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Pendulum with positive and negative dry friction. Continuum of homoclinic orbits*

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Abstract

A two-order differential equation of pendulum with dry friction is considered. The existence of a continuum of homoclinic orbits with various homotopic properties on the cylinder is proven.

Bernold Fiedler asked me about the double homoclinic orbit in concrete dynamical systems.
Here a pendulum–like systems with dry friction is considered for which the existence of a continuum of homoclinic orbits with various homotopic properties on the cylinder is proven.
Consider the equation

$$\ddot{\theta} + F(\theta, \dot{\theta}) + \sin\theta = 0 \tag{1}$$

or the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -F(x,y) - \sin x. \end{aligned} \tag{2}$$

Here $F(x+2\pi, y) = F(x,y)$ and

$$F(x,y) = \begin{cases} 0, & \text{for}\, y < 2, \quad x \in (-\pi, \pi), \\ \gamma_1, & \text{for}\, y > 2, \quad x \in (-\pi, 0), \\ -\gamma_2, & \text{for}\, y > 2, \quad x \in (0, \pi), \end{cases}$$

*This work has been completed in Institut für Angewandte Analysis und Stochastik, Berlin.

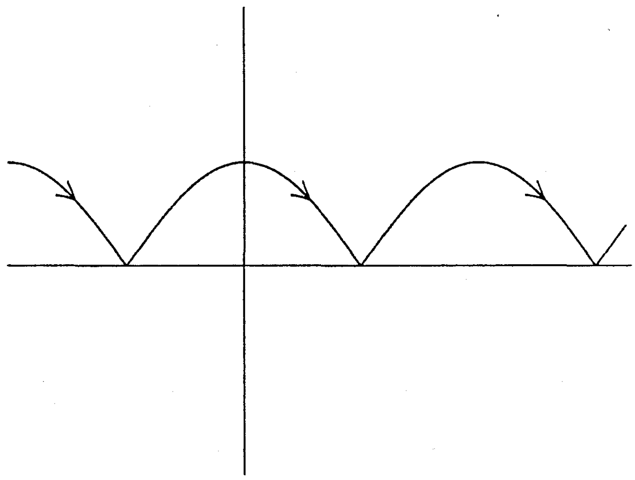

Figure 1: Classical homoclinic orbit

which corresponds to the classical homoclinic orbit in a cylindrical phase space. See figure 1.

Let us denote by Ω the following region in R^2:

$$\Omega = \{x \in R^1,\ G(x) < y \leq 2\}.$$

Definition. The trajectory $x(t)$, $y(t)$ of system (2) is called a *homoclinic orbit of degree* k if there exist the limits

$$\lim_{t\to+\infty} x(t), \quad \lim_{t\to+\infty} y(t), \quad \lim_{t\to-\infty} x(t), \quad \lim_{t\to-\infty} y(t)$$

and if

$$\left|\lim_{t\to+\infty} x(t) - \lim_{t\to-\infty} x(t)\right| = 2k\pi.$$

Of course this orbit is homoclinic with respect to the cylindrical phase space and heteroclinic with respect to R^2.

Proposition 1. *For every point* $(x_0, y_0) \in \Omega$ *and for every integer number* $k \geq 2$ *there exists a homoclinic orbit* γ *of degree* k *such that* $(x_0, y_0) \in \gamma$.

Proof. An important role in this proof is played by the sliding solution $y(t) \equiv 2$. See figure 2.

This solution is stable in the regions

$$U_{2j} = \{x \in ((2j-1)\pi, 2j\pi),\ y \in R^1\}$$

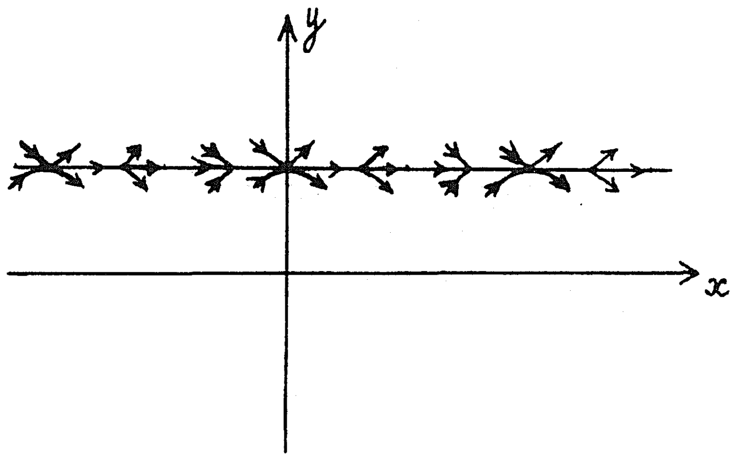

Figure 2: Sliding solution

Proof. An important role in this proof is played by the sliding solution $y(t) \equiv 2$. See figure 2.

This solution is stable in the regions

$$U_{2j} = \{x \in ((2j-1)\pi, 2j\pi),\ y \in R^1\}$$

and unstable in the regions

$$U_{2j+1} = \{x \in (2j\pi, (2j+1)\pi),\ y \in R^1\}.$$

In the regions U_{2j} we have unique solutions with respect to initial data and increase of time. In the regions U_{2j+1} we have unique solutions with respect to initial date and decrease of time.

Every point $x_0 \in ((2j-1)\pi, 2j\pi)$, $y_0 = 2$ is initial data of three solutions with respect to decrease of time. These solutions are the sliding solution, some solution in the region $y < 2$ and some solution in the region $y > 2$. Also every point $x_0 \in (2j\pi, (2j+1)\pi)$, $y_0 = 2$ is initial data of three solutions with respect to increase of time.

We fix now an integer $k \geq 2$ and a point $(x_0, y_0) \in \Omega$. It is easy to see now that there exist numbers $t_1 < t_2$ such that

$$y(t_1, x_0, y_0) = y(t_2, x_0, y_0) = 2, \quad x(t_1, x_0, y_0) = 2j\pi, \quad x(t_2, x_0, y_0) = 2(j+1)\pi$$

for some integer j. See figure 3.

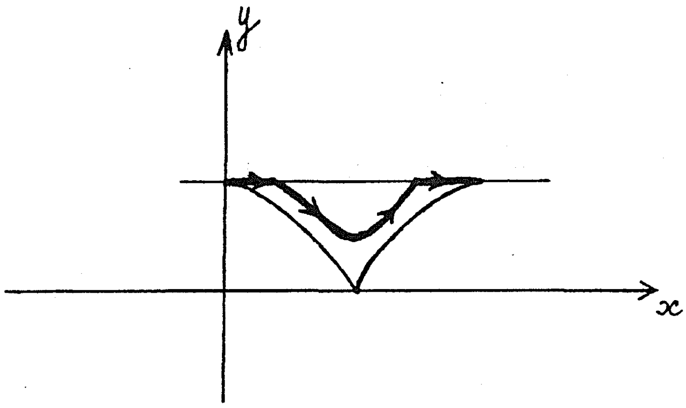

Figure 3: Homoclinic solution of degree k

We can consider the point $x(t_1, x_0, y_0) = 2j\pi$, $y(t_1, x_0, y_0) = 2$ as initial data for the classical homoclinic solution with respect to decrease of time. Hence it follows that

$$\lim_{t\to-\infty} x(t, x_0, y_0) = (2j-1)\pi, \quad \lim_{t\to-\infty} y(t, x_0, y_0) = 0.$$

See figure 4.

In the region

$$\{x \in (2(j+1)\pi, 2(j+k-1)\pi),\ y \in R^1\}$$

we can continue the solution under consideration as a sliding solution: $y(t, x_0, y_0) = 2$. Then we can consider the point $x = 2(j+k-1)\pi$, $y = 2$ as initial data for the classical homoclinic solution with respect to increase of time. Hence it follows that

$$\lim_{t\to+\infty} x(t, x_0, y_0) = (2j+2k-1)\pi, \quad \lim_{t\to+\infty} y(t, x_0, y_0) = 0.$$

See figure 5.

The proposition is proven.
Let us suppose that $\gamma_1 = \gamma_2 = \beta > 1$ and denote by $H(x)$ the function

$$H(x) = \sqrt{2(1 + \cos x + \beta|x|)}, \quad x \in [-\pi, \pi], \tag{3}$$

$H(x + 2\pi) \equiv H(x)$.

Let us denote by Φ the following region in R^2:

$$\Phi = \{x \in R^1, G(x) < y \leq H(x)\}.$$

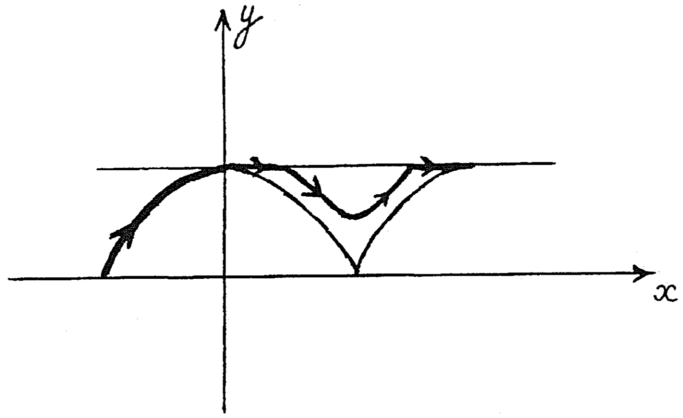

Figure 4: Homoclinic solution of degree k

Proposition 2. *For every point $(x_0, y_0) \in \Phi$ and for every integer number $k \geq 2$ there exists a homoclinic orbit γ of degree k such that $(x_0, y_0) \in \gamma$.*

Proof. Let us consider the function

$$V(x,y) = y^2 + H^2(x).$$

It is easy to see that for a solution $x(t), y(t)$ of system (2) such that $x(t) \neq j\pi$ the following equality is true:

$$\dot{V}(x(t), y(t)) = 0.$$

From this equality and from the form (3) of the function $H(x)$ we get that for every point $(x_0, y_0) \in \Phi$ there exist numbers $t_1 < t_2$ such that

$$y(t_1, x_0, y_0) = y(t_2, x_0, y_0) = 2,$$

$$x(t_1, x_0, y_0) = 2j\pi, \quad x(t_2, x_0, y_0) = 2(j+1)\pi.$$

for some integer j. See figure 6.

Now it remains to repeat the argumentation in the proof of proposition 1.
There exist various generalizations of propositions 1 and 2. Let us consider for example the following system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -Q(x,y) - f(x). \end{aligned} \tag{4}$$

Here $f(x)$ is continuously differentiable and 2π-periodic.We suppose also that $f(x)$ has exactly two zeros x_1 and x_2 on the interval $[0, 2\pi)$ such that $x_1 < x_2$,

$$f'(x_1) > 0, \; f'(x_2) < 0.$$

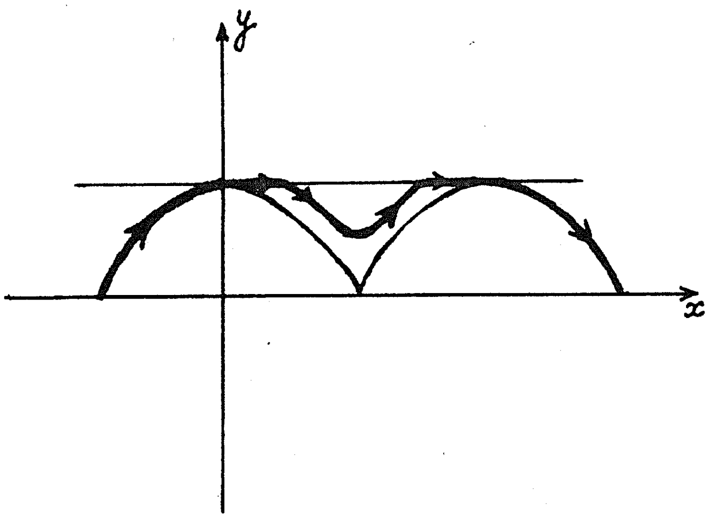

Figure 5: Homoclinic solution of degree 2

Here $Q(x+2\pi, y) = Q(x, y)$ and

$$Q(x,y) = \begin{cases} 0, & \text{for } y < \nu, \quad x \in (x_1, x_1 + 2\pi), \\ \gamma_1, & \text{for } y > \nu, \quad x \in (x_2, x_1 + 2\pi), \\ -\gamma_2, & \text{for } y > \nu, \quad x \in (x_1, x_2), \end{cases}$$

where ν, γ_1 and γ_2 are positive numbers such that

$$\gamma_1 > \max |f(x)|, \ \gamma_2 > \max |f(x)|,$$

$$\nu \leq \left(2 \int_{x_1}^{x_2} f(x) dx \right)^{1/2}.$$

Let us denote by $R(x)$ the 2π-periodic function

$$R(x) = \left(2 \int_{x}^{x_2} f(x) dx \right)^{1/2}$$

on the interval (μ, x_2) and $R(x) = 0$ on the interval $(x_2 - 2\pi, \mu)$. Here μ is a number such that

$$\int_{\mu}^{x_2} f(x) dx = 0.$$

Let us denote by Ψ the following region in R^2

$$\Psi = \{x \in R^1, \ R(x) < y \leq \nu\}.$$

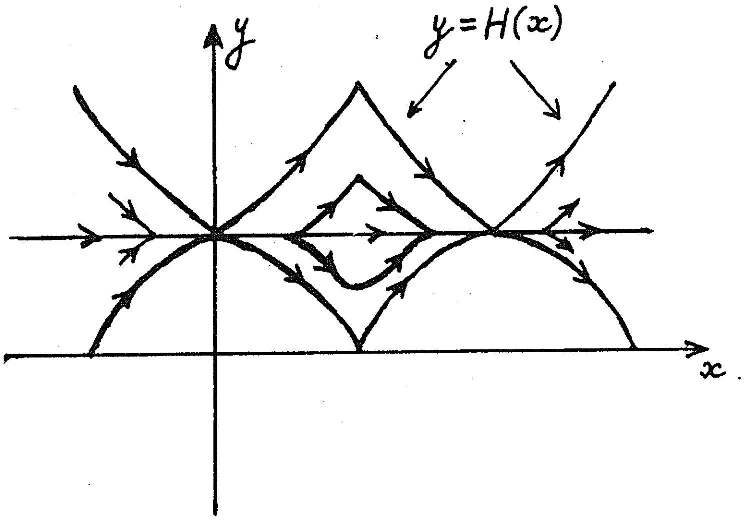

Figure 6: Region Φ

Proposition 3. *For every point $(x_0, y_0) \in \Psi$ and for every integer number $k \geq 2$ there exists a homoclinic orbit γ of degree k such that $(x_0, y_0) \in \gamma$.*

The proof of this proposition repeats in essence the argumentation in the proof of proposition 1.
Let us consider the following system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\alpha y - Q(x,y) - f(x). \end{aligned} \tag{5}$$

Here α is a positive number corresponding to viscous resistance. This system with $Q(x,y) = 0$ has been considered in the books [Andronov et al., 1965], [Barbashin and Tabueva, 1969], [Gelig et al., 1978], [Leonov et al., 1992], [Lindsey, 1972].

Conjecture. *For every $\alpha > 0$ and $f(x)$ there exists $Q(x,y)$ such that system (5) has a continuum of homoclinic orbits.*

This conjecture is true if we slightly change the definition of the function $Q(x,y)$:

$$Q(x,y) = \begin{cases} 0, & \text{for } y < \nu, \quad x \in (x_2 - 2\pi, x_2), \\ \gamma_1, & \text{for } y > \nu, \quad x \in (x_2 - 2\pi, x_3), \\ -\gamma_2, & \text{for } y > \nu, \quad x \in (x_3, x_2). \end{cases}$$

Here x_3 is a number on $(x_2 - 2\pi, x_2)$ such that

$$f(x_3) = \alpha\nu, \quad f(x) \neq \alpha\nu \ \forall x \in (x_3, x_1).$$

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