+1} - \theta^k \right|^2 \, dx \, d\tau \\
&\qquad \leq\; C_{12} \int_0^t \int_0^\tau \int_\Omega \Big(\left| \theta^k - \theta^{k-1} \right|^2 + \left| w_t^{k+1} - w_t^k \right|^2 \Big) \, dx \, ds \, d\tau \, , \qquad (4.29)
\end{aligned}
$$

whence, using (4.18),

$$\int_0^t \int_\Omega \left|\theta^{k+1} - \theta^k\right|^2 \, dx \, d\tau \;\le\; C_{13} \int_0^t \int_0^\tau \int_\Omega \left|\theta^k - \theta^{k-1}\right|^2 \, dx \, ds \, d\tau \, . \tag{4.30}$$

Using induction, we conclude that, for all $k \in \mathbb{N}$ and $t \in [0,T]$,

$$\int_0^t \int_\Omega \left|\theta^{k+1} - \theta^k\right|^2 \, dx \, d\tau \;\le\; \frac{(C_{13}\, t)^k}{k!} \, |\theta^1 - \theta^0|_2^2 \, . \tag{4.31}$$

Since the series $\sum_{k=0}^\infty ((C_{13}\, T)^k/(k!))^{1/2}$ is convergent, we have shown that $\{\theta^k\}$ is a Cauchy sequence in $L^2(\Omega_T)$. From Eq. (4.18) we infer that this also holds for $\{w_t^k\}$, and the claim is proved.

Having shown that $\{w_t^k\}$ and $\{\theta^k\}$ are Cauchy sequences in $L^2(\Omega_T)$, we can from this point exactly follow the lines of Theorem 3.2 in [8] to conclude that the limit pair (w,θ) is in fact a solution to the system (1.4), (2.10), (4.3), (4.4), having the asserted smoothness properties. It remains to show the uniqueness of the solution. To this, suppose (w_i,θ_i), $i = 1,2$, are solutions having the corresponding smoothness properties. Putting $w := w_1 - w_2$, $\theta := \theta_1 - \theta_2$, we see that Eqs. (4.14)–(4.18) are satisfied with (w^{k+1},θ^k) replaced by (w^1,θ^1), (w^k,θ^{k-1}) replaced by (w^2,θ^2), and z replaced by θ. Arguing in essentially the same way as in the derivation of the inequalities (4.18) and (4.29) (with obvious modifications), we find an estimate of the form

$$\int_0^t \int_\Omega \left(\theta^2 \,+\, w_t^2\right) dx \, d\tau \;\le\; C_{14} \int_0^t \int_0^\tau \int_\Omega \left(\theta^2 \,+\, w_t^2\right) ds \, dx \, d\tau \, , \tag{4.32}$$

whence, using Gronwall's lemma, $\theta = w_t = 0$ a.e. in Ω_T, from which the uniqueness follows.

Step 2: *There is some $\hat\varepsilon > 0$ such that $(w^{\hat\varepsilon},\theta^{\hat\varepsilon})$ is a solution to the original system (1.4), (2.7)–(2.10).*

We aim to show that there is some $\hat\varepsilon > 0$ such that $\theta^\varepsilon(x,t) \ge \hat\varepsilon$ a.e. in Ω_T for all $\varepsilon \in (0,\hat\varepsilon)$. It then follows that $T_{\hat\varepsilon}(\theta^{\hat\varepsilon}) = \theta^{\hat\varepsilon}$, and thus $\mu_{\hat\varepsilon}(\theta^{\hat\varepsilon}) = \mu(\theta^{\hat\varepsilon})$, which then implies that $(w^{\hat\varepsilon},\theta^{\hat\varepsilon})$ also satisfies (2.7), i.e. is a solution to (1.4), (2.7)–(2.10).

To this end, we test Eq. (2.9) by an arbitrary function $p \in H^1(\Omega_T)$ satisfying $p \le 0$ a.e. in Ω_T. Putting $h_\varepsilon := F_2'(\mathfrak{s}[w^\varepsilon]) + \mathcal{Q}[w^\varepsilon]$, we obtain

$$\int_\Omega \left(p\,\theta_t^\varepsilon \,+\, \nabla p \,\cdot\, \nabla\theta^\varepsilon\right)(x,t)\, dx \;=\; \int_\Omega \left(|p|\, h_\varepsilon\, (\mathfrak{s}[w^\varepsilon])_t\right)(x,t)\, dx \, . \tag{4.33}$$

We consider two cases. If $T_\varepsilon(\theta^\varepsilon) \le 1$, then $\mu_\varepsilon(\theta^\varepsilon) \ge \mu_0\, T_\varepsilon(\theta^\varepsilon)$, and we obtain from (4.4), using Young's inequality,

$$\begin{aligned} h_\varepsilon\,(\mathfrak{s}[w^\varepsilon])_t \;&=\; -\,\mu_\varepsilon(\theta^\varepsilon)\, w_t^\varepsilon\, (\mathfrak{s}[w^\varepsilon])_t \,-\, T_\varepsilon(\theta^\varepsilon)\, F_1'(\mathfrak{s}[w^\varepsilon])\, (\mathfrak{s}[w^\varepsilon])_t \\ &\le\; -\,\mu_\varepsilon(\theta^\varepsilon)\, (\mathfrak{s}[w^\varepsilon])_t^2 \,-\, T_\varepsilon(\theta^\varepsilon)\, F_1'(\mathfrak{s}[w^\varepsilon])\, (\mathfrak{s}[w^\varepsilon])_t \\ &\le\; \frac{1}{4\,\mu_\varepsilon(\theta^\varepsilon)} \left(F_1'(\mathfrak{s}[w^\varepsilon])\, T_\varepsilon(\theta^\varepsilon)\right)^2 \\ &\le\; \beta_1\, T_\varepsilon(\theta^\varepsilon) \, , \end{aligned} \tag{4.34}$$

where $\beta_1 := \|F_1'\|^2_{C[0,1]}/(4\,\mu_0)$ is a global constant, independent of ε.

In the case $T_\varepsilon(\theta^\varepsilon) > 1$ we have $\mu_\varepsilon(\theta^\varepsilon) \ge \mu_0$, and thus

$$\begin{aligned} h_\varepsilon\,(\mathfrak{s}[w^\varepsilon])_t \;&\le\; |h_\varepsilon|\,|w_t^\varepsilon| \;\le\; \frac{|h_\varepsilon|}{\mu_\varepsilon(\theta^\varepsilon)}\left|T_\varepsilon(\theta^\varepsilon)\,F_1'(\mathfrak{s}[w^\varepsilon]) \,+\, h_\varepsilon\right| \\ &\le\; \frac{|h_\varepsilon|}{\mu_0}\left|F_1'(\mathfrak{s}[w^\varepsilon]) \,+\, h_\varepsilon\right| T_\varepsilon(\theta^\varepsilon) \;\le\; \beta_2\,T_\varepsilon(\theta^\varepsilon), \end{aligned} \tag{4.35}$$

with the global constant

$$\beta_2 := \left(\|F_2'\|_{C[0,1]} \,+\, |K|_1\right)\,\left(\|F_1'\|_{C[0,1]} \,+\, \|F_2'\|_{C[0,1]} \,+\, |K|_1\right)/\mu_0\,. \tag{4.36}$$

In conclusion, taking $\beta > \max\{\beta_1, \beta_2\}$, we always have

$$h_\varepsilon\,(\mathfrak{s}[w^\varepsilon])_t \;\le\; \beta\,T_\varepsilon(\theta^\varepsilon)\,, \tag{4.37}$$

where $\beta > 0$ is independent of ε and T. Hence, by (4.33),

$$\int_\Omega (p\,\theta_t^\varepsilon \,+\, \nabla p\cdot\nabla\theta^\varepsilon)(x,t)\,dx \;\le\; \beta\int_\Omega \left(|p|\,T_\varepsilon(\theta^\varepsilon)\right)(x,t)\,dx\,, \quad \text{a.e. in } (0,T). \tag{4.38}$$

Now put $\hat\varepsilon := \delta\,e^{-\beta T}$, and

$$p(x,t) \;:=\; -\,\left(\delta\,e^{-\beta t} \,-\, \theta^{\hat\varepsilon}(x,t)\right)^+\,, \quad (x,t)\,\in\,\Omega_T\,. \tag{4.39}$$

Then we can infer from (4.38) that

$$\int_\Omega \left(p\,\left(p+\delta\,e^{-\beta t}\right)_t\right)(x,t)\,dx \;\le\; \beta\int_\Omega |p|\,\left(|p|+\delta\,e^{-\beta t}\right)(x,t)\,dx\,, \tag{4.40}$$

whence, in particular,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega p^2(x,t)\,dx \;\le\; \beta\int_\Omega p^2(x,t)\,dx\,. \tag{4.41}$$

By hypothesis we have $p(x,0)\equiv 0$, hence, by Gronwall's inequality, $p\equiv 0$. Therefore, $\theta^{\hat\varepsilon}(x,t) \ge \delta e^{-\beta t} \ge \hat\varepsilon$ a.e. We thus checked that $(w,\theta) := (w^{\hat\varepsilon}, \theta^{\hat\varepsilon})$ is a solution to (1.4), (2.7)–(2.10) satisfying the conditions of Theorem 2.2.

Step 3: Conclusion of the proof.

It remains to show that *any* solution (w,θ) to the system (1.4), (2.7)–(2.10) having the property that θ is positive a.e. in Ω_T automatically satisfies (2.14) with the constant $\beta > 0$ constructed above. Then (w,θ) is also a solution to the cutoff-system (1.4), (2.10), (4.3), (4.4) for $\varepsilon = \hat\varepsilon$, and by the unique solvability of the cutoff-system, coincides with its solution $(w^{\hat\varepsilon}, \theta^{\hat\varepsilon})$. In conclusion, the proof of Theorem 2.2 will be complete if we can show that θ satisfies (2.14).

To this end, suppose that (w,θ) is an arbitrary solution which enjoys the smoothness properties stated in Theorem 2.2 and satisfies $\theta > 0$, and thus $\mu(\theta) > 0$, a. e. in Ω_T. Apparently, replacing $(w^\varepsilon, \theta^\varepsilon)$ by (w,θ), we then can argue similarly as in the derivation of estimate (4.38) in Step 2 above to conclude that (4.38) holds with θ^ε replaced by θ. But then $\theta(x,t) \ge \delta\, e^{-\beta t} \ge \hat{\varepsilon}$ almost everywhere. With this, the assertion of Theorem 2.2 is proved. □

Remark 4.1 Theorem 2.2 may be generalized in different directions. Inspecting the above proof, we notice, for instance, that we only need to assume that $F_1, F_2 \in W^{2,\infty}(0,1)$. Also, the system (2.7)–(2.9) can be replaced by more general systems of the form (2.11), (2.12), where the stop operator in (2.7)–(2.9) may be substituted by any other bounded hysteresis operator $\mathcal{G}$ which is Lipschitz continuous on $W^{1,1}(0,T)$, satisfies a Lipschitz condition of the form (2.5) (with a global Lipschitz constant that may differ from 2) on $C[0,T]$, and fulfils a condition of the form (see (2.6))

$$K_1\,|(\mathcal{G}[w])_t(t)|^2 \;\le\; (\mathcal{G}[w])_t(t)\,w_t(t) \;\le\; K_2\,w_t^2(t) \quad \text{for all } w \in W^{1,1}(0,T)\,,$$

with given constants $K_1 > 0$ and $K_2 > 0$. The results on the long-time behaviour in the next section make also use of the clockwise convexity of the stop mentioned below in connection with Proposition 2.1 (iii), and more about the subject can be found in [7, 9]. Since we focus our attention to the system arising from nonlocal phase transitions, we do not elaborate on such possible extensions here.

5 Asymptotic behaviour as $t \to +\infty$

In this section, we prove the following result.

Theorem 5.1 *Let the hypotheses of Theorem 2.2 hold with (2.13) replaced by*

$$\exists\,\mu_0 > 0 : \mu(\theta) \ge \mu_0 \quad \forall\,\theta > 0. \tag{5.1}$$

Then there exists a constant $\hat{C} > 0$ such that the solution (w,θ) to the system (1.4), (2.7)–(2.10) satisfies the conditions

$$0 \,<\, \theta(x,t) \,\le\, \hat{C}\,, \quad |w_t(x,t)| \,\le\, \hat{C} \quad \text{a. e. in } \; \Omega\times(0,\infty)\,. \tag{5.2}$$

Moreover, if for $t > 0$ we put

$$\begin{cases} E_1(t) &:= \; \frac{1}{2}\int_\Omega |\nabla\theta(x,t)|^2 dx\,, \\ E_2(t) &:= \; \frac{1}{2}\int_\Omega |\mathfrak{s}[w]_t(x,t)|^2 dx\,, \\ E(t) &:= \; E_1(t) + E_2(t)\,, \end{cases} \tag{5.3}$$

then we have

$$\int_0^\infty E(t)\,dt \;\le\; \hat{C}\,, \quad \lim_{t\to\infty} E_1(t) \;=\; 0\,, \tag{5.4}$$

and there exists a function $E_2^* : [0,\infty) \to [0,\infty)$ *such that*

$$E_2(t) \;=\; E_2^*(t) \quad a.\,e.\;, \qquad \lim_{t\to\infty} E_2^*(t) \;=\; 0\,, \qquad \operatorname*{Var}_{[0,\infty)} \left((E_1 + E_2^*)^2\right) \;\le\; \hat{C}\,. \tag{5.5}$$

In particular, the function E_2 *satisfies the condition*

$$\lim_{t\to\infty} \ \sup \mathrm{ess}\, \{E_2(s)\,;\; s > t\} \;=\; 0\,. \tag{5.6}$$

The proof of Theorem 5.1 follows the lines of [9, Section 4] and is based on a series of estimates. Similarly as in the previous section, we denote by $C_1, C_2, \ldots$ any positive constant independent of x and t.

Estimate 1.

Eq. (1.9), the positivity of θ, and the boundedness of the stop yield

$$|\theta(\cdot,t)|_1 \;\le\; C_1\,. \tag{5.7}$$

From Eq. (2.7) it follows that

$$|w_t(x,t)| \;\le\; C_2\,(1+\theta(x,t)) \quad \text{a.e.}\;, \tag{5.8}$$

hence

$$|(F_2'(\mathfrak{s}[w]) + \mathcal{Q}[w])(\mathfrak{s}[w])_t(x,t)| \;\le\; C_3\,(1+\theta(x,t)) \quad \text{a.e.}\,, \tag{5.9}$$

and Theorem 3.1 of [9] applied to Eq. (2.9) enables us to conclude that

$$\left.\begin{aligned} \theta(x,t) &\;\le\; C_4 \\ |w_t(x,t)| &\;\le\; C_5 \end{aligned}\right\} \quad \text{a.e.} \tag{5.10}$$

Estimate 2.

Put $\lambda(x,t) := \log\,\theta(x,t)$. Then for a.e. $(x,t) \in \Omega\times(0,\infty)$ we have

$$\lambda_t - \Delta\lambda \;=\; \frac{1}{\theta}(\theta_t - \Delta\,\theta) + \left|\frac{\nabla\theta}{\theta}\right|^2\,, \tag{5.11}$$

where

$$\theta_t - \Delta\theta \;=\; -\;(F_2'(\mathfrak{s}[w]) + \mathcal{Q}[w])(\mathfrak{s}[w])_t \;=\; \mu(\theta)\,w_t\,\mathfrak{s}[w]_t \;+\; \theta\,F_1(\mathfrak{s}[w])_t\,, \tag{5.12}$$

hence, by (2.6),

$$\lambda_t - \Delta\lambda \;=\; F_1(\mathfrak{s}[w])_t \;+\; \frac{\mu(\theta)}{\theta}\,|\mathfrak{s}[w]_t|^2 \;+\; \left|\frac{\nabla\theta}{\theta}\right|^2 \tag{5.13}$$

a.e. in $\Omega\times(0,\infty)$. Integrating Eq. (5.13) with respect to x and t, we obtain for every $t > 0$ that

$$\int_0^t \int_\Omega \left(\frac{\mu(\theta)}{\theta} \, |\mathfrak{s}[w]_t|^2 \; + \; \left| \frac{\nabla \theta}{\theta} \right|^2 \right) (x,\tau) \, dx \, d\tau \tag{5.14}$$
$$\leq \; \int_\Omega \left(\log \theta(x,t) - \log \theta_0(x) - F_1(\mathfrak{s}[w])(x,t) + F_1(\mathfrak{s}[w])(x,0) \right) \, dx \; \leq \; C_6 \, .$$

From (5.10) and (5.1) it follows that

$$\int_0^t E(\tau) \, d\tau \; \leq \; C_7 \tag{5.15}$$

for every $t > 0$.

Estimate 3.

Test Eq. (2.9) with θ_t. This and the previous estimates in (5.10) yield for a. e. t that

$$|\theta_t(\cdot,t)|_2^2 + \frac{1}{2} \frac{d}{dt} |\nabla \theta(\cdot,t)|_2^2 \; \leq \; C_8 \, (1 + |\theta_t(\cdot,t)|_1) \; \leq \; \frac{1}{2} \left(|\theta_t(\cdot,t)|_2^2 + C_9 \right) \, , \tag{5.16}$$

hence

$$|\theta_t(\cdot,t)|_2^2 + \frac{d}{dt} |\nabla \theta(\cdot,t)|_2^2 \; \leq \; C_9 \quad \text{a. e.} \tag{5.17}$$

Thus, combining (5.17) with (5.15) and applying Lemma 3.1 of [11] yields that $E_1(t) = \int_\Omega |\nabla \theta|^2(x,t) \, dx$ tends to 0 as $t \to \infty$.

Estimate 4.

We differentiate the equation

$$w_t + \frac{\theta}{\mu(\theta)} \, F_1'(\mathfrak{s}[w]) + \frac{1}{\mu(\theta)} \, \left(F_2'(\mathfrak{s}[w]) \; + \; \mathcal{Q}[w] \right) = 0 \tag{5.18}$$

with respect to t and test with $\mathfrak{s}[w]_t$. This yields

$$(w_{tt} \, \mathfrak{s}[w]_t)(x,t) \; \leq \; C_{10} \, (1 + |\theta_t(x,t)|) \quad \text{a. e.} \, , \tag{5.19}$$

hence

$$\int_\Omega (w_{tt} \, \mathfrak{s}[w]_t)(x,t) \, dx \; \leq \; \frac{1}{2} \left(|\theta_t(\cdot,t)|_2^2 + C_{11} \right) \tag{5.20}$$

for a. e. $t > 0$. Combining (5.17) with (5.20) we obtain

$$\int_\Omega (w_{tt} \, \mathfrak{s}[w]_t)(x,t) \, dx \; + \; \frac{1}{2} \frac{d}{dt} |\nabla \theta(\cdot,t)|_2^2 \; \leq \; C_{12} \quad \text{a. e.} \tag{5.21}$$

For $t > 0$ put

$$q(t) \; := \; C_{12} t \; - \; E(t) \, . \tag{5.22}$$

We claim that for every $T > 0$ and every $\phi \in \overset{\circ}{W}{}^{1,1}(0,T)$ such that $\phi(t) \ge 0$ for every $t \in [0,T]$ we have

$$\int_0^T q(t)\, \phi_t(t)\, dt \ \le\ 0\,. \tag{5.23}$$

Indeed, let $T > 0$ and $\phi \in \overset{\circ}{W}{}^{1,1}(0,T)$ such that $\phi(t) \ge 0$ for every $t \in [0,T]$ be given. Then Ineq. (5.21) together with the Fubini theorem yield

$$\begin{aligned}
\int_0^T q(t)\, \phi_t(t)\, dt \ &= \ -\int_0^T \left(C_{12}\, \phi(t) + E(t)\, \phi_t(t)\right) dt \qquad (5.24)\\
&\le\ -\int_0^T \Bigg(\frac{d}{dt}\left(\frac{1}{2}\,\phi(t)\, \int_\Omega |\nabla\theta(x,t)|^2\, dx\right)\\
&\qquad + \phi(t)\, \int_\Omega (w_{tt}\, \mathfrak{s}[w]_t)(x,t)\, dx + \frac{1}{2}\,\phi_t(t)\, \int_\Omega (w_t\, \mathfrak{s}[w]_t)(x,t)\, dx\Bigg)\, dt\\
&=\ -\int_\Omega \int_0^T \left(\phi(t)\, (w_{tt}\, \mathfrak{s}[w]_t)(x,t) + \frac{1}{2}\,\phi_t(t)\, (w_t\, \mathfrak{s}[w]_t)(x,t)\right)\, dt\, dx\,.
\end{aligned}$$

By Proposition 2.1 (iii) and [9, Lemma 5.1] we have for a. e. $x \in \Omega$ that

$$\int_0^T \left(\phi(t)\, (w_{tt}\, \mathfrak{s}[w]_t)(x,t) + \frac{1}{2}\,\phi_t(t)\, (w_t\, \mathfrak{s}[w]_t)(x,t)\right)\, dt \ \ge\ 0\,, \tag{5.25}$$

and Ineq. (5.23) follows. Using once again Lemma 5.1 of [9] we conclude that there exists a non-decreasing function $q_* : [0,\infty) \to \mathbb{R}$ such that $q(t) = q_*(t)$ a. e. For $t \ge 0$ it now suffices to put $E_*(t) := C_{12} t - q_*(t)$. Proposition 5.2 of [9] with $y = E_*$, $Y = C_7$, $h \equiv 0$, $f(u) \equiv C_{12}$ entails that the function $(E_*)^2$ has bounded variation in $[0,\infty)$ and $\lim_{t\to\infty} E_*(t) = 0$. It now suffices to put $E_2^* := E_* - E_1$ and the assertion follows from Ineqs. (5.10) and (5.15).

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