Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

On a micro-macro transition for poroelastic Biot's model and corresponding Gassmann-type relations

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submitted: 2nd September 2003

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No. 868
Berlin 2003

2000 *Mathematics Subject Classification.* 74F10, 74L10, 74Q15, 74E30.

Key words and phrases. Micro-macro transitions, mechanics of poroelastic materials, Biot's model.

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Abstract

In the paper we consider a micro-macro transition for a linear thermodynamical model of poroelastic media which yields the Biot's model. We investigate a two-component poroelastic linear model in which a constitutive dependence on the porosity gradient is incorporated and this is compared with the classical Biot's model without added mass effects. We analyze three Gedankenexperiments: jacketed undrained, jacketed drained and unjacketed and derive a generalization of classical Gassmann relations between macroscopic material parameters and microscopic compressibility moduli of the solid, and of the fluid. Dependence on the porosity is particularly exposed due to its importance in acoustic applications of the model.

In particular we show that Gassmann relations follow as one of two physically justified solutions of the full set of micro-macro compatibility relations. In this solution the coupling to the porosity gradient is absent. Simultaneously, we demonstrate the second solution which lies near the Gassmann results but admits the coupling. In both models couplings are weak enough to admit, within the class of problems of acoustic wave analysis, an approximation by a "simple mixture" model in which coupling of stresses is fully neglected .

1 Introduction

Macroscopic modeling of saturated porous materials leads inevitably to a theory of immiscible mixtures considered as an extension of the classical Truesdell's theory of fluid mixtures (e.g. [1]). Such an extension was proposed under weak thermodynamical and mechanical restrictions by R. M. Bowen. In his contribution to the above quoted book one can find detailed references.

In recent years I was trying to incorporate into a similar frame the classical model of porous materials proposed by M. A. Biot (see the collection of Biot's papers [2]). As Biot's model is the most commonly used way to describe linear processes in poroelastic materials and, simultaneously, serves the purpose of a guideline in experiments on soils and many other porous materials it is important to know its limitations.

In particular, it is a rather straightforward matter to prove that the Biot's extension of the classical two-component model by adding relative accelerations violates the principle of material objectivity (see: [3]). Consequently, this way of describing added mass and tortuosity effects seems to be erroneous. The issue has an important practical bearing but it is not approached in this work.

The second important property of Biot's model is the interaction between components by partial volume changes. This interaction yields an additional contribution to partial stresses which in the linear model is characterized by a material parameter Q.

It has been shown within the classical thermodynamical theory of mixtures of fluids (e.g. see: [4]) that such couplings are eliminated in macroscopical models by the second law of thermodynamics if one does not account for gradients of partial mass densities. Such models without couplings are called *simple mixtures.*

The Biot's model should possess the same property. One can show by the direct exploitation of the second law of thermodynamics [5] that the Biot's coupling follows by linearization of a thermodynamical model in which a dependence on the gradient of porosity is included. Otherwise the model reduces to a model of simple mixtures with one of the components being solid. Simultaneously we demonstrate a peculiar property of the thermodynamical model that after such a linearization Biot's model without an influence of porosity gradient is admissible due to a vanishing material parameter. However one has to make one further extension related to changes of porosity. These are described by a porosity balance equation.

In this work we rely on such an extended linear model and compare it with the Biot's model. In particular, we show that the balance equation of porosity can be solved approximately and the solution agrees with rather intuitive results of soil mechanics. Bearing this in mind, one can exploit certain Gedankenexperiments and the following generalization of Gassmann relations shows that Biot's model is one of the admissible solutions of the micro-macro transition problem for the thermodynamical model with the gradient of porosity. In addition, it follows as well that a "simple mixture" model of porous materials, developed earlier (e.g. [6, 7, 8]) is a reasonable approximation of both Biot's model and the thermodynamical model with porosity gradient in application to analysis of acoustic waves.

We proceed in the following way.

In Sect. 2 we present a macroscopic linear poroelastic model of fully saturated, two-component media. All quantities appearing in this model have purely macroscopic, smeared-out interpretation. They are fields defined on a certain chosen constant domain of the three-dimensional configuration space. This means that they are functions of two variables: spatial position $\mathbf{x}$ and time t. As in the classical theory of mixtures, in each point $\mathbf{x}$ there appear both components of the system. This issue is important particularly in connection with frequent misunderstandings between those, who work in the frame of continuum approach and those, who work within classical soil mechanics with real grains and real fluids filling the channels. We return to this point in Sect. 3.

Sect. 3 is devoted to a particular class of static deformations described by the general two-component model – spherical deformations. In such cases components change solely their partial volumes but there is no shearing. This class is important for the exploitation of the so-called Gedankenexperiments. In the same section we introduce the Biot's notation for material parameters. The most important issue discussed in this section is the micro-macro transition. Under the assumption of homogeneity of the microstructure we demonstrate two classes of compatibility relations of a continuum macroscopic two-component model and a microscopic model of two true components. We show that a condition for partial stress tractions yields two dynamical compatibility relations and the homogeneity of microstructure yields two geometrical compatibility relations.

The set of relations consisting of a single equilibrium condition (static equilibrium in spherical deformation processes), four compatibility relations, two constitutive relations on micro- and two on macrolevel yields the solution of any static spherical homogeneous problem.

In Sect. 4 we present the so-called Gedankenexperiments. These are jacketed undrained, jacketed drained and unjacketed simple tests. They give rise to three additional relations. Together with the definition of the drained compressibility modulus K_d which is assumed to be given either experimentally or heuristically, we obtain four relations which specify a dependence of four macroscopic material parameters $\{K, M, C, N\}$ on two microscopic compressibility moduli $\{K_s, K_f\}$ as well as on the initial porosity n_0 and the drained compressibility modulus K_d. Here we use the Biot's notation and denote by K the bulk compressibility modulus of the skeleton, M is the compressibility modulus of the fluid, C is the coupling parameter which characterizes the influence of partial volume changes of one component on the pressure of the other component, and, finally, N denotes the parameter responsible for the coupling through the porosity gradient.

By means of the micro-macro transition procedure we obtain four algebraic relations for four material parameters which we solve numerically. One of those solutions contains the value $N \equiv 0$ which corresponds to classical Gassmann relations derived for the Biot's model.

The most important conclusion of the work is that, in spite of flaws of both Biot's model and classical Gassmann relations, they follow from the full thermodynamical model in the linear approximation as an admissible possibility. This statement should be qualified by the fact that Gedankenexperiments may be the right way to describe the microstructure of granular materials but it does not seem to be appropriate for rocks. In such materials the coupling through the porosity gradient may be stronger and this would be a reason to ignore models in which the coefficient N vanishes. This is, certainly, also the case in nonlinear materials such as clays or biological tissues where one should rather apply the full thermodynamical model.

The procedure of micro-macro transition used in the present work as a method of derivation of Gassmann-type relations has an advantage in comparison with classical *ad hoc* methods that it can be easily extended to more complicated problems. In particular, we investigate in a forthcoming paper similar relations for unsaturated poroelastic materials.

2 Linear model

The linear poroelastic two-component model of isothermal processes in porous materials is based on the following set of unknown fields

$$\left\{\rho^S, \rho^F, \mathbf{v}^S, \mathbf{v}^F, \mathbf{e}^S, n\right\}, \tag{1}$$

where ρ^S, ρ^F are macroscopic current partial mass densities of the skeleton and of the fluid, respectively, $\mathbf{v}^S, \mathbf{v}^F$ are macroscopic velocity fields of these components, $\mathbf{e}^S$ is the macroscopic deformation tensor and n denotes the current porosity, i.e. the fraction of the volume of voids in a chosen representative elementary volume (REV) to the corresponding total volume of REV. These fields must satisfy the following conditions which are the basis for the linearization of a nonlinear thermodynamical model (e.g. [6])

$$\max\left\{\left\|\mathbf{e}^S\right\|, |\varepsilon|, \left|\frac{n-n_0}{n_0}\right|\right\} \ll 1, \tag{2}$$

$$\left\|\mathbf{e}^S\right\| := \max\left\{\left|\lambda_e^{(1)}\right|, \left|\lambda_e^{(2)}\right|, \left|\lambda_e^{(3)}\right|\right\}, \quad \varepsilon := \frac{\rho_0^F - \rho^F}{\rho_0^F}, \tag{3}$$

where $\lambda_e^{(1)}, \lambda_e^{(2)}, \lambda_e^{(3)}$ are eigenvalues of the deformation tensor $\mathbf{e}^S$.

Fields (1) are functions of the spatial variable $\mathbf{x} \in \mathcal{B}$, and time $t \in \mathcal{T}$. They must satisfy field equations which follow from partial balance equations by a linear closure.

The partial balance equations appropriate for the chosen model are as follows

- mass conservation laws

$$\frac{\partial \rho^S}{\partial t} + \rho_0^S \operatorname{div} \mathbf{v}^S = 0, \quad \frac{\partial \rho^F}{\partial t} + \rho_0^F \operatorname{div} \mathbf{v}^F = 0, \tag{4}$$

- momentum balance equations

$$\rho_0^S \frac{\partial \mathbf{v}^S}{\partial t} = \operatorname{div} \mathbf{T}^S + \hat{\mathbf{p}}, \quad \rho_0^F \frac{\partial \mathbf{v}^F}{\partial t} = \operatorname{div} \mathbf{T}^F - \hat{\mathbf{p}}, \tag{5}$$

- balance equation of porosity

$$\frac{\partial (n - n_E)}{\partial t} + \operatorname{div} \mathbf{J} = \hat{n}, \tag{6}$$

- integrability condition for the deformation tensor

$$\frac{\partial \mathbf{e}^S}{\partial t} = \operatorname{sym} \operatorname{grad} \mathbf{v}^S. \tag{7}$$

This condition is related to the existence of a displacement vector $\mathbf{u}^S$. In the linear model such a vector leads to relations

$$\mathbf{v}^S = \frac{\partial \mathbf{u}^S}{\partial t}, \quad \mathbf{e}^S = \text{sym grad}\, \mathbf{u}^S.$$

Then the relation (7) becomes the identity. If we do not introduce the displacement vector, which is convenient in the wave analysis (e.g. [7, 8]), then the relation (7) becomes the part of the set of field equations.

The partial stress tensors satisfy the following constitutive relations

$$\begin{aligned} \mathbf{T}'^S &= \mathbf{T}_0^S + \lambda^S e\mathbf{1} + 2\mu^S \mathbf{e}^S + Q\varepsilon\mathbf{1} + \beta(n - n_E)\mathbf{1}, \quad e := \operatorname{tr} \mathbf{e}^S, \\ \mathbf{T}'^F &= -p'^F\mathbf{1}, \quad p'^F = p_0^F - \left(\rho_0^F \kappa\varepsilon + Qe\right) + \beta(n - n_E), \\ n_E &= n_0\left(1 + \delta e\right), \end{aligned} \tag{8}$$

where n_E denotes values of the porosity in the thermodynamical equilibrium which corresponds to vanishing sources $\hat{\mathbf{p}}$,$\hat{n}$. The constant tensor $\mathbf{T}_0^S$ is the initial partial stress in the skeleton, p_0^F – the initial partial pressure in the fluid, and these quantities as well as the material parameters $\lambda^S, \mu^S, \kappa, Q, \delta, \beta$ are functions of an initial porosity n_0. Certainly the parameters λ^S, μ^S correspond to classical Lamé constants while κ corresponds to the classical compressibility coefficient of an ideal fluid. The contribution with the parameter β is related to nonequilibrium changes of porosity and it may have an important bearing in the theory of nonlinear waves. It can be shown that it yields small contributions to volume changes of both components. We account for its presence only in the first part of our considerations.

The prime is used in the above relations to indicate a form of constitutive relations which follows directly by the linearization of the full nonlinear thermodynamical model. These relations shall be modified in the sequel.

The linear constitutive relations for the flux of porosity and for the sources have the following form

$$\mathbf{J} = \Phi\left(\mathbf{v}^F - \mathbf{v}^S\right), \quad \hat{\mathbf{p}} = \pi\left(\mathbf{v}^F - \mathbf{v}^S\right) - N \operatorname{grad} n, \quad \hat{n} = -\frac{n - n_E}{\tau}, \tag{9}$$

In the linear model the material parameters Φ, π, N, τ are constants depending solely on the initial porosity n_0. Due to this property the contribution of the gradient of porosity can be incorporated into the partial stresses. Namely if we define modified constitutive relations by

$$\mathbf{T}^S := \mathbf{T}'^S - N\left(n - n_0\right)\mathbf{1}, \quad \mathbf{T}^F := \mathbf{T}'^F + N\left(n - n_0\right)\mathbf{1}, \tag{10}$$

then the source in the equations (5) contains solely the contribution of the diffusive force $\pi\left(\mathbf{v}^F - \mathbf{v}^S\right)$.

It is easy to see that the mass density ρ^S can be eliminated from the set of fields by integration of the mass balance equation $(4)_1$. According to (7) we have

$$\frac{\partial e}{\partial t} = \operatorname{div} \mathbf{v}^S \quad \Rightarrow \quad \frac{\partial \rho^S}{\partial t} = -\rho_0^S \frac{\partial e}{\partial t} \quad \Rightarrow \quad e = \frac{\rho_0^S - \rho^S}{\rho_0^S}. \tag{11}$$

Simultaneously we can solve the porosity balance equation. We have

$$\frac{\partial(n-n_E)}{\partial t}+\frac{n-n_E}{\tau}=\Phi\frac{\partial(e-\varepsilon)}{\partial t}. \tag{12}$$

Hence

$$n=n_0[1+\delta e+\frac{\Phi}{n_0}(e-\varepsilon)-\frac{\Phi}{n_0\tau}\int_0^t\left.(e-\varepsilon)\right|_{t-s}e^{-s/\tau}ds]. \tag{13}$$

The last contribution describes *memory effects* which are similar to these caused by the viscosity. In the first approximation they can be neglected in models describing acoustic waves in soils which corresponds to the assumption $\tau\to\infty$. We shall rely on this assumption in this work.

Summing up the above considerations we see that the full linear thermodynamical model without memory effects contains the following essential fields

$$\left\{\mathbf{v}^S,\mathbf{v}^F,\mathbf{e}^S,\varepsilon\right\}, \tag{14}$$

which have to satisfy the field equations

$$\begin{aligned}\rho_0^S\frac{\partial\mathbf{v}^S}{\partial t} &= \operatorname{div}\left\{\lambda^S e\mathbf{1}+2\mu^S\mathbf{e}^S+Q\varepsilon\mathbf{1}+\beta(n-n_E)\mathbf{1}-N\left(n-n_0\right)\mathbf{1}\right\}+\\ &\quad+\pi\left(\mathbf{v}^F-\mathbf{v}^S\right),\\ \rho_0^F\frac{\partial\mathbf{v}^F}{\partial t} &= -\operatorname{grad}\left\{-\left(\rho_0^F\kappa\varepsilon+Qe\right)+\beta(n-n_E)-N\left(n-n_0\right)\right\}-\\ &\quad-\pi\left(\mathbf{v}^F-\mathbf{v}^S\right),\end{aligned} \tag{15}$$

and

$$\begin{aligned}\frac{\partial\mathbf{e}^S}{\partial t} &= \operatorname{sym}\operatorname{grad}\mathbf{v}^S,\quad \frac{\partial\varepsilon}{\partial t}=\operatorname{div}\mathbf{v}^F,\quad e\equiv\operatorname{tr}\mathbf{e}^S,\\ n &= n_0\left(1+\delta e+\frac{\Phi}{n_0}\left(e-\varepsilon\right)\right).\end{aligned} \tag{16}$$

If we assume $\beta\equiv 0, N\equiv 0$, then the set of equations (15) coincides with the set of Biot's equations in which the coupling through the added mass was neglected. The classical Biot's model does not contain any counterpart of the relation (16) for porosity. In soil mechanics one uses a kind of hybrid extension of Biot's model in order to specify n[1]. Results are identical with those following from (16) provided the parameters δ,Φ are given by the relations (30) which we derive by means of the micro-macro transition.

We proceed to investigate relations between the above introduced macroscopic material parameters and real properties of materials constituting the porous medium.

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[1] assuming that the partial fluid mass density ρ^F, and the true fluid mass density ρ^{FR} are given one can calculate the porosity from the relation $n=\rho^F/\rho^{FR}$. However such a statement is useful under the assumption of incompressibility of the real fluid: $\rho^{FR}=const.$ A similar statement can be formulated under the incompressibility assumption of skeleton. Both of them are useless in acoustics.

3 Spherical static deformations

Description of granular materials by means of a macroscopic model is particularly easy when a deformation is homogeneous, spherically symmetric and the mechanical reactions of the system reduce to pressures. We consider such a system in this section in order to construct a simple micro-macro transition procedure for identification of some macroscopic material parameters.

In our notation we have for such a deformation

$$\mathbf{e}^S = \frac{1}{3}e\mathbf{1}, \quad p^S = -\frac{1}{3}\operatorname{tr}\mathbf{T}^S, \quad p^F = -\frac{1}{3}\operatorname{tr}\mathbf{T}^F, \tag{17}$$

and the *macroscopic constitutive relations* for partial pressures are as follows

$$p^S - p_0^S = -\left(\lambda^S + \frac{2}{3}\mu^S - n_0\beta\delta\right)e - Q\varepsilon + (N-\beta)(n-n_0), \tag{18}$$

$$p^F - p_0^F = -\rho_0^F\kappa\varepsilon - (Q + n_0\beta\delta)e - (N-\beta)(n-n_0).$$

Here we have already eliminated the equilibrium porosity $n_E = n_0(1+\delta e)$.

In the static case the full pressure change must be in *equilibrium* with a given excess pressure Δp, i.e.

$$\Delta p = \left(p^S - p_0^S\right) + \left(p^F - p_0^F\right). \tag{19}$$

Clearly the problem to find volume changes e, ε, partial pressures p^S, p^F, and porosity n cannot be solved yet even if we account for the relation for porosity $(16)_4$. We have only 4 equations at the disposal and we need an additional equation specifying, for instance, a flow of the fluid through the boundary of the medium provided the problem remains homogeneous. This is natural for a two-component system and we present such equations in the sequel.

However we proceed with the closure of the above problem in a different way. Namely we specify properties of the microstructure and relations between microscopic and macroscopic models. Then the problem can be solved and additional equations mentioned above serve solely the purpose of relating microscopic and macroscopic properties (compatibility conditions!). Such an approach is possible for static problems due to the homogeneity. Then solutions of partial momentum equations are trivial and the problem becomes algebraic.

The microscopic model for spherical deformations is specified as follows. Volume changes on this microscopic level are denoted by e^R, ε^R. Then e^R describes relative true volume changes of grains, and ε^R describes changes of the true mass density of the fluid in channels of the granular material. The corresponding pressures are denoted by p^{SR}, p^{FR} and, of course, the latter is identical with the pore pressure p_f. For these quantities the following *microscopic constitutive relations* are assumed

$$p^{SR} - p_0^{SR} = -K_s e^R, \quad p^{FR} - p_0^{FR} = -K_f \varepsilon^R. \tag{20}$$

In these relations K_s, K_f denote real (true) compressibility moduli of the solid component (granulae) and of the fluid, respectively. In contrast to material parameters of the macroscopic model, these can be measured independently of a current morphology of the granular material. They are not influenced by a porosity, cohesivity, or any other property characteristic for the texture of the medium.

The above presented macroscopic and microscopic properties are related through two sets of compatibility conditions.

On the one hand we have *dynamic compatibility relations* which for pressures have the form

$$p^S = (1-n)\, p^{SR}, \quad p^F = np^{FR}, \tag{21}$$

where n denotes the current porosity of the medium. On a boundary of the porous material we require the following equivalence of local partial stress tractions

$$\mathbf{T}^S \mathbf{n} dA = \mathbf{T}^{SR} \mathbf{n} dA^S, \quad \mathbf{T}^F \mathbf{n} dA = \mathbf{T}^{FR} \mathbf{n} dA^F, \tag{22}$$

where $\mathbf{T}^{SR}, \mathbf{T}^{FR}$ denote true partial stresses in the skeleton and in the fluid, respectively, $\mathbf{n}$ is the unit outward normal to the surface, dA is an infinitesimal area of the surface, and dA^S, dA^F are contributions of the skeleton and of the fluid to this area: $dA = dA^S + dA^F$. If the surface fractions and volume fractions are the same, which follows under the assumption of randomness of the microstructure, we have $dA^S/dA = 1-n, dA^F/dA = n$. Then (21) follows from (22) if the partial stresses reduce to pressures.

On the other hand we have the following relations between partial mass densities

$$\begin{aligned} \rho^S(\mathbf{x},t) &= \frac{1}{V}\int_{REV(\mathbf{x},t)} \rho^{SR}(\mathbf{z},t)\, H^S(\mathbf{z},t)\, dV_{\mathbf{z}}, \\ \rho^F(\mathbf{x},t) &= \frac{1}{V}\int_{REV(\mathbf{x},t)} \rho^{FR}(\mathbf{z},t)\, H^F(\mathbf{z},t)\, dV_{\mathbf{z}}, \end{aligned} \tag{23}$$

where $REV(\mathbf{x},t)$ is the representative elementary volume located in the point $\mathbf{x}$ of the macroscopic continuum at the time t, V denotes its volume, H^S is the characteristic function of the microstructural skeleton, i.e. it is equal to 1 when the point is occupied by the solid and zero otherwise, and similarly for the characteristic function of the fluid H^F. If the microstructure is **homogeneous** we can write these relations in the form

$$\begin{aligned} \rho^S &= (1-n)\, \rho^{SR}, \quad \rho^F = n\rho^{FR}, \\ n &= \frac{1}{V}\int_{REV(\mathbf{x},t)} H^F(\mathbf{z},t)\, dV_z \equiv 1 - \frac{1}{V}\int_{REV(\mathbf{x},t)} H^S(\mathbf{z},t)\, dV_z. \end{aligned} \tag{24}$$

Relating changes of mass densities to volume changes introduced earlier we obtain

$$\begin{aligned} \rho^S &= \rho_0^S (1+e)^{-1}, \quad \rho^F = \rho_0^F (1+\varepsilon)^{-1}, \\ \rho^{SR} &= \rho_0^{SR}\left(1+e^R\right)^{-1}, \quad \rho^{FR} = \rho_0^{FR}\left(1+\varepsilon^R\right)^{-1}. \end{aligned} \tag{25}$$

Through the combination of these relations and linearization we arrive immediately at the following *geometrical compatibility relations*

$$e = e^R + \frac{n - n_0}{1 - n_0}, \quad \varepsilon = \varepsilon^R - \frac{n - n_0}{n_0}. \tag{26}$$

Consequently for the 9 unknown quantities of spherical homogeneous deformations

$$\left\{ e, \varepsilon, p^S, p^F, e^R, \varepsilon^R, p^{SR}, p^{FR}, n \right\}, \tag{27}$$

we have 9 equations at the disposal: 1 equilibrium condition (19), 2 macroscopic constitutive relations (18), 2 microscopic constitutive relations (20), 2 dynamical compatibility relations (21), and 2 geometrical compatibility conditions (26). This simple algebraic problem can be solved.

As an intermediate result we obtain the following relations between geometrical microscopic quantities n, e^R and ε^R and macroscopic geometrical quantities e, ε

$$\begin{aligned} n - n_0 &= -\frac{1}{K_s - K_f} \left\{ \left[\left(\lambda^S + \frac{2}{3}\mu^S + Q \right) - (1 - n_0) K_s \right] e + \right. \\ &\quad \left. + \left[\left(Q + \rho_0^F \kappa \right) - n_0 K_f \right] \varepsilon \right\}, \end{aligned} \tag{28}$$

$$\begin{aligned} e^R &= \frac{1}{(K_s - K_f)(1 - n_0)} \left\{ \left[\left(\lambda^S + \frac{2}{3}\mu^S + Q \right) - (1 - n_0) K_f \right] e + \right. \\ &\quad \left. + \left[\left(Q + \rho_0^F \kappa \right) - n_0 K_f \right] \varepsilon \right\}, \end{aligned}$$

$$\begin{aligned} \varepsilon^R &= -\frac{1}{(K_s - K_f) n_0} \left\{ \left[\left(\lambda^S + \frac{2}{3}\mu^S + Q \right) - (1 - n_0) K_s \right] e + \right. \\ &\quad \left. + \left[\left(Q + \rho_0^F \kappa \right) - n_0 K_s \right] \varepsilon \right\}. \end{aligned} \tag{29}$$

Consequently, comparing (28) and (16) we see that the material parameters δ, Φ, appearing in the relation for porosity can be expressed in terms of the other material parameters. We obtain

$$\delta = \frac{K_V - K}{n_0 (K_s - K_f)}, \quad \Phi = \frac{(Q + \rho_0^F \kappa) - n_0 K_f}{K_s - K_f}, \tag{30}$$

$$K := \lambda^S + \frac{2}{3}\mu^S + \rho_0^F \kappa + 2Q, \quad K_V := (1 - n_0) K_s + n_0 K_f. \tag{31}$$

The material parameter K denotes the bulk compressibility modulus and K_V is the Voigt compressibility modulus.

Now, for the determination of e and ε we have the following two equations. The dynamical compatibility conditions (21) yield the following relation between the macroscopic volume changes

$$\left\{ Q + \frac{K_f - N}{K_s - K_f} \left[\left(\lambda^S + \frac{2}{3}\mu^S + Q \right) - (1 - n_0) K_s \right] - \beta \frac{\left(Q + \rho_0^F \kappa \right) - n_0 K_f}{K_s - K_f} \right\} e =$$

$$= -\left\{ \left[\rho_0^F \kappa - n_0 K_f \right] + \frac{K_f - N}{K_s - K_f} \left[\left(Q + \rho_0^F \kappa \right) - n_0 K_s \right] + \beta \frac{\left(Q + \rho_0^F \kappa \right) - n_0 K_f}{K_s - K_f} \right\} \varepsilon. \tag{32}$$

Simultaneously the equilibrium condition (19) yields

$$\Delta p = -\left(\lambda^S + \frac{2}{3} \mu^S + Q \right) e - \left(\rho_0^F \kappa + Q \right) \varepsilon. \tag{33}$$

Hence substitution of ε calculated from (32) in (33) leads to e given in terms of the excess pressure Δp and, by means of (32), (29), (28), this determines in turn $\varepsilon, e^R, \varepsilon^R, n$ in terms of the excess pressure. Consequently the constitutive relations give rise to partial pressures in terms of the excess pressure. This completes the solution of the problem. We shall not quote these simple results in this work due to their lengthy form.

In the next section we present various forms of an additional equation which we have mentioned at the beginning of this section.

In the Biot's model it is customary to use a different field replacing the macroscopic volume change of the fluid ε. This variable, ζ, is called the increment of the fluid content. Its presence is related to the fact that, in contrast to the solid component, the representative elementary volume is not material for the fluid component. It means that REV consists always of the same particles of the real skeleton but the fluid may open and leave this domain due to the diffusion. The instantaneous mass of the fluid per macroscopic volume dV_0 is given by $\rho^F (1 + e) dV_0$ because e is the macroscopic measure of volume changes. Simultaneously in the same volume material with respect to the skeleton the initial amount of fluid mass was $\rho_0^F dV_0$. Consequently the change of the fluid mass contained in this volume is given by the difference of these two quantities. We introduce the dimensionless measure of this quantity

$$\zeta := \frac{1}{\rho_0^{FR}} \left[(1 + e) \rho^F - \rho_0^F \right] \approx n_0 (e - \varepsilon). \tag{34}$$

This is the *increment of the fluid content.* We refer to this variable frequently in the further considerations. However it is clear that the transformation $(e, \varepsilon) \rightarrow (e, \zeta)$ is smooth and one-to-one and it can be done whenever convenient.

As mentioned before we neglect further the influence of nonequilibrium changes of porosity described in constitutive relations by the material parameter β. A justification of this simplification is based on the analysis of linear acoustic waves and it seems to be indeed well found for soils. This paramter cannot be found by means of simple static experiments anyway.

Let us rewrite basic constitutive relations for spherical deformations (18) and the intermediate results (28), (29) using the material parameters introduced by Biot (e.g. see [9]). We have

$$\beta \equiv 0 \quad \Rightarrow$$

$$p - p_0 := (p^S - p_0^S) + (p^F - p_0^F) = -(K - n_0 C) e - n_0 C \varepsilon \equiv -K e + C \zeta,$$

$$
\begin{aligned}
p^{FR} - p_0^{FR} &\equiv p_f - p_f^0 := \frac{1}{n_0}\left(p^F - p_0^F\right) = -n_0 M \varepsilon - \left(C - n_0 M\right) e - N \frac{n - n_0}{n_0} \equiv \\
&\equiv M\zeta - Ce - N\frac{n - n_0}{n_0},
\end{aligned}
$$

$$
\begin{aligned}
n - n_0 &= -\frac{1}{K_s - K_f}\left\{\left[\left(K - n_0 C\right) - \left(1 - n_0\right) K_s\right] e + n_0 \left[C - K_f\right] \varepsilon\right\} \equiv \\
&\equiv -\frac{1}{K_s - K_f}\left\{\left[K - K_V\right] e - \left[C - K_f\right] \zeta\right\},
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
e^R &= \frac{1}{\left(K_s - K_f\right)\left(1 - n_0\right)}\left\{\left[\left(K - n_0 C\right) - \left(1 - n_0\right) K_f\right] e + n_0 \left[C - K_f\right] \varepsilon\right\} \equiv \\
&\equiv \frac{1}{(K_s - K_f)(1 - n_0)}\left\{\left[K - K_f\right] e - \left[C - K_f\right] \zeta\right\},
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon^R &= -\frac{1}{\left(K_s - K_f\right) n_0}\left\{\left[\left(K - n_0 C\right) - \left(1 - n_0\right) K_s\right] e + n_0 \left[C - K_s\right] \varepsilon\right\} \equiv \\
&\equiv -\frac{1}{(K_s - K_f) n_0}\left\{\left[K - K_s\right] e - \left[C - K_s\right] \zeta\right\},
\end{aligned}
$$

where K, K_V are defined by (31) and in addition

$$
C := \frac{1}{n_0}\left(Q + \rho_0^F \kappa\right), \quad M := \frac{\rho_0^F \kappa}{n_0^2}. \tag{36}
$$

This notation simplifies the comparison of the present results with those appearing in the literature on soil mechanics.

Making use of the relations (35) and $(20)_2$ for the pore pressure $p_f - p_f^0 \equiv p^{FR} - p_0^{FR}$ we obtain immediately the relation between volume changes e and increments of fluid content ζ

$$
\begin{aligned}
&\left\{C + \frac{K_f(K - K_s)}{n_0(K_s - K_f)} - \frac{N(K - K_V)}{n_0(K_s - K_f)}\right\} - \\
&- \left\{M + \frac{K_f(C - K_s)}{n_0(K_s - K_f)} - \frac{N(C - K_f)}{n_0(K_s - K_f)}\right\} \frac{\zeta}{e} = 0,
\end{aligned} \tag{37}
$$

which must hold for all spherical static deformations.

4 Gedankenexperiments for compressibilities

4.1 Jacketed and unjacketed tests

As we have indicated earlier the macroscopic deformation problem of a two-component medium requires an additional boundary condition. In the case of static, homogeneous, spherically symmetric problems such a condition can be easily formulated.

In Figure 1 we show schematically three simple tests considered commonly in soil mechanics which shall be considered in details.

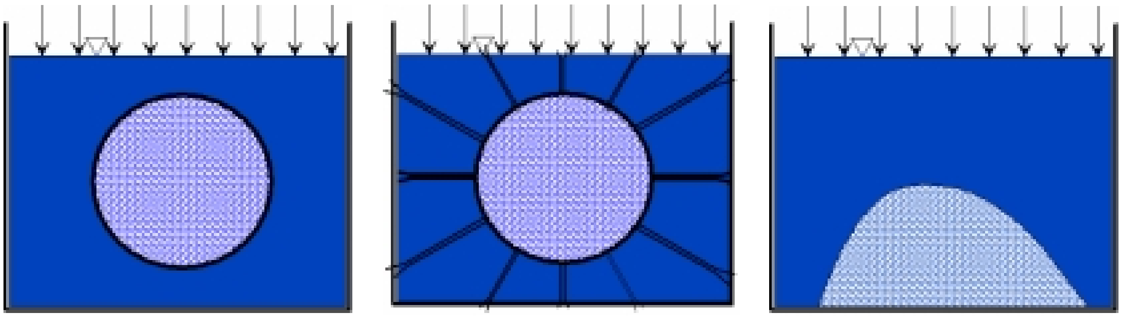

Figure 1: *Schemes of Gedankenexperiments: 1) jacketed undrained, 2) jacketed drained, 3) unjacketed.*

The additional conditions for those tests are as follows

1) jacketed undrained test in which the boundary of the sample is impermeable

$$\zeta = 0 \quad \Rightarrow \quad e = \varepsilon; \tag{38}$$

it means that there is no flow through the boundary and, consequently, macroscopic volume changes of both components must be equal,

2) jacketed drained test in which there is a drainage of the sample connecting the fluid component directly with the external world; then

$$p^{FR} = p_0^{FR} \quad \Rightarrow \quad \varepsilon^R = 0; \tag{39}$$

3) unjacketed test in which the pore pressure and the external fluid pressure must be the same, i.e.

$$p^{FR} - p_0^{FR} = \Delta p, \tag{40}$$

where Δp is a given excess pressure.

These tests are called Gedankenexperiments because they can be principally performed under ideal conditions (e.g. when capillary effects, viscosity, temperature changes etc. can be neglected), and, simultaneously, they lead to relations between physical properties – in our case, between macroscopic and microscopic material parameters.

Such procedures were proposed often for materials with microstructure. For instance, macroscopic properties of polycrystals were calculated by means of properties of grains by W. Voigt [11] (consistent stresses), A. Reuss [12] (consistent strains) or by means of some fully consistent micro-macro transition procedures, properties of composites were calculated by means of some homogenization procedures, etc. The above described simple tests appear in the literature on granular materials in a more or less explicit form since the works of F. Gassmann [13], J. Geertsma [14], M. A. Biot and D. G. Willis [15], and many others. The problem is still investigated and we refer to modern review works of J. E. White [16], R. D. Stoll [9], or J. G. Berryman [17] who present also various applications of the results of such an analysis.

The approach presented in this work differs from the previous presentations because the model contains an additional material parameter N. Simultaneously this systematic approach to the micro-macro transition enables generalizations to such systems in which we cannot assume full saturation or neglect capillary or viscous effects. This shall be the subject of a forthcoming publication.

The **jacketed undrained** test described by (38) yields immediately the *first compatibility relation* between material parameters

$$\mathcal{C}_1 := C + \frac{K_f(K-K_s) - N(K-K_V)}{n_0(K_s-K_f)} = 0. \tag{41}$$

We proceed to investigate the **jacketed drained** test.

It is commonly assumed that in addition to microscopic tests which deliver compressibilities K_s, K_f one can measure the so-called *drained compressibility modulus* K_d. This is defined as the negative fraction of the excess pressure Δp to the macroscopic volume change e in the drained jacketed test. Hence we have for $\beta \equiv 0$

$$\begin{aligned} K_d \quad : &= -\frac{\Delta p}{e} = -\frac{\left(p^S - p_0^S\right) + \left(p^F - p_0^F\right)}{e} = \\ &= -\frac{1}{e}\left\{-\left(K - n_0 C\right) e - n_0 C \varepsilon\right\} \equiv K - C\frac{\zeta}{e}. \end{aligned} \tag{42}$$

We can eliminate the fraction $\frac{\zeta}{e}$ either by means of the macroscopic condition for pore pressure $(39)_1$ or by means of the microscopic condition on volume changes of the fluid $(39)_2$. Bearing the relation $(35)_2$ in mind we obtain in the first case the following *second compatibility relation* between material parameters

$$\mathcal{C}_2 := n_0 - \frac{C}{M} - \frac{K_b}{K_s}\frac{1-(1-n_0)\frac{K_s}{K_b}}{1-\frac{1-n_0}{n_0}\frac{NC}{K_b M}}\left\{1 - \frac{N(K-n_0C)}{n_0 M K_b}\right\} = 0, \tag{43}$$

$$K_b := K - \frac{C^2}{M},$$

where we have used the relation $(35)_3$ for the porosity.

In the second case we substitute $(35)_5$ for ε^R and obtain the *third compatibility relation* between material parameters

$$\mathcal{C}_3 := K - K_d - C\frac{C-N}{M-\frac{N}{n_0}} = 0. \tag{44}$$

Finally for the **unjacketed** test we have due to the equilibrium condition (19) the equality of the full pressure given by $(35)_1$ and the pore pressure $(20)_2$. This yields the *fourth compatibility relation* between material parameters

$$\mathcal{C}_4 := \left(1-\frac{K}{K_W}\right)\left(M-C-N\frac{1-n_0}{n_0}\frac{C}{K_s}\right)+$$

$$+\left(1-\frac{C}{K_W}\right)\left(K-C-N\frac{1-n_0}{n_0}\left(1-\frac{K}{K_s}\right)\right)=0, \quad \frac{1}{K_W}:=\frac{1-n_0}{K_s}+\frac{n_0}{K_f}. \tag{45}$$

The set of relations

$$\{\mathcal{C}_1,\mathcal{C}_2,\mathcal{C}_3,\mathcal{C}_4\}=0 \tag{46}$$

forms four equations for four material parameters $\{K,C,M,N\}$, or equivalently for $\left\{\lambda^S+\frac{2}{3}\mu^S,\kappa,Q,N\right\}$ in terms of the material parameters $\{K_s,K_f,K_d\}$ and the initial porosity n_0. It is nonlinear and, consequently, possesses more than one solution. As it cannot be solved analytically we present further some typical numerical results. For data used in the example there exist two physically reasonable solutions in which all four quantities K,M,C,N are real and nonnegative.

However one analytical solution can be constructed in the fully general case. Namely, substitution of the parameter $N=0$ in $\{\mathcal{C}_1,\mathcal{C}_3,\mathcal{C}_4\}=0$ yields the following set of equations

$$K-K_s+n_0C\frac{K_s-K_f}{K_f}=0, \quad K_d-K+\frac{C^2}{M}=0,$$

$$K\left(1-\frac{C}{K}\right)-C+M-\frac{MK_d}{K_W}=0. \tag{47}$$

We see further that this set possesses two real positive solutions. One of them can be found analytically and it has the form

$$K=\frac{(K_s-K_d)^2}{\frac{K_s^2}{K_W}-K_d}+K_d, \quad C=\frac{K_s(K_s-K_d)}{\frac{K_s^2}{K_W}-K_d}, \quad M=\frac{K_s^2}{\frac{K_s^2}{K_W}-K_d}. \tag{48}$$

These are the famous *Gassmann relations* appearing in Biot's model (e.g. [9]). It is rather amazing that they satisfy **identically** the equation $\mathcal{C}_2=0$ with $N=0$. Consequently $\{K,C,M\}$ given by (48) and $N=0$ constitute a solution of the full system of compatibility conditions. It yields the important conclusion that

> *Biot's model whose material parameters satisfy the above Gassmann relations is thermodynamically admissible in spite of the fact that its constitutive relations for partial stresses do not contain a dependence on the porosity gradient.*

This property follows solely for the linear model in which the contribution of the porosity gradient can be incorporated in constitutive relations for stresses.

Let us note in passing that in the usual derivation of Gassmann relations the compatibility of these relations with the jacketed drained test is ignored – this test is solely used as a motivation of the definition of the drained compressibility modulus K_d – and, in addition, it is assumed that the porosity in the unjacketed experiment is constant. Details can be found, for instance, in the book of Stoll [9]. Substitution of Gassmann relations in the formula (35) for changes of porosity shows that $n = n_0$, i.e. the porosity follows as a constant in this experiment and does not have to be assumed to be constant as claimed by Stoll.

Let us comment on the condition $\mathcal{C}_2 = 0$. This condition is violated by the second solution of the set (47). We demonstrate it in the next section on a numerical example. However this solution can be used as a first step of an iteration procedure for calculating a solution with $N \neq 0$. It is essential in the full model with an additional contribution of the porosity gradient. If the parameter N is small in comparison with other material parameters of the model we can iterate the solution of the full set $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4\} = 0$ assuming that the second solution of Gassmann equations forms a zeroth step of iteration and $\mathcal{C}_2 = 0$ serves the purpose of determining N in this iteration level. Then we can proceed to the second step of iteration and so on. It is easy to show that such a procedure indeed converges to the full solution of the system. We shall not pursue any theoretical analysis of this method in this work and present rather a numerical example.

4.2 A numerical example

As mentioned above the set of equations $\{\mathcal{C}_1, C_2, C_3, \mathcal{C}_4\} = 0$ for material parameters $\{K, M, C, N\}$ cannot be solved analytically. However, it is a rather simple algebraic system which can be handled numerically with any standard package. Below we show such numerical results calculated by means of the package MAPLE 7.0. We have chosen the following data for these calculations

$$K_s = 48 \times 10^9 Pa, \quad K_f = 2.25 \times 10^9 Pa, \quad K_d = \frac{K_s}{1 + g n_0}, \tag{49}$$

where the last empirical relation is proposed after Geertsma (e.g. see: [16] where it is claimed that $g = 50$ gives a good fit with experimental data for many soils). We choose the values $g = 30$ and 50 for the illustration of Gassmann equations and solely $g = 50$ for the full model.

In Figures 2 and 3 we show numerical solutions of Gassmann equations (47). Solution 2 (denoted as sol.2 in Figures) satisfies relations (48) and, simultaneously, the equation $\mathcal{C}_2 = 0$ with $N = 0$. Consequently, this solution is also a solution of the full general set of compatibility relations. Solution 1 (denoted as sol.1 in Figures) in Figure 2 satisfies equations (47) but not the compatibility condition $\mathcal{C}_2 = 0$. Hence, it is not a solution of the full set and has no practical bearing. It is interesting to observe that solution 1 yields negative values of the coupling parameter Q for high values of porosity. This would yield instabilities (expansion under pressure). In addition to the violation of the compatibility relation this would eliminate an applicability of this solution.

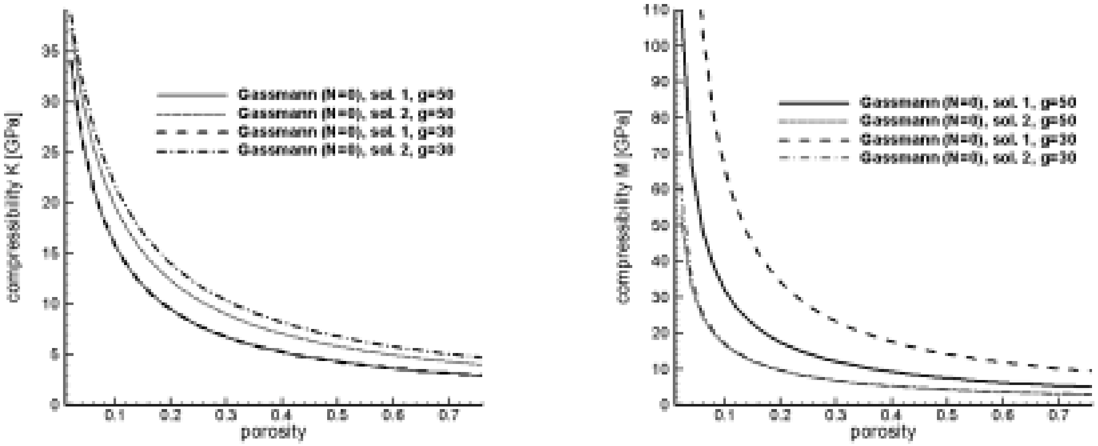

Figure 2: *Compressibility modulus K (left) and compressibility modulus M (right) according to Gassmann equations* (47) *for two values of Geertsma parameter* $g = 30, 50$.

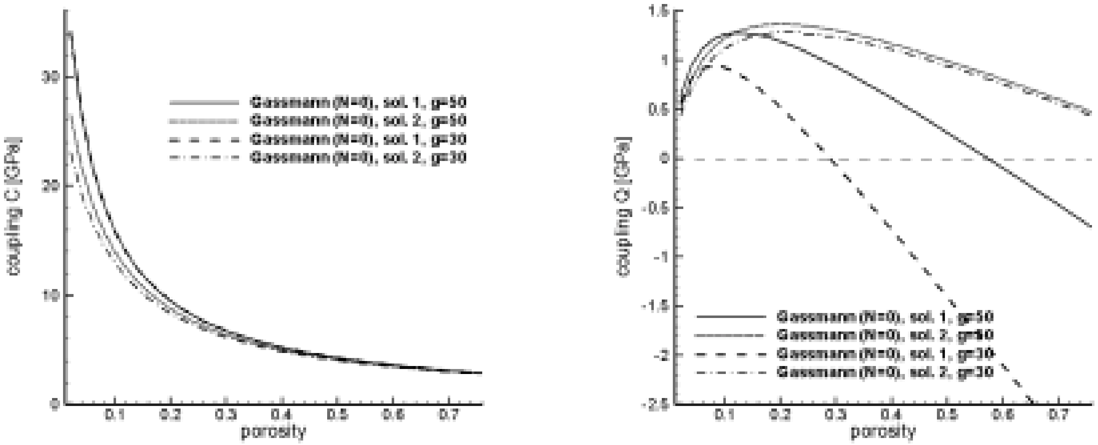

Figure 3: *Coupling parameters C (left) and Q (right) according to Gassmann equations* (47) *for two values of Geertsma parameter* $g = 30, 50$.

Let us make two observations essential for the further analysis. It is seen that the change of Geertsma parameter does not yield any qualitative changes in the behavior of the parameters. The model becomes stiffer for smaller values of g. For this reason we present further results for the full model choosing only one value of Geertsma parameter: $g = 50$. Secondly, it is seen in Figure 3 that values of the coupling parameter Q are much smaller than values of compressibility moduli K, M. This is an important conclusion for the construction of acoustics by means of the model without any coupling between stresses at all (e.g. [7], [10]).

We proceed to the presentation of results for the full model. The set of compatibility relations has been solved by iteration with respect to the parameter N. It has been found that the iteration converges very fast for small values of porosity and it slows down with growing porosity until, at approximately $n_0 = 0.58$, the procedure is not convergent anymore. One would have to use a different iteration procedure in order to find solutions for higher values of porosity. This has not been done because such results would have no physical bearing. Namely, for such values the coupling parameter Q becomes negative and this yields constitutive instability. Such an instability does not appear in Biot's model.

Let us inspect the numerical results. In Figure 4 we compare all material parameters K, M, C, N for the Biot's model (left) and for the full model (right). Clearly $N = 0$ in the case of Biot's model. It is seen that there is a good qualitative agreement of both models and quantitatively both models agree quite well for porosities bigger than 0.2. It is essential to notice that the curves for C and N intersect each other for the full model at $n_0 \approx 0.58$. As we see further this appears in the point of loss of stability. At this point the iteration was terminated.

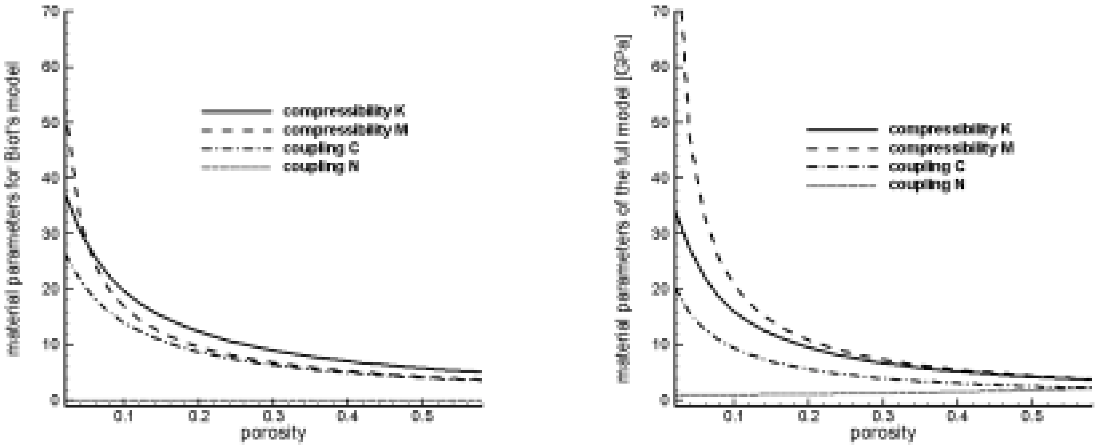

Figure 4: *Comparison of material parameters for Biot's model (left) and the full model with the porosity gradient (right) with Geertsma parameter* $g = 50$.

On the left hand side of Figure 5 we compare directly compressibilities K, M for both models. Obviously the compressibility modulus of the skeleton K is almost identical in both models. The compressibility modulus of fluid M is higher (app. 20%) for Biot's model than it is in the full model.

The coupling coefficient Q agrees qualitatively for both models and for the porosity $n_0 < 0.58$ (Figure 5, right) but the quantitative differences are essential. This coupling is much stronger in Biot's model and it becomes negative in the full model for $n_0 > 0.58$. It means, as we have already mentioned, that an increment of pressure would lead to an increment of volume (i.e. a nonconvex potential of stresses) which, according to classical arguments of continuum mechanics, yields an instability .

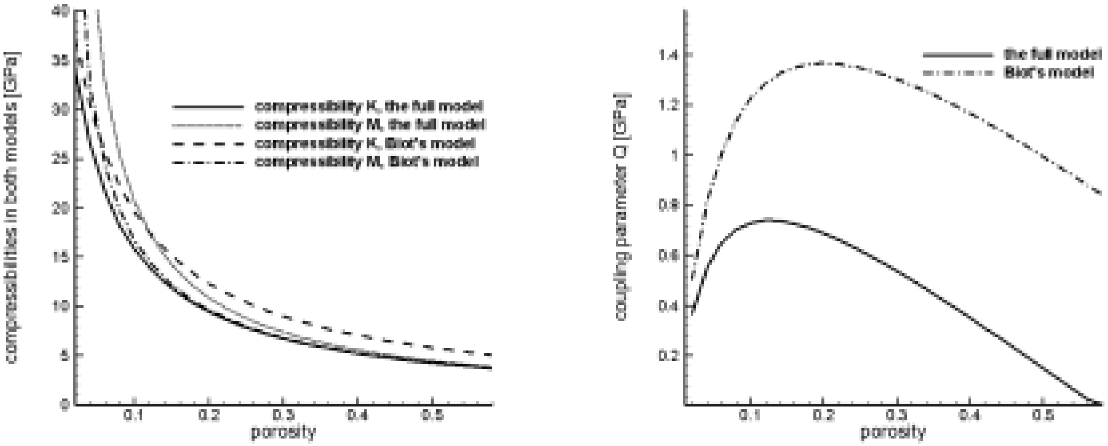

Figure 5: *Comparison of the compressibilities K, M (left) and of the coupling parameter Q (right) for the full model and Biot's model with Geertsma parameter $g = 50$.*

We proceed to present results for the coefficients δ and Φ describing the behavior of the current porosity n (see: relation $(16)_3$). These are shown in Figure 6. It is clear that equilibrium changes of porosity proportional to volume changes of the skeleton e are much bigger than nonequilibrium changes proportional to the difference of volume changes $e - \varepsilon$ (i.e. to the increment of fluid contents ζ). The parameter δ is app. 20 times bigger than the nonequilibrium parameter Φ/n_0. Changes of porosity are apparently more significant in the range of small initial porosities than for large initial porosities.

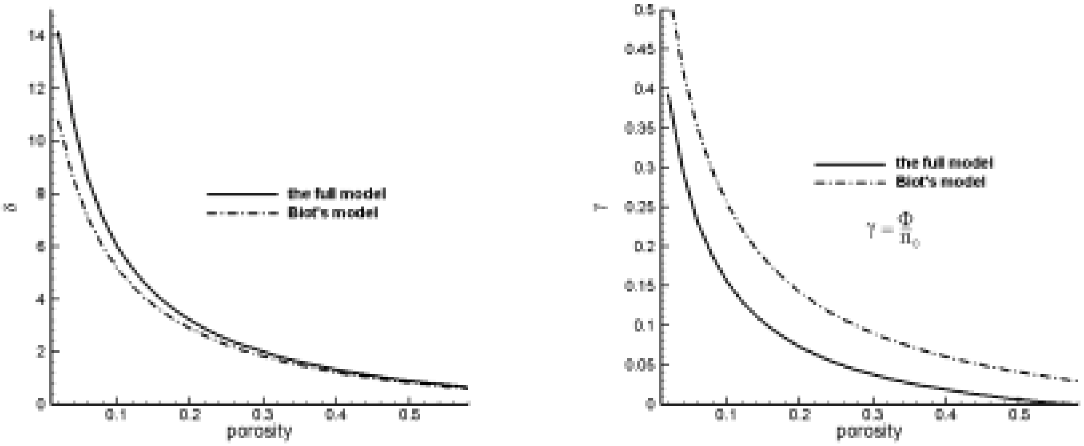

Figure 6: *Coefficients of porosity relation: equilibrium part δ, and nonequilibrium part $\gamma = \Phi/n_0$ with Geertsma parameter $g = 50$.*

5 Concluding remarks

The results presented in this work demonstrate the position of the classical Biot's model for linear poroelastic media among thermodynamically admissible models following as limit cases from a fully nonlinear model with a balance equation of porosity.

The most important conclusion of the above presented analysis of the micro-macro transition is that Biot's model follows as a special case of the full thermodynamical model in which the coupling with the porosity gradient vanishes ($N = 0$). Simultaneously the micro-macro transition yields compatibility relations which possess *two* solutions for material parameters. One of them coincides with classical Gassmann relations for Biot's model. The other yields a model in which the coupling with the porosity gradient is present. These two models seem to be possible due to the linearity assumption. The extension of the classical nonlinear two-component model of immiscible mixtures by adding a dependence on the porosity gradient is necessary for the existence of coupling between partial stresses and this dependence survives in nonlinear models not covered by the micro-macro transition applied in this work.

In order to obtain Biot's constitutive relations for stresses it is necessary as well to leave out relaxation properties of the porosity (memory effects) which may appear in the general thermodynamical model.

In the full thermodynamical model there appears a material instability for initial porosities higher than app. 0.58 – at least for the data used in the numerical analysis. This may be related to a real instability indicated by J. Dvorkin (e.g. [18]) which he assigns to the fluidization.

The numerical results presented in the previous section show that in practically relevant cases of analysis of acoustic waves both couplings – Biot's coupling of stresses with the parameter Q as well as the coupling through the porosity gradient with the parameter N – can be neglected without any qualitative consequences and very little quantitative influence on propagation conditions. The last conclusion is presented more extensively in the work [10].

Moreover, at least for granular materials for which the above micro-macro transition analysis is better justified than for rocks, changes of porosity in the wave analysis of linear acoustic waves can be assumed to be caused solely by volume changes of the skeleton. An influence of the increment of fluid content on these changes is of the order of a few percent of this following volume changes of the skeleton.

Acknowledgement

I would like to express my deepest appreciation to Dr. Carlo Lai (Milan, Italy) for his help, comments, critical remarks and encouragement during my work on this paper. Without his contributions and friendship this work would never be written. However, I carry, of course, alone the full responsibility for the above presented results.

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