Institut für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

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submitted: 11th February 1994

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Preprint No. 86
Berlin 1994

Edited by
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A method of constructing of dynamical systems with bounded nonperiodic trajectories *

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Abstract

A fifth-order system is considered for which the existence of a set of bounded trajectories that are neither periodic nor almost periodic is proven by means of analytical methods. The set is situated in the region of dissipation and has a positive Lebesgue measure.

Many numerical results of investigation of strange attractors of dynamical systems with quadratic nonlinearities have been received by now [Lorenz, 1963; Schuster, 1984; Sparrow, 1982; Dolzhansky et al., 1974].
In this paper a fifth-order autonomous system with nonlinearities of quadratic type is considered. It is demonstrated by analytical methods that the system has a set of bounded trajectories which are neither periodic nor almost periodic in any sense [Bohr, 1932; Osipov, 1992]. It is also shown that this set of trajectories is situated in dissipative region, i.e. in the region of contraction of volumes. We have used here certain ideas of O. Perron [Perron, 1930], which were applied by him in order to investigate of irregular systems.

Consider the system

$$\begin{aligned} \dot{u} &= -u^2, \\ \dot{z}_1 &= -uz_2, \\ \dot{z}_2 &= uz_1, \\ \dot{z}_3 &= -az_3, \\ \dot{y} &= (-2a + z_1 + z_2)y + z_3^2 - f(y). \end{aligned} \tag{1}$$

*This work was supported by Institut für Angewandte Analysis und Stochastik, Berlin and the Fund of Fundamental Investigations of the Russian Federation.

Here

$$f(y) = \begin{cases} 0, & \text{for } |y| \le \varepsilon, \\ \nu(y-\varepsilon)^2, & \text{for } y \ge \varepsilon, \\ -\nu(y+\varepsilon)^2, & \text{for } y \le -\varepsilon, \end{cases}$$

where a, ν and ε are positive numbers.

Proposition 1
All trajectories of system (1) with initial data $u(0) > 0$ are bounded on the interval $(0, +\infty)$.

Proof. Integrating the first three equations of system (1) we have

$$u(t) = \frac{1}{t + u(0)^{-1}},$$

$$z_1^2(t) + z_2^2(t) = z_1^2(0) + z_2^2(0).$$

Hence it follows that functions $u(t)$, $z_1(t)$, $z_2(t)$ are bounded for $t \ge 0$.

Since $z_3(t) = e^{-at} z_3(0)$ function $z_3(t)$ is bounded as well. For function $y^2(t)$ we have the relation

$$\frac{1}{2}\frac{d}{dt}\left(y^2(t)\right) = (-2a + z_1(t) + z_2(t))y^2(t) + z_3^2(t)y(t) - f(y(t))y(t). \tag{2}$$

From the fact that $z_1(t)$, $z_2(t)$, $z_3(t)$ are bounded and from the form of $f(y)$ it follows that the expression (2) is negative for great enough values $|y(t)| \ge \alpha$. That is why if $|y(t_0)| \ge \alpha$ for a certain $t_0 \ge 0$ then $|y(t)| \le |y(t_0)|$ for all $t \ge t_0$. Thus the proposition is proven. □

Note that the four first equations of the system may be integrated in a closed form

$$\begin{aligned} u(t) &= \frac{1}{t + u(0)^{-1}}, \\ z_1(t) &= A \cos \log(t + u(0)^{-1}) - B \sin \log(t + u(0)^{-1}), \\ z_2(t) &= A \sin \log(t + u(0)^{-1}) + B \cos \log(t + u(0)^{-1}), \\ z_3(t) &= e^{-at} z_3(0). \end{aligned}$$

where the constants A and B can be determined from the initial data $z_1(0)$, $z_2(0)$.

Note the following obvious properties of the solutions

$$\lim_{t \to +\infty} \dot{u}(t) = 0,$$
$$\lim_{t \to +\infty} \dot{z}_i(t) = 0, \quad i = 1, 2, 3. \tag{3}$$

Proposition 2
For the solutions of system (1) with initial data satisfying the inequalities $u(0) > 0$,

$z_3(0) > 0$, $y(0) > 0$, $A > 0$, $B > 0$, $A + 2B < 2a$ *there exists a sequence* $\{t_k\}$, $t_k \to +\infty$ *such that*

$$\lim_{k\to+\infty} y(t_k) = 0. \tag{4}$$

Proof. First of all we must note that from the inequalities $u(0) > 0$, $z_3(0) > 0$ it follows that $u(t) > 0$, $z_3(t) > 0$, $\forall t \geq 0$. Besides if $y(t) = 0$ we have $\dot{y}(t) = z_3^2(t) > 0$. That is why $y(t) > 0$, $\forall t \geq 0$.

We shall consider at first the case when $\nu = 0$, i.e. $f(y) \equiv 0$. Then a solution $y(t)$ may be written in the form

$$\begin{aligned} y(t) &= \exp[-2at + (t + u(0)^{-1})[A \sin\log(t + u(0)^{-1}) + B \cos\log(t + u(0)^{-1})]] \\ &\quad \times \left(C + \int_0^t \exp[-(\tau + u(0)^{-1})\,(A \sin\log(\tau + u(0)^{-1}) \right. \\ &\quad \left. + B \cos\log(\tau + u(0)^{-1}))]\; z_3^2(0)\, d\tau\right). \end{aligned} \tag{5}$$

Here C is a constant which depends on $y(0)$. Let us use an obvious inequality

$$\int_0^t \exp[-(\tau + u(0)^{-1}))(A \sin\log(\tau + u(0)^{-1}) + B \cos\log(\tau + u(0)^{-1}))]\, d\tau$$

$$\leq \int_0^t \exp[(A + B)(\tau + u(0)^{-1}))]\, d\tau$$

and let $t_k + u(0)^{-1} = \exp(2\pi k)$. Since in this case $A \sin\log(t_k + u(0)^{-1}) = 0$ we shall get

$$\begin{aligned} y(t_k) &\leq \exp[-2at_k + B(t_k + u(0)^{-1})] \\ &\quad \times \left(C + z_3^2(0) \int_0^t \exp[(A + B)(\tau + u(0)^{-1})]\, d\tau\right). \end{aligned}$$

Hence and from the inequality $2a > A + 2B$ it follows that (4) is true in case $\nu = 0$.

Let us compare now the solution $y_0(t)$ of system (1) corresponding to $\nu = 0$ and the solution $y_\nu(t)$ of system (1) corresponding to $\nu > 0$. We suppose here that $y_0(0) = y_\nu(0)$.

From the form of function $f(y)$ we get that $\dot{y}_0(t) \geq \dot{y}_\nu(t)$, $\forall t \geq 0$. Consequently, $y_0(t) \geq y_\nu(t)$, $\forall t \geq 0$. Hence and from the positiveness of $y_\nu(t)$ it follows that if the relation (4) holds for $\nu = 0$ then it holds for any $\nu > 0$ as well. The proposition is proven. □

Proposition 3
For the solutions of system (1) with initial data satisfying the inequalities $u(0) > 0$, $z_3(0) > 0$, $y(0) > 0$, $A > 0$, $B > 0$, $2a + B(1 + e^{-\pi}) < A(1 + e^{-\pi}/2)$ *there exists a sequence* $\{t_j\}$, $t_j \to +\infty$ *such that*

$$y(t_j) \geq \varepsilon, \qquad \forall t_j. \tag{6}$$

Proof. Suppose the opposite, i.e. suppose that the inequality (6) is false. Then without loss of generality we may affirm, that $y(t) \leq \varepsilon$, $\forall t \geq 0$. Then the solution $y(t)$ may be represented in the form of (5). Let $t_j + u(0)^{-1} = \exp(2\pi j + \pi/2)$. In this case the following estimates are true:

$$\exp[-2at_j + (t_j + u(0)^{-1})(A \sin\log(t_j + u(0)^{-1}) + B\cos\log(t_j + u(0)^{-1}))]$$
$$\geq \exp[u(0)^{-1}(A - B) + t_j(-2a + A - B)],$$

$$\int_0^{t_j} \exp[-(\tau + u(0)^{-1})(A\sin\log(\tau + u(0)^{-1}) + B\cos\log(\tau + u(0)^{-1}))]\, d\tau$$
$$\geq \int_{\alpha_j}^{\beta_j} \exp[(A/2 - B)(\tau + u(0)^{-1})]\, d\tau$$
$$\geq (t_j + u(0)^{-1}) \cdot \left(e^{-2\pi/3} - e^{-\pi}\right) \exp\left[(A/2 - B)(t_j + u(0)^{-1})e^{-\pi}\right].$$

Here $\alpha_j = (t_j + u(0)^{-1})e^{-\pi} - u(0)^{-1}$, $\beta_j = (t_j + u(0)^{-1})e^{-2\pi/3} - u(0)^{-1}$.

It follows from the estimates that in case $2a + B(1 + e^{-\pi}) < A(1 + e^{-\pi}/2)$ the relation

$$\lim_{j\to+\infty} y(t_j) = +\infty$$

is true. This fact contradicts the assumption that $y(t) \leq \varepsilon$, $\forall t \geq 0$ and therefore proves the affirmation (6). □

Let us denote the trace of Jacobi matrix of the right handside of system (1) by $Q(u, z_1, z_2, z_3, y)$. It is not difficult to see that

$$Q(u, z_1, z_2, z_3, y) = -2u - 3a + z_1 + z_2 - f'(y).$$

It is also easy to see that for positive functions $u(t)$ and $y(t)$ the following inequality is true

$$\int_0^t Q(u(\tau), z_1(\tau), z_2(\tau), z_3(\tau), y(\tau))\, d\tau$$
$$\leq -3at + (t + u(0)^{-1})(A\sin\log(t + u(0)^{-1}) + B\cos\log(t + u(0)^{-1}))$$
$$- u(0)^{-1}(A\sin\log(u(0)^{-1}) + B\cos\log(u(0)^{-1})).$$

Hence it follows that if $3a > A + B$ then

$$\lim_{t\to+\infty} \int_0^t Q(u(\tau), z_1(\tau), z_2(\tau), z_3(\tau), y(\tau))\, d\tau = -\infty,$$

and we watch the asymptotic contraction of the volume [Leonov & Boichenko, 1992] on these trajectories.

So for the trajectories with the initial data which satisfy the hypotheses of propositions 2 and 3, the conditions of dissipation are fulfilled and there exist sequences $\{t_k\}$ and $\{t_j\}$ for which the relations (3), (4) and (6) are true. It is evident that the trajectory, for which (3), (4) and (6) are satisfied, can be neither periodic nor almost periodic.

Note that conditions on initial data, which guarantee that the latter satisfy the hypotheses of propositions (2) and (3) simultaneously, single out in the phase space a set of positive Lebesgue measure.

Let us denote by $z_{31}(t)$ and $z_{32}(t)$ solutions of equations $\dot{z}_3 = -az_3$ with initial data $z_{31}(0)$ and $z_{32}(0)$. Let $y_1(t)$ be a solution of system (1) with positive intial data $u(0), A, B, z_{31}(0)$, $y_1(0)$ and $y_2(t)$ be a solution with positive initial data $u(0), A, B, z_{32}(0), y_2(0) = y_1(0)$.

Proposition 4 *(about strong instability)* *Suppose that following inequalities are fulfilled:*

$$2a + B(1 + e^{-\pi}) < A(1 + e^{-\pi}/2), \tag{7}$$

$$\mu z_{31}(0)^2 < z_{31}(0)^2 - z_{32}(0)^2, \tag{8}$$

where $\mu \in (0,1)$. *Suppose also that* $y_1(0)$ *sufficiently small with respect to parameters* a, ε, ν *and initial data* $u(0), A, B, z_{31}(0), z_{32}(0), \mu$.

Then there exists a number $T > 0$ *such that*

$$y_1(T) - y_2(T) \geq \mu\varepsilon.$$

Proof. Let us consider the time $T > 0$ such that $y_1(T) = \varepsilon, y_1(t) \in (0, \varepsilon), \quad \forall\, t \in (0, T)$. Existence of this T was proved in proposition 3.

From the expression (5) it follows that $y_2(t) \leq y_1(t) < \varepsilon, \forall\, t \in (0, T)$. Hence, from the inequality (8) and from the expression (5) it follows that

$$y_1(T) - y_2(T) > \mu y_1(T) - \mu C \exp[-2aT +$$
$$+(T + u(0)^{-1})(A \sin\log(T + u(0)^{-1}) + B \cos\log(T + u(0)^{-1}))]$$

Here $C \to 0$ as $y_1(0) \to 0$. Hence it follows that the proposition 4 is proven. □

From the propositions 1 – 4 it follows that B–attractor of the set of trajectories wich has been considered above is the set of equilibria and heteroclinic orbits on the cylinders

$$\left\{u = 0, z_1^2 + z_2^2 = A^2 + B^2, z_3 = 0, y \geq 0\right\}.$$

See Fig.1.

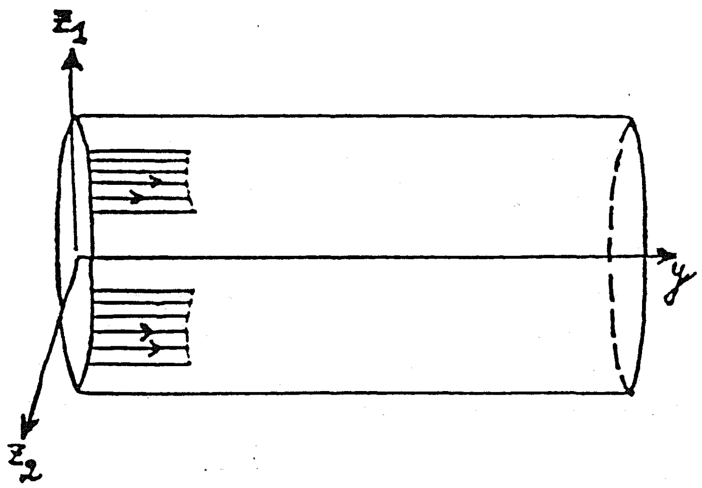

Figure 1. Element of B–attractor

Hence this B–attractor has simple trajectories. But in its small neighborhood there exist irregular trajectories.

Proposition 5

In ω-limit set of solution with initial data satisfying the inequalities from propositions 2,3 *there exists at least one heteroclinic orbit.*

Proof. Let us consider sufficiently large time T such that

$$2a = z_1(T) + z_2(T), \quad \dot{z}_1(T) + \dot{z}_2(T) > 0.$$

From the inequality for $y(t_k)$ in proof of proposition 2 it follows that there exists a positive number α such that

$$y(T) \leq z_3^2(0) \ \exp\left(-\alpha T\right).$$

From this inequality and from the expressions

$$\sqrt{\dot{z}_1^2 + \dot{z}_2^2} = u\sqrt{(A^2 + B^2)} = \frac{1}{t + u(0)^{-1}}\sqrt{(A^2 + B^2)}$$

it follows that there exists positive number β (which depend on α, A, B) and $T_K > T$ such that

$$z_1(T_K) + z_2(T_K) = 2a + \beta, \quad \dot{z}_1(T_K) + \dot{z}_2(T_K) > 0, \quad \lim_{K \to \infty} y(T_K) = 0.$$

Hence it follows that the heteroclinic orbit of equation

$$\dot{y} = \gamma y - f(y)$$

is for some γ contained in the ω-limit set of the solution under consideration. Thus the proposition is proven. □

Conjecture 1

In ω-limit set of solution with initial data satisfying the inequalities from propositions 2,3 there exists a lot of heteroclinic orbits.

Conjecture 2

Trajectories under consideration "fill densely" certain part of B-attractor.
We underline the difference in essence between behavior of trajectories in neighborhood of B-attractor under consideration and behavior of trajectories in neighborhood of homoclinic orbits. See for example [Wiggins, 1990; Guckenheimer and Holmes, 1986; Chua et al., 1986].

It is not difficult to construct various dynamical systems with similar properties. For example

$$
\begin{aligned}
\dot{u} &= -u^2, \\
\dot{z}_1 &= -uz_2, \\
\dot{z}_2 &= uz_1, \\
\dot{z}_3 &= -az_3, \\
\dot{y} &= (-a + z_1 + z_2)y + z_3 - f(y).
\end{aligned}
$$

or

$$
\begin{aligned}
\dot{u} &= -u^2 \text{sign } u, \\
\dot{z}_1 &= -uz_2 - f(z_1), \\
\dot{z}_2 &= uz_1 - f(z_2), \\
\dot{z}_3 &= -az_3, \\
\dot{y} &= (-b + z_1 + z_2)y + z_3 - f(y).
\end{aligned}
$$

Last system is dissipative also in sence of Levinson. In other words in this case there exists global compact attractor.
See for example various definitions of dissipation in [Leonov et al., 1992].

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