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Optimal design of mechanical structures

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Abstract

We prove new properties for the linear isotropic elasticity system and for thickness minimization problems. We also present very recent results concerning shape optimization problems for three-dimensional curved rods and for shells. The questions discussed in this paper are related to the control variational method and to control into coefficients problems.

1 Introduction

The analysis and the computation of various optimal mechanical structures has a long history and many applications. We just quote the recent books by Bendsoe [3], Cherkaev [6], Allaire [1], Zolesio and Delfour [19], where such topics are studied from various points of view and where numerous references may be found.

In this paper, we shall consider structures like plates, curved rods and shells under low regularity assumptions with respect to their geometry. In the first section we analyze the application of the control variational method, introduced by the authors in [11], [15], [16], to the general linear elasticity system and to linear elastic plates. Variational inequalities are also considered. It turns out that the approach is advantageous from the numerical point of view since the solution is reduced to sequential applications of Laplace's equation. In section 2, thickness minimization problems for plates are discussed. The last section contains a presentation of very recent results in shape optimization problems for curved rods and shells, obtained by the authors.

2 The linear elasticity system

We consider in $\Omega \in I\!\!R^3$ the weak formulation of the isotropic linear elasticity system,

$$\int_\Omega [\lambda e_{pp}(u) e_{qq}(v) + 2\mu e_{ij}(u) e_{ij}(v)]\, dx = \int_\Omega f_i v_i \, dx\,, \tag{2.1}$$

$$u = (u_1, u_2, u_3) \in V(\Omega)\,, \quad \forall v = (v_1, v_2, v_3) \in V(\Omega) = \left\{ v \in H^1(\Omega)^3\,,\ v|_{\Gamma_0} = 0 \right\}\,.$$

Above, it is assumed that the smooth boundary of Ω, $\partial\Omega = \Gamma_0 \cup \Gamma_1$, consists of two nonoverlapping open parts and (2.1) corresponds to homogeneous mixed boundary conditions, imposed for simplicity. The constants $\lambda \geq 0$, $\mu > 0$, are the Lamé coefficients, $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, $i, j = \overline{1,3}$, the summation convention is used, and $f = (f_1, f_2, f_3)$ gives the body forces. The existence of a unique solution $u = (u_1, u_2, u_3) \in V(\Omega)$ for (2.1) is wellknown, Ciarlet [7], [8]. We prove here that (2.1) admits an advantageous treatment via control theory. To this end, we consider the following problem:

$$\begin{aligned} &\text{Min} \left\{ \frac{1}{2} \int_\Omega \left\{ \mu |w|^2_{R^9} + \lambda [\text{div}\,(u)]^2 + \mu \left[\left(\frac{\partial u_1}{\partial x_1} \right)^2 + \left(\frac{\partial u_2}{\partial x_2} \right)^2 \right.\right.\right. \\ &+ \left.\left(\frac{\partial u_3}{\partial x_3} \right)^2 \right] + 2\mu \left(\frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \frac{\partial u_3}{\partial x_1} + \frac{\partial u_2}{\partial x_3} \frac{\partial u_3}{\partial x_2} \right) \Big\} \, dx \Big\} , \end{aligned} \tag{2.2}$$

subject to $w \in L^2(\Omega)^q$ and to

$$\int_\Omega \nabla u : \nabla v \, dx = \int_\Omega w : \nabla v \, dx + \frac{1}{\mu} \int_\Omega f \cdot v \, dx \,, \quad \forall v \in V(\Omega) \,, \tag{2.3}$$

where ∇u is the Jacobian of u and

$$\nabla u : \nabla v = \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \,.$$

Relation (2.3) is just the weak formulation of the system of the three decoupled Poisson equations

$$-\Delta u = -\text{div}\, w + \frac{1}{\mu} f \,, \quad \text{in } \Omega \,, \tag{2.4}$$

with homogeneous mixed boundary conditions. The divergence operator in (2.4) is applied to the rows of the 3×3 "matrix" $w \in L^2(\Omega)^9$.

We study briefly the problem (2.2)–(2.3) and we show that it provides exactly the solution of (2.1). The two problems are in fact equivalent.

Proposition 2.1 *Assume that* $[u^*, w^*] \in V(\Omega) \times L^2(\Omega)^q$ *is an optimal pair for the problem (2.2), (2.3). Then it holds*

$$\begin{aligned} &\int_\Omega \left[\mu w^* : q + \lambda \text{div}\,(u^*) \text{div}\,(z) + \mu \left(\frac{\partial u_1^*}{\partial x_1} \frac{\partial z_1}{\partial x_1} + \frac{\partial u_2^*}{\partial x_2} \frac{\partial z_2}{\partial x_2} + \frac{\partial u_3^*}{\partial x_3} \frac{\partial z_3}{\partial x_3} \right.\right. \\ &+ \left.\left. \frac{\partial u_1^*}{\partial x_2} \frac{\partial z_2}{\partial x_1} + \frac{\partial z_1}{\partial x_2} \frac{\partial u_2^*}{\partial x_1} + \frac{\partial u_1^*}{\partial x_3} \frac{\partial z_3}{\partial x_1} + \frac{\partial u_3^*}{\partial x_1} \frac{\partial z_1}{\partial x_3} + \frac{\partial u_2^*}{\partial x_3} \frac{\partial z_3}{\partial x_2} + \frac{\partial u_3^*}{\partial x_2} \frac{\partial z_2}{\partial x_3} \right) \right] dx = 0 \,, \end{aligned} \tag{2.5}$$

for any $z \in V(\Omega)$ *and for* $q \in L^2(\Omega)^9$ *with* $q = \nabla z$.

Proof. This is the usual Euler equation associated to (2.2), (2.3). As the control problem is unconstrained, we can take arbitrary variations of the form $u^* + sz$, $s \in I\!R$, around u^*, which correspond to variations $w^* + sq$ around the optimal control w^*, since z is the solution of the "equation in variations" corresponding to q:

$$\int_\Omega \nabla z : \nabla v\, dx = \int_\Omega q : \nabla v\, dx\,, \quad \forall v \in V(\Omega)\,.$$

One then writes that the cost corresponding to w^* is lower than the one corresponding to $w^* + sq$, then subtracts, divides by s (for $s > 0$ or $s < 0$) and takes the limit $s \to 0$ to obtain the result. □

Remark. Relation (2.5) is a characterization of optimality. The optimal pair, if it exists, is unique (by the strict convexity of (2.2)).

Next, we define the adjoint system for $p \in V(\Omega)$:

$$\begin{aligned}\int_\Omega \nabla p : \nabla z = \int_\Omega \Big[\lambda\, \mathrm{div}\,(u^*)\, \mathrm{div}\,(z) + \mu \Big(\frac{\partial u_1^*}{\partial x_1}\frac{\partial z_1}{\partial x_1} + \frac{\partial u_2^*}{\partial x_2}\frac{\partial z_2}{\partial x_2} \\ + \frac{\partial u_3^*}{\partial x_3}\frac{\partial z_3}{\partial x_3} + \frac{\partial u_1^*}{\partial x_2}\frac{\partial z_2}{\partial x_1} + \frac{\partial u_2^*}{\partial x_1}\frac{\partial z_1}{\partial x_2} + \frac{\partial u_1^*}{\partial x_3}\frac{\partial z_3}{\partial x_1} + \frac{\partial u_3^*}{\partial x_1}\frac{\partial z_1}{\partial x_3} \\ + \frac{\partial u_2^*}{\partial x_3}\frac{\partial z_3}{\partial x_2} + \frac{\partial u_3^*}{\partial x_2}\frac{\partial z_2}{\partial x_3}\Big)\Big]\, dx = 0\,, \quad \forall z \in V(\Omega)\,.\end{aligned} \tag{2.6}$$

Relation (2.6) is the weak form of a system of decoupled Poisson equations with homogeneous mixed boundary conditions. Existence and uniqueness of the solution $p \in V(\Omega)$ are obvious.

Proposition 2.2 *The optimality conditions for the problem* (2.2), (2.3) *are given by* (2.3), (2.6) *and the Pontryagin maximum principle:*

$$\int_\Omega (\mu w^* + \nabla p) : \nabla z\, dx = 0\,, \quad \forall z \in V(\Omega)\,. \tag{2.7}$$

Moreover, $p = \mu h - \mu u^*$ *in* Ω *with* h *defined in (2.8) below.*

Proof. By (2.6) and (2.5), we get

$$0 = \int_\Omega [\mu w^* : q + \nabla p : \nabla z]\, dx = \int_\Omega [\mu w^* : q + \nabla p : q]\, dx\,,$$

which is exactly the relation (2.7), as $q = \nabla z$. Notice that, by virtue of (2.3) and (2.7), we have

$$\begin{aligned}\int_\Omega \nabla u^* : \nabla z\, dx &= \int_\Omega w^* : \nabla z\, dx + \frac{1}{\mu}\int_\Omega f \cdot z\, dx \\ &= -\frac{1}{\mu}\left[\int_\Omega \nabla p : \nabla z\, dx - \int_\Omega f \cdot z\, dx\right].\end{aligned}$$

That is, if we denote by $h \in V(\Omega)$ the (weak) solution to the problem:

$$\int_\Omega \nabla h : \nabla z\, dx = +\frac{1}{\mu}\int_\Omega f \cdot z\, dx\,, \quad \forall z \in V(\Omega)\,, \tag{2.8}$$

then we obtain

$$\int_\Omega \nabla u^* : \nabla z \, dx = -\frac{1}{\mu} \int_\Omega \nabla p : \nabla z \, dx + \int_\Omega \nabla h : \nabla z \, dx \,, \quad \forall z \in V(\Omega) \,.$$

As u^*, p, h satisfy the same boundary conditions, the unique solvability of Laplace's problem concludes the proof. □

Again by (2.3), and by the definition of q in Proposition 2.1, we can write

$$\int_\Omega \mu w^* : q \, dx = \mu \int_\Omega w^* : \nabla z \, dx = \mu \int_\Omega \nabla u^* : \nabla z \, dx - \int_\Omega f \cdot z \, dx \,, \quad \forall z \in V(\Omega) \,.$$

By replacing this in (2.5), we have

$$\begin{aligned} \int_\Omega \Bigg[& \mu \nabla u^* : \nabla z \, dx + \lambda \operatorname{div}(u^*) \operatorname{div}(z) + \mu \left(\frac{\partial u_1^*}{\partial x_1} \frac{\partial z_1}{\partial x_1} + \frac{\partial u_2^*}{\partial x_2} \frac{\partial z_2}{\partial x_2} \right. \\ & + \frac{\partial u_3^*}{\partial x_3} \frac{\partial z_3}{\partial x_3} + \frac{\partial u_1^*}{\partial x_2} \frac{\partial z_2}{\partial x_1} + \frac{\partial u_2^*}{\partial x_1} \frac{\partial z_1}{\partial x_2} + \frac{\partial u_1^*}{\partial x_3} \frac{\partial z_3}{\partial x_1} + \frac{\partial u_3^*}{\partial x_1} \frac{\partial z_1}{\partial x_3} \\ & \left. + \frac{\partial u_2^*}{\partial x_3} \frac{\partial z_3}{\partial x_2} + \frac{\partial u_3^*}{\partial x_2} \frac{\partial z_2}{\partial x_3} \right) \Bigg] \, dx = \int_\Omega f \cdot z \, dx \,, \quad \forall z \in V(\Omega) \,. \end{aligned} \tag{2.9}$$

Regrouping the terms in (2.9) conveniently, we have thus proved:

Corollary 2.1 *$u^* \in V(\Omega)$ is the unique solution to (2.1).*

Remark. Relations (2.3), (2.6) and (2.7) provide a nonstandard decomposition of (2.1).

Remark. Corollary 2.1 provides a simple convenient method to solve (2.1) via (2.2), (2.3). In the setting of this control problem, we have to solve the state system (2.3) and the adjoint system (2.6) (both associated to the Laplace operator). Then, the gradient of the cost functional may be computed by Proposition 2.2 and gradient methods may be used. Notice also that the existence in (2.2), (2.3) follows from the result for (2.1), by Proposition 2.1 and Corollary 2.1.

Let us now consider the example of a linear elastic plate $(\Omega \subset I\!R^2!)$ submitted to unilateral restrictions:

$$\begin{aligned} a(y,v) = \int_\Omega & e^3 [y_{,11} v_{,11} + \tau y_{,11} v_{,22} \\ & + \tau y_{,22} v_{,11} + y_{,22} v_{,22} + 2(1-\tau) y_{,12} v_{,12}] \, dx \,, \\ & \forall y \in H_0^2(\Omega) \,, \quad \forall v \in H_0^2(\Omega) \,. \end{aligned} \tag{2.10}$$

$$a(y, y-v) \le \int_\Omega f(y-v) \, dx \,, \ y \in \mathcal{K} \,, \quad \forall v \in \mathcal{K} \,, \tag{2.11}$$

where, for $y \in H^2(\Omega)$, $y_{,ij} = \frac{\partial^2 y}{\partial x_i \partial x_j}$, $i,j = 1,2$.

Here $\mathcal{K} \subset H_0^2(\Omega)$ is a nonempty closed and convex set. The scalar functions $y \in H_0^2(\Omega)$, $e \in L^\infty(\Omega)_+$, $f \in L^2(\Omega)$, represent respectively the deflection, the positive thickness and the load of the plate, while $0 < \tau < \frac{1}{2}$ is the Poisson coefficient, Duvaut and Lions [9, Ch. 4].

We replace (2.10), (2.11) by the following optimal control problem:

$$\text{Min} \left\{ \frac{1}{2} \int_\Omega e^3 \left[w^2 + 2(1-\tau) y_{,12}^2 + 2(\tau - 1) y_{,11} y_{,22} \right] dx \right\} \tag{2.12}$$

subject to the state equation

$$\Delta y = w + e^{-3} g \quad \text{in } \Omega\,, \tag{2.13}$$

$$y = 0 \quad \text{on } \partial\Omega\,, \tag{2.14}$$

and to the state constraints

$$y \in \mathcal{K}\,. \tag{2.15}$$

Above, $g \in H^2(\Omega) \cap H_0^1(\Omega)$ is the solution to the Poisson problem with $\Delta g = f$ in Ω. We shall prove that the solution of (2.11) may be obtained via the control variational method given by (2.12)–(2.15). Notice the differences between (2.13) and (2.3) that show the flexibility of our approach. It is also clear that a numerical solution of (2.12)–(2.15) may be obtained by using first order finite elements which provides a simple way for the solution of (2.11).

Any pair $[y, w]$, $y \in \mathcal{K} \subset H_0^2(\Omega)$, $w = \Delta y - e^{-3} g$ is admissible for the problem (2.12)–(2.15).

In this special situation, one can prove directly the existence of optimal pairs:

Proposition 2.3 *The problem* (2.12)–(2.15) *has a unique optimal pair* $[y^*, w^*]$.

Proof. Let $[y^n, w_n]$ be a minimizing sequence for (2.12). Then $y_{,11}^n + y_{,22}^n = w_n + e^{-3} g$ and the cost functional is bounded from above:

$$\begin{aligned} c \geq \int_\Omega e^3 \big\{ & w_n^2 + 2(1-\tau)(y_{,12}^n)^2 \\ & + 2(\tau - 1)\left[y_{,11}^n w_n + e^{-3} g y_{,11}^n - (y_{,11}^n)^2 \right] \big\}\, dx\,. \end{aligned} \tag{2.16}$$

As $0 < \tau < \frac{1}{2}$, relation (2.16) shows that $\{w_n\}$, $\{y_{,12}^n\}$, $\{y_{,11}^n\}$ are bounded in $L^2(\Omega)$, and (2.13) yields that also $\{y_{,22}^n\}$ is bounded in $L^2(\Omega)$. That is, $\{y^n\}$ is bounded in $H_0^2(\Omega)$. One can take weakly convergent subsequences $y^n \to y^*$, $w_n \to w^*$ in $H_0^2(\Omega)$, $L^2(\Omega)$ respectively, pass to the limit in (2.13)–(2.15) as $\mathcal{K}$ is weakly closed and end the proof by the weak lower semicontinuity of the cost functional (2.12). Uniqueness is a clear consequence of the strict convexity of (2.12). □

Remark. Notice that, in this proof, $\Omega \subset \mathbb{R}^2$ plays an essential role.

The characterization of $[y^*, w^*]$ via the Euler (in)equation has to take the state constraints into account. We perform admissible variations of the form $y^*+s(z-y^*)$, $w^*+s(l-w^*)$, $s \in [0,1]$, $\forall z \in \mathcal{K}$, $l = \Delta z - e^{-3} g \in L^2(\Omega)$ to obtain that

$$0 \leq \int_\Omega e^3 \left\{ w^*(l-w^*) + 2(1-\tau) y^*_{,12}(z_{,12} - y^*_{,12}) \right. \\ \left. +2(\tau - 1)\left[y^*_{,11}(z_{,22} - y^*_{,22}) + y^*_{,22}(z_{,11} - y^*_{,11}) \right] \right\} dx\,, \tag{2.17}$$

for any $z \in \mathcal{K}$.

Using the fact that $w^* = \Delta y^* - e^{-3} g$, $l = \Delta z - e^{-3} g$, a convenient grouping of the terms in (2.17) and the partial integration

$$\int_\Omega e^3 (\Delta z - \Delta y^*) e^{-3} g\, dx = \int_\Omega f(z - y^*)\, dx$$

yield:

Corollary 2.2 $y^* \in H_0^2(\Omega)$ *is the unique solution to* (2.10), (2.11).

Remark. It is possible to compute directional derivatives and to write necessary conditions as in the previous case. Other boundary conditions may be studied as well, for instance partially clamped plates. Then, another artificial control has to be introduced in (2.14) which becomes $y = v \in H^{3/2}(\partial\Omega)$, $v = 0$ on the "clamped" part of $\partial\Omega$. A weak penalization $\varepsilon |v|^2_{H^{3/2}(\partial\Omega)}$, $\varepsilon > 0$, has to be added to (2.12). The analysis involves a limiting process for $\varepsilon \to 0$ and it is more technical. Finally, let us underline that cost functionals (2.2) or (2.12) represent the usual energy (up to a constant), after the substitution of the control by the state.

3 Thickness optimization of plates with unilateral conditions

We study the optimal design problem

$$\mathrm{Min}\, \{ J(e, y)\,,\ e \in E_{ad} \}\,, \tag{3.1}$$

subject to (2.10), (2.11), and with $J : L^\infty(\Omega) \times H_0^2(\Omega) \to I\!R$ a lower semicontinuous functional;

$$E_{ad} = \left\{ e \in L^\infty(\Omega)\,;\ 0 < \alpha \leq e \leq \beta \quad \text{a.d. } \Omega\,;\ |e|_{W^{1,t}(\Omega)} \leq \gamma \right\}. \tag{3.2}$$

Here, α, β γ, $t > 2$ are some given positive real numbers. One can also include other constraints in the definition of E_{ad}. For instance, the constant volume constraint

$$\int_\Omega e\, dx = const.$$

may be considered. Concerning the possible state constraints, as $\Omega \subset I\!R^2$, the solution y of (2.11) belongs to $C(\bar{\Omega})$ and one example of interest is given by the pointwise state constraint

$$y(x_0) \geq -\delta\,, \tag{3.3}$$

with $x_0 \in \Omega$ and $\delta > 0$ conveniently fixed.

An important case covered by (3.1)–(3.3) is the minimization of the volume (thickness) of the plate such that the deflection y remains above a given tolerance $-\delta$ (in one or in any point in Ω), for a prescribed load $f \in L^2(\Omega)$. This is a natural safety requirement.

In the sequel, we shall denote by $a(e,y,v)$ the functional (2.10), and we assume $0 \in \mathcal{K}$, just in order to simplify the writing.

Proposition 3.1 *Let $e_n \to e$ in $L^\infty(\Omega)$ strongly, and let y^n, y denote the corresponding solutions to* (2.11). *Then, $y^n \to y$ strongly in $H_0^2(\Omega)$.*

Proof. By Corollary 2.2 and (2.12), (2.13), we get

$$\int_\Omega e_n^3 (e_n^{-6} g^2)\,dx \geq \int_\Omega e_n^3 \left\{ w_n^2 + 2(1-\tau)(y^n_{,12})^2 + 2(\tau - 1) y^n_{,11} y^n_{,22} \right\} dx\,, \tag{3.4}$$

obtained by the admissible choice $\tilde{y}^n = 0$, $\tilde{w}_n = -e_n^{-3} g$. Then (3.2) and (3.4) show that, for any n:

$$\int_\Omega \left\{ w_n^2 + 2(1-\tau)(y^n_{12})^2 + 2(\tau-1) y^n_{11} y^n_{22} \right\} dx \leq c\,.$$

Arguing again as in (2.16), we see that $\{w_n\}$, $\{y^n\}$ are bounded in $L^2(\Omega)$, $H_0^2(\Omega)$, respectively. Denoting by $\tilde{y} \in \mathcal{K}$ the weak limit of y^n in $H_0^2(\Omega)$, on a subsequence, we can use the form (2.10), (2.11) of the variational inequality to see that $\tilde{y} = y$, by the weak lower semicontinuity of quadratic forms. By summing $a(e_n, y^n, y^n - y^m)$ and $a(e_m, y^m, y^m - y^n)$ according to (2.11) and to the uniform (in e) coercivity of $a(e,y,v)$ on $H_0^2(\Omega)$, we obtain, for some $c > 0$:

$$c|y^n - y^m|^2_{H_0^2(\Omega)} \leq a(e_m, y^m, y^m - y^n) - a(e_n, y^m, y^m - y^n)\,.$$

Using (2.10), and the uniform convergence of $\{e_n\}$, a short computation gives the strong convergence in $H_0^2(\Omega)$ for $\{y^n\}$, and the proof is finished. □

Corollary 3.1 *The optimization problem* (3.1)–(3.3) *has at least one optimal solution $e^* \in E_{ad}$ if it has admissible elements.*

This is a consequence of the compact embedding of $W^{1,t}(\Omega)$ in $C(\bar{\Omega})$, $t > 2$, by the Sobolev theorem and of Proposition 3.1.

Remark. Corollary 3.1 is a partial extension of results obtained by Hlavacek, Bock and Lovišek [10], Bendsoe [3], Sprekels and Tiba [14]. If (2.11) is the obstacle problem, Sokolowski and Rao [13] have studied its sensitivity with respect to variations around e^*.

In the present more general setting, we prove a weaker differentiability-type property. We fix some $b \in L^\infty(\Omega)$, and we denote by y^λ the solution of (2.11) associated to $e+\lambda b$, $\lambda \in I\!R$. By Proposition 3.1, $y^\lambda \to y$ strongly in $H_0^2(\Omega)$ as $\lambda \to 0$. Denote by $v^\lambda = \frac{y^\lambda - y}{\lambda} \in H_0^2(\Omega)$.

Proposition 3.2 *$\{v^\lambda\}$ is bounded in $H_0^2(\Omega)$. If $\hat{v}$ is a limit point of $\{v^\lambda\}$, then it satisfies:*

$$a(e, y, \hat{v}) = \int_\Omega f \hat{v}\, dx\,, \tag{3.5}$$

$$\begin{aligned} 0 \geq a(e, \hat{v}, \hat{v} - l) + \int_\Omega 3e^2 b\big[y_{,11}(\hat{v}_{,11} - l_{,11}) + \tau y_{,11}(\hat{v}_{,22} - l_{,22}) \\ + \tau y_{,22}(\hat{v}_{,11} - l_{,11}) + y_{,22}(\hat{v}_{,22} - l_{,22}) + 2(1-\tau) y_{,12}(\hat{v}_{,12} - l_{,12})\big]\, dx\,, \quad \forall l \in \hat{Z}\,, \end{aligned} \tag{3.6}$$

with $\hat{Z} \subset H_0^2(\Omega)$ a closed convex nonvoid set defined in the proof.

Proof. By adding $a(e, y, y - y^\lambda)$ and $a(e + \lambda b,\ y^\lambda, y^\lambda - y)$ and by (2.11), we get

$$\begin{aligned} 0 \geq a(e, y^\lambda - y, y^\lambda - y) + \lambda \int_\Omega (3e^2 b + 3\lambda e b^2 + \lambda^2 b^3)\big[y^\lambda_{,11}(y^\lambda_{,11} - y_{,11}) \\ + \tau y^\lambda_{,11}(y^\lambda_{,22} - y_{,22}) + \tau y^\lambda_{,22}(y^\lambda_{,11} - y_{,11}) + y^\lambda_{,22}(y^\lambda_{,22} - y_{,22}) \\ + 2(1-\tau) y^\lambda_{,12}(y^\lambda_{,12} - y_{,12})\big]\, dx\,. \end{aligned} \tag{3.7}$$

Dividing by λ^2 in (3.7), and using the coercivity of $a(e,\cdot,\cdot)$ and the convergence of y^λ, we find that $\{v^\lambda\}$ is bounded in $H_0^2(\Omega)$. Let $\hat{v}$ be a limit point of $\{v^\lambda\}$, on some subsequence. Passing to the limit $\lambda \searrow 0+$ in

$$a(e, y, -v^\lambda) \leq -\int_\Omega f v^\lambda dx\,,$$

$$a(e + \lambda b, y^\lambda, v^\lambda) \leq \int_\Omega f v^\lambda dx\,,$$

we get (3.5).

Consider now test functions $l^\lambda \in Z_\lambda = \big[\frac{1}{\lambda}(\mathcal{K} - y) \cap \frac{1}{\lambda}(y^\lambda - \mathcal{K})\big] \subset H_0^2(\Omega)$, $\lambda > 0$. Notice that Z_λ is a nonvoid closed convex set and $0 \in Z_\lambda$, $v^\lambda \in Z_\lambda$. If $l^\lambda \in Z_\lambda$, then $y + \lambda l^\lambda \in \mathcal{K}$, $y^\lambda - \lambda l^\lambda \in \mathcal{K}$, $\lambda > 0$. We use these test functions in (2.11) to obtain:

$$a(e, y, y - y^\lambda + \lambda l^\lambda) \leq -\int_\Omega f(y - y^\lambda + \lambda l^\lambda)\, dx\,,$$

$$a(e + \lambda b, y^\lambda, y^\lambda - y - \lambda l^\lambda) \leq \int_\Omega f(y^\lambda - y - \lambda l^\lambda)\, dx\,.$$

Adding these inequalities, and dividing by λ^2, we have

$$0 \geq a(e, v^\lambda, v^\lambda - l^\lambda) + \int_\Omega (3e^2 b + 3\lambda e b^2 + \lambda^2 b) \left[y^\lambda_{,11}(v^\lambda_{,11} - l^\lambda_{,11}) + \tau y^\lambda_{,11}(v^\lambda_{,22} - l^\lambda_{,22}) \right.$$
$$+ \tau y^\lambda_{,22}(v^\lambda_{,11} - l^\lambda_{,11}) + y^\lambda_{,22}(v^\lambda_{,22} - l^\lambda_{,22})$$
$$\left. + 2(1-\tau) y^\lambda_{,12}(v^\lambda_{,12} - l^\lambda_{,12}) \right] dx \,, \quad \forall l^\lambda \in Z_\lambda \,. \tag{3.8}$$

If $\lambda_n \searrow 0$ is chosen such that $v_{\lambda_n} \to \hat{v}$ weakly in $H^2_0(\Omega)$, we denote by $\hat{Z} = \liminf_{\lambda \to 0} Z_{\lambda_n} = \left\{ p \in H^2_0(\Omega) \,;\ \exists p_{\lambda_n} \in Z_{\lambda_n} \,,\ p_{\lambda_n} \to p \text{ in } H^2_0(\Omega) \right\}$. This is a nonvoid closed convex subset of $H^2_0(\Omega)$. Passing to the limit in (3.8) gives (3.6) which ends the proof. □

Remark. The dependence of $\hat{Z}$ and of $\hat{v}$ on the way we choose a convergent subsequence of $\{v^\lambda\}$ shows that they may be not uniquely determined.

4 Curved rods and shells

For the three-dimensional curved rods, we relax the usual regularity hypotheses on the parametrization, of type $W^{3,\infty}(0,L)$, by avoiding the use of the classical Frenet or Darboux frames, Cartan [5]. A new local system of axes valid for $C^1(0,L)$ or $W^{2,\infty}(0,L)$ curves was introduced in Ignat, Sprekels and Tiba [12]. As, here, we are mainly interested in optimization questions, we perform a direct parametrization of the tangent vector,

$$\bar{t}(\cdot) = (\sin\tau(\cdot)\cos\psi(\cdot) \,,\ \sin\tau(\cdot)\sin\psi(\cdot) \,,\ \cos\tau(\cdot)) \,. \tag{4.1}$$

The curve is then parametrized by

$$\bar{\theta}(x_3) = \int_0^{x_3} \bar{t}(s) ds \,,\ x_3 \in [0, L] \,. \tag{4.2}$$

Notice that in this way a unit speed curve $\bar{\theta}$ in $I\!R^3$ with fixed length $L > 0$ is automatically generated. Moreover, the local frame can be obtained by algebraic means,

$$\bar{n} = (\cos\tau\cos\psi \,,\ \cos\tau\sin\psi \,,\ -\sin\tau) \,, \tag{4.3}$$
$$\bar{b} = (-\sin\psi \,,\ \cos\psi \,,\ 0) \,. \tag{4.4}$$

The mappings $\tau, \psi \in C^1(0,1)$ give the real parametrization. If $\omega(x_3) \subset I\!R^2$ is a bounded domain, not necessarily simply connected, we define the open set

$$\Omega = \bigcup_{x_3 \in]0,L[} (\omega(x_3) \times \{x_3\}) \subset I\!R^3 \,. \tag{4.5}$$

The curved rod $\tilde{\Omega}$ associated to $\bar{\theta}$ is then obtained by the transformation

$$\begin{aligned} \bar{x} = (x_1, x_2, x_3) \subset \Omega \mapsto F\bar{x} = \tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \\ = \bar{\theta}(x_3) + x_1 \bar{n}(x_3) + x_2 \bar{b} \in \tilde{\Omega}\,, \ \forall \bar{x} \in \Omega\,. \end{aligned} \tag{4.6}$$

The Jacobian J of the transformation F satisfies $\det J(\bar{x}) \geq c > 0$, $\forall \bar{x} \in \Omega$, if the sets $\omega(x_3)$ are all contained in a sufficiently small disk in $\mathbb{R}^2$. In Ciarlet [8] it is proved that F is one-to-one and that $\tilde{\Omega}$ is well defined.

We make the geometrical assumption that the displacement has the following form for $\tilde{x} \in \tilde{\Omega}$:

$$\bar{y}(\tilde{x}) = \bar{\rho}(x_3) + x_1 \bar{N}(x_3) + x_2 \bar{B}(x_3)\,, \ \bar{x} = F^{-1}(\tilde{x})\,. \tag{4.7}$$

The unknowns are $\bar{\rho}$, $\bar{N}$, $\bar{B} \in H_0^1(0,L)^3$, and (4.7) enters the category of polynomial models. Comparing with the shell model considered later in this section, we may say that (4.7) gives a generalized Naghdi model for curved rods. By introducing (4.7) into the elasticity system, we get the following variational equation (here $(h_{ij}) = J^{-1}$ and $\tilde{\lambda}$, $\tilde{\mu}$ are the Lamé coefficients) for $\bar{\rho}$, $\bar{N}$, $\bar{B}$:

$$\begin{aligned} &\tilde{\lambda} \int_\Omega \sum_{i,j=1}^3 \Big[N_i(x_3) h_{1i}(\bar{x}) + B_i(x_3) h_{2i}(\bar{x}) + \big(\rho_i'(x_3) + x_1 N_i'(x_3) \\ &+ x_2 B_i'(x_3)\big) h_{3i}(\bar{x}) \Big] \Big[M_j(x_3) h_{1j}(\bar{x}) + D_j(x_3) h_{2j}(\bar{x}) + \big(\mu_j'(x_3) + x_1 M_j'(x_3) \\ &+ x_2 D_j'(x_3)\big) h_{3j}(\bar{x}) \Big] |\det J(\bar{x})|\, d\bar{x} + \tilde{\mu} \int_\Omega \sum_{i<j} \Big[N_i(x_3) h_{1j}(\bar{x}) + B_i(x_3) h_{2j}(\bar{x}) \\ &+ \big(\rho_i'(x_3) + x_1 N_i'(x_3) + x_2 B_i'(x_3)\big) h_{3j}(\bar{x}) + N_j(x_3) h_{1i}(\bar{x}) + B_j(x_3) h_{2i}(\bar{x}) \\ &+ \big(\rho_j'(x_3) + x_1 N_j'(x_3) + x_2 B_j'(x_3)\big) h_{3i}(\bar{x}) \Big] \Big[M_i(x_3) h_{1j}(\bar{x}) + D_i(x_3) h_{2j}(\bar{x}) \\ &+ \big(\mu_i'(x_3) + x_1 M_i'(x_3) + x_2 D_i'(x_3)\big) h_{3j}(\bar{x}) + M_j(x_3) h_{1i}(\bar{x}) + D_j(x_3) h_{2i}(\bar{x}) \\ &+ \big(\mu_j'(x_3) + x_1 M_j'(x_3) + x_2 D_j'(x_3)\big) h_{3j}(\bar{x}) \Big] |\det J(\bar{x}|\, d\bar{x} \\ &+ 2\tilde{\mu} \int_\Omega \sum_{i=1}^3 \Big[N_i(x_3) h_{1i}(\bar{x}) + B_i(x_3) h_{2i}(\bar{x}) + \big(\rho_i'(x_3) + x_1 N_i'(x_3) + x_2 B_i'(x_3)\big) h_{3i}(\bar{x}) \Big] \\ &\Big[M_i(x_3) h_{1i}(\bar{x}) + D_i(x_3) h_{2i}(\bar{x}) + \big(\mu_i'(x_3) + x_1 M_i'(x_3) + x_2 D_i'(x_3) h_{3i}(\bar{x}) \Big] |\det J(\bar{x})|\, d\bar{x} \\ = &\sum_{l=1}^3 \int_\Omega f_l(\bar{x}) \big(\mu_l(x_3) + x_1 M_l(x_3) + x_2 D_l(x_3)\big) |\det J(\bar{x})|\, d\bar{x} \\ &+ \sum_{i,j=1}^3 \sum_{l=1}^3 \int_{\partial\Omega} g_l(\bar{x}) \big(\mu_l(x_3) + x_1 M_l(x_3) + x_2 D_l(x_3)\big) |\det J(\bar{x})| \sqrt{\nu_i g^{ij} \nu_j}\, d\tau\,. \end{aligned} \tag{4.8}$$

Above, $\bar{\mu}$, $\bar{M}$, $\bar{D} \in H_0^1(0,L)^3$ are test functions, (ν_i) is the normal vector to $\partial\Omega$, $(g^{ij}) = J^{-1}(J^T)^{-1}$, and $\bar{f} \in L^2(\Omega)^3$, $\bar{g} \in L^2(\partial\Omega)^3$ are the acting forces.

The coercivity of the bilinear form is established under the assumption that $\omega(x_3) \supset \omega$, $\forall x_3 \in [0, L]$, and

$$0 = \int_\omega x_1 \, dx_1 dx_2 = \int_\omega x_2 \, dx_1 dx_2 = \int_\omega x_1 x_2 \, dx_1 dx_2 \, .$$

The argument in Ignat, Sprekels and Tiba [12] is a direct one. It is based on the algebraic identity

$$\begin{aligned} &\frac{1}{2}\Big[(z_1 h_{32} + z_2 h_{31})^2 + (z_2 h_{33} + z_3 h_{32})^2 + (z_1 h_{33} + z_3 h_{31})^2\Big] \\ &+ \frac{3}{2}\,(z_1^2 h_{31}^2 + z_2^2 h_{32}^2 + z_3^2 h_{33}^2) = \frac{1}{2}\,(z_1^2 + z_2^2 + z_3^2)(h_{31}^2 + h_{32}^2 + h_{33}^2) \\ &+ \frac{1}{2}\,(z_1 h_{31} + z_2 h_{32})^2 + \frac{1}{2}\,(z_1 h_{31} + z_3 h_{33})^2 + \frac{1}{2}\,(z_2 h_{32} + z_3 h_{33})^2 \, . \end{aligned}$$

A general formulation of optimization problems associated to curved rods is (see (4.7)):

$$\operatorname*{Min}_{\tau,\psi} \{\Pi(\tau, \psi) = j(\bar\theta, \bar y)\} \, , \tag{4.9}$$

subject to (4.8) and to constraints $\bar\theta \in \mathcal{K} \subset C^2(0, L)^3$, bounded closed subset. A typical example for (4.9) is the quadratic case, for instance $j(\bar\theta, \bar y) = \sum_{i=1}^3 |\rho_i|^2_{H_0^1(0,L)}$ (minimization of the displacement of the line of centroids). Notice that our construction eliminates degenerate cases like rods of length zero. By imposing the constraint $0 \le \tau(x_3) \le \frac{\pi}{2} - \varepsilon$, $x_3 \in [0, L]$, self-intersecting curves are also eliminated. The partial periodicity constraint

$$\int_0^L t_1 \, dx_3 = \int_0^L t_2 \, dx_3 = 0$$

can be used for the optimization of spirals, etc.

Theorem 4.1 *If the set of admissible* $\{\tau, \psi\}$ *is compact in* $C^1(0, L)^2$, *and if* $j : C^2(0, L)^3 \times H_0^1(0, L)^9 \to I\!R$ *is lower semicontinuous, then the problem* (4.9), (4.8) *admits at least one optimal curved rod* $\bar\theta^*$.

In Arnăutu, Sprekels and Tiba [2] it is also proved that the mapping $\{\tau, \theta\} \mapsto y$ is Gâuteaux differentiable from $C^1(0, L)^2$ to $H_0^1(0, L)^9$ and the directional derivative for the cost (4.9) are computed together with the first order optimality conditions. Many numerical examples may be found in Ignat, Sprekels and Tiba [12] and in Arnăutu, Sprekels and Tiba [2]. Some of them have a clear physical meaning, which may be interpreted as a validation of the model.

In the case of shells, we consider an open bounded set $\omega \subset I\!R^2$, not necessarily simply connected and $\varepsilon > 0$, "small". We denote by $\Omega = \omega \times] - \varepsilon, \varepsilon[$ and by $p : \omega \to I\!R$ a $C^2(\bar\omega)$ mapping whose graph represents the middle surface of the shell. The shell $\hat{\Omega}$ is obtained via the transformation $\hat{F} : \Omega \to \hat{\Omega}$, $\hat{F}(x_1, x_2, x_3) = (x_1, x_2, p(x_1, x_2)) + x_3 \bar{n}(x_1, x_2, x_3)$ where $\bar{n}$ is the normal vector:

$$\bar{n} = (n_1, n_2, n_3) = \frac{1}{\sqrt{1 + p_1^2 + p_2^2}} (-p_1, -p_2, 1)$$

and where p_1, p_2 are the partial derivatives of p. The shell is assumed to be partially clamped along $\hat{\Gamma}_0 = \hat{F}(\Gamma_0)$, with $\Gamma_0 = \gamma_0 \times] - \varepsilon, \varepsilon[$ and $\gamma_0 \subset \partial\omega$ being some open part. The displacement $\hat{u} \in V(\hat{\Omega}) = \{\hat{v} \in H^1(\hat{\Omega})^3 ;\ \hat{v}|_{\hat{\Gamma}_0} = 0\}$ is supposed to be of the form

$$\hat{u}(\hat{x}) = \bar{u}(x_1, x_2) + x_3 \bar{r}(x_1, x_2)\,,\ (x_1, x_2, x_3) = \hat{F}^{-1}(\hat{x})\,.$$

The unknowns $\bar{u}, \bar{r} \in V(\omega) = \{\bar{v} \in H^1(\omega)^3 ;\ \bar{v}|_{\gamma_0} = 0\}$ represent the displacement of the middle surface of the shell, respectively the modification of the normal vector. This is allowed to change the length as well (that is the elastic material can dilate or contract), which is a generalization of the classical Naghdi model, studied for instance by Blouza [4] under similar regularity conditions. For ε "small", we get $\det J(\bar{x}) \geq c > 0$, $J = \nabla \hat{F}$, which justifies the above construction. If we denote by $(h_{ij}(\bar{x})) = J(\bar{x})^{-1}$, the same approach as for the curved rods, based on the linear elasticity system, generates the following BVP:

$$\begin{aligned}
&\tilde{\lambda} \int_\Omega \Bigg\{ \sum_{i=1}^3 \Bigg[\left(\frac{\partial u_i}{\partial x_1} + x_3 \frac{\partial r_i}{\partial x_1} \right) h_{1i} + \left(\frac{\partial u_i}{\partial x_2} + x_3 \frac{\partial r_i}{\partial x_2} \right) h_{2i} \\
&+ r_i h_{3i} \Bigg] \Bigg\} \Bigg\{ \sum_{j=1}^3 \Bigg[\left(\frac{\partial \mu_j}{\partial x_1} + x_3 \frac{\partial \varrho_j}{\partial x_1} \right) h_{1j} + \left(\frac{\partial \mu_j}{\partial x_2} + x_3 \frac{\partial \varrho_j}{\partial x_2} \right) h_{2j} + \varrho_j h_{3j} \Bigg] \Bigg\} \\
&|\det J(\bar{x})|\, d\bar{x} + 2\tilde{\mu} \int_\Omega \sum_{i=1}^3 \Bigg[\left(\frac{\partial u_i}{\partial x_1} + x_3 \frac{\partial r_i}{\partial x_1} \right) h_{1i} + \left(\frac{\partial u_i}{\partial x_2} + x_3 \frac{\partial r_i}{\partial x_2} \right) h_{2i} \\
&+ r_i h_{3i} \Bigg] \Bigg[\left(\frac{\partial \mu_i}{\partial x_1} + x_3 \frac{\partial \varrho_i}{\partial x_1} \right) h_{1i} + \left(\frac{\partial \mu_i}{\partial x_2} + x_3 \frac{\partial \varrho_i}{\partial x_2} \right) h_{2i} + \varrho_i h_{3i} \Bigg] |\det J(\bar{x})|\, d\bar{x} \\
&+ \tilde{\mu} \int_\Omega \sum_{i<j} \Bigg\{ \Bigg[\left(\frac{\partial u_i}{\partial x_1} + x_3 \frac{\partial r_i}{\partial x_1} \right) h_{1j} + \left(\frac{\partial u_i}{\partial x_2} + x_3 \frac{\partial r_i}{\partial x_2} \right) h_{2j} + r_i h_{3j} + \Bigg(\frac{\partial u_j}{\partial x_1} \\
&+ x_3 \frac{\partial r_j}{\partial x_1} \Bigg) h_{1i} + \left(\frac{\partial u_i}{\partial x_2} + x_3 \frac{\partial r_j}{\partial x_2} \right) h_{2i} + r_j h_{3i} \Bigg] \Bigg[\left(\frac{\partial \mu_i}{\partial x_1} + x_3 \frac{\partial \varrho_i}{\partial x_1} \right) h_{1j} \\
&+ \left(\frac{\partial \mu_i}{\partial x_2} + x_3 \frac{\partial \varrho_i}{\partial x_2} \right) h_{2j} + \varrho_i h_{3j} + \left(\frac{\partial \mu_j}{\partial x_1} + x_3 \frac{\partial \varrho_j}{\partial x_1} \right) h_{1i} + \left(\frac{\partial \mu_j}{\partial x_2} + x_3 \frac{\partial \varrho_j}{\partial x_2} \right) h_{2i} \\
&+ \varrho_j h_{3i} \Bigg] \Bigg\} |\det J(\bar{x})|\, d\bar{x} = \int_\Omega \sum_{l=1}^3 f_l (\mu_l + x_3 \varrho_l) |\det J(\bar{x})|\, d\bar{x} \\
&+ \int_{\partial\Omega - \Gamma_0} \sum_{l=1}^3 \sum_{i,j=1}^3 g_l (\mu_l + x_3 \varrho_l) |\det J(\bar{x})| \sqrt{\nu_i(\bar{x}) g^{ij}(\bar{x}) \nu_j(\bar{x})}\, d\tau\,.
\end{aligned} \tag{4.10}$$

Here, the notations are similar to (4.8). To prove the existence and the uniqueness of the solution $(\bar{u}, \bar{r}) \in V(\omega)^2$ in (4.10), we have established the coercivity of the bilinear form by applying Korn's inequality, Sprekels and Tiba [17]. Moreover, in Arnăutu, Sprekels and Tiba [2], by using an extension technique to $H^1(\mathbb{R}^3)$, it is shown that this coercivity constant is independent of the geometry (of p) in some given classes. We associate to (4.10) the shape optimization problem

$$\operatorname*{Min}_{p \in \mathcal{K}} \{\Pi(p) = j(\bar{y}, \bar{p})\} \tag{4.11}$$

with $\bar{y} = (\bar{u}, \bar{r}) \in H^1(\omega)^6$ and $\mathcal{K} \subset C^2(\bar{\omega})$ closed. The mapping $j : H^1(\omega)^6 \times C^2(\bar{\omega}) \to \mathbb{R}$ is of general type. Some well-known examples of cost functionals and of constraints $\mathcal{K}$ are:

$$j(\bar{y}, p) = |u_1|^2_{H^1(\omega)} + |u_2|^2_{H^1(\omega)} + |u_3|^2_{H^1(\omega)}$$

(minimization of the displacement of the middle surface of the shell), respectively

$$\int_\omega \sqrt{1 + p_1^2 + p_2^2}\, dx_1 dx_2 \leq const$$

(area limitation for the shell).

Theorem 4.2 *If $\mathcal{K} \subset C^2(\bar{\omega})$ is compact and $j : H^1(\omega)^6 \times C^2(\bar{\omega}) \to \mathbb{R}$ is lower semicontinuous, then the shape optimization problem* (4.10), (4.11) *has at least one optimal solution.*

Remark. It is possible to compute directional derivatives of the mapping $p \mapsto \bar{y}$ and to write optimality conditions, Arnăutu, Sprekels and Tiba [2]. However, numerical experiments seem very difficult to perform as the coercivity constant is of the order ε^3 which shows the lack of stability properties in the computations.

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