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Strong uniqueness for cyclically symbiotic branching diffusions

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ABSTRACT. A uniqueness problem raised in 2001 for critical cyclically catalytic super-Brownian motions is solved in the simplified space-less case, that is, for cyclically catalytic branching diffusions $\mathbf{X}$. More precisely, $\mathbf{X}$ is characterized as the unique strong solution of a singular stochastic equation.

1. INTRODUCTION

1.1. **Motivation.** In Fleischmann and Xiong [FX01], a critical cyclically catalytic super-Brownian motion $\mathbf{X} = (X^0, \ldots, X^{K-1})$ in R was constructed as a strong Markov solution to a martingale problem involving $K \geq 2$ continuous function-valued processes $t \mapsto X_t^k$. But uniqueness in law of solutions is known only in the special case $K = 2$ of only two species, that is, for the mutually catalytic branching process of Dawson and Perkins [DP98] and Mytnik [Myt98]. The main reason for this is that only in the strongly symmetric case of two species the model has an exponential self-duality property, and if $K > 2$, no dual (or approximate dual) process has been found so far, carrying enough information to characterize the process. It is true that there are moment dual processes of $\mathbf{X}$, but the moments of $\mathbf{X}$ seem to grow so fast [for the space-less case, see Proposition A2 in the appendix] that Carleman's (sufficient) condition for the moment problem to be well-posed is not satisfied.

In the present paper, we simplify the problem by restricting to the "zero-dimensional" case, that is, we drop the space coordinate in the model. Then, $\mathbf{X}$ is a diffusion in R_+^K [see (1) below] with some local non-Lipschitz coefficients. On the other hand, in our space-less case, we allow non-criticality terms in the equation as well as correlations between the noises, the latter in the spirit of Etheridge and Fleischmann [EF03].

1.2. **Model and result.** Fix an integer $K \geq 1$, and denote by $\mathsf{K} := \{0, \ldots, K-1\}$ the cyclic group with addition modulo K. Consider the following stochastic equation

$$(1)\quad \begin{cases} \mathrm{d}X_s^k \;=\; \displaystyle\sum_{l\,\in\,\mathsf{K}\setminus\{k\}} \alpha_{l,k} X_s^l \,\mathrm{d}s + \beta_k X_s^k \,\mathrm{d}s + \sqrt{\gamma_k X_s^{k-1} X_s^k}\,\mathrm{d}W_s^k, \quad s > 0,\; k \in \mathsf{K}, \\ \qquad\qquad \text{with initial condition} \quad \mathbf{X}_0 \,=\, \mathbf{a} \,=\, (a_0, \ldots, a_{K-1}) \in \mathsf{R}_+^K \end{cases}$$

for a diffusion process $\mathbf{X} = (X^k)_{k\in\mathsf{K}}$ in R_+^K. Here $\boldsymbol{\alpha} = (\alpha_{l,k})_{l,k\in\mathsf{K},\, l\neq k} \geq 0$, $\boldsymbol{\beta} = (\beta_k)_{k\in\mathsf{K}}$ and $\boldsymbol{\gamma} = (\gamma_k)_{k\in\mathsf{K}} > 0$ have constant entries. Moreover, $\mathbf{W} = (W^k)_{k\in\mathsf{K}}$ denotes a vector of standard Wiener processes in R, where *any correlation* between the components is allowed. Note that for $K = 1$, equation (1) reduces to $\mathrm{d}X_s^0 = \beta_0 X_s^0 \,\mathrm{d}s + \sqrt{\gamma_0 X_s^0}\,\mathrm{d}W_s^0$, which is *Feller's branching diffusion* with branching rate γ_0 and *non-criticality* $\beta_0 \in \mathsf{R}$. In general, X_t^k can be interpreted as the mass of species k at time t of a continuous-state branching population. *Intuitively,* the subpopulation X^k of $\mathbf{X}$ of species k evolves as Feller's branching diffusion with branching rate $\gamma_k X_s^{k-1}$ changing with time s, with non-criticality β_k, and with a *cross species drift* caused by $\alpha_{l,k} X_s^l$, $l \neq k$. Hence, the subpopulation X^{k-1} serves as a *catalyst* for the branching of X^k, for each $k \in \mathsf{K}$. But note that by this cyclic interaction over *all* the species (also if $\boldsymbol{\alpha} = \mathbf{0} = \boldsymbol{\beta}$ and if the noises $\dot{\mathbf{W}}$ are uncorrelated) the basic independence assumption in branching theory is violated, so that neither $\mathbf{X}$ nor any of its components X^k is a superprocess (that is, continuous-state branching process) according to the usual definition. If $\boldsymbol{\alpha} = \mathbf{0} = \boldsymbol{\beta}$ and the noises $\dot{\mathbf{W}}$ are uncorrelated, we get the cyclically catalytic branching process from [FX01] in the space-less case, whereas for $\boldsymbol{\alpha} = \mathbf{0} = \boldsymbol{\beta}$, $K = 2$, and $\gamma_0 = \gamma_1$ we have the space-less variant of the symbiotic branching model of [EF03].

The *construction* of a weak solution $\mathbf{X} = (X^k)_{k \in \mathsf{K}} \in \mathcal{C}(\mathsf{R}_+, \mathsf{R}_+^K)$ to (1) can be provided via a standard tightness argument using a fourth moment estimate, starting from (correlated) catalytic Feller's branching diffusions with drifts and with piecewise constant (frozen) catalysts X^{k-1} on small time intervals. These approximating equations have unique strong solutions, for each $k \in \mathsf{K}$, (the correlation of the driving Wiener processes is irrelevant for strong solutions). We skip any further details to this construction.

If $K > 2$ and $\boldsymbol{\alpha} > \mathbf{0}$, the uniqueness seems still to be open. Also, if $\boldsymbol{\alpha} = \mathbf{0} = \boldsymbol{\beta}$ and $\mathbf{W}$ is uncorrelated, the recent uniqueness result [BP03] of Bass and Perkins for certain degenerate diffusions in R_+^K does not apply due to the singularity caused by the fact that the catalysts can hit zero.

Our *main result* is the strong uniqueness of $\mathbf{X}$, provided that $\boldsymbol{\alpha} = \mathbf{0}$:

Theorem 1 (Strong uniqueness of X). *For fixed $K \geq 1$, $\boldsymbol{\alpha} = \mathbf{0}$, $\boldsymbol{\beta} \in \mathsf{R}^K$, $\boldsymbol{\gamma} > \mathbf{0}$, and $\mathbf{a} \in \mathsf{R}_+^K$, there is a unique strong solution $\mathbf{X} \in \mathcal{C}(\mathsf{R}_+, \mathsf{R}_+^K)$ to (1) satisfying $\mathbf{X}_0 = \mathbf{a}$.*

We call this process $\mathbf{X} = (\mathbf{X}, P_{\mathbf{a}}, \mathbf{a} \in \mathsf{R}_+^K)$ the *cyclically symbiotic branching diffusion* in R_+^K with *interaction vector* $\boldsymbol{\gamma}$ and *non-criticality* $\boldsymbol{\beta}$. As already mentioned, main emphasis concerns the case $K > 2$. In fact, $K = 1$ is the classical Feller's branching diffusion, where strong uniqueness is well-known, and $K = 2$ with $\boldsymbol{\beta} = \mathbf{0}$ and $\gamma_0 = \gamma_1$ can be seen as a zero-dimensional version of the symbiotic branching model of [EF03], where uniqueness in law follows from an exponential self-duality in the spirit of [Myt98], except for extreme correlation cases of $\mathbf{W}$.

There is actually a very simple *idea behind* this uniqueness in the space-less case. Indeed, away from the zero boundary, uniqueness holds by a local Lipschitz condition. On the other hand, once a component, say X^k, reaches zero, it is trapped there (recall that we assumed $\boldsymbol{\alpha} = \mathbf{0}$). But after this trapping, the model simplifies drastically. Indeed, let's restrict for the moment to the case of uncorrelated noises $\dot{\mathbf{W}}$. Then, X^k is the catalyst for X^{k+1}, therefore X^{k+1} does not fluctuate anymore, hence it is *trapped* at its present stage if $\beta_{k+1} = 0$, or it drifts deterministically, otherwise. In any case, X^{k+1} is unique. Now, given X^{k+1}, and since the noises $\dot{\mathbf{W}}$ are uncorrelated in the present consideration, the component X^{k+2} is the (only one-sided interacting) catalytic Feller's branching diffusion with branching rate $\gamma_{k+1} X^{k+1}$, hence, X^{k+2} is strongly unique. This way one can continue within the cycle until returning to the trapped X^k. Consequently, once a component hits the boundary, the true interaction in the cyclically catalytic model brakes down to only one-sided interactions where strong uniqueness holds. In the general case, where additionally correlations within $\mathbf{W}$ are allowed, the previous argument can be modified to get the pathwise uniqueness, see Section 2 below.

Note that this kind of approach is quite natural and has been used in proving uniqueness for diffusions in a simplex, see Sato [Sat78, Section 4] and Swart [Swa99, Example 3.1.8]. However, note that in the original spatial case of [FX01], these ideas do not work, since there, as a rule, a component hits zero (that is, it enters the singularity region) only in a part of space and is not trapped there by a possible migration of mass from other regions of space. So the spatial case remains *open* (if $K \geq 3$) even in the case of uncorrelated noises. Similarly, if $\boldsymbol{\alpha} > \mathbf{0}$ in (1), this trapping method does not work due to the drift caused by other species.

In an appendix, we add some discussion on the moments of $\mathbf{X}$, mainly restricting to the case of uncorrelated noises $\dot{\mathbf{W}}$.

2. Proof of Theorem 1

With c we denote a positive constant which might change from place to place. Let $|\cdot|$ denote the Euclidean norm.

Without loss of generality, we may assume that $K \geq 2$. Fix $k \in \mathsf{K}$ and $\mathbf{a} \in \mathsf{R}_+^K$. Clearly, from Itô's formula, $t \mapsto \mathrm{e}^{-\beta_k t} X_t^k$ is a non-negative martingale (recall we assumed $\boldsymbol{\alpha} = \mathbf{0}$), implying that the zero state is a trap for this martingale. Hence, also X^k is *trapped* at 0 once it reaches it.

Suppose we have two solutions $\mathbf{X}$ and $\mathbf{Y}$ to (1) with the same $\mathbf{W}$ and satisfying $\mathbf{X}_0 = \mathbf{a} = \mathbf{Y}_0$ (and $\boldsymbol{\alpha} = \mathbf{0}$). We have to show that $\mathbf{X} = \mathbf{Y}$. It suffices to do this on a finite interval $[0, T]$, for any fixed $T > 0$.

2.1. Pathwise uniqueness in the Lipschitz region.

Fix $\mathbf{a} > 0$. Choose any $\varepsilon > 0$ such that $\varepsilon < a_k < \varepsilon^{-1}$, $k \in \mathsf{K}$. Introduce the (possibly infinite) stopping time

$$\tau_\varepsilon \;:=\; \inf\Big\{ t \in [0,T] :\; \exists k \in \mathsf{K} \text{ with } X_t^k \wedge Y_t^k \leq \varepsilon \text{ or } X_t^k \vee Y_t^k \geq \varepsilon^{-1} \Big\}. \tag{2}$$

Lemma 2 (Pathwise uniqueness in the Lipschitz region). *We have* $\mathbf{X} = \mathbf{Y}$ *on* $[0, T \wedge \tau_\varepsilon]$.

Proof. We start by mentioning the elementary inequality

$$\big|\sqrt{bc} - \sqrt{de}\big| \;\leq\; \frac{1}{2\varepsilon^2}\big(|b-d| \,+\, |c-e|\big) \quad \text{if} \quad \varepsilon \,\leq\, b,c,d,e \,\leq\, \frac{1}{\varepsilon}\,. \tag{3}$$

(To see this, multiply and divide by $\sqrt{bc} + \sqrt{de}$.) For $\mathbf{Z} := \mathbf{X} - \mathbf{Y}$, $k \in \mathsf{K}$, and $t \leq T$, from equation (1) we have

$$Z_{t\wedge\tau_\varepsilon}^k \;=\; \beta_k \int_0^{t\wedge\tau_\varepsilon} \mathrm{d}s \; Z_s^k \,+\, \int_0^{t\wedge\tau_\varepsilon} \mathrm{d}W_s^k \left(\sqrt{\gamma_k X_s^{k-1} X_s^k} - \sqrt{\gamma_k Y_s^{k-1} Y_s^k} \right). \tag{4}$$

Hence, combined with (3) we obtain the second moment estimate

$$\begin{aligned} P_{\mathbf{a}}\big[|Z_{t\wedge\tau_\varepsilon}^k|^2\big] \;&\leq\; \Big(2\,T\,|\boldsymbol{\beta}| \,+\, \frac{1}{2\varepsilon^4}\,|\boldsymbol{\gamma}|\Big)\, P_{\mathbf{a}} \int_0^{t\wedge\tau_\varepsilon} \mathrm{d}s \;\big(|Z_s^{k-1}|^2 + |Z_s^k|^2\big) \\ &\leq\; c \int_0^t \mathrm{d}s \; P_{\mathbf{a}}\big(|Z_{s\wedge\tau_\varepsilon}^{k-1}|^2 + |Z_{s\wedge\tau_\varepsilon}^k|^2\big)\,, \qquad t \leq T. \end{aligned} \tag{5}$$

Summing over $k \in \mathsf{K}$, Gronwall's inequality gives $P_{\mathbf{a}}\big[|\mathbf{Z}_{t\wedge\tau_\varepsilon}|^2\big] = 0$, for $t \leq T$. This proves the claim in the lemma. □

As $\varepsilon \downarrow 0$, we have the non-decreasing convergence of τ_ε to some stopping time $\tau \leq \infty$. On $\{\tau = \infty\}$, we clearly got $\mathbf{X} = \mathbf{Y}$ on the considered interval $[0, T]$. On the other hand, on $\{\tau < \infty\}$, we have $\tau \leq T$, and there exists a $k \in \mathsf{K}$ such that $X_\tau^k = 0 = Y_\tau^k$. Indeed, note that $\mathbf{X}$ cannot explode on a finite time interval (for instance, use that $\mathbf{X}$ has a finite variance, see Lemma A1 in the appendix). Consequently, $\mathbf{X} = \mathbf{Y}$ on $[0, \tau]$ under $\{\tau < \infty\}$. It remains to study what happens after a trapping event.

2.2. **Pathwise uniqueness after a trapping event.** By the strong Markov property, from now on we may assume that $a_{k-2}=0$ for some $k\in\mathsf{K}$. Then, since 0 is a trap, $X^{k-2}=0$. This implies that $\mathrm{d}X^{k-1}_s=\beta_{k-1}X^{k-1}_s\,\mathrm{d}s$, that is,

$$X^{k-1}_t \;=\; a_{k-1}\,\mathrm{e}^{\beta_{k-1}t}, \qquad t\ge 0. \tag{6}$$

Consequently, X^{k-2} and X^{k-1} are pathwise uniquely determined. If $K=2$, the proof is finished.

Assume now that $K\ge 3$. Then

$$X^k_t \;=\; a_k \,+\, \beta_k\int_0^t \mathrm{d}s\ X^k_s\,\mathrm{d}s \,+\, \int_0^t \mathrm{d}W^k_s\ f_s\sqrt{X^k_s}, \qquad t\ge 0, \tag{7}$$

with the random continuous function

$$f_s \;:=\; \sqrt{\gamma_k\,X^{k-1}_s}, \qquad s\ge 0, \tag{8}$$

and X^{k-1} from (6). For $\varepsilon>0$, introduce the stopping time

$$\tau^k_\varepsilon \;:=\; \inf\Bigl\{t\in[0,T]:\ X^k_t\le\varepsilon \ \text{ or } \ X^k_t\ge\varepsilon^{-1}\Bigr\}. \tag{9}$$

On $[0,T\wedge\tau^k_\varepsilon]$ we again get pathwise uniqueness of X^k by the Lipschitz property. Letting $\varepsilon\downarrow 0$, implying $\tau^k_\varepsilon\uparrow$ some τ^k, we obtain pathwise uniqueness of X^k on $[0,T\wedge\tau^k]$. Now $\tau^k<\infty$ implies $X^k_{\tau^k}=0$, and X^k is trapped from there on. Altogether, X^k is pathwise unique on $[0,T]$.

Finally, if $K\ge 4$, we repeat the previous argument for $k+1$ instead of k [without having an explicit representation of X^k as we had with (6) for X^{k-1}]. This way the argument can be repeated until the cycle is closed. This completes the proof of the theorem. □

Appendix: On the moments of $\mathbf{X}$

A.1. **Finite moments of all orders.** Our uniqueness proof was based on the finiteness of variances which is a special case of the following lemma.

Lemma A1 (Finite moments of all orders). *Each solution* $\mathbf{X}$ *of* (1) *has finite moments of all orders.*

Proof. Fix $\mathbf{a}\in\mathsf{R}^K_+$ and $n\ge 1$. Clearly, from Itô's formula, for $k\in\mathsf{K}$ and $t\ge 0$,

$$\begin{aligned} P_{\mathbf{a}}\bigl[(X^k_t)^n\bigr] \;&=\; a^n_k \,+\, n\sum_{l\neq k}\alpha_{l,k}\int_0^t \mathrm{d}s\ P_{\mathbf{a}}\bigl[X^l_s(X^k_s)^{n-1}\bigr] \\ &\quad+\, n\beta_k\int_0^t\mathrm{d}s\ P_{\mathbf{a}}\bigl[(X^k_s)^n\bigr] \,+\, \tbinom{n}{2}\gamma_k\int_0^t\mathrm{d}s\ P_{\mathbf{a}}\bigl[X^{k-1}_s(X^k_s)^{n-1}\bigr]. \end{aligned} \tag{A1}$$

Summing over $k\in\mathsf{K}$, using that $\max_k\le\sum_k\le K\max_k$, that

$$\max_{l,k}\bigl(X^l_s(X^k_s)^{n-1}\bigr) \;\le\; \bigl(\max_k X^k_s\bigr)^n, \tag{A2}$$

and abbreviating $g_t:=P_{\mathbf{a}}\bigl[(\max_{k\in\mathsf{K}}X^k_t)^n\bigr]$, we obtain

$$g_t \;\le\; Kg_0 \,+\, \Bigl[nK^2\,|\boldsymbol{\alpha}| \,+\, nK\,|\boldsymbol{\beta}| \,+\, \tbinom{n}{2}K\,|\boldsymbol{\gamma}|\Bigr]\int_0^t\mathrm{d}s\ g_s\,, \qquad t\ge 0. \tag{A3}$$

Now the claim follows from Gronwall's inequality. □

A.2. **Failure of Carleman's Condition.** From now on, we *assume* that $\dot{\mathbf{W}}$ is a vector of *uncorrelated noises* (otherwise the situation is much more complicated), and suppose $\boldsymbol{\beta} \geq \mathbf{0}$. Consequently, we restrict our attention to a *cyclically catalytic branching diffusion* $\mathbf{X}$ in R_+^K with interaction vector $\boldsymbol{\gamma}$, super-criticality $\boldsymbol{\beta} \geq 0$, and cross species drift matrix $\boldsymbol{\alpha} \geq \mathbf{0}$.

The moments grow so fast that Carleman's condition for the moment problem for $\mathbf{X}$ to be well-posed is not satisfied (see Remark A3 below). Similar claims are without proof in several papers on mutually catalytic, cyclically catalytic, or symbiotic branching. In the present case, this follows from the following result.

Proposition A2 (Growth of Moments). *Let $\dot{\mathbf{W}}$ be a vector of uncorrelated noises. Consider $\mathbf{a} \in \mathsf{R}_+^K$ such that $a_{\min} := \min_{k \in \mathsf{K}} a_k > 0$, and $\boldsymbol{\beta} \geq 0$. Then there is a constant $c = c(\mathbf{a}, \boldsymbol{\gamma}) > 0$ such that for all $n \geq 1$,*

$$m_t^{(2n)} \;:=\; P_{\mathbf{a}}\left[(X_t^k)^{2n}\right] \;\geq\; (c\,t^2)^n\,(n!)^2, \qquad t \geq 0, \quad k \in \mathsf{K}. \tag{A4}$$

Proof. From Itô's formula, for $n \geq 1$,

$$\begin{aligned}(X_t^k)^n \;&\geq a_k^n \,+\, n\int_0^t \mathrm{d}W_s^k\; (X_s^k)^{n-1}\sqrt{\gamma_k\, X_s^{k-1} X_s^k} \\ &\quad+\; \tbinom{n}{2}\gamma_k \int_0^t \mathrm{d}s\; X_s^{k-1}(X_s^k)^{n-1}.\end{aligned} \tag{A5}$$

Switching to $n-1 \geq 1$, and then multiplying by X_t^{k-1}, we obtain (by dropping the first term)

$$\begin{aligned}X_t^{k-1}(X_t^k)^{n-1} \;&\geq (n-1)\, X_t^{k-1}\int_0^t \mathrm{d}W_s^k\; (X_s^k)^{n-2}\sqrt{\gamma_k\, X_s^{k-1} X_s^k} \\ &\quad+\; \tbinom{n-1}{2}\gamma_k \int_0^t \mathrm{d}s\; X_t^{k-1}\, X_s^{k-1}(X_s^k)^{n-2}.\end{aligned} \tag{A6}$$

Using the conditional expectation formula $P_{\mathbf{a}}\left\{X_t^k \,|\, X_s^k\right\} \;=\; X_s^k\,\mathrm{e}^{\beta_k(t-s)}$, taking expectation in (A5) and (A6) amounts to

$$m_t^{(n)} \;\geq\; \tbinom{n}{2}\gamma_{\min}\int_0^t \mathrm{d}s\; P_{\mathbf{a}}\left[X_s^{k-1}(X_s^k)^{n-1}\right], \qquad n \geq 1, \tag{A7}$$

and

$$P_{\mathbf{a}}\left[X_t^{k-1}(X_t^k)^{n-1}\right] \;\geq\; \tbinom{n-1}{2}\gamma_{\min}\int_0^t \mathrm{d}s\; \mathrm{e}^{\beta_k(t-s)} P_{\mathbf{a}}\left[(X_s^{k-1})^2\,(X_s^k)^{n-2}\right], \tag{A8}$$

for $n \geq 2$, respectively. In fact, the stochastic integral term in (A6) vanishes if $t = 0$ and it is driven by W^k, whereas the factor X^{k-1} is a martingale driven by W^{k-1}. But W^k and W^{k-1} are uncorrelated by assumption, hence the expectation of the product of both martingales vanishes.

Applying (A5) to $(X_s^{k-1})^2$ and dropping the last term there, from (A8) we get

$$P_{\mathbf{a}}\left[X_t^{k-1}(X_t^k)^{n-1}\right] \;\geq\; \tbinom{n-1}{2}\gamma_{\min}\, a_{\min}^2\int_0^t \mathrm{d}s\; \mathrm{e}^{\beta_k(t-s)}\, m_s^{(n-2)}, \qquad n \geq 2. \tag{A9}$$

Inserting (A9) into (A7) results to

$$m_t^{(n)} \;\geq\; \tbinom{n}{2}\tbinom{n-1}{2}\, a_{\min}^2\, \gamma_{\min}^2 \int_0^t \mathrm{d}s \int_0^s \mathrm{d}r\; \mathrm{e}^{\beta_k(s-r)}\, m_r^{(n-2)}, \tag{A10}$$

$n \geq 2$. Now $\mathrm{e}^{\beta_k(s-r)} \geq 1$. Therefore,

$$m_t^{(n)} \;\geq\; c n^4 \int_0^t \mathrm{d}r \; m_r^{(n-2)}(t-r), \qquad t \geq 0, \quad n \geq 2, \tag{A11}$$

with $c = c(\mathbf{a}, \boldsymbol{\gamma}) > 0$. Setting $g_t^{(n)} := m_t^{(2n)}$, $n \geq 0$, (A11) reads as

$$g_t^{(n)} \;\geq\; c n^4 \int_0^t \mathrm{d}r \; (t-r)\, g_r^{(n-1)}, \qquad t \geq 0, \quad n \geq 1, \tag{A12}$$

with $g_t^{(0)} \equiv 1$ (and changing c). This recursive system implies that

$$g_t^{(n)} \;\geq\; c^n\, (n!)^4\, \frac{t^{2n}}{(2n)!}, \qquad t \geq 0, \quad n \geq 1. \tag{A13}$$

Indeed, this follows by induction employing the identity

$$\int_0^t \mathrm{d}r \; (t-r)\, \frac{r^{2n}}{(2n)!} \;=\; \frac{t^{2(n+1)}}{\big(2(n+1)\big)!}, \qquad n \geq 0. \tag{A14}$$

Next we use that $(2n)! \leq 2 \cdot 2 \cdot 4 \cdot 4 \cdots 2n \cdot 2n = 2^{2n}\,(n!)^2$. Therefore, changing the constant $c = c(\mathbf{a}, \boldsymbol{\gamma})$, from (A13) we obtain

$$g_t^{(n)} \;\geq\; (n!)^2\, (c\, t^2)^n, \tag{A15}$$

which finishes the proof. □

Remark A3 (**Failure of Carleman's Condition**). The moment estimate (A4) implies that the Carleman (sufficient) condition for the well-posedness of the moment problem (see, for instance, [Chu74, Section 4.5]), namely

$$\sum_{n \geq 1} \big(m_t^{(2n)}\big)^{-1/2n} \;=\; \infty, \qquad t > 0, \tag{A16}$$

does not hold. ◇

A.3. **Moment equation system.** For simplicity, assume in addition that $\boldsymbol{\alpha} = \mathbf{0}$. As in related models, for a fixed order, the moments are uniquely determined by a closed system of linear ordinary differential equations:

Proposition A4 (**Moment equation system**). *Let $\dot{\mathbf{W}}$ be a vector of uncorrelated noises, and suppose that $\boldsymbol{\alpha} = \mathbf{0}$. Fix an initial condition $\mathbf{a} \in \mathsf{R}_+^K$ and $n \geq 1$. Then the n^{th} moments*

$$m^{\mathbf{k}}(s) \;=\; m^{\mathbf{k}}_{\mathbf{a}}(s) \;:=\; P_{\mathbf{a}} \mathbf{X}_s^{\mathbf{k}}, \qquad \mathbf{k} = (k_1, \ldots, k_n) \in \mathsf{K}^n, \quad s \geq 0, \tag{A17}$$

of the cyclically catalytic branching diffusion $\mathbf{X}$, where we set $\mathbf{X}_s^{\mathbf{k}} := X_s^{k_1} \cdots X_s^{k_n}$, solve uniquely

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} m^{\mathbf{k}}(s) \;=\; m^{\mathbf{k}}(s) \displaystyle\sum_{i=1}^{n} \beta_{k_i} \;+\; \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \delta_{k_i, k_j}\, \frac{\gamma_{k_i}}{2}\, m^{\sigma_j(\mathbf{k}, k_i - 1)}(s), \\[2ex] \qquad\qquad \textit{with initial condition} \quad m^{\mathbf{k}}(0) \;=\; \mathbf{a}, \end{cases} \tag{A18}$$

$\mathbf{k} = (k_1, \ldots, k_n) \in \mathsf{K}^n$, $s > 0$, where $\delta_{k,l}$ denotes the Kronecker symbol, and where by definition, $\sigma_j(\mathbf{k}, l) \in \mathsf{K}^n$ arises from $\mathbf{k} = (k_1, \ldots, k_n)$ by replacing the j^{th} component k_j of $\mathbf{k}$ by $l \in \mathsf{K}$.

Proof. By Itô's formula,

$$\begin{aligned} \mathrm{d}(X_s^{k_1}\cdots X_s^{k_n}) \;&=\; \sum_i \Big(\prod_{j\neq i} X_s^{k_j}\Big)\mathrm{d}X_s^{k_i} \;+\; \frac{1}{2}\sum_{i\neq j}\Big(\prod_{j'\neq i,j} X_s^{k_{j'}}\Big)\mathrm{d}\langle\!\langle X^{k_i},X^{k_j}\rangle\!\rangle_s \\ &\overset{\text{emt}}{=}\; \mathbf{X}_s^{\mathbf{k}}\sum_i \beta_{k_i}\,\mathrm{d}s \;+\; \frac{1}{2}\sum_{i\neq j}\delta_{k_i,k_j}\gamma_{k_i}X_s^{k_i-1}X_s^{k_i}\prod_{j'\neq i,j}X_s^{k_{j'}}\,\mathrm{d}s, \end{aligned} \tag{A19}$$

where $\overset{\text{emt}}{=}$ means equality except a martingale term, starting from zero. Taking expectations gives claim (A18), finishing the proof. □

A.4. **Particle system moment dual.** Suppose additionally that $\boldsymbol{\alpha}=\mathbf{0}=\boldsymbol{\beta}$. Another tool is a *particle system moment dual process* $\mathbf{N}$, we want to introduce now. For the case $K=2$ and $\gamma_0=\gamma_1$ it follows from [EF03].

Let $\mathcal{N}_{\mathrm{f}}$ denote the set of all vectors $\mathbf{n}=[n_0,\dots,n_{K-1}]$ with entries $n_k\geq 0$, describing a finite system of n_k particles of species k, etc. Then $\mathbf{N}$ will be an $\mathcal{N}_{\mathrm{f}}$–valued Markov jump process with càdlàg paths. The *generator* G of $\mathbf{N}$ is given by

$$Gf(\mathbf{n}) \;:=\; \sum_{k\in\mathsf{K}}\gamma_k\tbinom{n_k}{2}\big[f(\sigma^k\mathbf{n})-f(\mathbf{n})\big], \qquad \mathbf{n}\in\mathcal{N}_{\mathrm{f}}, \tag{A20}$$

where $\sigma^k\mathbf{n}$ (provided that $n_k\geq 2$) denotes that element of $\mathcal{N}_{\mathrm{f}}$ which is obtained from $\mathbf{n}$ by *switching the species* of one of the n_k particles of species k to species $k-1$. Consequently, each pair of particles of species k may experience a jump with rate γ_k, and upon a jump, exactly one of the particles gets the species $k-1$. Write $\mathbf{P}_{\mathbf{n}}$ for the law of $\mathbf{N}$ starting from $\mathbf{N}_0=\mathbf{n}\in\mathcal{N}_{\mathrm{f}}$.

Next we want to introduce a *duality function* $\mathfrak{N}$ of the generating function type. For $\mathbf{a}\in\mathsf{R}_+^K$ and $\mathbf{n}\in\mathcal{N}_{\mathrm{f}}$, set

$$\mathfrak{N}(\mathbf{a},\mathbf{n}) \;:=\; \mathbf{a}^{\mathbf{n}} \;:=\; \prod_{k\in\mathsf{K}} a_k^{n_k}. \tag{A21}$$

In the duality relation (A23) below, we will use the following notation:

$$\|\mathbf{n}\|_{=} \;:=\; \sum_{k\in\mathsf{K}}\gamma_k\tbinom{n_k}{2}. \tag{A22}$$

Thus, $\|\mathbf{n}\|_{=}$ is the weighted number of pairs of particles in $\mathbf{n}$ having the same species.

Proposition A5 (**Particle system moment duality relation**). *Assume $\dot{\mathbf{W}}$ is a vector of uncorrelated noises and $\boldsymbol{\alpha}=\mathbf{0}=\boldsymbol{\beta}$. Fix $\mathbf{a}\in\mathsf{R}_+^K$ and $\mathbf{n}\in\mathcal{N}_{\mathrm{f}}$. Consider the cyclically catalytic branching process $(\mathbf{X},P_{\mathbf{a}})$ and the particle system moment dual $(\mathbf{N},\mathbf{P}_{\mathbf{n}})$ in $\mathbb{Z}^d$. Then, for all $t\geq 0$,*

$$E_{\mathbf{a}}\mathbf{X}_t^{\mathbf{n}} \;=\; \mathbf{E}_{\mathbf{n}}\,\mathbf{a}^{\mathbf{N}_t}\exp\Big[\int_0^t\mathrm{d}s\,\|\mathbf{N}_s\|_{=}\Big]. \tag{A23}$$

Proof. The generator $\mathcal{G}$ of $\mathbf{X}$ is given by

$$\mathcal{G}f(\mathbf{a}) \;:=\; \sum_k\frac{\gamma_k}{2}\,a_{k-1}a_k\,\frac{\partial^2}{\partial a_k^2}f(\mathbf{a}), \tag{A24}$$

where f is a twice continuously differentiable function on $\mathbf{a} = (a_0, \dots, a_{K-1}) \in \mathsf{R}_+^K$. Hence,

$$\mathcal{G}\mathfrak{N}(\,\cdot\,, \mathbf{n})\,(\mathbf{a}) \;=\; \gamma \sum_k \gamma_k \tbinom{n_k}{2} \mathbf{a}^{\sigma^k \mathbf{n}} \tag{A25}$$

[with σ^k defined after (A20)]. On the other hand, by (A20),

$$G\mathfrak{N}(\mathbf{a},\,\cdot\,)\,(\mathbf{n}) \;=\; \sum_k \gamma_k \tbinom{n_k}{2} \mathbf{a}^{\sigma^k \mathbf{n}} \;-\; \|\mathbf{n}\|_{=}\, \mathbf{a}^{\mathbf{n}}. \tag{A26}$$

Therefore,

$$\mathcal{G}\mathfrak{N}(\,\cdot\,, \mathbf{n})\,(\mathbf{a}) \;=\; G\mathfrak{N}(\mathbf{a},\,\cdot\,)\,(\mathbf{n}) \;+\; \|\mathbf{n}\|_{=}\, \mathfrak{N}(\mathbf{a}, \mathbf{n}). \tag{A27}$$

The claimed duality relation (A23) now follows by standard arguments; see [EK86, Corollary 4.4.13]. □

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