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On a decay rate for a Landau-Ginzburg system with viscosity for martensitic phase transitions in shape memory alloys

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Abstract

In this paper, we investigate the decay rate of stabilization of the solution of the system of partial differential equations governing the dynamics of martensitic phase transitions in shape memory alloys under the presence of a viscous stress. The corresponding free energy is assumed in Landau-Ginzburg form and nonconvex as a function of the order parameter. We prove that for appropriate constants, which appear in the above-mentioned model, we can decide upon the exponential decrease of the solution to its attractor for time tending to infinity.

1 Introduction

In the present paper, we continue the study of the asymptotic behaviour of the solutions to a system that arises in the thermo-mechanical developments in a one-dimensional heat-conducting viscous solid of constant mass density ρ (assumed $\rho = 1$). The solid is subjected to heating and loading. We think of metallic solids that not only respond to a change of the strain ϵ by an elastic stress $\sigma = \sigma(\epsilon)$, but also to a change of the curvature of their metallic lattice by a couple stress $\mu = \mu(\epsilon_x)$.

We assume that the Helmholtz free energy density F is a potential of Landau-Ginzburg form, i.e.

$$F = F(\epsilon, \epsilon_x, \theta), \tag{1.1}$$

where θ denotes the absolute temperature. To cover systems modelling first-order, stress-induced and temperature-induced solid-solid phase transitions accompanied by hysteresis phenomena, we do not assume that F is a convex function of the order parameter ϵ.

A particular class of materials, in which both temperature-induced and stress-induced first-order phase transitions leading to a rather spectacular hysteretic behaviour occur, are the so-called shape memory alloys. For an account of the physical properties of shape memory alloys, we refer the reader to Chapter 5 in the monograph [1]. The model we investigate has the following form

$$u_{tt} - \gamma u_{xxt} - \frac{\partial}{\partial x}(f_1(u_x)\theta + f_2(u_x)) + \delta u_{xxxx} = 0, \tag{1.2}$$

$$C_V\theta_t - k\theta_{xx} - f_1(u_x)\theta u_{xt} - \gamma u_{xt}^2 = 0 \tag{1.3}$$

for $(x,t) \in (0,1) \times (0,\infty)$, with positive physical constants α_i, $i = 1,2,3$, k, γ, δ, θ_1, and C_V, and where

$$\begin{gathered} f_1(z) = F_1'(z), \text{ with } F_1(z) = \alpha_1 z^2, \\ f_2(z) = F_2'(z), \text{ with } F_2(z) = \alpha_3 z^6 - \alpha_2 z^4 - \alpha_1\theta_1 z^2. \end{gathered} \tag{1.4}$$

The system is endowed with the boundary conditions

$$u|_{x=0} = u_{xx}|_{x=0} = 0,\ u_x|_{x=1} = (\gamma u_{xt} - \delta u_{xxx} + \sigma_1)|_{x=1} = 0, \tag{1.5}$$

with

$$\sigma_1 = f_1(u_x)\theta + f_2(u_x), \tag{1.6}$$

and

$$\theta_x|_{x=0,1} = 0. \tag{1.7}$$

We prescribe the initial state by the initial data

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x) > 0,\ x \in [0,1]. \tag{1.8}$$

We can divide the existing literature on the system (1.2)-(1.7) into three parts. The first part covers the case when $\gamma > 0$ and $\delta > 0$. For the existence of the solution of the model (1.2)-(1.7), we refer the reader to Hoffmann and Zochowski [5]. Sprekels, Zheng and Zhu in [10] investigated the asymptotic behaviour of the solution of the system (1.2)-(1.7). In [9], the existence of the maximal compact attractor was derived. The second case $\delta = 0$ and $\gamma > 0$ was studied in Racke and Zheng [7] from the point of view of the global existence, uniqueness and asymptotic behaviour of weak solutions if both ends of the rod are insulated and if at least one end is stress-free. The third part covers the case when $\delta > 0$ and $\gamma = 0$. For the global existence and uniqueness, we refer the reader to Sprekels and Zheng [8]. We also refer to the works [2] and [3].

But one question still unanswered is whether we can obtain additional information about the decay rate for the model (1.2)-(1.7). The main theorem of this paper provides an answer for the case $\delta > 0$ and $\gamma > 0$. But in Section 4, we give sufficient conditions that also cover the case $\delta = 0$. The main result of this paper is the following.

Theorem 1.1 *Suppose that the solution u of the system* (1.2)-(1.7) *satisfies the compatibility conditions $u_t|_{x=0} = (u_0)_{xx}|_{x=0} = u_{xt}|_{x=1} = 0$. Let the constants θ_1, δ, α_i, $i = 1,2,3$, C_V, k, γ be positive. In addition, let there exist constants $\epsilon > 0$ and $K > 1$ such that*

$$\epsilon + \frac{\gamma\|\theta(t)\|_\infty}{K} + \frac{1}{2K} < \frac{\gamma}{2},\ \epsilon + \alpha_1\theta_1 + \frac{\alpha_2^2}{\alpha_3} < \frac{\delta}{2} \tag{1.9}$$

and

$$\frac{4\alpha_2^2}{\alpha_3} + \frac{2\|\theta(t)\|_\infty^2\alpha_1^2}{\gamma\epsilon} + 2\|\theta(t) - \theta_1\|_\infty\alpha_1 + \|\theta(t)\|_\infty^4\frac{2\alpha_1^2}{\epsilon K} < \delta \tag{1.10}$$

for a.a. $t \in (0,\infty)$. Then

$$\|u_t(t)\|_2^2 + \|u_{xx}(t)\|_2^2 + \|\theta(t) - \overline{\theta}(t)\|_2^2 \le k_1 e^{-k_2 t}, \tag{1.11}$$

where the constants k_1, $k_2 > 0$ only depend on the initial state.

Remark 1.2 From the main result in [10] it may be concluded that $\|\theta_x(t)\|_2 \to 0$ for $t \to \infty$ and $\|\theta\|_{L^\infty(0,\infty,L^1(0,1))} \leq c_0$. Combining these two facts, we can assert that

$$\|\theta(t)\|_\infty \leq \|\theta_x(t)\|_1 + \|\theta\|_{L^\infty(0,\infty;L^1(0,1))} \leq c_1(t) + c_0 \leq c_2, \tag{1.12}$$

where the constant c_0 only depends on the initial state, and where $c_1(t) \to 0$ for $t \to \infty$. Moreover, the constant c_2 is of the form $c_2 = c(1+\alpha_1^2)$, which is a consequence of Lemma 2.5 in [10].

It is worth pointing out that the method used here also provides similar properties for the solutions to the 1-D Navier-Stokes equations cf. [11].

In Section 2, we will look more closely at the energy conservation of the equations (1.2)-(1.7). Section 3 is devoted to the proof of our main result. Section 4 provides other conditions which lead to an exponential decrease in connection with the paper [4].

The notation in this paper will be as follows: $L^p(0,1)$, $1 \leq p \leq \infty$, and $L^p(0,T;L^q(0,1))$, $1 \leq p,q \leq \infty$, $0 < T \leq \infty$, respectively, denote the standard Lebesgue and Bochner spaces. By $\|\cdot\|_p$ and $\|\cdot\|_{L^p(0,T;L^q(0,1))}$, we denote the corresponding norms. We also use the denotation $\overline{\theta}(t)$ for the integral $\int_0^1 \theta(x,t)\,dx$.

2 Energy inequalities

In this section, we proceed with the study of the energy identity, and we derive new forms of the energy inequality for the solution to the system (1.2)-(1.7).

Lemma 2.1 *Let the couple* (θ, u) *be the classical solution to the model* (1.2)-(1.7). *Then*

$$\frac{d}{dt}\left[\int_0^1 \left(\frac{u_t^2}{2} + \frac{\delta u_{xx}^2}{2} + F_2(u_x) + C_V\theta\right)\,dx\right] = 0, \tag{2.13}$$

$$\frac{d}{dt}\left[\int_0^1 \left(\frac{u_t^2}{2} + \frac{\delta u_{xx}^2}{2} + F_2(u_x)\right)\,dx\right] - \|\theta(t)\|_\infty^2 \frac{2\alpha_1^2}{\gamma}\int_0^1 u_x^2\,dx + \frac{\gamma}{2}\int_0^1 u_{xt}^2\,dx \leq 0, \tag{2.14}$$

and

$$\frac{d}{dt}\left[\int_0^1 \left(\frac{u_t^2}{2} + \frac{\delta u_{xx}^2}{2} + F_2(u_x) + \frac{C_V(\theta-\overline{\theta})^2}{2K}\right)\,dx\right] + \frac{k}{K}\int_0^1 (\theta-\overline{\theta})_x^2\,dx$$
$$-\frac{1}{K}\int_0^1 f_1(u_x)\theta(\theta-\overline{\theta})u_{xt}\,dx - \frac{\gamma}{K}\int_0^1 u_{xt}^2(\theta-\overline{\theta})\,dx$$
$$-\|\theta(t)\|_\infty^2 \frac{2\alpha_1^2}{\gamma}\int_0^1 u_x^2\,dx + \frac{\gamma}{2}\int_0^1 u_{xt}^2\,dx \leq 0. \tag{2.15}$$

P r o o f: To deduce (2.13), take u_t as a test function for the equation (1.2). This gives

$$\frac{d}{dt}\left[\int_0^1 \left(\frac{u_t^2}{2} + \frac{\delta u_{xx}^2}{2} + F_2(u_x)\right)\, dx\right] + \int_0^1 f_1(u_x)\theta u_{xt}\, dx + \gamma \int_0^1 u_{xt}^2\, dx = 0 \quad (2.16)$$

because the boundary terms satisfy the identity

$$-\gamma[u_{xt}u_t]_0^1 - [f_1(u_x)\theta u_t + f_2(u_x)u_t]_0^1 + \delta[u_{xxx}u_t]_0^1 - \delta[u_{xx}u_{xt}]_0^1 = 0.$$

This is a consequence of (1.5)-(1.8). Integrating (1.3), and adding this equation to the identity (2.16), we get (2.13).

Now, we can derive the estimate

$$\left|\int_0^1 f_1(u_x)\theta u_{xt}\, dx\right| = \left|\int_0^1 2\alpha_1 u_x \theta u_{xt}\, dx\right| \le 2\|\theta(t)\|_\infty \int_0^1 |\alpha_1 u_x u_{xt}|\, dx$$
$$\le \|\theta(t)\|_\infty^2 \frac{2\alpha_1^2}{\gamma}\int_0^1 u_x^2\, dx + \frac{\gamma}{2}\int_0^1 u_{xt}^2\, dx. \quad (2.17)$$

Combining (2.16) and (2.17), we deduce (2.14).

Using $\frac{\theta-\overline{\theta}}{K}$ as a test function in (1.3) yields the identity

$$\frac{d}{dt}\int_0^1 \frac{C_V(\theta-\overline{\theta})^2}{2K}\, dx + \frac{k}{K}\int_0^1 (\theta-\overline{\theta})_x^2\, dx - \frac{1}{K}\int_0^1 f_1(u_x)\theta(\theta-\overline{\theta})u_{xt}\, dx$$
$$-\frac{\gamma}{K}\int_0^1 u_{xt}^2(\theta-\overline{\theta})\, dx = 0 \quad (2.18)$$

for some K. After adding to (2.14) we conclude (2.15). □

From Lemma 2.1, we obtain the following result.

Consequence 2.1 *Under the assumptions of Theorem 1.1, we have*

$$\|u_t\|_{L^\infty(0,\infty;L^2(0,1))}^2 + \|\theta\|_{L^\infty(0,\infty;L^1(0,1))} \le c_0, \quad (2.19)$$

as well as

$$\left(\frac{\delta}{2} - \alpha_1\theta_1 - \frac{\alpha_2^2}{\alpha_3}\right)\|u_{xx}\|_{L^\infty(0,\infty;L^2(0,1))}^2 + \|F_3(u_x)\|_{L^\infty(0,\infty;L^1(0,1))} \le c_0, \quad (2.20)$$

where the constant c_0 only depends on the initial state, and where the function $F_3(z)$, $F_3(z) \ge 0$, is defined by

$$F_3(z) := \begin{cases} \alpha_3 z^6 - \alpha_2 z^4 & \text{if } |z| \ge \sqrt{\frac{\alpha_2}{\alpha_3}} \\ \alpha_3 z^6 & \text{if } |z| < \sqrt{\frac{\alpha_2}{\alpha_3}}. \end{cases} \quad (2.21)$$

P r o o f: This is an obvious consequence of (2.13) and the inequality $\|u_x\|_2 \leq \|u_{xx}\|_2$, because we can easily deduce that

$$F_2(z_i) = 0 \text{ for } z_1 = \sqrt{\frac{\alpha_2}{\alpha_3}},\ z_2 = -\sqrt{\frac{\alpha_2}{\alpha_3}},\ z_3 = 0. \qquad \square$$

3 Proof of Theorem 1.1

First we complete the form of the energy inequality. Using u as a test function for the equation (1.2) we derive the identity

$$\frac{d}{dt}\left[\int_0^1 u_t u + \frac{\gamma u_x^2}{2}\ dx\right] - \int_0^1 u_t^2\ dx$$

$$+\int_0^1 \left(f_1(u_x)\theta u_x + f_2(u_x)u_x\right)\ dx + \delta\int_0^1 u_{xx}^2\ dx = 0, \tag{3.22}$$

since

$$-\gamma[u_{xt}u]_0^1 - [f_1(u_x)\theta u + f_2(u_x)u]_0^1 + \delta[u_{xxx}u]_0^1 - \delta[u_{xx}u_x]_0^1 = 0.$$

This is a consequence of the boundary and compatibility conditions.

We multiply the equation (3.22) with ϵ, $\epsilon > 0$, and we add this expression to (2.15). This leads to the inequality

$$\frac{d}{dt}V_\epsilon(t) + W_\epsilon(t) \leq 0,\ t \in (0,\infty), \tag{3.23}$$

where

$$V_\epsilon(t) := \int_0^1 \left(\frac{u_t^2}{2} + \frac{\delta u_{xx}^2}{2} + F_2(u_x) + \frac{C_V(\theta-\overline{\theta})^2}{2K} + \epsilon u_t u + \epsilon\frac{\gamma u_x^2}{2}\right)\ dx, \tag{3.24}$$

and

$$W_\epsilon(t) := -\|\theta(t)\|_\infty^2 \frac{2\alpha_1^2}{\gamma}\int_0^1 u_x^2\ dx + \frac{\gamma}{2}\int_0^1 u_{xt}^2\ dx$$

$$+\frac{k}{K}\int_0^1 (\theta-\overline{\theta})_x^2\ dx - \frac{1}{K}\int_0^1 f_1(u_x)\theta(\theta-\overline{\theta})u_{xt}\ dx - \frac{\gamma}{K}\int_0^1 u_{xt}^2(\theta-\overline{\theta})\ dx$$

$$-\epsilon\int_0^1 u_t^2\ dx + \epsilon\int_0^1 2\alpha_1(\theta-\theta_1)u_x^2 + 6\alpha_3 u_x^6 - 4\alpha_2 u_x^4\ dx + \epsilon\delta\int_0^1 u_{xx}^2\ dx. \tag{3.25}$$

The only point remaining concerns the behaviour of the functionals $W_\epsilon(t)$ and $V_\epsilon(t)$. If we can prove that

$$W_{\epsilon_0}(t) \geq k(\epsilon_0)V_{\epsilon_0}(t) \geq 0 \tag{3.26}$$

for some constant $k(\epsilon_0) > 0$ and all $t \in (0,\infty)$, then the inequality (1.11) follows.

First we estimate the integrals contained in $V_\epsilon(t)$ one by one. It is easy to check that

$$\epsilon \int_0^1 |u_t u| \; dx \leq \epsilon \left(\int_0^1 \frac{u_t^2}{2} \; dx + \int_0^1 \frac{u^2}{2} \; dx \right)$$

$$\leq \epsilon \int_0^1 \frac{u_t^2}{2} \; dx + \epsilon \int_0^1 \frac{u_{xx}^2}{2} \; dx \tag{3.27}$$

and

$$\epsilon \frac{\gamma}{2} \int_0^1 u_x^2 \; dx \leq \epsilon \frac{\gamma}{2} \int_0^1 u_{xx}^2 \; dx. \tag{3.28}$$

This follows immediately from the Young inequality and from the fact that $\|u\|_2 \leq \|u_x\|_2 \leq \|u_{xx}\|_2$. Hence, we have that

$$\int_0^1 \left((1-\epsilon) \frac{u_t^2}{2} + (\delta - \epsilon) \frac{u_{xx}^2}{2} + F_2(u_x) + \frac{C_V(\theta - \overline{\theta})^2}{2K} \right) \; dx \leq V_\epsilon(t)$$

$$\leq \int_0^1 \left((1+\epsilon) \frac{u_t^2}{2} + (\delta + \epsilon + \gamma\epsilon) \frac{u_{xx}^2}{2} + F_3(u_x) + \frac{C_V(\theta - \overline{\theta})^2}{2K} \right) \; dx. \tag{3.29}$$

In the next step, we investigate the functional $W_\epsilon(t)$. We conclude from the compatibility conditions that

$$\epsilon \int_0^1 u_t^2 \; dx \leq \epsilon \int_0^1 u_{xt}^2 \; dx. \tag{3.30}$$

It is obvious that the inequality

$$2\epsilon\alpha_1 \int_0^1 u_x^2 (\theta - \theta_1) \; dx \leq 2\epsilon \|\theta(t) - \theta_1\|_\infty \alpha_1 \int_0^1 u_{xx}^2 \; dx \tag{3.31}$$

is fulfilled. It is also easy to check that

$$f_2(z)z = 6\alpha_3 z^6 - 4\alpha_2 z^4 \geq 4(\alpha_3 z^6 - \alpha_2 z^4) + 2\alpha_3 z^6$$

$$\geq 4F_3(z) - \frac{4\alpha_2^2}{\alpha_3} F_4(z) + 2\alpha_3 z^6, \tag{3.32}$$

with

$$F_4(z) := \begin{cases} z^2 & \text{if } z \leq \sqrt{\frac{\alpha_2}{\alpha_3}} \\ 0 & \text{if } z > \sqrt{\frac{\alpha_2}{\alpha_3}}. \end{cases} \tag{3.33}$$

According to the inequality above, we have

$$\epsilon \int_0^1 f_2(u_x) u_x \; dx \geq 4\epsilon \int_0^1 F_3(u_x) \; dx - \epsilon \frac{4\alpha_2^2}{\alpha_3} \int_0^1 u_{xx}^2 \; dx. \tag{3.34}$$

We easily deduce that the estimate

$$\|\theta(t)\|_\infty^2 \frac{2\alpha_1^2}{\gamma} \int_0^1 u_x^2 \; dx \leq \|\theta(t)\|_\infty^2 \frac{2\alpha_1^2}{\gamma} \int_0^1 u_{xx}^2 \; dx \tag{3.35}$$

holds.

The Poincare inequality provides that

$$\frac{k}{K}\int_0^1 (\theta-\overline{\theta})_x^2\ dx \geq \frac{k}{K}\int_0^1 (\theta-\overline{\theta})^2\ dx. \tag{3.36}$$

Applying the Young inequality, we obtain the estimate

$$\frac{1}{K}\int_0^1 |f_1(u_x)\theta(\theta-\overline{\theta})u_{xt}| \leq \|\theta(t)\|_\infty^2 \frac{2\alpha_1}{K}\int_0^1 |u_x u_{xt}|\ dx$$

$$\leq \|\theta(t)\|_\infty^4 \frac{2\alpha_1^2}{K}\int_0^1 u_{xx}^2\ dx + \frac{1}{2K}\int_0^1 u_{xt}^2\ dx. \tag{3.37}$$

In the same manner, we can derive that

$$\frac{\gamma}{K}\int_0^1 u_{xt}^2|\theta-\overline{\theta}|\ dx \leq \frac{\gamma\|\theta(t)\|_\infty}{K}\int_0^1 u_{xt}^2\ dx. \tag{3.38}$$

Combining the above estimates, we have

$$W_\epsilon(t) \geq \left(\frac{\gamma}{2}-\epsilon-\frac{\gamma\|\theta(t)\|_\infty}{K}-\frac{1}{2K}\right)\int_0^1 u_{xt}^2\ dx + \frac{k}{K}\int_0^1 (\theta-\overline{\theta})^2\ dx$$

$$+\epsilon\left(\delta-\frac{4\alpha_2^2}{\alpha_3}-\frac{2\|\theta(t)\|_\infty^2\alpha_1^2}{\gamma\epsilon}-2\|\theta(t)-\theta_1\|_\infty\alpha_1-\|\theta(t)\|_\infty^4\frac{2\alpha_1^2}{\epsilon K}\right)\times$$

$$\int_0^1 u_{xx}^2\ dx + 4\epsilon\int_0^1 F_3(u_x)\ dx. \tag{3.39}$$

We conclude from (3.29) and (3.39) that there exists a constant $k(\epsilon_0)$ such that (3.26) is fulfilled, and therefore

$$\frac{d}{dt}V_\epsilon(t) + k(\epsilon_0)V_\epsilon(t) \leq 0. \tag{3.40}$$

An easy computation shows that $V_\epsilon(t) \geq 0$, and this finishes the proof. □

Consequence 3.1 *Suppose that the constants from Theorem 1.1 satisfy*

$$\epsilon+\frac{\gamma\|\theta(t)\|_\infty}{K}+\frac{1}{2K}<\frac{\gamma}{2},\ \ \epsilon+\alpha_1\theta_1+\frac{\alpha_2^2}{\alpha_3}<\frac{\gamma\epsilon}{2} \tag{3.41}$$

and

$$\frac{4\alpha_2^2}{\alpha_3}+\frac{2\|\theta(t)\|_\infty^2\alpha_1^2}{\gamma\epsilon}+2\|\theta(t)-\theta_1\|_\infty\alpha_1+\|\theta(t)\|_\infty^4\frac{2\alpha_1^2}{\epsilon K}<\delta \tag{3.42}$$

for a.a. $t\in(0,\infty)$. *Then*

$$\|u_t(t)\|_2^2+\|u_{xx}(t)\|_2^2+\|\theta(t)-\overline{\theta}(t)\|_2^2 \leq k_3 e^{-k_4 t}, \tag{3.43}$$

where the constants $k_3,\ k_4>0$ *depend only on the initial state.*

P r o o f: We can use the term $\frac{\epsilon\gamma}{2}\int_0^1 u_x^2\,dx$, which appears in the functional $V_\epsilon(t)$, to verify (3.43). $\square$

Consequence 3.2 *If the relations*

$$\epsilon < \frac{\gamma}{2},\ \epsilon + \alpha_1\theta_1 + \frac{\alpha_2^2}{\alpha_3} < \frac{\delta}{2} \tag{3.44}$$

and

$$\frac{4\alpha_2^2}{\alpha_3} + \frac{2\|\theta(t)\|_\infty^2\alpha_1^2}{\gamma\epsilon} + 2\|\theta(t)-\theta_1\|_\infty\alpha_1 < \delta \tag{3.45}$$

hold for a.a. $t \in (0,\infty)$*, then*

$$\|u_t(t)\|_2^2 + \|u_{xx}(t)\|_2^2 \le k_5 e^{-k_6 t}, \tag{3.46}$$

where the constants $k_5,\ k_6 > 0$ *depend only on the initial state.*

P r o o f: We can easily construct the functionals $\widetilde{V}_\epsilon(t)$ and $\widetilde{W}_\epsilon(t)$ adding (2.14) to (3.22). The rest of the proof proceeds in a similar way as above. $\square$

4 Remarks

Now, we show possible modifications of the proof, which lead to simpler forms of (1.9) and (1.10). We recommend the reader to compare these new results with [4]. There it was proved that the only solution to the steady state equation is $u = 0$ if inequality (4.49) bellow holds.

Lemma 4.1 *Suppose positive constants* $\delta,\ \gamma,\ \alpha_i,\ i = 1,2,3,\ \theta_1$*, and constants* $\epsilon > 0$ *and* $K > 1$ *are given such that*

$$\epsilon + \frac{\gamma\|\theta(t)\|_\infty}{K} + \frac{1}{2K} < \frac{\gamma}{2},\ \epsilon + \frac{\alpha_2^2}{\alpha_3} < \frac{\delta}{2}. \tag{4.47}$$

If, in addition, the inequalities

$$\frac{2\|\theta(t)\|_\infty^2\alpha_1^2}{\gamma\epsilon} + \|\theta(t)\|_\infty^4\frac{2\alpha_1^2}{\epsilon K} < \delta \tag{4.48}$$

and

$$\theta(x,t) > \theta_1 + \frac{\alpha_2^2}{3\alpha_3\alpha_1} \tag{4.49}$$

hold for all $x \in [0,1]$ *and all* $t \ge t_0$*, then*

$$\|u_t(t)\|_2^2 + \|u_{xx}(t)\|_2^2 + \|\theta(t) - \overline{\theta}(t)\|_2^2 \le k_7 e^{-k_8 t} \tag{4.50}$$

for $t \ge t_0$*, where the constants* $k_7,\ k_8 > 0$ *depend only on the initial state.*

P r o o f: We rewrite (3.31) and (3.32) as

$$6\alpha_3 z^6 - 4\alpha_2 z^4 + 2(\theta(x,t) - \theta_1)\alpha_1 z^2 \geq 4F_3(z) + F_5(z), \tag{4.51}$$

where

$$F_5(z) := \begin{cases} 6\alpha_3 z^6 - 4\alpha_2 z^4 + 2(\theta(x,t) - \theta_1)\alpha_1 z^2 & \text{if } |z| \leq \sqrt{\frac{\alpha_2}{\alpha_3}} \\ 0 & \text{if } |z| > \sqrt{\frac{\alpha_2}{\alpha_3}} \end{cases}$$

It is easily seen that

$$4\alpha_2 z^4 = 4\alpha_2 \frac{\sqrt{12\alpha_3}}{4\alpha_2} z^3 \frac{4\alpha_2}{\sqrt{12\alpha_3}} z \leq 6\alpha_3 z^6 + \frac{2\alpha_2^2}{3\alpha_3} z^2. \qquad \square$$

Remark 4.2 If we look more closely at the functional

$$\mathcal{F}_1(\theta_1, u_x, u_{xx}) := \int_0^1 F_6(\theta_1, u_x, u_{xx})\, dx,$$

where

$$F_6(\theta_1, u_x, u_{xx}) := -\alpha_1\theta_1 u_x^2 - 4\alpha_2 u_x^4 + 6\alpha_3 u_x^2 + \delta u_{xx}^2,$$

we can see that, using the condition (1.9) and the inequality $\|u_x\|_2^2 \leq \|u_{xx}\|_2^2$, we can assert that the functional $\mathcal{F}_1$ has only global minimum for $u_x = u_{xx} = 0$.

Remark 4.3 If we define the function

$$F_8(\theta, u_x, u_{xx}) := \alpha_1(\theta - \theta_1)u_x^2 - \alpha_2 u_x^2 + \alpha_3 u_x^6 + \delta u_{xx}^2$$

and the functional

$$\mathcal{F}_2(\theta, u_x, u_{xx}) := \int_0^1 F_8(\theta, u_x, u_{xx})\, dx,$$

then the conditions (4.47)-(4.49) imply that this functional has only one global minimum for $u_x = u_{xx} = 0$ using the estimate $\|u_x\|_2^2 \leq \|u_{xx}\|_2^2$.

Lemma 4.4 *Suppose that there exist constants γ and K such that that the estimates (3.41) and*

$$\theta(x,t) > \max\left\{\theta_1 + \frac{2\alpha_2^2}{3\alpha_3\alpha_1}, \frac{2\|\theta(t)\|_\infty^2\alpha_1}{\gamma\epsilon} + \|\theta(t)\|_\infty^4 \frac{2\alpha_1}{\epsilon K}\right\} \tag{4.52}$$

hold for all $x \in [0,1]$ and all $t \geq t_0$. Then

$$\|u_t(t)\|_2^2 + \|u_x(t)\|_2^2 + \|\theta(t) - \overline{\theta}(t)\|_2^2 \leq k_9 e^{-k_{10}t}. \tag{4.53}$$

This result also holds for the case $\delta = 0$.

P r o o f: It is easy to check the assertion of this remark using (3.35), (3.37) and a modification of the estimate (4.51). $\square$

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