Bifurcation Analysis for Spherically Symmetric Systems Using Invariant Theory

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1 Introduction

We reconsider steady–state bifurcation in the five dimensional irreducible representation of $\mathbf{O}(3)$. This problem has been studied by Golubitsky & Schaeffer [11]. They used a geometric approach reducing the five dimensional problem to a two dimensional D_3–equivariant problem. This reduction process is very special to the five dimensional representation. A similar process does not exist in the higher dimensional representations. In our study we derive all the results from the Poincaré–series, yielding the degrees of the generators of the ring of invariant functions and the module of equivariant polynomial maps respectively. The special representation is used only to show that a certain scaling is natural. The final results do not depend on this scaling.

General results about bifurcation with higher representations are due to Ihrig & Golubitsky [12]. Only in a few cases the bifurcations and the local dynamics are understood, see Chossat, Lauterbach & Melbourne [6], Fiedler & Mischaikow [10] for the seven and nine dimensional representations. The fact that the local bifurcation scenario in the nine dimensional case is complete is shown in Chossat & Lauterbach [5]. In all these papers the main emphasis is the use of the equivariant branching lemma, which guarantees the existence of solutions with isotropy subgroup Σ if Σ has a one–dimensional fixed point subspace. With the exception of Fiedler & Mischaikow [10], Lauterbach [13], [14] the dynamics of the problem was not considered. In this paper we use invariant theory to understand the dynamics near the bifurcation point more completely. We restrict ourselves to the simplest case, the five dimensional irreducible representation of $\mathbf{O(3)}$. Among other things, we find the existence of a *heteroclinic cycle*. Heteroclinic cycles have been observed in mode interactions involving the

$\ell = 2$ representation. In the interaction of the $\ell = 1$ and $\ell = 2$ modes ARMBRUSTER & CHOSSAT [2] and CHOSSAT & ARMBRUSTER [4] have shown the existence of a structurally stable heteroclinic cycle. In CASTRO [7] a heteroclinic cycle occurs in the interaction of the $\ell = 0$ and the $\ell = 2$ modes. The heteroclinic cycle in CASTRO [7] seems to be closely related to our cycle, however, the precise relation is not clear.

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2 Invariant theory

We use the standard notation: $\mathbf{O(3)}$ is the group of all orthogonal linear mappings on $\mathbb{R}^3$. As is well known there exist irreducible representations of this group in all odd dimensions $2\ell + 1$. They can be visualized as the action on the space of homogeneous, harmonic polynomials $P : \mathbb{R}^3 \to \mathbb{R}$ by

$$(\gamma, P)(x) = P(\gamma^{-1}x). \tag{1}$$

Let V_ℓ be the corresponding space. There is a litle subtlety concerning the actions of $\mathbf{O(3)}$. If the elements of $\mathbf{SO(3)}$ act as in (1) then $-\mathbb{1}$ can act as plus or minus identity. If the action of $-\mathbb{1}$ is given by (1), then we call it the natural representation of $\mathbf{O(3)}$. Given such a representation $\mathcal{R}$ denotes the ring of invariant polynomials $p : V_\ell \to \mathbb{R}$. We write $\mathcal{M}$ for the module of equivariant polynomial mappings $V_\ell \to V_\ell$. Hilbert's theorem guarantees that $\mathcal{R}$ and $\mathcal{M}$ are finitely generated. Observe that a theorem of SCHWARZ [18] proves that the generators for the polynomial invariants also generate the smooth invariants. The precise number of generators can be read off from the Poincaré–series, see SPRINGER [19]. The Poincaré–series for the ring of invariant functions is defined as the formal power series

$$\mathcal{P}^{\mathcal{R}}(t) = \sum_{j=0}^{\infty} \alpha_j t^j,$$

where $\alpha_j = \dim_{\mathbb{R}} \mathcal{R}_j$ and $\mathcal{R}_j$ is the real vector space of homogeneous invariant polynomials of degree j. Similar the formal power series

$$\mathcal{P}^{\mathcal{M}}(t) = \sum_{j=0}^{\infty} \beta_j t^j,$$

with $\beta_j = \dim_{\mathbb{R}} \mathcal{M}_j$, where $\mathcal{M}_j$ is the real vectorspace of homogeneous equivariant polynomial mappings of V_ℓ into itself, is called the Poincaré–series for $\mathcal{M}$.

Theorem 2.1 (Springer [19]) *We have*

$$\mathcal{P}^{\mathcal{R}}(t) = \int_{\mathbf{O(3)}} \frac{d\gamma}{\det(1 - t\gamma)}$$

and

$$\mathcal{P}^{\mathcal{M}}(t) = \int_{\mathbf{O}(3)} \frac{\operatorname{tr}(\gamma)d\gamma}{\det(1 - t\gamma)}$$

In order to simplify the calculations of these integrals one can use the Weyl integral formula, see Bröcker & tom Dieck [3] IV 1.11. Since determinant and trace are class functions one is left with an integral over the maximal torus, here $\mathbf{SO}(2)$. Identifying this group with the unit circle in $\mathbb{C}$ one has to compute some residues to obtain the Poincaré series. Applying this procedure to representations of $\mathbf{O(3)}$ we obtain for the irreducible representation on V_ℓ the following formulae for the respective Poincaré–series

$$P^{\mathcal{R}}(t) = \frac{1}{|W|} \int_T \det\left(\mathbb{1}_{G/T} - Ad_{G/T}(h^{-1})\right) \int_G \frac{1}{\det(\mathbb{1} - tg^{-1}hg)} dg dh$$

and

$$P^{\mathcal{M}}(t) = \frac{1}{|W|} \int_T \det\left(\mathbb{1}_{G/T} - Ad_{G/T}(h^{-1})\right) \int_G \frac{\operatorname{tr}(g^{-1}hg)}{\det(\mathbb{1} - tg^{-1}hg)} dg dh.$$

In our case the maximal torus is $\mathbf{SO(2)}$ its normalizer is $\mathbf{O(2)}$ and the Weyl group has two elements. Let T be a chosen maximal torus, H be its infinitesimal generator and set

$$h = \exp(i\theta H) \text{ and } z = e^{i\theta}.$$

Then for a root system R_+ we have (compare Lemma 1.8 in Chapter VI of [3])

$$\det\left(\mathbb{1}_{G/T} - Ad_{G/T}(h^{-1})\right) = \Pi_{a\in R_+}(1 - z^a)(1 - z^{-a}).$$

For the group $\mathbf{SO(3)}$ this last expression equals

$$(1 - z)(1 - z^{-1}).$$

For the $2\ell + 1$ dimensional representation of $\mathbf{SO(3)}$ we have

$$\mathrm{tr}(h) = \sum_{k=-\ell}^{\ell} z^k$$

and

$$\det(\mathbb{1} - th) = \Pi_{k=-\ell}^{\ell}(1 - tz^k).$$

Altogether we have shown the following theorem.

Theorem 2.2 *The Poincaré–series for the ring of invariant functions is given by*

$$\mathcal{P}^{\mathcal{R}}(t) = \frac{1}{2\pi i}\frac{1}{2}\oint\limits_{|z|=1} \frac{(1-z)(1-z^{-1})}{\Pi_{k=-\ell}^{\ell}(1-tz^{\ell-k})}\frac{dz}{z}$$

and the series for the module of equivariant mappings is equal to

$$\mathcal{P}^{\mathcal{M}}(t) = \frac{1}{2\pi i}\frac{1}{2}\oint\limits_{|z|=1} \frac{(1-z^{-1})(z^{-\ell}-z^{\ell+1})}{\Pi_{k=-\ell}^{\ell}(1-tz^{\ell-k})}\frac{dz}{z}.$$

Therefore one can compute the Poincaré series in either case by calculating some residues and adding up some expressions involving roots of unity. Specializing to the case $\ell = 2$ we find

$$\mathcal{P}^{\mathcal{R}}(t) = \frac{1}{(1-t^2)(1-t^3)}$$

and

$$\mathcal{P}^{\mathcal{M}}(t) = \frac{t+t^2}{(1-t^2)(1-t^3)}.$$

We can read off that there are precisely two generators π_1, π_2 of the ring of invariant functions of degrees 2 and 3, respectively. Moreover there are two generators of the module of equivariant mappings over this ring. These generators e_1, e_2 have degree 1 and 2, respectively. Therefore we can choose $e_i = \nabla\pi_i,\ i = 1, 2$. Obviously e_1 is a linear mapping, from the fact that the action is absolutely irreducible we conclude e_1 is the identity mapping. Up to a multiple π_1 is a Hilbert space norm on V_2, such that the group action is orthogonal. Let $< \cdot, \cdot >$ denote the corresponding $\mathbf{O(3)}$ invariant inner product. Let $x = (x_1, \ldots, x_5) \in V_2$ denote the elements in V_2 and write

$$\Pi : V_2 \to \mathbb{R}^2.$$

The range depends on the scaling of π_1 and π_2. Therefore we have to choose a certain scaling and we shall see that this choice also determines the precise form of the reduced equation. Since π_1 is a norm it is always positive and the range of Π is determined if we find the maximal and minimal value of π_2 for a fixed value of π_1. In order to find a natural choice for these scalings we look at the following representation of $\mathbf{O}(3)$. Let S denote the set of symmetric, traceless 3×3 matrices. S forms a five dimensional real vector space. $\mathbf{O}(3)$ acts by conjugation on this space. Observe that GOLUBITSKY & SCHAEFFER [11] made extensive use of this particular representation to study this problem. We use it only to find an appropriate scaling of the invariants, all other scalings would do as well. Let D denote the two dimensional subspace of S of diagonal matrices. By linear algebra any element of S can be conjugated into D, and therefore $\Pi(S) = \Pi(D)$. A natural choice for the invariants comes from the observation that the characteristic polynomial χ_A of a matrix A is invariant under conjugation, i.e. (observe $\mathrm{tr}(A) = 0$)

$$\chi_A(\lambda) = \lambda^3 + \pi_1\lambda + \pi_2.$$

Let $\mu_1,\ \mu_2,\ -(\mu_1 + \mu_2)$ denote the eigenvalues of a matrix in D. A short calculation yields

$$\pi_1(A) = -\mu_1\mu_2 + (\mu_1 + \mu_2)^2 = \mu_1^2 + \mu_2^2 + \mu_1\mu_2$$

and

$$\pi_2(A) = \det(A) = \mu_1\mu_2(\mu_1 + \mu_2).$$

Maximizing π_2 on $\pi_1 = const.$ gives the condition that A has a double eigenvalue, i.e. $\Delta = \mathrm{discr}(\chi(A)) = 0$, and

$$\Delta = \pi_1^3 - 27\pi_2^2.$$

Summarizing we have that this choice of $\pi_1,\ \pi_2$ gives

$$R(\Pi) = \{(\pi_1, \pi_2) \in \mathbb{R}^2 \mid \Delta \geq 0\}$$

Remark 2.3 *We want to point out that due to the construction of a global section to the group orbit, which at the same time is the fixed point subspace under a subgroup, we have an isomorphism between the* $\mathbf{SO(3)}$ *and the* D_3 *theory.*

3 Equivariant equations

Let us briefly describe an abstract setting and then specialize to our current situation of an $\mathbf{O(3)}$–equivariant bifurcation problem on V_2. Given a compact

Lie group Γ acting linearly on a space V, we denote by $\pi_1, \ldots, \pi_s$ a set of generators of the ring of Γ invariant functions and $e_1, \ldots, e_t$ generators of the module of equivariant, polynomial mappings $V \to V$ over this ring. We are interested in smooth equivariant mappings $f : V \to V$. It is a theorem of SCHWARZ [18] that all smooth equivariant mappings have the form

$$f(x) = \sum_{i=1}^{t} g_i(\pi_1(x), \ldots, \pi_s(x)) e_i(x).$$

In general we have $t \geq s$ and one can choose the generators such that $e_i = \nabla \pi_i$ for $i = 1, \ldots, s$.

We want to investigate the flow of the differential equation

$$\dot{x} = f(x).$$

In order to reduce the dimension we derive equations for the invariant functions. By the chain rule we have

$$\dot{\pi}_j(x) = < \nabla \pi_j(x), f(x) > = < e_j(x), f(x) > = \sum_{i=1}^{t} g_i(\Pi(x)) < e_i(x), e_j(x) > .$$

Therefore we want to calculate the inner products $< e_i, e_j >$, in our case, for $i, j = 1, 2$. We get

Lemma 3.1 (a) $< e_1, e_1 > = 2\pi_1$,
(b) $< e_1, e_2 > = 3\pi_2$,
(c) *The inner product* $< e_2, e_2 > = \frac{1}{6}\pi_1^2$ *if and only if the boundary of the range of* Π *is given by* $\Delta = 0$.

Proof: Let us first introduce the Euler operator $E(x) = \sum_i x_i \frac{\partial}{\partial x_i}$. Recall that applying E to a homogeneous polynomial p counts the degree of p, i.e. $Ep = (\deg p)p$.

(a) $< e_1(x), e_1(x) > = < x, \nabla \pi_1(x) > = E\pi_1(x) = 2\pi_1(x)$.

(b) $< e_1(x), e_2(x) > = < x, \nabla \pi_2(x) > = E\pi_2(x) = 3\pi_2(x)$.

(c) $< e_2(x), e_2(x) >$ is a quartic invariant and therefore it is a multiple of π_1^2, as we see from the Poincaré series. This constant can be computed for some special value of x. Assume that π_2 is maximal on the surface where π_1 is constant. Then $\nabla \pi_2(x) = \lambda x$ for some real λ. Therefore we get

$$< e_2(x), e_2(x) > = \lambda < e_2(x), e_1(x) > = \lambda^2 < e_1(x), e_1(x) > .$$

We conclude

$$\lambda = \frac{3\pi_2(x)}{2\pi_1(x)}$$

and

$$< e_2, e_2 > = \frac{9\pi_2^2}{2\pi_1}.$$

On $\Delta = 0$ this implies $< e_2, e_2 > = \frac{1}{6}\pi_1^2$.

A similar calculation gives, that $< e_2, e_2 > = \frac{1}{6}\pi_1^2$ implies that the extremal values for π_2 on a level surface of π_1 satisfy the equation $\Delta = 0$. $\square$

Corollary 3.2 *The reduced equation has the form*

$$\begin{aligned} \dot{\pi}_1 &= 2\pi_1 f_1(\pi_1, \pi_2) + 3\pi_2 f_2(\pi_1, \pi_2) \\ \dot{\pi}_2 &= 3\pi_2 f_1(\pi_1, \pi_2) + \frac{1}{6}\pi_1^2 f_2(\pi_1, \pi_2). \end{aligned}$$

Proof: Follows immediately from the foregoing. $\square$

4 Geometry of the phase space

The phase space of the reduced differential equation is the set

$$G_\Delta = \{(\pi_1, \pi_2) \in \mathbb{R}^2 \mid \Delta(\pi_1, \pi_2) \geq 0\}.$$

Let us briefly mention the stratification of the phase space into orbit types with respect to action of $\mathbf{O(3)}$ on V_2.

Theorem 4.1 *There are three orbit types with respect to the action of* $\mathbf{SO(3)}$ *on V_2. They correspond to isotropy subgroups D_2,* $\mathbf{O(2)}$*, and* $\mathbf{SO(3)}$*. 0 has orbit type* $\mathbf{SO(3)}$*. The nonzero points on the locus $\Delta = 0$ have isotropy type* $\mathbf{SO(2)}$ *and finally all other points have isotropy type D_2.*

Proof: By the irreducibility of the action it is obvious that 0 is the only point in V_2 having isotropy type $\mathbf{SO(3)}$. Since Π separates orbits (compare Poénaru [17]), $\Pi(x) = 0$ if and only if $x = 0$. (Of course, this follows already from $\pi_1 = \|x\|^2$.) The nonzero points on $\Delta = 0$ correspond to double eigenvalues of our matrix A above. Such a matrix commutes with $\mathbf{O(2)}$. Finally all other diagonal, traceless matrices commute with the four element group generated by the matrices

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

$\square$

The set of points in G_Δ with isotropy type $\mathbf{O(2)}$ consists of two components, the upper component, where π_2 is positive and the lower component, i.e. where $\pi_2 < 0$. Similarly we speak of the upper and the lower sheet, when we consider the set $(\pi_1, \pi_2, \lambda) \in \mathbb{R}^3$ with $\Delta = 0$. This is the boundary of the phase space for our parameter dependent equation.

It is a simple matter to check that the boundary of the phase space, i.e. $\{(\pi_1, \pi_2) \in \Pi(\mathbb{R}^5) \mid \Delta(\pi_1, \pi_2) = 0\}$ is invariant under the flow. In fact any vectorfield which is constructed that way has to respect the stratification into orbit types. That means that the points of the same orbit type form an invariant set, and the vectorfield is tangent to each stratum. Especially the origin is always a rest point for such an equation. In general, the existence of a Lyapunov function simplifies the analysis of a differential equation significantly. In our context we do not find such a Lyapunov function, but Δ comes very close to being a Lyapunov function.

Lemma 4.2 *Δ satisfies the following simple differential equation*

$$\dot{\Delta} = 6 f_1 \Delta.$$

Proof: Differentiate the defining relation. □

In order to find the equilibria of the reduced equation one has to solve the following algebraic system

$$\begin{aligned} 0 &= 2\pi_1 f_1(\pi_1, \pi_2) + 3\pi_2 f_2(\pi_1, \pi_2) \\ 0 &= 3\pi_2 f_1(\pi_1, \pi_2) + \frac{1}{6}\pi_1^2 f_2(\pi_1, \pi_2). \end{aligned}$$

One can rewrite this as

$$\begin{pmatrix} 2\pi_1 & 3\pi_2 \\ 3\pi_2 & \frac{1}{6}\pi_1^2 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = 0.$$

The determinant of the matrix on the left hand side is $\frac{1}{3}\Delta$. Therefore this system is equivalent to

$$\begin{aligned} 2\pi_1 f_1(\pi_1, \pi_2) + 3\pi_2 f_2(\pi_1, \pi_2) &= 0 \\ \Delta &= 0 \end{aligned}$$

or

$$f_1(\pi_1, \pi_2) = f_2(\pi_1, \pi_2) = 0.$$

In the next two sections we investigate these problems separately.

5 Equilibria on the boundary

In the following we make the simplifying assumption

$$f_1 = \lambda + a_1\pi_1 + \varepsilon_1\pi_2 \quad (2)$$
$$f_2 = c + a_2\pi_1 + \varepsilon_2\pi_2, \quad (3)$$

where λ is supposed to be a bifurcation parameter, a_1, a_2, c are nonzero constants, while $\varepsilon_{1,2}$ are supposed to be small. This is not a complete analysis, however it provides insight in the behavior of the dynamical system in an open region in parameter space.

To solve the equation on $\Delta = 0$ we have to combine $\Delta = 0$ with the first equation. Plugging the form of our mapping into it we get

$$\begin{aligned} 2\lambda\pi_1 + 2a_1\pi_1^2 + 2\varepsilon_1\pi_1\pi_2 + 3c\pi_2 + 3a_2\pi_1\pi_2 + 3\varepsilon_2\pi_2^2 &= 0 \\ \pi_1^3 - 27\pi_2^2 &= 0. \end{aligned}$$

Solving the second equation for π_2^2 and squaring the first equation we obtain

$$\pi_1(3c + \pi_1(2\varepsilon_1 + 3a_2))^2 = 27(2\lambda + 2a_1\pi_1 + \frac{1}{9}\varepsilon_2\pi_1^2)^2.$$

This yields a fourth order polynomial in π_1, namely

$$Q_{\varepsilon_1,\varepsilon_2}(\pi_1, \lambda) = \frac{1}{3}\varepsilon_2\pi_1^4 + A_1\pi_1^3 + A_2\pi_1^2 + A_3\pi_1 + 108\lambda^2 = 0, \quad (4)$$

with $A_1 = (12(a_1\varepsilon_2 - \varepsilon_1 a_2) - 4\varepsilon_1^2 - 9a_2^2)$, $A_2 = 108a_1^2 + 12(\lambda\varepsilon_2 - c\varepsilon_1) - 18a_2c$ and $A_3 = (216\lambda a_1 - 9c^2)$.
Concerning the zero set of this polynomial we have the following result.

Theorem 5.1 *If $\varepsilon_1 = \varepsilon_2 = 0$ and $a_2c < 0$ the solution set has the following features:*

(i) *the connected component $C_{0,0}$ of the zero set of $Q_{0,0}$ containing $(0,0)$ is contained in the set where $\pi_1 \geq 0$.*

(ii) *for each $\pi_1 \geq 0$, $\pi_1 \neq 0$ and $a_2\pi_2 + c \neq 0$ it has precisely two solutions with $\lambda \neq 0$.*

(iii) *there exist numbers $\lambda_{\min} < 0 < \lambda_{\max}$, $\lambda_c \in (\lambda_{\min}, \lambda_{\max})$ such that for $\lambda \in (\lambda_{\min}, \lambda_{\max})$, $\lambda \neq \lambda_c$ there exist three solutions with $\pi_1 \geq 0$, for $\lambda \notin [\lambda_{\min}, \lambda_{\max}]$ there exists precisely one such solution. For $\lambda = \lambda_c$, $\lambda_{\min}$, $\lambda_{\max}$ there exist two solutions as before.*

Proof: If $\varepsilon_1 = \varepsilon_2 = 0$, the fourth order polynomial (4) reduces to a cubic polynomial. It has the form

$$Q_{0,0}(\pi_1, \lambda) = 9a_2^2\pi_1^3 + (18a_2c - 108a_1^2)\pi_1^2 + (9c^2 - 216a_1\lambda)\pi_1 - 108\lambda^2 = 0. \quad (5)$$

Let us first look at the equation $Q_{0,0}(\pi_1, 0) = 0$. This yields $\pi_1 = 0$ or the second order polynomial

$$a_2^2\pi_1^2 + (2a_2c - 12a_1^2)\pi_1 + c^2 = 0. \quad (6)$$

Its discriminant is

$$48a_1^2(3a_1^2 - a_2c).$$

According to our assumption this quantity is positive, therefore this polynomial has two solutions, since c^2 is positive, they have the same sign. Since $a_2c - 6a_1^2 = -(3a_1^2 - a_2c) - 3a_1^2 < 0$ both solutions are positive. For small λ nonzero, by the Newton diagram, the small solution (that means the solution near $\pi_1 = 0$ for the cubic polynomial) solves

$$9c\pi_1 - 108\lambda^2 = 0,$$

i.e. it is positive. For $\lambda \neq 0$ the cubic polynomial has no zero at $\pi_1 = 0$ and therefore all solutions have to stay in the positive half plane. This proves (i).

To prove (ii) we look at the discriminant with respect to λ. It is given by

$$216^2 a_1\pi_1^2 + 432\left(9a_2^2\pi_1^3 + (18a_2c - 108a_1\pi_1^2 + 9c^2\pi_1)\right).$$

Since $216^2 = 108 \cdot 432$ this expression is

$$9 \cdot 432\pi_1(a_2\pi_1^2 + 2a_2c\pi_1 + c^2) = 9 \cdot 432\pi_1(a_2\pi_1 + c)^2.$$

This proves (ii).

In order to prove (iii) we look at the discriminant $D(\lambda)$ of the cubic polynomial with respect to λ. It turns out to be a quartic polynomial in λ with negative leading coefficient, i.e

$$D(\lambda) = \sum_{i=0}^{4} p_i\lambda^i,$$

with $p_0 = 944784a_1^4c^4 - 314928a_1^2a_2c^5 = 314928(3a_1^2 - a_2c)$ and $p_4 < 0$. With $\lambda_{\min} = \min\{\lambda \in \mathbb{R} \mid D(\lambda) = 0\}$ and $\lambda_{\max} = \max\{\lambda \in \mathbb{R} \mid D(\lambda) = 0\}$, we have $\lambda_{\min} < 0 < \lambda_{\max}$ and $D(\lambda)$ is negative for $\lambda \notin (\lambda_{\min}, \lambda_{\max})$. Define

$$\lambda_c = \frac{a_1c}{a_2}, \quad (7)$$

then we get $D(\lambda_c) = D'(\lambda_c) = 0$ and $D''(\lambda_c) = -629856(12a_1^2 + a_2c)^2a_2c$. The sign condition on a_2c yields that λ_c is a minimum. Therefore $\lambda_c \in (\lambda_{\min}, \lambda_{\max})$. $\square$

Remark 5.2 *The following picture shows schematically the zero set of the cubic polynomial. Observe that eliminating the variable π_2 identifies the upper and the lower sheet of the boundary. Therefore we get the projection of the zero set on the boundary onto the $(\pi_1, \lambda)-$ plane. For each point on the zero set we shall identify the sheet on which it is located. At λ_c the upper and the lower branch cross each other in the projection onto the (π_1, λ)–plane.*

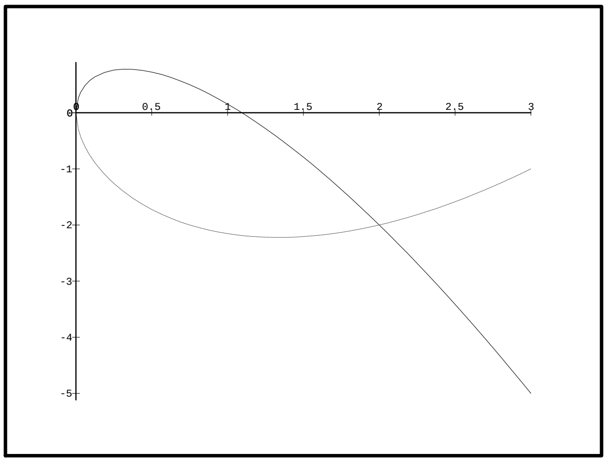

Figure 1: The zero set of the cubic polynomial for $a_1 = 1, a_2 = -4, c = 8$

Before we do that let us return to the original problem and assume $\varepsilon_1, \varepsilon_2$ are nonzero. From continuous dependence of the zeros of a polynomial on the coefficients we conclude the following theorem. In order to give a precise statement we need some notation. We consider compact subsets K of G_Δ of the following form $K = K_\Delta^\rho \times I$, where $I \subset \mathbb{R}$ is a closed interval $I = [\lambda_-, \lambda_+]$ with $\lambda_- < \lambda_{\min} < \lambda_{\max} < \lambda_+$ and $K_\Delta^\rho = G_\Delta \cap \{(\pi_1, \pi_2) \mid \pi_1 \leq \rho\}$. If ρ is chosen sufficiently large, then the connected component $\mathcal{C}_{0,0}$ of the zero set of the cubic polynomial intersects ∂K in two nontrivial points (i.e. other than $(0, 0)$) in the faces $\lambda = \lambda_\pm$. Let $\mathcal{C}_{\varepsilon_1,\varepsilon_2}$ denote the connected component of $(0, 0)$ in the zero set of of the quartic $Q_{\varepsilon_1,\varepsilon_2}$ in (4). Let $B_\varepsilon(\pi_1, \pi_2)$ be the ball of radius ε about (π_1, π_2). Then we have:

Theorem 5.3 *For each compact subset K of G_Δ, as above, and for each $\varepsilon > 0$ there exists a number $\delta > 0$, such that for $\varepsilon_i < \delta$, for $i = 1, 2$ the connected component $\mathcal{C}_{\varepsilon_1,\varepsilon_2}$ is contained in $K \cap T_\varepsilon$, where T_ε is the tubular neighborhood*

$$T_\varepsilon = \bigcup_{(\pi_1,\pi_2)\in\mathcal{C}_{0,0}} B_\varepsilon(\pi_1, \pi_2)$$

of $\mathcal{C}_{0,0}$ of radius ε.

Proof: This theorem follows from continuous dependence of the zeros of a polynomial on its coefficients. □

So far we have described some global properties of the set of singular points of our vectorfield. In order to discuss the stability of these equilibria we are going to study the local bifurcation scenario. It is clear from our previous discussion that $\pi_1 = \pi_2 = 0$ is a solution for all $\lambda \in \mathbb{R}$. For $\lambda \neq 0$ the linearization of the vectorfield is regular and therefore this branch is locally unique. At $\lambda = 0$ the linearization becomes singular. Near the bifurcation point we had

$$\pi_1 = \frac{12}{c}\lambda^2.$$

The Newton polynomial for the (π_2, λ) scaling comes from the first equation, i.e.

$$8\lambda^3 + c^2\pi_2 = 0.$$

Therefore for $\lambda > 0$ the branch is on the lower sheet, for $\lambda < 0$ it is on the upper sheet. At $\lambda_{\min}$ or $\lambda_{\max}$, respectively the upper, or the lower branch, respectively undergo a turning point bifurcation.

In the next section we shall see, among other things, that there are no further equilibria near the bifurcation point.

However, let us first look at the stability of the bifurcating branch near the bifurcation point. From the classical principle of exchange of stability, applied on the $\Delta = 0$ surface, we conclude that the bifurcating solution is unstable for $\lambda < 0$ and has a stable direction for λ positive. However from the differential equation for Δ we infer that this surface (near the bifurcating branch, i.e where f_1 is approximately $\lambda + \frac{12}{c}\lambda^2$) is unstable for $\lambda > 0$ and stable for $\lambda < 0$. This implies instability of this branch with one stable and one unstable direction on both sides.

At the turning point bifurcation the stability of the equilibria on the boundary changes. The eigenvalue corresponding to the eigenvector tangent to the surface $\Delta = 0$ changes sign. Therefore, after the turning point the subcritical branch is stable, the supercritical branch becomes completely unstable. This remains true until further bifurcations occur.

6 Internal equilibria

In order to find the internal equilibria we have to solve the equation

$$\begin{aligned} f_1(\pi_1, \pi_2) &= 0 \\ f_2(\pi_1, \pi_2) &= 0 \end{aligned}$$

We assume the hypotheses of theorem 5.1, especially, $a_2 c < 0$. Taking again $\varepsilon_i = 0$ for $i = 1, 2$ we get the line $\pi_1 = -\frac{c}{a_2}$ and $\lambda = \lambda_c$ in (π_1, π_2, λ)–space of solutions. It intersects the two sheets of $\Delta = 0$ at the double zero of the cubic polynomial (5). Therefore we have a singular line L connecting the upper and the lower branch of equilibria on the boundary. The complete picture is described in the next theorem.

Theorem 6.1 *Assume $a_1\varepsilon_2 - a_2\varepsilon_1 \neq 0$. Near L there exists a line $L_{\varepsilon_1,\varepsilon_2}$, parametrized over λ, of steady state solutions connecting the lower and the upper branch of the boundary equilibria.*

Proof: The system $f_1 = f_2 = 0$ is linear in all variables and we can solve it:

$$\begin{aligned} \pi_1(\lambda) &= \frac{c\varepsilon_1 - \lambda\varepsilon_2}{a_1\varepsilon_2 - a_2\varepsilon_1} \\ \pi_2(\lambda) &= \frac{\lambda a_2 - a_1 c}{a_1\varepsilon_2 - a_2\varepsilon_1}. \end{aligned}$$

Let λ_c be defined as before and set

$$\pi_1^c = -\frac{c}{a_2} \tag{8}$$

and $\pi_2^c = \pm\sqrt{\frac{1}{27}(\pi_1^c)^3}$. We have to show that the line $\{(\pi_1(\lambda), \pi_2(\lambda)) \mid \lambda \in \mathbb{R}\}$ intersects the surface $\Delta = 0$ near the values λ_c, π_1^c, π_2^c. Define

$$F : \mathbb{R}^5 \to R^3 : (\lambda, \pi_1, \pi_2, \varepsilon_1, \varepsilon_2) \mapsto (f_1, f_2, \Delta).$$

Obviously $F(\lambda_c, \pi_1^c, \pi_2^c, 0, 0) = 0$. Let us look at the partial derivative with respect to the first three variables at this point. It is represented by the matrix

$$\begin{pmatrix} 1 & a_1 & 0 \\ 0 & a_2 & 0 \\ 0 & 3\pi_1^2 & -54\pi_2 \end{pmatrix},$$

which is regular. Therefore the implicit function theorem yields the existence of points $\lambda_{\varepsilon_1,\varepsilon_2}, \pi_1^{\varepsilon_1,\varepsilon_2}, \pi_2^{\varepsilon_1,\varepsilon_2}$ near $\lambda_c, \pi_1^c, \pi_2^c$ solving $f_1 = f_2 = \Delta = 0$. Since all solutions have to be on the line $L_{\varepsilon_1,\varepsilon_2}$, we have shown that this line intersects the domain of our differential equation. Now we have established, that it connects the upper and the lower branch. $\square$

Remark 6.2 *Of course at the intersection of $L_{\varepsilon_1,\varepsilon_2}$ with the boundary, defined by $\Delta = 0$, the line hits the solutions on the boundary. Therefore this line can be viewed as a secondary bifurcation.*

Let $\Delta(\lambda)$ denote $\Delta(\pi_1(\lambda), \pi_2(\lambda))$. $\Delta(\lambda)$ is a cubic polynomial in λ. For large λ we have $\Pi(\lambda) \in G_\Delta$ on one side and $\Pi(\lambda) \notin G_\Delta$ on the other side.

In order to discuss the stability of the solutions on $L_{\varepsilon_1,\varepsilon_2}$ we have to study the local bifurcation at $\lambda_{\varepsilon_1,\varepsilon_2}, \pi_1^{\varepsilon_1,\varepsilon_2}, \pi_2^{\varepsilon_1,\varepsilon_2}$ and the corresponding exchange of stability. The proofs of the corresponding bifurcation results will be given in the next section.

Theorem 6.3 (i) *If $a_1\varepsilon_2 - a_2\varepsilon_1 < 0$ then the branch of internal solutions is unstable.*

(ii) *If $a_1\varepsilon_2 - a_2\varepsilon_1 > 0$ we have the following two cases*

(a) If $12a_1^2 > -a_2c$, then the trace of the linearization is of one sign along the branch of secondary solutions, it is given by $\operatorname{sign} a_1$*, i.e. the solutions are unstable if $a_1 > 0$ and stable otherwise.*

(b) If $12a_1^2 < -a_2c$ we get the following cases

(1) *If $a_1 > 0,\ a_2 > 0$ then the solutions on $L_{\varepsilon_1,\varepsilon_2}$ are unstable for $\pi_2 > 0$ and stable near the lower sheet.*

(2) *If $a_1 > 0,\ a_2 < 0$ then the internal solutions are unstable for $\pi_2 < 0$ and stable near the upper sheet.*

(3) *If $a_1 < 0,\ a_2 > 0$ then the internal solutions are stable for $\pi_2 < 0$ and unstable near the upper sheet.*

(4) *Finally, If $a_1 < 0,\ a_2 < 0$ then the internal solutions are stable for $\pi_2 > 0$ and unstable near the lower sheet.*

7 Hopf bifurcation, heteroclinic bifurcation

In order to find Hopf bifurcation in our system we have to study the stability of the steady state solutions. The linearization of the vectorfield is given by

$$\begin{pmatrix} 2\lambda + 4a_1\pi_1 + 2\varepsilon_1\pi_2 + 3a_2\pi_2 & 2\varepsilon_1\pi_1 + 3c + 3a_2\pi_1 + 6\varepsilon_2\pi_2 \\ 3a_1\pi_2 + \frac{1}{3}\pi_1(c + a_2\pi_1 + \varepsilon_2\pi_2) + \frac{1}{6}a_2\pi_1^2 & 3\lambda + 3a_1\pi_1 + 6\varepsilon_1\pi_2 + \frac{1}{6}\varepsilon_2\pi_1^2 \end{pmatrix}.$$

In order to prove Hopf bifurcation we have to show the existence of a branch of steady state solutions $(\Pi(\lambda), \lambda)$ and a point $(\Pi(\lambda_0), \lambda_0)$ where the trace of the linearization changes sign and its determinant is positive. We begin with the following lemma. We need a nondegeneracy condition which essentially says, that the branch of internal solutions can be parameterized over λ.

Lemma 7.1 *Suppose the nondegeneracy condition*

$$a_1\varepsilon_2 - a_2\varepsilon_1 \neq 0 \tag{9}$$

holds, then the determinant of the linearization along the branch of internal solutions is of one sign.

Proof: Suppose the lemma was not true, then we had a change of sign along the internal branch, leading to bifurcation from this branch. Since we know that the possible steady state bifurcations occur only at the boundary of our domain, this cannot happen. Since the determinant is quadratic in λ it has no other zeros than those on the boundary. □

Therefore it suffices to get an estimate on the determinant at a single point along the line of internal solutions.

Lemma 7.2 *If* $a_2 c < 0$ *and*

$$a_1\varepsilon_2 - a_2\varepsilon_1 > 0 \tag{10}$$

then the determinant of the linearization along the internal solution is positive.

Proof: Compute the determinant for $\lambda = \lambda_c$, $\pi_1 = \pi_1^c$ at $\pi_2 = 0$ to obtain

$$-\frac{1}{3}\frac{c^3}{a_2^3}(a_2\varepsilon_2 - a_2\varepsilon_1).$$

□

The following lemma gives a condition when the trace undergoes a change of sign along the branch of internal solutions.

Lemma 7.3 *If*

$$a_2 c + 12a_1^2 < 0, \tag{11}$$

then the trace changes sign along the branch of internal solutions.

Proof: We calculate the trace of the linearization for $\pi_2 = 0$ and on the locus $\Delta = 0$ for $\varepsilon_1 = \varepsilon_2 = 0$. For $\pi_2 = 2$ we find the value $-2\lambda_c$ and on the upper or lower sheet respectively we get $-2\lambda_c \pm 3a_2\pi_2^c$. A change of sign along the part of the line where π_2 is positive or negative occurs if $12a_1^2 < -a_2 c$ and persists for $\varepsilon_{1,2}$ nonzero. □

Theorem 7.4 *If the conditions (9) and (11) are satisfied, then there a exists a value* $\lambda_h \in (\lambda_{\min}, \lambda_{\max})$, *and a point* (π_1^h, π_2^h) *on* $L_{\varepsilon_1,\varepsilon_2}$ *such that a Hopf bifurcation occurs at* (π_1^h, π_2^h) *for* $\lambda = \lambda_h$. *The branch of periodic solutions is unique. Moreover, we find*

$$\operatorname{sign}(\pi_2^h) = -\operatorname{sign}(a_1)\operatorname{sign}(a_2).$$

Proof: The existence of the Hopf point follows immediately from the lemmata 7.1, 7.3. We only have to show the uniqueness of the branch of periodic solutions. It follows trivially from the fact that the trace $t(\lambda)$ of the linearization along the branch of internal solutions is quadratic in λ and therefore a change of sign at λ_h means that $t'(\lambda_h) \neq 0$. Since the eigenvalues are complex conjugate the eigenvalues cross the imaginary axis with nonzero speed, yielding the uniqueness of the Hopf branch. □

In order to get more refined statements on the local Hopf bifurcation one has to compute the point on L where the trace changes sign. This is a first approximation of the Hopf points on $L_{\varepsilon_1,\varepsilon_2}$. At this point a degenerate bifurcation occurs and provides some information on the nearby Hopf points. The line L is characterized by $\varepsilon_1 = \varepsilon_2 = 0$, $\lambda = \lambda_c$ and $\pi_1 = \pi_1^c$. Along L the right lower entry of the linearization, given in section 7 is zero. The critical value for π_2 is where the left upper entry vanishes. We have

$$\pi_2^{crit} = -\frac{1}{3a_2}(2\lambda_c + 4a_1\pi_1^c) = \frac{2a_1 c}{3a_2^2}. \tag{12}$$

The linearization at this point is nilpotent. A normal form analysis near this point is rather complicated. The branching direction of the Hopf solutions will be determined differently and discussed later. The following two figures show schematically the flow near the nilpotent point and how it fits into the global flow. The last figure indicates a singular heteroclinic cycle, the singular part consisting of a family of equilibria on L. The heteroclinic cycle becomes a real heteroclinic cycle for certain parameter values. The branch of periodic solutions approaches this heteroclinic cycle and disappears. We will not prove this picture completely. In the next section we show that the family of periodic solutions disappears because its period approaches infinity. Then we show that at this instance a heteroclinic cycle occurs. In the last section we prove a certain nondegeneracy. For $\varepsilon_{1,2}$ sufficiently small we show that the Hopf bifurcation is not degenerate in the sense, that not all periodic solutions appear for the same parameter value.

8 Global Behavior

We recall the global Hopf bifurcation theorem [1], [9], [15] stating that along a branch of periodic solutions coming from a regular Hopf point (that is that the dimension of the unstable manifold changes by pairs of purely imaginary eigenvalues crossing the imaginary axis) one of the following alternatives has to occur

(i) the amplitude goes to infinity

(ii) the period goes to infinity

(iii) another Hopf point occurs.

This theorem will be crucial for the proof of the following result.

Theorem 8.1 *The Hopf branch disappears through a infinite period bifurcation to a heteroclinic cycle with two boundary equilibria on it.*

Proof: A periodic solution of a 2–dimensional system always has to wind around an equilibrium. After the disappearance of the internal no such solution is available and therefore the periodic solution cannot exist any more. Since we have only one Hopf point in our system the branch of periodic solutions cannot connect to another Hopf point. (Observe that the fact that the trace along the internal solutions is quadratic in λ and the fact that the trace changes sign between $\pi_2 = 0$ and $\pm\pi_2^c$ means that there can only be one Hopf point.) Therefore either the amplitude or the period have to go to infinity. Let us first exclude that the amplitude becomes arbitrarily large.

First we note, that due to the fact that Δ satisfies the differential equation $\dot{\Delta} = 6f_1\Delta$ the region $\Delta \geq \mu$ is positively invariant if $f_1 > 0$ on this region and negatively invariant if f_1 has the other sign. In any case the region $\Delta \geq \mu$ cannot contain and cannot be transversed by a periodic solution if f_1 is of one sign on this region. For any $\mu > 0$ the asymptotics of the curve $\Delta = \mu$ is the same as for $\Delta = 0$. Due to the form of f_1 there exists a $\mu^* > 0$ such that f_1 is of one sign on $\Delta > \mu^*$. Therefore we have to exclude the existence of an unbounded family of periodic solutions in $\{(\pi_1, \pi_2) \in G_\Delta \mid \Delta(\pi_1, \pi_2) < \mu^*\}$. Each periodic solution has to have a maximal π_1–value, where $\dot{\pi}_1 = 0$. Therefore we compute the $\dot{\pi}_1 = 0$ near infinity. This curve is given by

$$2\lambda\pi_1 + 2a_1\pi_1^2 + (2\varepsilon_1 + 3a_2)\pi_1\pi_2 + 3c\pi_2 + 3\varepsilon_2\pi_2^2 = 0.$$

To get the asymptotics near infinity we have to consider the Newton polynomial near infinity, it is given by

$$2a_1\pi_1^2 + (2\varepsilon_1 + 3a_2)\pi_1\pi_2 + 3\varepsilon_2\pi_2^2 = 0.$$

As a result we get two arcs which are asymptotically linear near infinity. Therefore they cannot be contained in the set $\{(\pi_1, \pi_2) \in G_\Delta \mid 0 \leq \Delta \leq \mu^*\}$. Therefore there cannot be a family of periodic solutions with its amplitudes going to infinity.

The next step is to investigate how the period can get large. This happens if the periodic solution approaches one or more equilibria giving rise to a homoclinic loop or a heteroclinic cycle.

We exclude the case of a homoclinic loop. If there were a homoclinic loop the saddle had to be on the boundary. If the saddle would be hyperbolic then either the stable or the unstable manifold would be part of the boundary. Since the boundary is a one–dimensional manifold a solution on the boundary cannot converge to the same point for $t \to \infty$ and $t \to -\infty$. So a homoclinic loop can only occur if the equilibrium on it is not hyperbolic. A simple calculation yields that the points on the boundary are hyperbolic with the

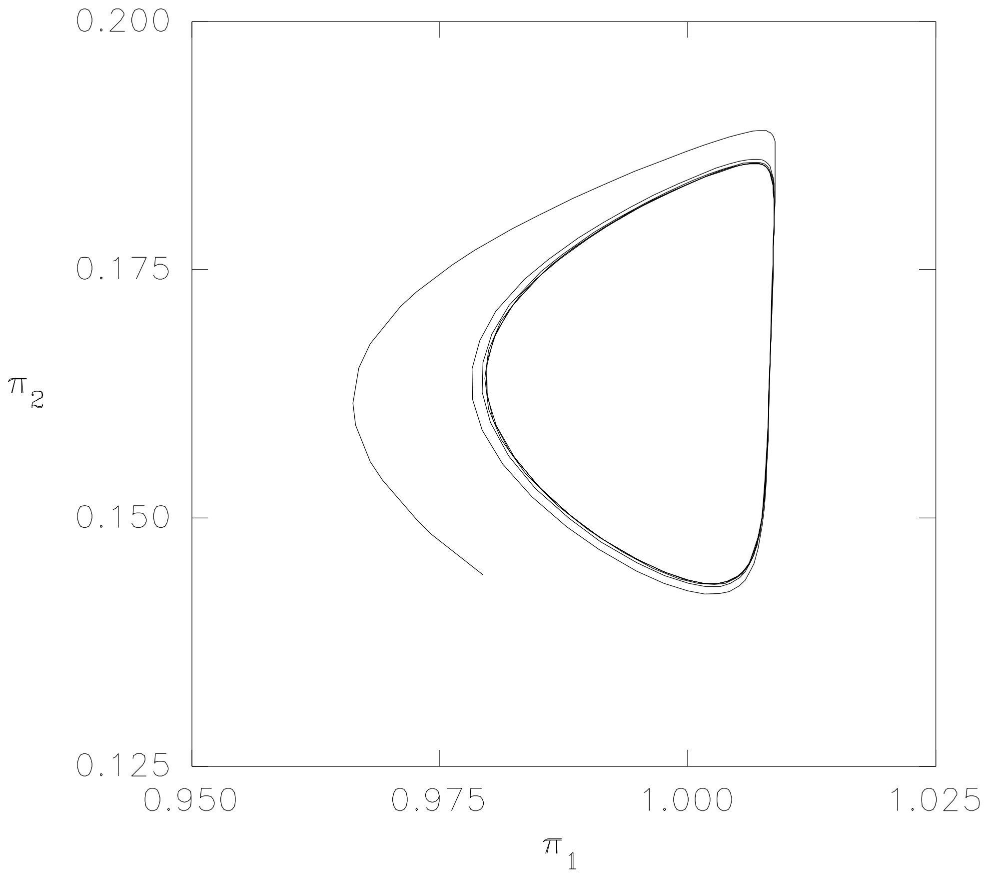

Figure 2: Stable periodic solutions

only exception of the point where the line $L_{\varepsilon_1,\varepsilon}$ intersects the boundary. But there the linearization has only one zero eigenvalue, since the trace is nonzero, according to lemma 7.3. Then the generalized Hartman–Grobman theorem, see PALMER [16] applies and shows that this point cannot be the α and ω limit set of a solution. Therefore there must be a heteroclinic cycle, involving at least two points on the boundary. If $\lambda \neq 0$, then the the origin is either stable or completely unstable (i.e. has unstable dimension 2) and therefore it cannot be part of the heteroclinic cycle. This means, that only the solutions on one sheet can be on this cycle, and we have shown that it contains precisely two equilibria.

9 Nondegeneracy of the Hopf Bifurcation

As indicated above we show, that for $\varepsilon_{1,2}$ sufficiently small the Hopf bifurcation is not vertical, i.e. for $\lambda = \lambda_h$ there exists a neighborhood of the Hopf point which does not contains any periodic solution. This proof relies on

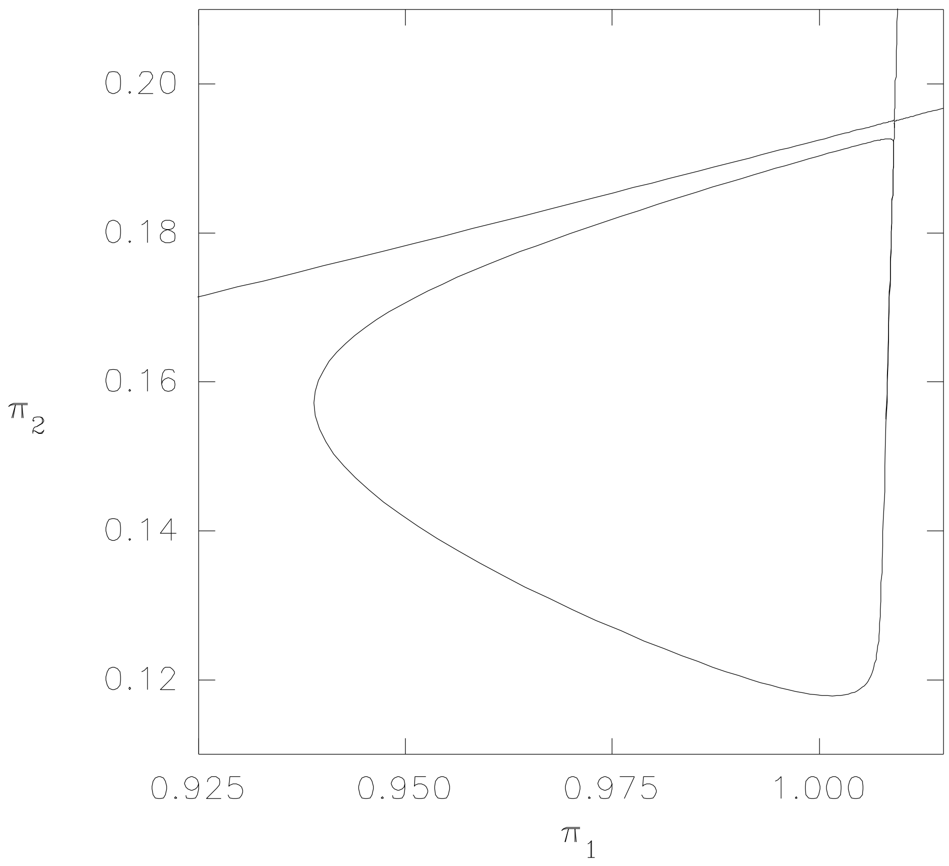

Figure 3: Stable periodic solutions naer a saddle

analyticity of our vector field. Of course the result remains true if we perturb the vectorfield keeping only its k-jet, for k sufficiently large. We use the following two basic properties of flows of analytic vectorfields in the plane:

(i) If v is an analytic vectorfield on a two dimensional smooth manifold, with a singular point p, such that any neighborhood of p contains a periodic orbit of v, then there exists a neighborhood U of p, such that $U \setminus \{p\}$ is filled with periodic orbits.

(ii) If γ is a periodic orbit of v, then there is either a neighhood W, such that $W \setminus \gamma$ contains no periodic orbits, or there exists a neighborhood which is filled with periodic orbits.

We need one more result on planar flows, due to DOS REISS [8]. For the sake of completeness let us recall it.

Lemma 9.1 *Let p_i, γ_i $i = 1, \dots, n$ be a heteroclinic cyle, i.e. pi_i are hyperbolic saddle points and γ_i are heteroclinic solutions connecting p_i with p_{i+1}, tacitly assuming $n + 1 = 1$. Let μ_i denote the stable eigenvalue at p_i and τ_i be the unstable eigenvalue. If k a transverse section to one of the γ_i and y is a coordinate on k, let $P(y)$ denote the coordinate of the next intersection of the trajectory through y with k (if defined). Then there exists a continuous function ρ on k, bounded away from zero, such that*

$$P(y) = \rho(y)|y|^{\frac{\mu_1 \cdot \mu_2 \cdots \mu_n}{\tau_1 \cdot \tau_2 \cdots \tau_n}}.$$

Theorem 9.2 *For $\varepsilon_1 \varepsilon_2 \neq 0$, $\varepsilon_{1,2}$ sufficiently small and if conditions (9), (11) are satisfied the Hopf bifurcation is not vertical, i.e. there exists a number λ_1 near λ_h such that on the interval (λ_h, λ_1) (or (λ_1, λ_h) if $\lambda_1 < \lambda_h$) respectively) there exists an continuous and injective mapping assigning to each λ in this interval an initial value of a periodic orbit.*

Proof: Suppose the theorem were not true. Then, for a sequence of pairs $(\varepsilon_1^n, \varepsilon_2^n)$ approaching zero for $n \to \infty$ the Hopf bifurcation is vertical. It follows that for each pair $(\varepsilon_1^n, \varepsilon_2^n)$ there exists a neighborhood U_n of the Hopf point $(\pi_1^{h,n}, \pi_2^{h,n})$ for $\lambda = \lambda_h^n$ such that $U \setminus \{(\pi_1^{h,n}, \pi_2^{h,n})$ is filled with periodic solutions. Then, according to our previous observations the whole Hopf branch is contained in $\mathbb{R}^2 \times \{\lambda_h^n\}$. Especially the heteroclinic cycle exists for $\lambda = \lambda_h^n$. Since the interior of this cycle is filled with periodic solutions it follows that it is neutrally stable.

Let us compute the stability according to lemma 9.1. We do the computations for $\varepsilon_1 = \varepsilon_2 = 0$ and conclude the stability for possible heteroclinic cycles existing for sufficiently small values of the parameters. This will give a contradiction.

Let us first compute the candidates for the two equilibria on the cycle, i.e. choose $\varepsilon_{1,2} = 0, \lambda = \lambda_c$. One of the points is among the intersections of L with the boundary, i.e. one of the points $(\pi_1^c, \pm\pi_2^c)$ and the other one is the nontrivial solution on the boundary between $(0,0)$ and the first point. The sign of $\pm\pi_2^c$ is determined by the sign of the π_2 value of this other point, since they have to be on the same sheet. A short computation yields the coordinates of this point, call it π_1^{sp}, π_2^{sp} as

$$\pi_1^{sp} = 12\frac{a_1^2}{a_2^2}$$

and

$$\pi_2^{sp} = -8\frac{a_1^3}{a_2^3}.$$

The determinant of the linearization at this point is given by

$$-6\frac{a_1^2}{a_2^2}(ca_2 + 12a_1^2)^2.$$

By (11) this point has two nontrival eigenvalues $\mu_1 < 0 < \tau_1$. At the other point $(\pi_1^c, \pm\pi_2^c)$, $\lambda = \lambda_c$ we have one nonzero eigenvalue and one zero eigenvalue. Therefore one of the products $\mu_1\mu_2$ or $\tau_1\tau_2$ is zero, the other one nonzero. For $\varepsilon_{1,2}$ small, one of the products will be small, the other one is far away from zero, contradicting the neutral stability, by lemma 9.1. □

The following two pictures show the typical shape along solution branches. The first picture depicts an axisymmetric point. The second presents a D_2 symmetric one. The heteroclinic cycle consists of two arcs, one containing axisymmetric points, the other one D_2 points. Along the axisymmetric arc the shape remains constant, only the size of the solution changes. On the other part the solution starts almost axisymmetric. As it travels away from the boundary the saddle becomes more and more distinct. Finally it returns to an almost axisymmetric state of different size. On a nearby periodic orbit we expect to see periodically a similar change in size and shape.

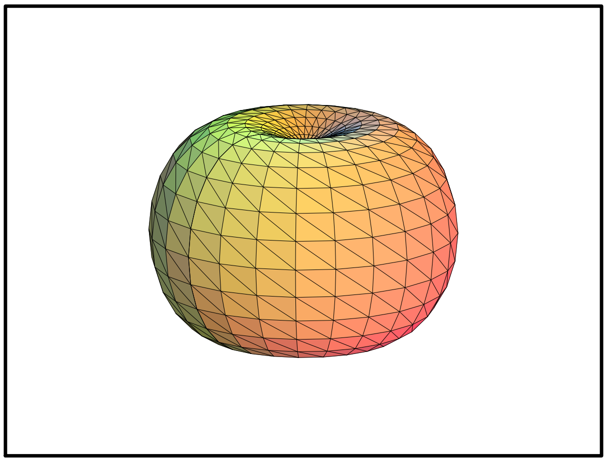

Figure 4: The shape of an axisymmetric solution

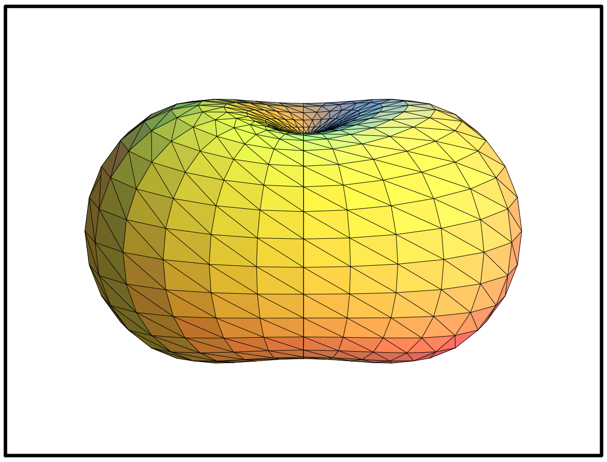

Figure 5: The shape of a typical non-axisymmetric solution

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