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Stabilization of weak solutions of compressible Navier-Stokes equations for isothermal fluids with a nonlinear stress tensor

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Abstract

The aim of this paper is to study the stabilization of solutions to the Navier-Stokes equations for isothermal fluids with a nonlinear stress tensor. We study stabilization from the point of view of the method used in [17], where the authors studied the asymptotic behaviour of solutions to barotropic compressible Navier-Stokes equations.

1 Introduction

First, we introduce the model for the isothermal fluids with a nonlinear stress tensor. This model includes two equations:

Continuity equation

$$\rho_t + \operatorname{div}(\rho\mathbf{u}) = 0, \tag{1.1}$$

Balance of momentum

$$(\rho\mathbf{u})_t + \operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\rho - \operatorname{div}P(\mathbf{u}) = \rho\mathbf{f},\ x\in\Omega,\ t\in(0,\infty), \tag{1.2}$$

where the operator P represents the nonlinear dependence of the stress tensor on the velocity field. The stabilization will be studied under the boundary condition of Dirichlet's type,

$$\mathbf{u}(x,t) = 0,\ x\in\partial\Omega,\ t\in(0,\infty), \tag{1.3}$$

and the initial state is prescribed by

$$\rho(x,0) = \rho_0(x) \geq 0,\ x\in\Omega, \tag{1.4}$$

$$(\rho\mathbf{u})(x,0) = \mathbf{q}_0(x),\ x\in\Omega, \tag{1.5}$$

for a bounded domain Ω. The proof of the global existence of a solution to this problem was given by A. E. Mamontov in [14], [15]. Moreover, the existence theorem was proved independently of the dimension. This requirement led to the special form of the stress tensor, and this was the reason why the existence of the solution of the problem (1.1)–(1.5) was only proved in appropriate Orlicz spaces.

There are a lot of related results not only in one space variable (see e.g. [1], [2], [3], [22]) but also in several space dimensions (see [16], [19]), when the data is a small perturbation of a constant equilibrium. In [18], the unconditional stabilization of solutions of barotropic compressible Navier-Stokes equations on the space periodic problem with a certain symmetry was investigated. This paper was followed by [8] and [17], where the Dirichlet boundary condition was considered and a different method was used.

By the stabilization of solutions in this context we mean that, given a weak solution to the problem (1.1)–(1.5), for any sequence $t_n \to \infty$, and for a Young function Φ such that its complementary function Ψ satisfies

$$\sup_w \int_\Omega \Psi(|w^{\frac{1}{\alpha}}|)\ dx \leq c,$$

with $\alpha \in (0,1)$, where $w \in \widetilde{L}_{\Psi_1}(\Omega)$ is such that $\int_\Omega \Psi_1(w)\ dx \leq 1$, there exists a function ρ_∞ such that

$$\lim_{n\to\infty} \|\rho(t_n) - \rho_\infty\|_\Phi = 0, \tag{1.6}$$

where the equilibrium density ρ_∞ is a solution to the rest state equations

$$\nabla \rho_\infty = \rho_\infty \mathbf{f} \quad \text{ a.e. in } \Omega, \tag{1.7}$$

$$\int_\Omega \rho_\infty\ dx = \int_\Omega \rho_0\ dx,\ \ \rho_\infty \geq 0. \tag{1.8}$$

Our technique of proof for the stabilization of solutions to the problem (1.1)–(1.5) is motivated by the method which was given in [17] for the first time. Let us mention here some distinctions and difficulties. The purpose of this method is to find a function $\overline{\rho}(t)$ which is close to the density $\rho(t)$ and at the same time $\overline{\rho}(t_n)$ converges to ρ_∞ strongly in appropriate spaces. The construction of the function $\overline{\rho}$ is based on the solvability of the Neumann problem

$$\int_\Omega \nabla w_{\epsilon k}(s) \cdot \nabla \eta\ dx = \int_\Omega R_\epsilon(\rho(x,\cdot))(s)\mathbf{f} \cdot \nabla \eta\ dx,\ \ \forall \eta \in W^{1,p}(\Omega),$$

$$\int_\Omega w_{\epsilon k}(x,s)\ dx = 0,$$

where R_ϵ means the regularization in time variable.

But we are not able to decide upon the solvability of the Neumann problem if we only know that $\rho \in L^\infty(0,\infty; L_{\Phi_1}(\Omega))$, with $\Phi_1(z) = z\ln(1+z)$. In this case the method breaks down. We overcome this difficulty by using a cut-off function $T_k(\rho)$. The second problem is that we cannot improve the global estimate for the density ρ, as was shown in [17]. The next open question is whether we can test the equation (1.2) in the pressure term with the function $\mathbf{v}$ solving the problem

$$\operatorname{div} \mathbf{v} = f,\ \ f \in L^\infty(\Omega),$$

$$\mathbf{v}|_{\partial\Omega} = 0,\ \ \int_\Omega f\ dx = 0.$$

This question will be answered in Section 5. The main difficulty in carrying out this construction is that we must verify that the function $\overline{\rho}(t)$ is sufficiently close to the function $\overline{\rho}_k(t)$ generated by the Neumann problem with the right-hand side containing the function $T_k(\rho)$ instead of ρ.

In Section 2, we establish the basic notation used. In Section 3, we summarize all assumptions on external force, initial state and stress tensor. In addition, we give an example for the stress tensors considered in this paper, there. In Section 4, we give a brief outline of properties of appropriate Young functions, and we prove auxiliary lemmas. Before beginning the proof of stabilization, we must complete the theory about renormalized solutions of the equation (1.1). This is the aim of Section 6. In Sections 7-9, we will be concerned with stabilization, and our main result will be stated and proved.

2 Preliminaries

In this section, we adopt the notation. Let us denote by D the symmetric part of the velocity gradient, i.e.,

$$D_{ij}\mathbf{u} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Further, M stands for a Young function with a growth given by the estimates $c^{-1}e^{c^{-1}z} \leq M(z) \leq ce^{cz}$, for $z \geq z_0 \geq 0$, with some constant $c > 1$. Φ_β denotes the Young function having the form $(1+z)\ln^\beta(1+z)$ in case $\beta > 1$, and $z\ln(1+z)$ in case $\beta = 1$. Let Ψ_β and $\overline{M}$ denote the complementary functions to the Young functions Φ_β and M, respectively. There is no problem to verify that the growth of the functions $\Psi_\beta(z)$ is of the type $e^{z^{1/\beta}}$, and that the function $\overline{M}$ is equivalent to the Young function Φ_1. $L_M(\Omega)$ and $L_{\Phi_\beta}(\Omega)$ denote the Orlicz spaces generated by the Young functions M and Φ_β. These spaces are endowed with the norm $\|v\|_\Phi = \sup \int_\Omega vw\, dx$, where supremum is taken over all functions w such that

$$\int_\Omega \Psi(|w|)\, dx \leq 1.$$

For simplicity of notation, we used Φ instead of Φ_β or M. Sometimes it is convenient to take into account the Luxemburg norm defined by the expression $|||v|||_\Psi :=$ $\inf\{\lambda > 0;\ \int_\Omega \Psi(|v/\lambda|)\, dx \leq 1\}$. This norm is equivalent to the Orlicz norm generated by the same Young function. It is suitable to define the set $\widetilde{L}_\Phi(\Omega)$. This set contains all the functions v satisfying $\int_\Omega \Phi(|v|)\, dx < \infty$. Next, we establish the appropriate Orlicz spaces to which the velocity field belongs. Thus,

$$X := \{\mathbf{u},\ D\mathbf{u} \in L_M(\Omega),\ \mathbf{u}|_{\partial\Omega} = 0\},\ \|\mathbf{u}\|_X = \|D\mathbf{u}\|_{M,\Omega},$$

and

$$Y := \{\mathbf{v},\ D\mathbf{v} \in L_M(Q_T),\ \mathbf{v}(t)|_{\partial\Omega} = 0\},\ \|\mathbf{v}\|_Y = \|D\mathbf{v}\|_{M,Q_T},$$

where $Q_T = \Omega \times (0,T)$. $E_\Phi(\Omega)$ denotes the Orlicz space which is defined as the closure of the space $C_0^\infty(\Omega)$ in the Orlicz norm $\|\cdot\|_\Phi$. Let us remark that, unlike Lebesgue spaces, the spaces $E_\Phi(\Omega)$ and $L_\Phi(\Omega)$ do not coincide. We will use the following notation for dual spaces,

$$W^{-1}L_\psi(\Omega) = [W_0^1 L_\Phi(\Omega)]',$$

where $W_0^1 L_\Phi(\Omega)$ is the closure of the space $C_0^\infty(\Omega)$ in the norm

$$\|v\|_{1,\Phi} = \sqrt{\|v\|_\Phi^2 + \|\nabla v\|_\Phi^2}.$$

The notation of Lebesgue spaces and Sobolev spaces is standard, i.e., $L^p(\Omega)$ and $W^{1,p}(\Omega)$ for the spaces and $\|\cdot\|_p$ and $\|\cdot\|_{1,p}$ for their appropriate norms.

Definition 2.1 *The sequence $\{v_n\}_{n=1}^\infty \subset L_\Phi(\Omega)$ is said to be E_Ψ-weak convergent to the function v if*

$$\int_\Omega v_n \psi \, dx \to \int_\Omega v\psi \, dx, \quad \forall \psi \in E_\Psi.$$

Let us remark that all the bounded sets in the space $L_\Phi(\Omega)$ are E_Ψ-weakly compact. For more details about Orlicz spaces we refer the reader to [9] and [11].

There will be a short mentioning of Hardy spaces and BMO-spaces in Section 5. Therefore, we introduce here the Hardy space $\mathcal{H}^1(R^N)$ as a space of distributions such that $f \in \mathcal{H}^1(R^N)$ if, for some $\phi \in \mathcal{S}$ with $\int_{R^N} \phi \, dx = 1$, the maximal function

$$(M_\phi f)(x) := \sup_{t>0} |(f * \phi_t)(x)|$$

is in $L^1(R^N)$, with $\phi_t(x) = t^{-N}\phi(x/t)$. Here $\mathcal{S}$ is the usual space of infinitely differentiable functions which together with all their derivatives are rapidly decreasing, and $\|f\|_{\mathcal{H}^1} := \int_{R^N} |(M_\phi f)(x)| \, dx$. $BMO(R^N)$ is a space of locally integrable functions such that there is an $A < \infty$ such that

$$\frac{1}{|B|} \int_B |f(x) - f_B| \, dx \le A$$

holds for all balls B and $f_B := |B|^{-1} \int_B f \, dx$. The smallest such A will denote the norm of f in $BMO(R^N)$. We refer the reader to [21, pp. 87-228] for more details about $\mathcal{H}^1$- and BMO-spaces.

We will use the usual mollifier with respect to the variable t given by

$$(R_\epsilon v)(t) := \int_{-\infty}^\infty \phi_\epsilon(t-s)v(s) \, ds := \frac{1}{\epsilon} \int_{-\infty}^\infty \phi_0\left(\frac{t-s}{\epsilon}\right) v(s) \, ds,$$

where *supp* $\phi_0 \subset (-1,1)$, $\int_{-\infty}^\infty \phi_0(s) \, ds = 1$, $\phi_0 \ge 0$, $\phi_0 \in C^\infty(R^1)$.

3 Fundamental assumptions

The definition of appropriate spaces enables us to establish the fundamental assumptions, and these assumptions will be needed throughout the paper. We assume that:

1. $\mathbf{f} = \nabla g, \; g \in W^{2,\infty}(\Omega), \; \partial\Omega \in C^2$;

2. $\rho_0 \in L_{\Phi_\beta}(\Omega),\ \beta > 3,\ \sqrt{\rho_0}\mathbf{u}_0 \in L^2(\Omega)$;

3. the operator P is coercive, i.e.,

$$\int_\Omega P(\mathbf{v}) : D\mathbf{v}\ dx \geq \int_\Omega M(|D\mathbf{v}|)\ dx \tag{3.1}$$

for all $\mathbf{v} \in X$;

4. $P(\cdot)$ acts boundedly from X into $L_{\overline{M}}(\Omega)$, i.e.,

$$\int_\Omega \overline{M}(|P(\mathbf{v})|)\ dx \leq c\left(1 + \int_\Omega M(|D\mathbf{v}|)\ dx\right), \tag{3.2}$$

and the estimate

$$2^m \|P(\mathbf{v})\|_{\overline{M}} \leq c\left(k^m \int_\Omega M(|D\mathbf{v}|)\ dx + 1\right) \tag{3.3}$$

holds for all $m \in N_0$ $(N_0 = \{0, 1, 2, \ldots\})$, $\mathbf{v} \in X$, and for some fixed $k > 2$.

Now, it is convenient to present an example for the operator P.

Example: The example of the operator P can be given by the expression

$$P(\mathbf{v}) := \frac{M(|D\mathbf{v}|)D\mathbf{v}}{|D\mathbf{v}|^2} \text{ if } D\mathbf{v} \neq 0,\ \ P(\mathbf{v}) := 0 \text{ if } D\mathbf{v} = 0.$$

Remark 3.1 From now on, $M(z)$ denotes the Young function defined by the expression

$$M(z) := e^z - z - 1.$$

We can afford this definition of the function M without loss of generality, since the function $e^z - z - 1$ belongs to the class of equivalent Young functions generated by the estimate above.

4 Basic lemmas

In this section, we formulate lemmas which give us basic information about the used Young functions.

Lemma 4.1 *Let the function M be established as in Remark 3.1 and Ψ_2 be the complementary function to Φ_2. Then the inequality*

$$2^m\|v\|_{\Psi_2} \leq c\left(\max\left\{6 \cdot 2^{3m}, \frac{11}{4}\sqrt{2^{7m}}, \frac{2^{4m}}{4}\right\} e^{\frac{2^m+1}{2}} \int_\Omega M(|v|)\ dx + 1\right) \tag{4.1}$$

holds for $m \in N_0$.

P r o o f: We begin by proving the existence of a constant $K(c) > 0$ such that

$$c^3 z^3 e^{\sqrt{cz}} \le K(c)(e^z - z - 1),\ c \ge 1,\ z \ge 0,$$

with $\Psi_2(z) = z^3 e^{\sqrt{z}}$ being a function in the class of Young functions having the growth $e^{\sqrt{z}}$. We can easily derive one of the possible values for this constant in the form $K(c) = 24 \max\left\{6c^3, \frac{11}{4}c^3\sqrt{c}, \frac{c^4}{4}\right\} e^{\frac{c+1}{2}}$. The proof is based on studying the derivatives of the functions on both sides of the above inequality. Then the assertion follows from the estimate

$$2^m \|v\|_{\Psi_2} \le \int_\Omega \Psi_2(|2^m v|)\ dx + 1. \qquad \square$$

Lemma 4.1 may be summarized by saying that the function v is small enough in an appropriate Orlicz norm on the condition that this function is small in M-mean sense.

Lemma 4.2 *Let a sequence $\{v_m\}_{m=1}^\infty \subset L_M(\Omega)$, $m \in N_0$, be given such that*

$$\|v_m\|_\infty \le c,\ \|v_m\|_2 \le \frac{1}{\sqrt{M(\max\{1,c\}2^m)}},\quad m \in N_0.$$

Then the inequality

$$\|v_m\|_M \le \frac{K}{2^m} \tag{4.2}$$

is fulfilled for all $m \in N_0$.

P r o o f: It is a well known fact that the estimate

$$\|2^m v_m\|_p^p \le \|v_m\|_2^2 (2^m)^p \|v_m\|_\infty^{p-2} \le \|v_m\|_2^2 (2^m \max\{1,c\})^p,\ p \ge 2,$$

holds.

By Taylor's formula, we obtain that

$$\int_\Omega M(|2^m v_m|)\ dx \le \|v_m\|_2^2 M(2^m \max\{1,c\}).$$

For the rest of the proof it is enough to realize that the Young inequality implies the estimate

$$2^m \|v_m\|_M \le \int_\Omega M(|2^m v_m|)\ dx + 1,$$

and that equivalence of the norms, which are generated by the Young functions with the growth e^z, holds. $\square$

Proposition 4.3 [13] *The inequality*

$$\|v\|_{1,p} \le \frac{cp^2}{p-1}\|Dv\|_p \tag{4.3}$$

holds for all $p > 1$ and $v \in W_0^{1,p}(\Omega)$ with a constant $c > 0$ independent of p and v.

Lemma 4.4 *Let $v \in L_{\Psi_2}(\Omega)$ and $w \in L_M(\Omega)$. Assume that the inequality*

$$\|v\|_p \leq cp\|w\|_p \tag{4.4}$$

is fulfilled for all $p > 1$ with c independent of p. Then, using the inequality (4.3), we can deduce the estimate

$$\|v\|_{\Psi_2} \leq c\|w\|_M. \tag{4.5}$$

P r o o f: Let us examine the Young function Ψ defined by the expression

$$\Psi(z) := \begin{cases} \frac{e^{z_0^{1/2}}}{z_0^2} z^2 & \text{if } z \in [0, z_0),\ z_0 > 1, \\ e^{z^{1/2}} & \text{if } z \in [z_0, \infty). \end{cases} \tag{4.6}$$

By virtue of Taylor's formula for $e^{\sqrt{z}}$, we find that the estimate

$$\Psi(z) = \sum_{q=1}^{\infty} \frac{1}{q!} z^{q/2} \leq 2 \sum_{p=1}^{\infty} \frac{1}{(2p)!} z^p \tag{4.7}$$

holds, since

- either $\frac{z^{1/2}}{q+1} \geq 1$,
- or $\frac{z^{1/2}}{2q} < \frac{z^{1/2}}{q+1} < 1$,

with q being an odd number. The remaining part of the proof is based on the existence of an appropriate λ (see [11, p. 149]) such that the following integrals exist. The Taylor formula for the function Ψ, Fatou's lemma, and Lebesgue's dominated convergence theorem now lead to the estimate

$$\int_\Omega \Psi\left(\left|\frac{v}{\lambda}\right|\right)\, dx \leq K \int_\Omega \tilde{\Psi}\left(\left|\frac{cw}{\lambda}\right|\right)\, dx, \tag{4.8}$$

with

$$\tilde{\Psi}(z) := \begin{cases} \frac{e^{z_1}}{z_1^2} z^2 & \text{if } z \in [0, z_1),\ z_1 > 1, \\ e^z & \text{if } z \in [z_1, \infty). \end{cases} \tag{4.9}$$

Using the Luxemburg definition of the norm and the imbedding theorem for Orlicz spaces, we get (4.5). □

Lemma 4.5 *Let $v \in W_0^1 L_{\Psi_\gamma}(\Omega)$, $\gamma \geq 1$. Then the estimate*

$$\sup_{z \in R^N \setminus \{0\}} \left\| \frac{v(\cdot - z) - v(\cdot)}{|z|} \right\|_{\Psi_\gamma} \leq \|v\|_{1,\Psi_\gamma} \tag{4.10}$$

holds.

P r o o f: Our proof starts with derivation of the inequality

$$|v(x-z)-v(x)| = \left| \int_0^1 \frac{d}{dr} v(x-rz)\, dr \right| \le \int_0^1 |\nabla v(x-rz)||z|\, dr, \; v \in C_0^\infty(\Omega).$$

Using the density of the space $C_0^\infty(\Omega)$ in $W_0^1 L_{\Psi_\gamma}(\Omega)$, Jensen's inequality, and definition of the Luxemburg norm, we finish our proof. □

Lemma 4.6 *Let $v(x,s) \in W_0^{1,p}(\Omega)$ for almost all $s > 0$, with $p > N$. Suppose that $\|v(s)\|_{1,p} \le K$ for almost all $s > 0$, where the constant K is independent of s. Then there exists a set of functions $\psi_h(x,s) \in C_0^\infty(\Omega)$, with $h > 0$, such that*

$$\|v(s) - \psi_h(s)\|_\infty < \epsilon \tag{4.11}$$

for a.a. $s > 0$ and for all $\epsilon > 0$ with $h = h(\epsilon)$, where $h = h(\epsilon) \to 0$ for $\epsilon \to 0$. Moreover,

$$\|\psi_h(s)\|_{1,p} \le C \qquad \textit{for a.a. } s > 0, \tag{4.12}$$

where the constant C does not depend on s.

P r o o f: The proof consists in the construction of appropriate functions $\psi_h(x,s)$. We define a domain Ω_h such that $\Omega_h \subset \Omega$ and $dist(\partial\Omega, \partial\Omega_h) = h$, with $h \in (0,1)$. Now, we take a function $\xi_h \in C_0^\infty(\Omega_h)$ such that $\xi_h = 1$ in Ω_{2h} and $|\nabla \xi_h| < c/h$. Then the functions $\widetilde{u}_h$, which are defined by $\widetilde{u}_h := u\xi_h$, are bounded in the space $L^\infty(0,\infty; W_0^{1,p}(\Omega))$ independently of h, and for p_1 such that $N < p_1 < p$ the estimate

$$\|u - \widetilde{u}_h\|_{L^\infty(0,\infty;W_0^{1,p_1}(\Omega))} \le |\Omega \setminus \Omega_{2h}|^{\frac{p-p_1}{p}} \|u\|_{L^\infty(0,\infty;W_0^{1,p}(\Omega))}$$

holds. Using the imbedding theorem, we get

$$\|u - \widetilde{u}_h\|_{L^\infty(0,\infty;C(\overline{\Omega}))} \le c_1(h),$$

where $c_1(h) \to 0$ for $h \to 0$.

Finally, we define the function $\psi_h := \vartheta_h * \widetilde{u}_h$, where $\vartheta \in C_0^\infty(-1,1)$, $\vartheta \ge 0$, $\int_{-\infty}^\infty \vartheta(z)\, dz = 1$ and $\vartheta_h(|x|) = \frac{1}{h^N}\vartheta\left(\frac{|x|}{h}\right)$. It is obvious that $\psi_h \in L^\infty(0,\infty; W_0^{1,p}(\Omega))$, and these functions are bounded in this space, independently of $h \in (0,1)$. Using the Arzelà-Ascoli theorem and theorem about compact imbedding, we obtain

$$|\psi_h(x,s) - \widetilde{u}_h(x,s)| = \left| \int_{B_1(0)} \vartheta(z)(\widetilde{u}_h(x-hz,s) - \widetilde{u}_h(x,s))\, dz \right| \le c_2(h)$$

for a.a. $s \in (0,\infty)$ and $x \in \Omega$, where $c_2(h) \to 0$ for $h \to 0$. □

Definition 4.7 *Let us define the cut-off function $T \in C^\infty(R_0^+)$ by*

$$T(z) = z,\ z \in [0,1],\ T(z) \le z,\ z \in [1,3],\ T(z) = 2,\ z \ge 3,$$

$$T'(z) \le C,\ z \in [0,\infty),$$

$$T_k(z) = kT\left(\frac{z}{k}\right),\ k = 1,2,\dots$$

The following lemma yields information about the behaviour of the cut-off function $T_k(w)$ for $k \to \infty$.

Lemma 4.8 *Let a function $w \in L^\infty(0,\infty; L_{\Phi_1}(\Omega))$ be such that $w(x,t) \geq 0$ on $\Omega \times (0,\infty)$ and $0 < \|w\|_{L^\infty(0,\infty;L_{\Phi_1}(\Omega))} \leq K$. Then the estimate*

$$\|w - T_k(w)\|_{L^\infty(0,\infty;L^1(\Omega))} \leq \frac{cK^2}{\ln(k)} \qquad k = 2, 3, \ldots \tag{4.13}$$

holds.

P r o o f: Our proof starts with the observation that the inequality

$$\int_{\Omega_k(t)} |w(t) - T_k(w(t))| \, dx \leq \frac{cK^2}{|\Omega_k(t)| \ln(3k)} \|\chi_{\Omega_k(t)}\|_{\Psi_1} \|\chi_{\Omega_k(t)}\|_{\Phi_1},$$

is fulfilled, with $\chi_{\Omega_k(t)}$ being the characteristic function of the set $\Omega_k(t)$, where $w(x,t) > 3k$ for $x \in \Omega_k(t)$. Substituting the representation (see [11, p. 149])

$$\|\chi_{\Omega_k(t)}\|_{\Phi} = |\Omega_k(t)| \Phi^{-1}\left(\frac{1}{|\Omega_k(t)|}\right)$$

into the inequality above, we transform our proof to the verification of the relation

$$\lim_{z \to 0+} z \Phi_1^{-1}\left(\frac{1}{z}\right) \Psi_1^{-1}\left(\frac{1}{z}\right) \leq c, \; c > 1.$$

The rest of the proof is obvious. □

Proposition 4.9 [17] *Let Ω be of class C^2. Then, for any sufficiently small $\eta > 0$, there exists a domain $\Omega_\eta \subset \Omega$ such that $\overline{\Omega}_\eta \subset \Omega$, $|\Omega \setminus \Omega_\eta| \leq c\eta$, and if $x \in \partial\Omega$ then there is a unique $y = y(x) \in \partial\Omega_\eta$ such that $\nu(x) = \nu(y(x))$ and $|x - y(x)| = \eta$ for all $x \in \partial\Omega$. In addition, there is a function $\kappa \in W^{1,\infty}(\Omega)$ such that $\kappa(x) = 1$ for $x \in \Omega_\eta$, $\kappa(x) = 0$ for $x \in \Omega \setminus \Omega_{\frac{\eta}{2}}$, $|\nabla\kappa| \leq \frac{c}{\eta}$ for $x \in \Omega_{\frac{\eta}{2}} \setminus \Omega_\eta$ and $\frac{d\kappa}{d\tau}|_{\partial\Omega_\eta} = 0$, where τ is the tangential unit vector to the boundary.*

5 On the problem $\operatorname{div} \mathbf{v} = f$ with $f \in L^\infty(\Omega)$

Since the main idea of the proof of our main result is based on testing (1.2) in the pressure term with an appropriate function $\mathbf{v}$ satisfying the equation $\operatorname{div} \mathbf{v} = f$, with $f \in L^\infty(\Omega)$, we must prove that this problem has a solution whose regularity is sufficient for using this function as a test function in the equation (1.2).

Lemma 5.1 *Let $\{u_n\}_{n=1}^\infty \in L_\Psi(\Omega)$ be an E_Φ-weakly convergent sequence. Assume that the function Φ satisfies the Δ_2-condition. Then*

$$\lim_{n \to \infty} \inf \|u_n\|_\Psi \geq \|u\|_\Psi.$$

P r o o f: Let us first mention that the Young function Φ satisfies the Δ_2-condition if its complementary function Ψ has an exponential growth. This is the case which is interesting for the proof of Theorem 5.2. As Φ satisfies the Δ_2-condition, $E_\Phi(\Omega) = L_\Phi(\Omega) = \widetilde{L}_\Phi(\Omega)$ (see [11]). Then there exists for each $\epsilon \in (0,1)$ a function $v_\epsilon \in E_\Phi(\Omega)$ such that $\int_\Omega \Phi(|v_\epsilon|)\, dx \le 1$, and the estimate

$$\|u\|_\Psi \le \int_\Omega u v_\epsilon \, dx + \epsilon = \lim_{n\to\infty} \inf \int_\Omega u_n v_\epsilon \, dx + \epsilon \le \lim_{n\to\infty} \inf \|u_n\|_\Psi + \epsilon$$

holds. $\square$

We now consider the following problem: Let $f \in L^\infty(\Omega)$ and

$$\int_\Omega f \, dx = 0. \tag{5.1}$$

Find a vector field $\mathbf{v}$ such that

$$\operatorname{div} \mathbf{v} = f, \tag{5.2}$$

$$\mathbf{v} \in W_0^1 L_M(\Omega), \tag{5.3}$$

$$\|\mathbf{v}\|_{1,M} \le c\|f\|_\infty, \tag{5.4}$$

with the Young function M defined in Remark 3.1.

Theorem 5.2 *Let Ω be a bounded domain with Lipschitzian boundary. Then the problem (5.1)–(5.4) has a solution $\mathbf{v} \in W_0^1 L_M(\Omega)$.*

P r o o f: A suitable solution of problem (5.1)–(5.4) we are interested in is represented by a weakly singular integral (see [5]),

$$\mathbf{v}(x) = \int_{\Omega'} \widetilde{f}(y) \left[\frac{x-y}{|x-y|^N} \int_{|x-y|}^\infty \omega\left(y + \xi \frac{x-y}{|x-y|} \right) \xi^{N-1} \, d\xi \right] dy \tag{5.5}$$

after the change of variables

$$x \to x' = \frac{x - x_0}{R}$$

and decomposition Ω on star-shaped domains with respect to an open ball. Here, $\omega \in C_0^\infty(B_1(0))$ and $\int_{B_1(0)} \omega \, dx = 1$.

Then the expression of $\frac{\partial}{\partial x_j} v_i$ is formed by a singular integral. Now we study the behaviour of the singular integrals on the space of bounded functions more precisely. We begin with the observation that the singular integral maps $L^\infty(R^N)$ into $BMO(R^N)$ (see [21, p. 155, 178]), and a maximal operator maps $\widetilde{L}_{\Phi_1}(\Omega)$ into $L_1(\Omega)$ (see [20]). It follows from above and from the inequalities

$$\|w\|_{\mathcal{H}^1} \le \int_\Omega \Phi_1(|w|)\, dx + c$$

for $w \in \widetilde{L}_{\Phi_1}(\Omega)$, and

$$\left| \int_{R^N} vw \, dx \right| \le \|v\|_{BMO} \|w\|_{\mathcal{H}^1}, \tag{5.6}$$

that $BMO(R^N) \hookrightarrow L_{\Psi_1}(R^N)$, where Ψ_1 is the complementary function to Φ_1. Thus we conclude that for each $f \in C_0^\infty(\Omega)$ there exists a solution $\mathbf{v}$ to problem (5.1)–(5.4) such that

$$\|\mathbf{v}\|_{1,\Psi_1} \leq c\|f\|_\infty.$$

Having disposed of these preliminary steps, we can now associate to each $f \in L^\infty(\Omega)$ a sequence $\{f_m\}_{m=1}^\infty$ such that $f_m \in C_0^\infty(\Omega)$ and $f_m \to f$ in $L^p(\Omega)$ for all $p \in [1,\infty)$. Then the sequence $\{g_m\}_{m=1}^\infty$ defined by $g_m := f_m - \phi \int_\Omega f_m \, dx$, where $\int_\Omega \phi \, dx = 1$, converges to f as well, and the inequality $\|g_m\|_\infty \leq c\|f\|_\infty$ holds. The last inequality is a consequence of the fact that we construct the functions f_m using the mollifier. By the E_{Φ_1}-weak convergence of the sequence $\{\nabla \mathbf{v}_m\}_{m=1}^\infty$, we thus get

$$\int_\Omega \operatorname{div} \mathbf{v} \, \varphi \, dx = \int_\Omega f \varphi \, dx$$

for all $\varphi \in C_0^\infty(\Omega)$, and hence

$$\operatorname{div} \mathbf{v} = f \text{ a.e in } \Omega.$$

We conclude from Lemma 5.1 that estimate (5.4) holds. □

Let us mention an important consequence of Theorem 5.2 .

Consequence 5.3 *Let* $\mathbf{g} \in L^\infty(\Omega)$, $\mathbf{g} \cdot \nu|_{\partial\Omega} = 0$ *and* $\operatorname{div} \mathbf{g} \in L_{\Psi_1}(\Omega)$. *Let* S *be the weakly singular operator (5.5) generated by the problem (5.1)–(5.4). Then the operator* $S(\operatorname{div} \mathbf{g})$ *is well-defined in* $L_M(\Omega)$, *and the following estimate*

$$\|S(\operatorname{div} \mathbf{g})\|_M \leq c\|\mathbf{g}\|_\infty \tag{5.7}$$

is fulfilled.

P r o o f: The operator $S(\operatorname{div} \mathbf{g})$ is well-defined in the space $L^p(\Omega)$, $p \in (1,\infty)$. As a consequence of the theory in [5], we can construct a sequence $\{\mathbf{g}_n\}_{n=1}^\infty \subset C_0^\infty(\Omega)$ such that $\operatorname{div} \mathbf{g}_n \to \operatorname{div} \mathbf{g}$ in $L^p(\Omega)$. According to Theorem 5.2, we have

$$\|S(\operatorname{div} \mathbf{g}_n)\|_M \leq c\|\mathbf{g}\|_\infty, \ \forall n \in N.$$

Passing to an E_{Φ_1}-weakly convergent subsequence, $S(\operatorname{div} \mathbf{g}_{n_k}) \to \overline{S}$, and using the linearity of the operator S, we find $S(\operatorname{div} \mathbf{g}_n) \to S(\operatorname{div} \mathbf{g})$ in $L^p(\Omega)$, which leads to $S(\operatorname{div} \mathbf{g}) = \overline{S}$ a.e. in Ω. Lemma 5.1 clearly forces estimate (5.7). □

6 On renormalized solutions to (1.1)

This section deals with the existence of the renormalized solution to equation (1.1).

Lemma 6.1 *Let* $\mathbf{u} \in L^1(0,T;X)$ *and* $\rho \in L^\infty(0,T;L_{\Phi_\beta}(\Omega))$, $\beta > 2$, *be a weak (distributional) solution to (1.1) in* $\mathcal{D}'(Q_T)$. *Prolonging* ρ *and* $\mathbf{u}$ *by zero outside* Ω, *we have*

$$\rho_t + \mathrm{div}\,(\rho \mathbf{u}) = 0 \ \mathit{in}\ \mathcal{D}'(R^N \times (0,T)). \tag{6.1}$$

P r o o f: Let us follow the idea of the proof from [7]. Thus, we consider a regularizing sequence

$$\vartheta_\epsilon(x) := \frac{1}{\epsilon^N}\vartheta\left(\frac{|x|}{\epsilon}\right), \tag{6.2}$$

where

$$\vartheta \in C^\infty(R^1),\ supp[\vartheta] \subset (-1,1),\ \vartheta \geq 0,\ -\int_0^1 \vartheta'(z) z\ dz = 1,$$
$$\vartheta(-z) = \vartheta(z) \text{ and } \vartheta'(z) \leq 0 \text{ for all } z \geq 0,$$

and any positive parameter $\epsilon > 0$. In the same manner as in [7], we can construct the functions ϕ_m with the properties

$$0 \leq \phi_m \leq 1 \text{ on } \Omega,\ \phi_m = 1 \text{ if } dist(x, \partial\Omega) \geq \frac{1}{m},\ \phi_m \in \mathcal{D}(\Omega),$$

and

$$|\nabla \phi_m(x)| \leq 3m \text{ for all } x \in \Omega.$$

Now taking $\varphi \in \mathcal{D}(R^N \times (0,T))$ arbitrary, one has

$$0 = \int_0^T \int_\Omega \rho\varphi_t\phi_m + \phi_m \rho \mathbf{u}\cdot\nabla\varphi + \varphi\rho\mathbf{u}\cdot\nabla\phi_m\ dxdt.$$

Consequently, it is enough to show that

$$\left|\int_0^T \int_\Omega \rho\mathbf{u}\cdot\nabla\phi_m\ dxdt\right| \to 0 \text{ for } m \to \infty.$$

But from Theorem 8.4 from [10, p. 69] for $k = 1$, $\epsilon = 0$ and $\eta = -p$, Proposition 4.3 and Lemma 4.4, it follows that

$$\left|\int_0^T \int_{dist(x,\partial\Omega)\leq\frac{1}{m}} \rho\mathbf{u}\cdot\nabla\phi_m\ dxdt\right| \leq \int_0^T \int_{dist(x,\partial\Omega)\leq\frac{1}{m}} \rho\frac{|\mathbf{u}|}{dist(x,\partial\Omega)}\ dxdt$$
$$\leq c\left(\frac{1}{m}\right)\int_0^T \|\rho(t)\|_{\Phi_\beta, dist(x,\partial\Omega)\leq\frac{1}{m}} \left\|\frac{\mathbf{u}(t)}{dist(x,\partial\Omega)}\right\|_{\Psi_2, dist(x,\partial\Omega)\leq\frac{1}{m}} dt$$
$$\leq c\left(\frac{1}{m}\right)\int_0^T \|D\mathbf{u}(t)\|_{M, dist(x,\partial\Omega)\leq\frac{1}{m}}\ dt,$$

where $c\left(\frac{1}{m}\right) \to 0$ for $m \to \infty$. $\square$

Lemma 6.2 *Let $\rho \in L^\infty(0,T;L_{\Phi_\beta}(\Omega))$, $\beta > 3$, and $\mathbf{u} \in Y$ be a weak solution to (1.1) in $\mathcal{D}'(\Omega \times (0,T))$. Then prolonging ρ and $\mathbf{u}$ by zero outside Ω and taking $\rho_h = \rho * \vartheta_h$, where $*$ means the convolution, with ϑ_h defined by (6.2), we have*

$$(\rho_h)_t + \mathrm{div}\,(\rho_h \mathbf{u}) = r_h \ \textit{a.e. on}\ R^N \times (0,T) \tag{6.3}$$

with $r_h \to 0$ in $L_{\Phi_{\beta-2}}(Q_T)$. Finally, the function ρ is a renormalized solution to (1.1), i.e.,

$$(b(\rho))_t + \mathrm{div}\,(b(\rho)\mathbf{u}) + (\rho b'(\rho) - b(\rho))\mathrm{div}\,\mathbf{u} = 0 \ \textit{in}\ \mathcal{D}'(\Omega \times (0,T)), \tag{6.4}$$

for any continuously differentiable function b such that b and b' are uniformly bounded.

P r o o f: We can verify by the same method as in [7] that after taking $\varphi(x,t) = \psi(t)\vartheta_h(x-y)$ as a test function of the equation (1.1), we derive

$$(\rho_h)_t + \mathrm{div}\,(\rho_h \mathbf{u}) = r_h \ \text{a.e. on}\ R^N \times (0,T), \tag{6.5}$$

with

$$r_h := \mathrm{div}\,(\rho_h \mathbf{u}) - \nabla\vartheta_h * (\rho\mathbf{u})$$

or, equivalently,

$$r_h = \rho_h \mathrm{div}\,\mathbf{u} + \int_{R^N} (\mathbf{u}(x) - \mathbf{u}(y)) \cdot \nabla\vartheta_h(x-y)\rho(y)\ dy.$$

So,

$$\left| \int_\Omega w(x)\rho_h(x)\mathrm{div}\,\mathbf{u}(x)\ dx \right| \le \|D\mathbf{u}(t)\|_M \|\rho_h(t) w\|_{\Phi_1}.$$

Let $w_1 \in \widetilde{L}_{\Psi_{\beta-2}}(\Omega)$ and $\int_\Omega \Psi_{\beta-2}(|w_1|)\ dx \le 1$. Using Lemma 4.5 and the prolongation of the density ρ by zero, we get

$$\left| \int_\Omega w_1(x) \int_{R^N} \frac{|\mathbf{u}(x) - \mathbf{u}(x-z)|}{|z|} |\nabla\vartheta_h(z)||z|\rho(x-z) dz dx \right|$$
$$\le c\|D\mathbf{u}(t)\|_M |||\rho|||_{L^\infty(0,T;L_{\Phi_\beta}(\Omega))}. \tag{6.6}$$

Thus $r_h \in L_{\Phi_{\beta-2}}(Q_T)$ if we replace the function $w_1(x)$ with $w_1(x,t)$.

The task is now to verify that

$$\|\mathrm{div}\,(\rho_h\mathbf{u}) - (\rho\mathbf{u}) * \nabla\vartheta_h\|_{\Phi_\gamma} \to 0 \text{ for } h \to 0+$$

with $\gamma \in [1, \beta-2]$. But the proof of the above mentioned limit proceeds almost in the same way as in [12, p. 43].

From (1.1) it may be concluded that

$$\int_\Omega \rho(x,t)\phi(x)\ dx$$

is continuous for any $\phi \in C_0^\infty(\Omega)$ which implies $\rho \in C(0,T; L^{weak}_{\Phi_\beta}(\Omega))$.

Multiplying the equation (6.3) by $b'(\rho_{h_i})$, and subtracting (6.3) with $i = 1$ from (6.3) with $i = 2$, we can derive

$$\int_\Omega b(\rho_h(x,t))\phi(x)\ dx \to \int_\Omega b(\rho(x,t))\phi(x)\ dx$$

uniformly in $C(0,T)$. $\square$

Consequence 6.3 *Let $\zeta \in W^1 L_M(\Omega)$. Then by the weak version of the renormalized continuity equation we have*

$$\int_\Omega (R_\epsilon(\theta(\rho(s))))_t \zeta\ dx = \frac{1}{\epsilon}\int_\Omega \int_0^\infty \phi_0'\left(\frac{s-\tau}{\epsilon}\right)\theta(\rho(\tau))\zeta\ d\tau dx$$

$$= \int_\Omega (R_\epsilon(\theta(\rho)\mathbf{u}))\cdot\nabla\zeta\ dx - \int_\Omega R_\epsilon((\rho\theta'(\rho)-\theta(\rho))\mathrm{div}\,\mathbf{u})\zeta\ dx$$

$$+\frac{1}{\epsilon}\phi_0\left(\frac{s}{\epsilon}\right)\int_\Omega \rho_0\zeta\ dx, \quad s > 0,$$

and hence, in the sense of distributions,

$$(R_\epsilon(\theta(\rho)))_t = -\mathrm{div}\,(R_\epsilon(\theta(\rho)\mathbf{u})) + R_\epsilon((\theta(\rho)-\rho\theta'(\rho))\mathrm{div}\,\mathbf{u}) + \frac{1}{\epsilon}\phi_0\left(\frac{s}{\epsilon}\right)\rho_0. \qquad (6.7)$$

7 On the long-time regularity of functions ρ and $D\mathbf{u}$

Lemma 7.1 *Let ρ and $\mathbf{u}$ be a solution of the problem (1.1)–(1.5). Then $\rho \in L^\infty(0,\infty;\widetilde{L}_{\Phi_1}(\Omega))$ and*

$$\int_0^\infty \int_\Omega M(|D\mathbf{u}(x,s)|)\ dxds < \infty.$$

In particular,

$$\lim_{t\to\infty}\int_{t-a}^{t+a}\int_\Omega M(|D\mathbf{u}(x,s)|)\ dxds = 0 \text{ for any } a > 0. \qquad (7.1)$$

P r o o f: Taking the function g ($\nabla g = \mathbf{f}$ from the assumption 1) as a test function in (1.1), we can rewrite the energy identity as

$$\int_\Omega \left(\rho\frac{|\mathbf{u}|^2}{2} + \rho\ln\rho - \rho g\right)\ dx\Big|_0^t + \int_0^t\int_\Omega P(\mathbf{u}) : D\mathbf{u}\ dxds = 0. \qquad (7.2)$$

Using (3.1) finishes the proof. $\square$

Lemma 7.2 *Let (7.1) hold. Then*

$$\lim_{t\to\infty}\int_{t-a}^{t+a}\|P(\mathbf{u})(s)\|_{\overline{M}}\,ds=0, \tag{7.3}$$

$$\lim_{t\to\infty}\int_{t-a}^{t+a}\|D\mathbf{u}(s)\|_{\Psi_2}\,ds=0, \tag{7.4}$$

and

$$\lim_{t\to\infty}\int_{t-a}^{t+a}\|D\mathbf{u}(s)\|_{\Psi_2}^2\,ds=0. \tag{7.5}$$

P r o o f: We start with the observation that the estimate

$$2^m\int_{t-a}^{t+a}\|P(\mathbf{u})(s)\|_{\overline{M}}\,ds\le c\left(k^m\int_{t-a}^{t+a}\int_\Omega M(|D\mathbf{u}(s)|)\,dxds+1\right)\le c$$

holds (see (3.3)) on the condition that

$$\int_{t-a}^{t+a}\int_\Omega M(|D\mathbf{u}(s)|)\,dxds\le\frac{1}{k^m},\ m\in N_0.$$

To prove (7.4) and (7.5), we use Lemma 4.1 and the same idea as in case (7.3). □

Roughly speaking, the above-mentioned lemmas provide information about the behaviour and global properties of the solution of the problem (1.1)–(1.5).

8 Global uniform estimates

Let us start with the important proposition which ensures the existence of a bounded function θ with appropriate properties.

Proposition 8.1 [17] *There exist a positive constant c_0 and a bounded increasing continuously differentiable function θ on R with $\lim_{r\to\infty} r\theta'(r)=0$ and $\theta'(r)>0$ such that*

$$(r_1-r_2)(\theta(r_1)-\theta(r_2))\ge c_0(\theta(r_1)-\theta(r_2))^2. \tag{8.1}$$

For the reader's convenience, we mention here the modification of the construction of the function $\overline{\rho}_{\epsilon k}$ from [17], but we omit the detailed derivation. For each $s>0$ we can define $w_{\epsilon k}(s)$ as a unique generalized solution to the Neumann problem

$$\int_\Omega\nabla w_{\epsilon k}(s)\cdot\nabla\eta\,dx=\int_\Omega R_\epsilon(T_k(\rho)(x,\cdot))(s)\mathbf{f}\cdot\nabla\eta\,dx,\ \forall\eta\in W^{1,p}(\Omega),$$

$$\int_\Omega w_{\epsilon k}(x,s)\,dx=0, \tag{8.2}$$

where $s \geq 0$. The cut-off function T_k was defined by Definition 4.7, and the solution to this problem satisfies the estimate

$$\|w_{\epsilon k}(s)\|_{1,p'} \leq c\|R_\epsilon(T_k(\rho))\|_{L^\infty(0,\infty;L^{p'}(\Omega))} \leq ck < +\infty, \; p' > 1, \tag{8.3}$$

for $k = 1, 2, \ldots$ arbitrary but fixed.

Now, let us introduce

$$G_{\epsilon k}(s, m) := \int_\Omega \theta(w_{\epsilon k}(s) + m) \; dx, \; s > 0, \; m \in R, \; \epsilon > 0, \; k = 1, 2, \ldots. \tag{8.4}$$

There is no problem to verify that the integral $\int_\Omega R_\epsilon(\theta(\rho(x, \cdot)))(s) \; dx$ lies in the range of $G_{\epsilon k}(s, \cdot)$, and for any fixed $s \geq 0$, $\epsilon > 0$ and $k = 1, 2, \ldots$ the equation

$$G_\epsilon(s, m) = \int_\Omega R_\epsilon(\theta(\rho(x, \cdot)))(s) \; dx \tag{8.5}$$

has a unique solution $m = m_{\epsilon k}(s)$. Now, let us define

$$\overline{\rho}_{\epsilon k}(x, s) := w_{\epsilon k}(x, s) + m_{\epsilon k}(s). \tag{8.6}$$

Then, from (8.5), (8.6), we have

$$\int_\Omega \theta(\overline{\rho}_{\epsilon k}(x, s)) \; dx = \int_\Omega R_\epsilon(\theta(\rho(x, \cdot)))(s) \; dx \tag{8.7}$$

for $s > 0$, $\epsilon > 0$, $k = 1, 2, \ldots$. Finally, we define another auxiliary function $\psi_{\epsilon k}(s)$ as a solution to

$$\begin{aligned} \operatorname{div} \psi_{\epsilon k}(s) &= R_\epsilon(\theta(\rho))(s) - \theta(\overline{\rho}_{\epsilon k}(s)) \quad \text{in } \Omega, \\ \psi_{\epsilon k}(x, s) &= 0, \; x \in \partial\Omega, \; s > 0, \; \epsilon > 0, \; k = 1, 2, \ldots. \end{aligned} \tag{8.8}$$

This problem is not uniquely solvable, but it is known that one possible solution is given by

$$\begin{aligned} \psi_{\epsilon k}(x, s) &= S(R_\epsilon(\theta(\rho))(s) - \theta(\overline{\rho}_{\epsilon k}(s))) \\ &= \int_\Omega K(x, y)(R_\epsilon(\theta(\rho))(s) - \theta(\overline{\rho}_{\epsilon k}(s))) \; dy, \end{aligned} \tag{8.9}$$

where K is explicitly defined by a weakly singular kernel. It is known that the operator S maps from $L^\infty(\Omega)$ into $W^1 L_M(\Omega))$, according to Theorem 5.2. With our particular choice of θ we have

$$\|\psi_{\epsilon k}\|_{L^\infty(0,\infty;W^1 L_M(\Omega))} \leq C < \infty, \tag{8.10}$$

with C independent of ϵ, s, k and x. For the proof of uniform boundedness of $m_{\epsilon k}$, i.e.,

$$|m_{\epsilon k}(s)| \leq C < \infty, \quad \epsilon > 0, \; k = 1, 2, \ldots, \; s > 0, \tag{8.11}$$

we refer the reader to [17].

Lemma 8.2 *Under the assumptions above, there exists the limit*

$$\lim_{\epsilon \to 0+} w_{\epsilon k} = w_k \text{ in } L^r_{loc}([0,\infty); W^{1,p'}(\Omega)) . \tag{8.12}$$

The set $\{m_{\epsilon k}(s)\}_{\epsilon \in (0,1)}$ *is bounded in* $W^{1,2}_{loc}(0,T)$, *and passing to a subsequence there exists a limit*

$$\lim_{n \to \infty} m_{\epsilon_n k} = m_k \text{ in } L^\infty_{loc}(0,\infty) \tag{8.13}$$

with some $\epsilon_n \to 0+$ *and* $r \in [1,\infty),\ p' \in [1,\infty),\ k = 1,2,\ldots$. *In particular,*

$$\overline{\rho}_{\epsilon_n k} \to w_k + m_k =: \overline{\rho}_k \tag{8.14}$$

in the above sense for $k = 1,2,\ldots$.

P r o o f: The first part of the proof concerning the convergence of $w_{\epsilon k}$ is obvious. In the same way as in [17], we can verify that

$$c_0 := \inf_{\epsilon, s} \int_\Omega \theta'(w_{\epsilon k}(s) + m_\epsilon(s))\ dx > 0.$$

Consequently, by the Implicit Function Theorem there exists the derivative $m'_{\epsilon k}(s)$, and we have

$$m'_{\epsilon k}(s) = \left(\int_\Omega \theta'(w_{\epsilon k}(s) + m_{\epsilon k}(s))\ dx \right)^{-1} \times$$
$$\left(\int_\Omega (R_\epsilon \theta(\rho)(s))_t\ dx - \int_\Omega \theta'(w_{\epsilon k}(x,s) + m_{\epsilon k}(s))(w_{\epsilon k})_t(x,s)\ dx \right),$$

which yields the estimate

$$|m'_{\epsilon k}(s)| \le c \left(\|(w_{\epsilon k})_t(s)\|_1 + \|D\mathbf{u}(s)\|_{\Psi_2} + \frac{1}{\epsilon}\phi_0\left(\frac{s}{\epsilon}\right) \right) . \tag{8.15}$$

Since functions $\eta \in C^\infty_0(\Omega)$ such that $\eta - \frac{1}{|\Omega|}\int_\Omega \eta\ dy = \triangle \xi$, with $\xi \in C^2(\overline{\Omega})$, where $\frac{\partial \xi}{\partial \nu} = 0$, are dense in $L^r(\Omega)$, $r \in (1,\infty)$, we obtain

$$\int_\Omega (w_{\epsilon k})_t \eta\ dx = \int_\Omega (w_{\epsilon k})_t \left(\eta - \frac{1}{|\Omega|}\int_\Omega \eta\ dy \right)\ dx = -\int_\Omega (R_\epsilon(T_k(\rho)))_t \mathbf{f} \cdot \nabla \xi\ dx$$
$$\le \|\mathbf{f}\|_{1,\infty} \|\nabla \xi\|_{1,p'} \|(R_\epsilon(T_k(\rho)))_t\|_{-1,p}$$
$$\le c\left(\|R_\epsilon(T_k(\rho)\mathbf{u})\|_p + \|R_\epsilon((\rho T'_k(\rho) - T_k(\rho))\operatorname{div}\mathbf{u})\|_p\right)\|\eta\|_{p'}, \tag{8.16}$$

where the last part of the above estimate is a consequence of Consequence 6.3 and allows us to see that the solution of Neumann problem (8.2) is differentiable and $\partial_t w_{\epsilon k} \in L^p(\Omega)$ for all $p > 1$, $\epsilon > 0$ and $t > 0$. Now, we are ready to prove boundedness of the set $\{m_{\epsilon k}(s)\}_{\epsilon \in (0,1)}$ in $W^{1,2}(\tau, T)$, with $\tau > 0$ arbitrary but fixed.

Combining (8.15) with (8.16), we conclude

$$\int_\tau^T |m'_{\epsilon k}(s)|^2\ ds \le c\Bigg(\int_\tau^T \|D\mathbf{u}(s)\|^2_{\Psi_2}\ ds + \int_\tau^T \frac{1}{\epsilon^2}\phi_0^2\left(\frac{s}{\epsilon}\right)\ ds$$

$$+\int_\tau^T \|(w_{\epsilon k})_t(s)\|_1^2\, ds\Bigg) \le kc_4(T)+c_2(\tau),\ \tau>0. \qquad \square$$

Put now

$$Q_k(t) := \int_{t-1}^t \int_\Omega (\rho(s)-\overline{\rho}_k(s))(\theta(\rho(s))-\theta(\overline{\rho}_k(s)))\, dxds, \quad t\ge 1. \tag{8.17}$$

By the monotonicity of $\rho(\cdot)$ and θ, we have $Q_k(t)\ge 0$. Our intention is to prove the following global property of $Q_k(t)$.

Lemma 8.3 *Let $\overline{\rho}_k$ be a function established in (8.14). Then the limit state of the function $Q_k(t)$ defined by (8.17) fulfils the estimate*

$$0 \le \lim_{t\to\infty} Q_k(t) \le \delta_1(k), \tag{8.18}$$

with a function $\delta_1(k)$ satisfying $\delta_1(k)\to 0$ for $k\to\infty$.

P r o o f: Let $a>1,\ \varphi\in C_0^\infty(-a,a),\ \varphi\ge 0,\ \varphi(\sigma)=1$ for $\sigma\in(-1,0)$. Put

$$Q_{k,a}^\epsilon(t) := \int_{t-a}^{t+a} \varphi(s-t)\int_\Omega (\rho(s)-\overline{\rho}_{\epsilon k}(s))(R_\epsilon(\theta(\rho(s)))-\theta(\overline{\rho}_{\epsilon k}(s)))\, dxds. \tag{8.19}$$

Then clearly

$$Q_{k,a}^\epsilon(t) = \int_{t-a}^{t+a} \varphi(s-t)\int_\Omega (\rho(s)-\overline{\rho}_{\epsilon k}(s))(\theta(\rho(s))-\theta(\overline{\rho}_{\epsilon k}(s)))\, dxds$$

$$+\int_{t-a}^{t+a} \varphi(s-t)\int_\Omega (\rho(s)-\overline{\rho}_{\epsilon k}(s))(R_\epsilon(\theta(\rho(s)))-\theta(\rho(s)))\, dxds, \tag{8.20}$$

where the last term on the right-hand side of (8.20) tends to zero as $\epsilon\to 0+$. By Lemma 8.2,

$$\lim_{n\to\infty}\int_{t-1}^t \varphi(s-t)\int_\Omega (\rho(s)-\overline{\rho}_{\epsilon_n k}(s))(\theta(\rho(s))-\theta(\overline{\rho}_{\epsilon_n k}(s)))\, dxds = Q_k(t) \tag{8.21}$$

for $t>1$ and for some $\epsilon_n\downarrow 0$. Now we wish to estimate $Q_{k,a}^\epsilon(t)$. Let us denote $V_a(t):=\{(x,s);\ x\in\Omega,\ t-a<s<t+a\}$. Using (8.8) and (1.2), we can write

$$\int_{V_a(t)} \varphi(s-t)\rho(s)(R_\epsilon(\theta(\rho(s)))-\theta(\overline{\rho}_{\epsilon k}(s)))\, dxds$$

$$= \int_{V_a(t)} \varphi(s-t)\Big((-\rho\mathbf{u}(\psi_{\epsilon k}(s))_t - \rho\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\psi_{\epsilon k}(s) + P(\mathbf{u}) : D\psi_{\epsilon k}(s)$$

$$-\rho\mathbf{f}\cdot\psi_{\epsilon k}(s)\Big)\, dxds - \int_{V_a(t)} \varphi'(s-t)\rho\mathbf{u}\psi_{\epsilon k}(s)\, dxds. \tag{8.22}$$

Now, take the Helmholtz-Weyl decomposition of $\psi_{\epsilon k}(s)$, that is,

$$\psi_{\epsilon k}(s) = \nabla z_{\epsilon k}(s) + \mathbf{v}_{\epsilon k}(s), \quad \operatorname{div}\mathbf{v}_{\epsilon k}(s)=0 \text{ in } \Omega, \quad \mathbf{v}_{\epsilon k}(s)\cdot\nu = 0 \text{ in } \partial\Omega.$$

By the usual construction of the decomposition, and by (8.10), we have $\int_\Omega z_{\epsilon k}\ dx = 0$, $\mathbf{v}_{\epsilon k} \in W^{1,r}(\Omega)$, $z_{\epsilon k} \in W^{2,r}(\Omega)$, $\frac{\partial z_{\epsilon k}}{\partial \nu}|_{\partial\Omega} = 0$, $r \in (1,\infty)$. Taking into account the generalized formulation (8.2), we find that

$$\int_{V_a(t)} \varphi(s-t)\overline{\rho}_{\epsilon k}(s)(R_\epsilon(\theta(\rho(s))) - \theta(\overline{\rho}_{\epsilon k}(s)))\ dxds$$

$$= -\int_{V_a(t)} \varphi(s-t)\nabla w_{\epsilon k}(s)\cdot(\nabla z_{\epsilon k}(s) + \mathbf{v}_{\epsilon k}(s))\ dxds$$

$$= -\int_{V_a(t)} \varphi(s-t)\nabla w_{\epsilon k}(s)\cdot\nabla z_{\epsilon k}(s)\ dxds$$

$$= -\int_{V_a(t)} \varphi(s-t)R_\epsilon(T_k(\rho(s)))\mathbf{f}\cdot\nabla z_{\epsilon k}(s)\ dxds. \tag{8.23}$$

Subtracting (8.23) from (8.22), we obtain that

$$\int_{V_a(t)} \varphi(s-t)(\rho(s) - \overline{\rho}_{\epsilon k}(s))(R_\epsilon(\theta(\rho(s))) - \theta(\overline{\rho}_{\epsilon k}(s)))\ dxds$$

$$= \int_{V_a(t)} \varphi(s-t)(P(\mathbf{u}) : D\psi_{\epsilon k}(s) - \rho\mathbf{u}\cdot(\mathbf{u}\cdot\nabla)\psi_{\epsilon k}(s)\ dxds$$

$$- \int_{V_a(t)} \varphi'(s-t)\rho\mathbf{u}\psi_{\epsilon k}(s)\ dxds$$

$$- \int_{V_a(t)} \varphi(s-t)(\rho\mathbf{u}(\psi_{\epsilon k})_t(s) - (\rho - R_\epsilon(T_k(\rho(s))))\mathbf{f}\cdot\nabla z_{\epsilon k}(s))\ dxds$$

$$- \int_{V_a(t)} \varphi(s-t)\rho\mathbf{f}\cdot\mathbf{v}_{\epsilon k}(s)\ dxds =: \sum_{j=1}^{6} I^\epsilon_{jk}(t).$$

Define

$$\sigma_a(t) := \max\left\{ \int_{t-a}^{t+a} \|D\mathbf{u}(s)\|_{\Psi_2}\ ds,\ \int_{t-a}^{t+a} \|D\mathbf{u}(s)\|^2_{\Psi_2}\ ds,\right.$$
$$\left. \int_{t-a}^{t+a} \|P(\mathbf{u})(s)\|_{\overline{M}}\ ds\right\}. \tag{8.24}$$

We estimate the integrals $I^\epsilon_{jk}(t)$ individually. Thus,

$$|I^\epsilon_{1k}(t)| \le \|\varphi\|_\infty \int_{t-a}^{t+a} \|P(\mathbf{u})(s)\|_{\overline{M}}\|D\psi_{\epsilon k}(s)\|_M\ ds \le c\sigma_a(t). \tag{8.25}$$

$$|I^\epsilon_{2k}(t)| \le \|\varphi\|_\infty \int_{V_a(t)} \rho(s)|\mathbf{u}(s)|^2|\nabla\psi_{\epsilon k}(s)|\ dxds$$

$$\le c\int_{t-a}^{t+a} \|\rho(s)\|_{\Phi_1}\|\nabla\psi_{\epsilon k}(s)\|_M\|D\mathbf{u}(s)\|^2_{\Psi_2}\ ds$$

$$\le c\|\rho\|_{L^\infty(0,\infty;L_{\Phi_1}(\Omega))}\|\nabla\psi_{\epsilon k}\|_{L^\infty(0,\infty;L_M(\Omega))}\sigma_a(t). \tag{8.26}$$

$$|I^\epsilon_{3k}(t)| \le \|\varphi'\|_\infty \int_{t-a}^{t+a} \|\mathbf{u}(s)\|_\infty \int_\Omega \rho(s)|\psi_{\epsilon k}(s)|\ dxds \le c\sigma_a(t). \tag{8.27}$$

Using the properties of the kernel K from (8.9), we can show in the same manner as in [17] that $\partial_t \psi_{\epsilon k}(x,t)$ exists in the sense of $W^{1,p}(\Omega)$, with $p \in [1,\infty)$ arbitrary.

We proceed to show an appropriate estimate of the function $(\psi_{\epsilon k})_t$. Consequence 6.3 and (8.9) enable us to represent the function $(\psi_{\epsilon k})_t$ in the form

$$(\psi_{\epsilon k})_t = S \operatorname{div} \mathbf{z} + Sq - S\theta(\overline{\rho}_{\epsilon k})_t, \tag{8.28}$$

where

$$\mathbf{z} = -R_\epsilon(\theta(\rho)\mathbf{u}), \ q = R_\epsilon((\theta(\rho) - \rho\theta'(\rho))\operatorname{div} \mathbf{u}) + \frac{1}{\epsilon}\phi\left(\frac{s}{\epsilon}\right)\rho_0. \tag{8.29}$$

It can be easily checked that $\mathbf{z}$ belongs to $\{\mathbf{w} \in C^\infty(0,\infty; C(\overline{\Omega})),\ \mathbf{w}\cdot\nu|_{\partial\Omega} = 0,$ $\operatorname{div}\mathbf{w} \in C^\infty(0,\infty; L_{\Psi_1}(\Omega))\}$. Thus, we have

$$\|S \operatorname{div} \mathbf{z}\|_M + \|Sq\|_M \le c(\|\mathbf{z}\|_\infty + \|q\|_p),$$

with $p > N$, and the problem with regularity of ρ_0 can be overcome by regularization of ρ_0, since it disappears for ϵ sufficiently small, and so

$$\|S \operatorname{div} \mathbf{z}\|_M + \|Sq\|_M \le c\left(\|D\mathbf{u}(s)\|_{\Psi_2} + \frac{1}{\epsilon}\phi_0\left(\frac{s}{\epsilon}\right)\right). \tag{8.30}$$

Let us remind the fact that the operator S is well-defined according to Consequence 5.3. Now, the Young theorem for convolutions yields the estimate

$$\|S(\theta(\overline{\rho}_{\epsilon k})_t(s))\|_\infty \le c\|\theta(\overline{\rho}_{\epsilon k})_t(s)\|_p$$

for $p > N$. The inequality

$$\|\theta(\overline{\rho}_{\epsilon k})_t(s)\|_p \le c(\|(w_{\epsilon k})_t(s)\|_p + |m'_{\epsilon k}(s)|)$$

is easy to verify using the definition of $\overline{\rho}_{\epsilon k}$. Hence, we get

$$|I^\epsilon_{4k}(t)| \le c\|\rho\|_{L^\infty(0,\infty;L_{\Phi_1}(\Omega))}\int_{t-a}^{t+a} \|\mathbf{u}(s)\|_\infty \|(\psi_{\epsilon k}(s))_t\|_M \, ds \le ck\sigma_a(t). \tag{8.31}$$

On account of Lemma 4.8, we have

$$|I^\epsilon_{5k}(t)| \le c\|\varphi\|_\infty\|\mathbf{f}\|_\infty \sup_{s,\epsilon,k}\|\psi_{\epsilon k}(s)\|_\infty \int_{t-a}^{t+a}\|R_\epsilon(T_k(\rho))(s) - \rho(s)\|_1 \, ds$$

$$=: \delta^1_\epsilon(k), \tag{8.32}$$

where $\delta^1_\epsilon(k) \to 0$ for $k \to \infty$. Moreover $\lim_{\epsilon\to 0+}|I^\epsilon_{5k}(t)| \le \delta^1(k)$, where $\delta^1(k) \to 0$ for $k \to \infty$.

It remains to estimate the integral

$$|I^\epsilon_{6k}(t)| = \left|\int_{V_a(t)} \varphi(s-t)\rho(s)\mathbf{f}\cdot\mathbf{v}_{\epsilon k} \, dxds\right|. \tag{8.33}$$

But we do not know whether $D\mathbf{v}_{\epsilon k}(s) \in L_M(\Omega)$ for a.a. $s \geq 0$ and $\epsilon \in (0,1)$, which is necessary for using the weak formulation of the equation (1.2). This is the reason why we must approximate the function $\psi_{\epsilon k}$ with the help of Lemma 4.6. Define the integral

$$I^{\epsilon}_{6kh}(t) := \int_{V_a(t)} \varphi(s-t)\rho(s)\mathbf{f}\cdot\mathbf{v}_{\epsilon kh}\, dxds,$$

where $\mathbf{v}_{\epsilon kh} \in C^1(\overline{\Omega})$ is the function from the Helmholtz-Weyl decomposition of $\psi_{\epsilon kh}$. The functions $\psi_{\epsilon kh}$ belong to $C_0^\infty(\Omega)$ and $\|\psi_{\epsilon kh} - \psi_{\epsilon k}\|_{L^\infty(0,\infty;L^\infty(\Omega))} \to 0$ for $h \to 0+$. Moreover, $\|\psi_{\epsilon kh}\|_{L^\infty(0,\infty;W_0^{1,p}(\Omega))} \leq C$, where $p \in [1,\infty)$ is arbitrary, and the constant C is independent of k, ϵ and h (see Lemma 4.6 for details).

The estimates

$$\|\nabla z_{\epsilon kh}(s) - \nabla z_{\epsilon k}(s)\|_\infty \leq \|\nabla\psi_{\epsilon kh}(s) - \nabla\psi_{\epsilon k}(s)\|_p \leq C$$

for $p > N$ and

$$\|\nabla z_{\epsilon kh}(s) - \nabla z_{\epsilon k}(s)\|_2 \leq C\|\psi_{\epsilon kh}(s) - \psi_{\epsilon k}(s)\|_M$$

follow from the Helmholtz-Weyl decomposition of $\psi_{\epsilon kh}(s)$.

By the boundedness of the gradient of $\psi_{\epsilon k}(s)$ and Lemma 4.2, we get that the sequences $|\mathbf{v}_{\epsilon k}(s) - \mathbf{v}_{\epsilon kh}(s)|$ and $|\nabla z_{\epsilon k}(s) - \nabla z_{\epsilon kh}(s)|$ converge to zero in $L^\infty(0,\infty;L_M(\Omega))$ for $h \to 0+$, and the above yields the estimate

$$|I^{\epsilon}_{6k}(t) - I^{\epsilon}_{6kh}(t)| \leq \delta_2(h),$$

with $\delta_2(h) \to 0$ for $h \to 0+$ and δ_2 independent of t and ϵ.

We can estimate $\mathbf{v}_{\epsilon kh}$ in this way,

$$\|\mathbf{v}_{\epsilon kh}\|_{L^\infty(0,\infty;W^{1,\infty}(\Omega))} \leq \Delta(h), \tag{8.34}$$

where $\Delta(h) \to \infty$ for $h \to 0$.

Let $\eta > 0$ and $\kappa \in C_0^\infty(\Omega_\eta)$ be such that $|supp(1-\kappa)| \leq \eta$. The existence of such a function κ was shown in Proposition 4.9. Then,

$$\begin{aligned}\int_{V_a(t)} \varphi(s-t)\rho(s)\mathbf{f}\cdot\mathbf{v}_{\epsilon kh}\, dxds &= \int_{V_a(t)} \varphi(s-t)\rho(s)\mathbf{f}\cdot\mathbf{v}_{\epsilon kh}\kappa\, dxds \\ &+ \int_{V_a(t)} \varphi(s-t)\rho(s)\mathbf{f}\cdot\mathbf{v}_{\epsilon kh}(1-\kappa)\, dxds =: J_1^h + J_2^h,\end{aligned} \tag{8.35}$$

and clearly

$$|J_2^h| \leq ca\|\rho\|_{L^\infty(0,\infty;L_{\Phi_1}(\Omega))}\|\mathbf{f}\|_\infty\|\mathbf{v}_{\epsilon kh}\|_{L^\infty(0,\infty;L_M(\Omega))}|supp(1-\kappa)| \leq c\eta. \tag{8.36}$$

Since $(\rho,\, \mathbf{u})$ is a solution of (1.1)–(1.5), we can rewrite J_1^h in the form

$$J_1^h = \int_{t-a}^{t+a} \varphi(s-t)\int_\Omega \Big(\kappa P(\mathbf{u}) : D\mathbf{v}_{\epsilon kh} - \kappa\rho\mathbf{u}\cdot((\mathbf{u}\cdot\nabla)\mathbf{v}_{\epsilon kh})$$

$$-\rho\mathbf{u}\kappa(\mathbf{v}_{\epsilon kh})_t\Big) \, dxds - \int_{t-a}^{t+a} \varphi'(s-t) \int_\Omega \kappa\rho\mathbf{u}\cdot\mathbf{v}_{\epsilon kh} \, dxds$$

$$+ \int_{t-a}^{t+a} \varphi(s-t) \int_\Omega \Big(P(\mathbf{u}) : Sym(\nabla\kappa \otimes \mathbf{v}_{\epsilon kh})$$

$$-\rho(\mathbf{u}\cdot\nabla\kappa)(\mathbf{u}\cdot\mathbf{v}_{\epsilon kh}) - \rho\nabla\kappa\cdot\mathbf{v}_{\epsilon kh}\Big) \, dxds =: \sum_{i=1}^{7} \overline{J}_i^h. \tag{8.37}$$

Here, Sym means the symmetric part of the tensor.

The integrals $\overline{J}_1^h, \ldots, \overline{J}_4^h$ are estimated by the term $c\Delta(h)\sigma_a(t)$, and the integrals $\overline{J}_5^h$ and $\overline{J}_6^h$ by $\frac{c}{\eta}\sigma_a(t)$ in the same manner as above.

Given $x \in \Omega_{\eta/2} \setminus \Omega_\eta$, issue from x the ray which is a normal to $\partial\Omega_\eta$ at x_1 and to $\partial\Omega$ at x_2. Then $|x - x_2| \le \eta$. Further, since $\mathbf{v}_{\epsilon kh}(x_2)\cdot\nu(x_2) = 0$, $\nu(x_2) = \nu(x_1)$ and $\nabla\kappa(x_1) \perp \mathbf{v}_{\epsilon kh}(x_2)$, by Proposition 4.9, we have $\nabla\kappa(x)\cdot\mathbf{v}_{\epsilon kh}(x_2) = 0$. Indeed, we might construct Ω_α with $\alpha = |x - x_2|$ and use the same argument as in Proposition 4.9 for Ω_η to show $\nabla\kappa(x)\cdot\tau(x) = 0$ for any vector τ tangential to $\partial\Omega_\alpha$ at x. Consequently, we find that

$$|\nabla\kappa(x)\cdot\mathbf{v}_{\epsilon kh}(x)| = |\nabla\kappa(x)\cdot(\mathbf{v}_{\epsilon kh}(x) - \mathbf{v}_{\epsilon kh}(x_2))|$$

$$\le \frac{c}{\eta}\eta\|\mathbf{v}_{\epsilon kh}\|_{L^\infty(0,\infty;W^{1,\infty}(\Omega))} \le c\Delta(h). \tag{8.38}$$

Having disposed the preliminary steps, we can return to the estimate of the integral J_1^h and J_2^h, namely,

$$|J_1^h + J_2^h| \le c\eta + c\Delta(h)\sigma_a(t) + c\Delta(h)w(\eta) + \frac{c}{\eta}\sigma_a(t), \tag{8.39}$$

where $w(\eta) \to 0$ for $\eta \to 0+$, using the estimate

$$\left| \int_{V_a(t)} \rho\nabla\kappa\cdot\mathbf{v}_{\epsilon kh} \, dxds \right| \le c\Delta(h)\|\rho\|_{L^\infty(0,\infty;L_{\Phi_1}(\Omega))}\|\chi_{\Omega_{\eta/2}\setminus\Omega_\eta}\|_M. \tag{8.40}$$

We are now in a position to verify the estimate

$$\lim_{t\to\infty}\lim_{\epsilon\to 0}|I_{6kh}^\epsilon(t)| \le \delta_2(h) + w(\eta)\Delta(h),$$

with $\delta_2(h) \to 0$ for $h \to 0$ and $w(\eta) \to 0$ for $\eta \to 0$.

We have proved that

$$0 \le Q_k(t) \le \lim_{n\to\infty} Q_{ak}^{\epsilon_n}(t)$$

$$\le c\sigma_a(t)\left(1 + \Delta(h) + \frac{1}{\eta}\right) + w(\eta)\Delta(h) + \delta_1(k) + \delta_2(h),$$

and thus (8.18) holds. □

9 Convergence of density

In this section, we examine the relation between the density ρ and the function $\overline{\rho}_k$ more closely.

Proposition 9.1 [17] *Let $q \in W^{1,1}_{loc}(a,\infty)$, $a \in R$, be such that $q(s) \geq 0$ for $s \geq a$ and $\lim_{t\to\infty} \int_{t-1}^{t} (q(s) + |q'(s)|)\ ds \leq \delta_1(k)$. Then*

$$\lim_{t\to\infty} q(t) \leq \delta_1(k). \tag{9.1}$$

Put

$$q_k(t) := \|\theta(\rho(t)) - \theta(\overline{\rho}_k(t))\|_2^2, \quad t > 1. \tag{9.2}$$

Then, by Proposition 8.1,

$$\int_{t-1}^{t} q_k(s)\ ds \leq Q_k(t), \tag{9.3}$$

and hence

$$\lim_{t\to\infty} \int_{t-1}^{t} q_k(s)\ ds \leq \delta_1(k). \tag{9.4}$$

It remains to verify that

$$\lim_{t\to\infty} \int_{t-1}^{t} |q_k'(s)|\ ds = 0,$$

which is a consequence of the following lemma.

Lemma 9.2 *The inequality*

$$\sqrt{\int_{t-1}^{t} \left|\frac{d}{ds}\|\theta(\rho(s)) - \theta(\overline{\rho}_k(s))\|_2^2\right|\ ds} \leq c\sqrt{\sigma(t)}, \tag{9.5}$$

is satisfied for the function $\sigma(t)$ defined by

$$\sigma(t) := \max\left\{ \int_{t-1}^{t} \|D\mathbf{u}(s)\|_{\Psi_2}\ ds,\ \int_{t-1}^{t} \|D\mathbf{u}(s)\|_{\Psi_2}^2\ ds, \right.$$
$$\left. \int_{t-1}^{t} \|P(\mathbf{u})(s)\|_{\overline{M}}\ ds \right\}.$$

P r o o f: Since we can prove this lemma in much the same way as in [17], we refer the reader to this article. □

Lemma 9.3 *Under the assumptions and definitions above, the limit*

$$\lim_{t\to\infty} \|\theta(\rho(t)) - \theta(\overline{\rho}(t))\|_r = 0 \tag{9.6}$$

exists for each $r \in [1,\infty)$, where $\overline{\rho} = \overline{w} + \overline{m}$.

P r o o f: Arguments similar to those used in (8.16) lead to the estimate

$$\|w_{k_1}(s) - w_{k_2}(s)\|_{p'} \le c\|T_{k_1}(\rho(s)) - T_{k_2}(\rho(s))\|_1, \; p' \in \left(1, \frac{N}{N-1}\right), \tag{9.7}$$

for a.a. $s > 0$, where w_{k_i} are the solutions of the problem (8.2). Thus inequality (9.7) implies that the sequence $\{w_k\}_{k=1}^{\infty}$ is a Cauchy sequence in the space $L^\infty(0,\infty; L^{p'}(\Omega))$, with $p' \in \left(1, \frac{N}{N-1}\right)$, and hence convergent in the same space. Let us denote its limit by $\overline{w}$. The identities

$$\int_\Omega \theta(w_{\epsilon k}(s) + m_{\epsilon k}(s)) - \theta(w_{\epsilon k+q}(s) + m_{\epsilon k+q}(s)) \; dx = 0 \tag{9.8}$$

and

$$\int_\Omega \theta(w_k(s) + m_k(s)) - \theta(w_{k+q}(s) + m_{k+q}(s)) \; dx = 0, \tag{9.9}$$

which hold for a.a. $s > 0$, and for each $q \in N_0$, yield

$$\limsup_{k\to\infty} \int_\Omega \theta(w_k(s) + m_k(s)) \; dx = \liminf_{k\to\infty} \int_\Omega \theta(w_k(s) + m_k(s)) \; dx \tag{9.10}$$

for a.a. $s > 0$, and hence $\lim_{k\to\infty} m_k(s) = \overline{m}(s)$ almost everywhere in $[0,\infty)$. The proof will be complete if we show that $m_k(s) \to \overline{m}(s)$ in $L^\infty(0,\infty)$. Let us suppose for the moment that for each $k_0 \in N$ there exists a k, $k \ge k_0$, such that $\|m_k - \overline{m}\|_{L^\infty(0,\infty)} \ge \delta > 0$. Since $\int_\Omega \theta(w_k + m_k)\; dx$ is independent of k, and $m_k \to \overline{m}$ a.e., we get

$$\begin{aligned}
0 &= \left\| \int_\Omega \theta(w_k(\cdot) + m_k(\cdot)) - \theta(\overline{w}(\cdot) + \overline{m}(\cdot)) \; dx \right\|_{L^\infty(0,\infty)} \\
&= \Bigg\| \int_\Omega \left(\int_0^1 \theta'(\alpha(w_k(\cdot) + m_k(\cdot)) + (1-\alpha)(\overline{w}(\cdot) + \overline{m}(\cdot))) \; d\alpha \right) \times \\
&\qquad (w_k(\cdot) + m_k(\cdot) - \overline{w}(\cdot) - \overline{m}(\cdot)) \; dx \Bigg\|_{L^\infty(0,\infty)} \\
&= \Bigg\| \int_\Omega \left(\int_0^1 \theta'(\alpha(w_k + m_k) + (1-\alpha)(\overline{w} + \overline{m})) \; d\alpha \right) (w_k - \overline{w}) \; dx \\
&\quad + \int_{\Omega_{\epsilon\cdot}} \left(\int_0^1 \theta'(\alpha(w_k + m_k) + (1-\alpha)(\overline{w} + \overline{m})) \; d\alpha \right) (m_k - \overline{m}) \; dx \\
&\quad + \int_{\Omega\setminus\Omega_{\epsilon\cdot}} \left(\int_0^1 \theta'(\alpha(w_k + m_k) + (1-\alpha)(\overline{w} + \overline{m})) \; d\alpha \right) (m_k - \overline{m}) \; dx \Bigg\|_{L^\infty(0,\infty)} \\
&= \|I_1^k(\cdot) + I_2^k(\cdot) + I_3^k(\cdot)\|_{L^\infty(0,\infty)} \\
&\ge \|I_2^k(\cdot)\|_{L^\infty(0,\infty)} - \|I_1^k(\cdot)\|_{L^\infty(0,\infty)} - \|I_3^k(\cdot)\|_{L^\infty(0,\infty)},
\end{aligned}$$

where $\Omega_{\epsilon s} := \{x \in \Omega;\; |w_k(x,s) - \overline{w}(x,s)| \le \epsilon\}$. The inequality (9.7) implies that

$$\text{ess} \inf_{s\in(0,\infty)} |\Omega_{\epsilon s}| \ge \delta_1(\epsilon) \text{ and ess} \sup_{s\in(0,\infty)} |\Omega \setminus \Omega_{\epsilon s}| \le \delta_2(\epsilon),$$

where the functions δ_i are such that $\delta_i > 0$, δ_i do not depend on s for ϵ sufficiently small. Moreover, $\delta_1(\epsilon) \to |\Omega|$ and $\delta_2(\epsilon) \to 0$ for $\epsilon \to 0$. We conclude the first case from the contradiction based on the fact that the sequence $\{w_k\}_{k=1}^{\infty}$ is convergent in $L^\infty(0,\infty; L^{p'}(\Omega))$, with $p' \in \left(1, \frac{N}{N-1}\right)$. Similar arguments applied to the second case enable us to finish this part of the proof. Boundedness of $\theta'(r)$ gives $\|I_1^k(\cdot)\|_{L^\infty(0,\infty)} \to 0$, and it is easy to check that

$$|I_3^k(s)| \le c|\Omega \setminus \Omega_{\epsilon s}| \le c\delta_2(\epsilon), \text{ for a.e. } s \in (0,\infty).$$

This means that $\|I_3^k(\cdot)\|_{L^\infty(0,\infty)}$ is arbitrary small. But

$$\left\| \int_{\Omega_{\epsilon\cdot}} \int_0^1 \theta'(\alpha(w_k + m_k) + (1-\alpha)(\overline{w} + \overline{m}))\ d\alpha\ dx \right\|_{L^\infty(0,\infty)} > \delta_3 > 0$$

for $k \to \infty$ because $\theta'(s) > 0$, and for all k there exists a small set with a nonzero measure such that $|m_k(s) - \overline{m}(s)| \ge \delta$ on this set. Hence we get that

$$\lim_{k\to\infty} \|I_2^k(\cdot)\|_{L^\infty(0,\infty)} > \delta_3 > 0,$$

which is a contradiction.

This means nothing but

$$\lim_{t\to\infty} \|\theta(\rho(t)) - \theta(\overline{\rho}(t))\|_r = 0 \tag{9.11}$$

for each $r \in [1,\infty)$, where $\overline{\rho} = \overline{w} + \overline{m}$. □

We conclude this contribution with our main result.

Theorem 9.4 *Under the assumptions stated in section 3., there exists a unique function $\rho_\infty \in L_{\Phi_1}(\Omega)$, with $\int_\Omega \rho_\infty = \int_\Omega \rho_0$, such that*

$$\lim_{t\to\infty} \|\rho(t) - \rho_\infty\|_\Phi = 0, \tag{9.12}$$

where Φ is such a Young function that its complementary function Ψ satisfies the inequality

$$\sup_w \int_\Omega \Psi(|w^{\frac{1}{\alpha}}|)\ dx \le c \tag{9.13}$$

for $\alpha \in (0,1)$, where $\int_\Omega \Psi_1(|w|)\ dx \le 1$, and ρ_∞ satisfies the equations (1.7) and (1.8).

P r o o f: Let $\{t_n\}_{n=1}^{\infty}$, $t_n \to \infty$, be an arbitrary sequence. Then we can select $\{s_n\}_{n=1}^{\infty} \subset \{t_n\}_{n=1}^{\infty}$ such that $\rho(s_n) \to \rho_\infty$ E_{Ψ_1}-weakly, $\overline{m}(s_n) \to m_\infty$ and $\overline{w}(s_n) \to w_\infty$ in $L^{p'}(\Omega)$, $p' \in \left(1, \frac{N}{N-1}\right)$, as a result of the estimate

$$\int_\Omega (\overline{w}(s_n) - \overline{w}(s_m))\Delta\xi\ dx = \int_\Omega (\rho(s_n) - \rho(s_m))\mathbf{f} \cdot \nabla\xi\ dx$$

$$\leq \|\rho(s_n) - \rho(s_m)\|_{-1,p'} \|\nabla\xi \cdot \mathbf{f}\|_{1,p}, \ p > N.$$

For more details we refer the reader to the estimate (8.16). We would like to point out, here, the fact that an E_{Ψ_1}-weak convergence implies the strong convergence in $W^{-1,p'}(\Omega)$. From (9.6) it may be concluded that $\theta(\rho(s_n)) - \theta(\overline{\rho}(s_n)) \to 0$ a.e in Ω. Hence, $\theta(\overline{\rho}(s_n)) \to \theta(w_\infty + m_\infty)$ and $\theta(\rho(s_n)) \to \theta(w_\infty + m_\infty)$ a.e. in Ω, and thus $\rho(s_n) \to w_\infty + m_\infty$ a.e. in Ω. By boundedness of $\|\rho(s_n)\|_{\Phi_1}$, and by the estimate

$$\int_\Omega uw\ dx = \int_\Omega u^{1-\alpha} u^\alpha w\ dx \leq \|u\|_1^{1-\alpha} \left(\int_\Omega \Phi_1(|u|)\ dx + \int_\Omega \Psi_1(|w^{\frac{1}{\alpha}}|)\ dx \right)^\alpha$$

for $\alpha \in (0,1)$, it follows that the sequence $\{\rho(s_n)\}_{n=1}^\infty$ converges in $L_\Phi(\Omega)$, where Ψ satisfies (9.13).

It remains to prove that the equilibrium density ρ_∞ satisfies the identity (1.7). Since $\rho(s_n) \to \rho_\infty$, we find that

$$\int_\Omega \rho_\infty \Delta\xi\ dx = \int_\Omega (w_\infty + m_\infty)\Delta\xi\ dx = \int_\Omega \rho_\infty \mathbf{f} \cdot \nabla\xi\ dx,$$

with the function ξ from the Helmholtz-Weyl decomposition of $\eta \in C_0^\infty(\Omega)$. In particular, $\eta = \nabla\xi + \mathbf{z}$. As a consequence of the fact that

$$\int_\Omega \rho_\infty \operatorname{div} \mathbf{z}\ dx = 0$$

it remains to verify that $\int_\Omega \rho_\infty \mathbf{f} \cdot \mathbf{z}\ dx = 0$. It is clear that the proof will be complete if we show that

$$\lim_{n\to\infty} \int_\Omega \rho(s_n)\mathbf{f} \cdot \mathbf{z}\ dx = 0 \tag{9.14}$$

for $\mathbf{z} \in C^\infty(\overline{\Omega})$, $\operatorname{div} \mathbf{z} = 0$ and $\mathbf{z} \cdot \nu|_{\partial\Omega} = 0$, since then

$$0 = \lim_{n\to\infty} \int_\Omega \rho(s_n)\mathbf{f} \cdot \mathbf{z}\ dx = \int_\Omega \rho_\infty \mathbf{f} \cdot \mathbf{z}\ dx,$$

by the convergence of $\rho(s_n)$ to ρ_∞. To prove (9.14), it suffices to show that

$$\int_{t-1}^t \left| \int_\Omega \rho(s)\mathbf{f} \cdot \mathbf{z}\ dx \right| ds \to 0 \text{ for } t \to \infty \tag{9.15}$$

and

$$\left| \int_{t-1}^t \varphi'(s) \int_\Omega \rho(s)\mathbf{f} \cdot \mathbf{z}\ dxds \right| \to 0 \text{ for } t \to \infty, \tag{9.16}$$

with $\varphi \in C_0^\infty(t-1,t)$. The following estimate is almost a repetition of the estimate of the integral $I_{6kh}^\epsilon(t)$. Thus,

$$\int_{t-a}^{t+a} |v(s)| \left| \int_\Omega \rho(s)\mathbf{f} \cdot \mathbf{z}\ dx \right| ds \leq c\|v\|_\infty \|P\mathbf{u}\|_{L^1(t-a,t+a;L_{\overline{M}}(\Omega))} \|\nabla\mathbf{z}\|_\infty$$

$$+c\|\nabla\mathbf{z}\|_\infty \|v\|_\infty \|\rho\|_{L^\infty(0,\infty;L_{\Phi_1}(\Omega))} \int_{t-a}^{t+a} \|D\mathbf{u}(s)\|_{\Psi_2}^2\ ds$$

$$+\|\mathbf{z}\|_\infty \int_{t-a}^{t+a} |v'(s)| \, \|\rho \mathbf{u}(s)\|_1 \; ds \to 0$$

for $t \to \infty$ and $v \in C_0^\infty(t-a, t+a)$ such that $v(s) \equiv 1$ for $s \in [t-1, t]$. Further, from the weak equation of continuity we obtain that

$$\left| \int_{t-1}^{t} \varphi'(s) \int_\Omega \rho(s) \mathbf{f} \cdot \mathbf{z} \; dxds \right| = \left| \int_{t-1}^{t} \varphi(s) \int_\Omega \rho(s)(\mathbf{u}(s) \cdot \nabla)(\mathbf{f} \cdot \mathbf{z}) \; dxds \right|$$

$$\leq c\|\mathbf{f}\|_{1,\infty}\|\mathbf{z}\|_{1,\infty}\|\varphi\|_{L^2(t-1,t)} \sqrt{\int_{t-1}^{t} \|D\mathbf{u}(s)\|_{\Psi_2}^2 \; ds}.$$

By Proposition 9.1 for $\delta(k) \equiv 0$, we can finish the proof of (9.14). We have shown that

$$\int_\Omega \rho_\infty \mathrm{div}\, \eta \; dx = \int_\Omega \rho_\infty \mathbf{f} \cdot \eta \; dx, \;\; \forall \eta \in C_0^\infty(\Omega), \tag{9.17}$$

which implies that ρ_∞ satisfies the equations (1.7), (1.8). But since, according to [6], this problem has a unique solution, the convergence of $\rho(t)$ is not only restricted to subsequences but it is complete, i.e. (9.12) holds true. $\square$

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