Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Snake representation of a super-Brownian reactant in the catalytic region

Klaus Fleischmann[1], Jie Xiong[2]

submitted: 15th May 2003

[1] Weierstrass Institute
for Applied Analysis
and Stochastis,
Mohrenstr.
D-10117 Berlin, Germany
E-Mail: fleischm@wias-berlin.de

[2] Department of Mathematics,
University of Tennessee,
Knoxville, Tennessee 37996-1300,
USA
and
Department of Mathematical and
Statistical Sciences, University of Alberta,
Edmonton, Alberta T6G 2G1, Canada
E-Mail: jxiong@math.utk.edu

No. 847
Berlin 2003

2000 *Mathematics Subject Classification.* 60K35, 60G57, 60J80.

Key words and phrases. Admissible catalyst, part of reactant, superprocess, collision measure, collision local time, Markov branching process, catalytic Brownian snake, modified hitting measure.

Partly supported by the research program "Interacting Systems of High Complexity" and the graduate program "Probabilistic Analysis and Stochastic Processes" of the German Science Foundation, the Humboldt Foundation, and by an NSA research grant.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

ABSTRACT. For a continuous super-Brownian reactant X in R^d with general catalyst ϱ a Brownian snake representation is derived for the part X^{c} of X in the catalyst region. This extends results of Dawson et al. (2002) and Klenke (2003) in that it allows the collision local time of an intrinsic reactant particle with the catalyst to have flat pieces, caused by catalyst-free regions.

CONTENTS

1. Introduction and main result

1.1. **Background, motivation, and purpose.** A lot of interesting *open interface questions* had been raised for a mutually catalytic branching model in R^2, see Dawson et al. [DEF$^+$02, Section 7]. The present paper was motivated by this, but pays attention only to the model of catalytic continuous super-Brownian motion in R^d, $d \geq 2$, which is simpler since it allows only a "one-sided interaction".

Using earlier ideas of Le Gall (see [LG99], e.g.) and Bertoin et al. [BLGLJ97], a Brownian snake approach for a continuous super-Brownian reactant with a stable catalyst in R had been introduced in Dawson et al. [DFM02], where it turned out to be a powerful tool in deriving a functional scaling limit theorem for the reactant. This Brownian snake representation in a singular medium had been generalized by Klenke [Kle03] for a broad class of catalytic superprocesses having the property that the collision local time between the catalyst and an intrinsic reactant particle is (strictly) increasing in time. In a sense, here the catalyst has to be present "everywhere" in space.

In the present paper we attack the problem of a Brownian snake representation of a continuous super-Brownian reactant in R^d if catalyst-free regions are allowed, leading to possibly flat parts of the mentioned collision local time. For this purpose, we introduce the concept of the *part of the reactant in the catalyst region* (see Subsection 1.5) for which we construct a *Brownian snake representation* (Theorem 15).

We hope that the derived catalytic snake representation can be used to attack interface questions for catalytic super-Brownian motion.

For an introduction to the Brownian snake we recommend Le Gall [LG99], and for recent surveys on catalytic branching models we refer to Dawson and Fleischmann [DF02] and Klenke [Kle00].

1.2. **Preliminaries: notation and spaces.** With $c = c(q)$ we always denote a positive constant which (in the present case) might depend on a quantity q and might also change from place to place. Moreover, an index on c as $c_{(\#)}$ or $c_{\#}$ will indicate that this constant first occurred in formula line (#) or (for instance) Lemma #, respectively.

For $\lambda \in \mathsf{R}$, introduce the reference function

$$\phi_\lambda(x) := \mathrm{e}^{-\lambda|x|}, \qquad x \in \mathsf{R}^d. \tag{1}$$

For $f : \mathsf{R}^d \to \mathsf{R}$, set

$$|f|_\lambda := \|f/\phi_\lambda\|_\infty \tag{2}$$

where $\|\cdot\|_\infty$ refers to the supremum norm. Denote by $\mathcal{C}_\lambda$ the separable Banach space of all continuous functions $f : \mathsf{R}^d \to \mathsf{R}$ such that $|f|_\lambda$ is finite and that $f(x)/\phi_\lambda(x)$ has a finite limit as $|x| \uparrow \infty$. Introduce the space

$$\mathcal{C}_{\exp} = \mathcal{C}_{\exp}(\mathsf{R}^d) := \bigcup_{\lambda>0} \mathcal{C}_\lambda \tag{3}$$

of *exponentially decreasing* continuous functions on R^d. For a closed interval $I \subset \mathsf{R}_+$, write $\mathcal{C}^I_{\exp}$ for the set of all functions $\psi : I \times \mathsf{R}^d \to \mathcal{C}_{\exp}$ such that there is a constant $c = c(\psi)$ and a $\lambda = \lambda(\psi) > 0$ such that $|\psi|(t,x) \leq c\,\phi_\lambda(x)$, $(t,x) \in I \times \mathsf{R}^d$.

Let $\mathcal{M} = \mathcal{M}(\mathsf{R}^d)$ denote the set of all (non-negative) Radon measures μ on R^d and d_0 a complete metric on $\mathcal{M}$ which induces the vague topology. Introduce the space $\mathcal{M}_{\text{tem}} = \mathcal{M}_{\text{tem}}(\mathsf{R}^d)$ of all measures μ in $\mathcal{M}$ such that $\langle \mu, \phi_\lambda \rangle < \infty$, for all $\lambda > 0$. We topologize this set $\mathcal{M}_{\text{tem}}$ of *tempered* measures by the metric

$$\text{(4)} \qquad \mathrm{d}_{\text{tem}}(\mu,\nu) \;:=\; \mathrm{d}_0(\mu,\nu) + \sum_{n=1}^{\infty} 2^{-n} \left(|\mu-\nu|_{1/n} \,\wedge\, 1 \right), \qquad \mu,\nu \in \mathcal{M}_{\text{tem}}\,.$$

Here $|\mu-\nu|_\lambda$ is an abbreviation for $\bigl|\langle \mu, \phi_\lambda\rangle - \langle \nu, \phi_\lambda\rangle\bigr|$. Note that $(\mathcal{M}_{\text{tem}}\,, \mathrm{d}_{\text{tem}})$ is a Polish space (that is, a complete separable metric space), and that $\mu_n \to \mu$ in $\mathcal{M}_{\text{tem}}$ if and only if

$$\text{(5)} \qquad \langle \mu_n\,, \varphi \rangle \underset{n\uparrow\infty}{\longrightarrow} \langle \mu, \varphi \rangle \quad \text{for all } \varphi \in \mathcal{C}_{\exp}\,.$$

If (E, d_E) is a Polish space, we write $\mathcal{C}(\mathsf{R}_+, E)$ for the space of all continuous functions $f : \mathsf{R}_+ \to E$. Equipped with the metric

$$\text{(6)} \qquad \mathrm{d}^{\mathcal{C}}(f,\tilde{f}) \;:=\; \sum_{n=1}^{\infty} 2^{-n} \Bigl(\sup_{0\le t\le n} \mathrm{d}_E(f_t\,,\tilde{f}_t) \wedge 1 \Bigr), \qquad f,\tilde{f} \in \mathcal{C}(\mathsf{R}_+\,, E),$$

we get a Polish space $\bigl(\mathcal{C}(\mathsf{R}_+, E), \mathrm{d}^{\mathcal{C}}\bigr)$.

Let $\mathbb{C}$ denote the Polish space of all *non-decreasing* functions in $\mathcal{C}(\mathsf{R}_+, \mathsf{R}_+)$. For $t \ge 0$ fixed, write $\mathbb{C}^t$ for the closed set of all *stopped paths* $f^t := f_{t\wedge(\cdot)}\,,\ f \in \mathbb{C}$. We identify $\mathbb{C}^0$ with R_+.

For $t \ge 0$, define $\mathcal{M}^t_{\text{tem}} = \mathcal{M}^t_{\text{tem}}(\mathsf{R}^d \times \mathbb{C}^t)$ as the set of all measures ν on $\mathsf{R}^d \times \mathbb{C}^t$ with marginal measure $\nu\bigl((\cdot) \times \mathbb{C}^t\bigr)$ in $\mathcal{M}_{\text{tem}}$. Analogously, we identify $\mathcal{M}^0_{\text{tem}}$ with $\mathcal{M}_{\text{tem}}$.

By an abuse of notation, if (E, d_E) is a Polish space, we write $\mathcal{D}(\mathsf{R}_+, E)$ for the space of all *càglàd* functions $w : \mathsf{R}_+ \to E$, that is, left-continuous functions with right hand limits. Endowed with a metric $\mathrm{d}_{\mathcal{D}}$ analogously to the Skorohod metric, $\mathcal{D}(\mathsf{R}_+, E)$ is again a Polish space, we call it the *Skorohod space of càglàd functions.* $\mathcal{D}^t(\mathsf{R}_+, E)$ is the subset of all paths $w^t := w\bigl(t \wedge (\cdot)\bigr)$ stopped at time $t \ge 0$. (The use of càglàd functions will be very convenient for our modelling, see, for instance, formula (51) below.)

The one-point compactification of a locally compact non-compact space E is denoted by $\dot{E}$, where the additional point is written as $\dagger$ (cemetery point). In such situation, we always fix a metric $\mathrm{d}_{\dot{E}}$ on $\dot{E}$ inducing the same topology and making $(\dot{E}, \mathrm{d}_{\dot{E}})$ a Polish space. Functions φ on E are considered also as functions on $\dot{E}$ by setting $\varphi(\dagger) := \lim_{e\to\dagger} \varphi(e)$, provided the limit exists. In particular, $\mathcal{C}_{\exp}(\dot{\mathsf{R}}^d)$ is the space of all functions φ defined on $\dot{\mathsf{R}}^d$ vanishing at $\dagger$ and, restricted to R^d, belonging to $\mathcal{C}_{\exp}$. Analogously, denote by $\mathcal{M}_{\text{tem}}(\dot{\mathsf{R}}^d)$ the space of all measures μ defined on $\dot{\mathsf{R}}^d$ vanishing at the cemetery point and, restricted to R^d, belonging to $\mathcal{M}_{\text{tem}}$. In this sense, we can identify $\mathcal{M}_{\text{tem}}(\dot{\mathsf{R}}^d)$ with $\mathcal{M}_{\text{tem}}$.

Random objects are always thought of as defined over a large enough stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, \mathcal{P})$ satisfying the usual hypotheses.

Let p denote the (standard) heat kernel in R^d related to $\frac{1}{2}\Delta$:

$$\text{(7)} \qquad \mathrm{p}_t(x) \;:=\; (2\pi t)^{-d/2} \exp\Bigl[-\frac{|x|^2}{2t} \Bigr], \qquad t > 0, \quad x \in \mathsf{R}^d.$$

Write $B = (B,\, P_{r,x}\,,\, r \ge 0,\, x \in \mathsf{R}^d)$ for the corresponding (standard) Brownian motion in R^d. Here $P_{r,x}$ refers to the law of B if it starts at time r at $B_r = x$ (using for convenience this time-inhomogeneous setting for the time-homogeneous

Brownian motion). B with law $P_{r,x}$ is extended to a (in general non-continuous) process $\dot{B} : [r,\infty] \to \dot{\mathsf{R}}^d$ by requiring that $\dagger \in \dot{\mathsf{R}}^d$ is a trap and by setting $\dot{B}_\infty := \dagger$.

Write θ_x for the shift operators acting on functions φ defined on R^d :

$$\theta_x \varphi\,(y) \;:=\; \varphi(y-x), \qquad x,y \in \mathsf{R}^d. \tag{8}$$

Similarly, time-shift operators θ_t , $t \in \mathsf{R}_+$, are defined.

ℓ denotes the Lebesgue measure, and ℓ_A its restriction to a measurable subset A.

1.3. **Basic model: continuous super-Brownian reactant X with catalyst ϱ.** We need to recall the notion of collision local time. For $\varrho \in \mathcal{C}(\mathsf{R}_+, \mathcal{M}_{\text{tem}})$ and each $\varepsilon > 0$, define a continuous additive functional $L^\varepsilon_{[\varrho,B]}$ of Brownian motion B under $P_{r,x}$ by

$$L^\varepsilon_{[\varrho,B]}(\mathrm{d}s) \;:=\; \langle \varrho_s\,,\, \theta_{B_s}\mathrm{p}_\varepsilon\rangle\, \mathrm{d}s \quad \text{on} \quad [r,\infty) \tag{9}$$

(approximating collision local time).

Definition 1 (Collision local time). Let $\varrho \in \mathcal{C}(\mathsf{R}_+, \mathcal{M}_{\text{tem}})$. If there is a continuous additive functional $L_{[\varrho,B]}$ of Brownian motion B such that for all $T > 0$ and $\psi \in \mathcal{C}^{[0,T]}_{\text{exp}}$,

$$\sup_{r\in[0,T],\; x\in\mathsf{R}^d} P_{r,x} \sup_{t\in[r,T]} \left| \int_r^t L^\varepsilon_{[\varrho,B]}(\mathrm{d}s)\, \psi(s,B_s) - \int_r^t L_{[\varrho,B]}(\mathrm{d}s)\, \psi(s,B_s) \right|^2 \underset{\varepsilon\downarrow 0}{\longrightarrow} 0,$$

then $L_{[\varrho,B]}$ is called the *collision local time* (of ϱ and B). We identify the (random) measure $L_{[\varrho,B]}(\mathrm{d}s)$ under $P_{r,x}$ also with the (a.s. finite) non-decreasing function $t \mapsto L_{[\varrho,B]}\bigl([r,t]\bigr)$. For $t \geq r$, write $L^t_{[\varrho,B]}$ for the *stopped collision time process* $s \mapsto L_{[\varrho,B]}\bigl([r, s\wedge t]\bigr)$, stopped at time t. $\diamond$

Interpreting ϱ as a catalyst and B as a reactant particle's path, $L_{[\varrho,B]}$ measures the time the reactant particle spends on the catalyst, weighted by the catalyst's "density", in short terms:

$$L_{[\varrho,B]}(\mathrm{d}s) \;=\; \mathrm{d}s \int_{\mathsf{R}^d} \varrho_s(\mathrm{d}x)\; \delta_x(B_s). \tag{10}$$

Definition 2 (Admissible catalyst). A path $\varrho \in \mathcal{C}(\mathsf{R}_+, \mathcal{M}_{\text{tem}})$ is called an *admissible catalyst* in R^d, if the collision local time $L_{[\varrho,B]}$ exists and there is a $\delta \in (0,1)$ and to each $T > 0$ a constant $c = c(\varrho,\delta,T)$ such that

$$P_{r,x} \int_r^t L_{[\varrho,B]}(\mathrm{d}s)\, \phi^2_\lambda(B_s) \;\leq\; c\,|t-r|^\delta\, \phi_\lambda(x), \qquad 0 \leq r \leq t \leq T,\;\; x \in \mathsf{R}^d. \quad \diamond \tag{11}$$

Example 3 (Admissible catalysts). Here are some well-studied examples of admissible catalysts:

- (i) (Everywhere uniform catalyst in R^d): For $c > 0$ fixed, $\varrho_r(\mathrm{d}x) \equiv c\,\mathrm{d}x$, resulting into $L_{[\varrho,B]}(\mathrm{d}s) = c\,\mathrm{d}s$.
- (ii) (Single point catalyst in R): For $c \in \mathsf{R}$ fixed, $\varrho_r(\mathrm{d}x) \equiv \delta_c(\mathrm{d}x)$ with δ_c the delta measure at c. Then $L_{[\varrho,B]}(\mathrm{d}s)$ reduces to the Brownian local time at level c (which is non-trivial only in dimension 1).
- (iii) (Stable catalyst in R): For $\Gamma(\mathrm{d}x)$ the stable random measure on R with index $0 < \gamma < 1$, let $\varrho_r \equiv \Gamma$, for a fixed sample Γ. Then, $L_{[\varrho,B]}(\mathrm{d}s)$ is an infinite weighted sum of Brownian local times.

(iv) (Super-Brownian catalyst in R^d, $d \le 3$): ϱ is sampled from an ordinary continuous super-Brownian motion in R^d started from "nice" ϱ_0. Then $L_{[\varrho,B]}(\mathrm{d}s)$ exists non-trivially exactly in dimensions $d \le 3$ ([BEP91, EP94, DF97]).

(v) (Hyperplanes in R^d weighted by a stable measure): Suppose

$$\varrho_t(\mathrm{d}x) \;:=\; \Gamma(\mathrm{d}x_1)\,\mathrm{d}x_2\cdots\mathrm{d}x_d\,, \qquad x=(x_1,\dots,x_d)\in\mathsf{R}^d,$$

with Γ as in (iii) ([DF95]).

But our purpose is to deal with more general admissible catalysts ϱ, covering the following examples:

(vi) (Catalyst with bounded density function): Here we assume that

$$\varrho_t(\mathrm{d}x) \;:=\; \varrho_t(x)\,\mathrm{d}x \quad \text{with} \quad \operatorname*{ess\,sup}_{t,x}\varrho_t(x) < \infty,$$

implying, for almost all $0 \le r \le t$, that

$$L_{[\varrho,B]}\big([r,t]\big) \;=\; \int_r^t \mathrm{d}s\; \varrho_s(B_s), \;\; P_{r,x}\text{–a.s.}, \quad x\in\mathsf{R}^d$$

(vii) (Isolated hyperplanes in R^d): Suppose

$$\varrho_t(\mathrm{d}x) \;:=\; \sum_i \delta_{a_i}(\mathrm{d}x_1)\,\mathrm{d}x_2\cdots\mathrm{d}x_d\,, \qquad x=(x_1,\dots,x_d)\in\mathsf{R}^d,$$

where the a_i form a countable subset of R^d without accumulation. ◇

Now we are ready to introduce the model on which the present paper is based on (see [DF97]).

Proposition 4 (Continuous super-Brownian reactant X with catalyst ϱ). *Let ϱ be an admissible catalyst in R^d $(d \ge 1)$. Then there exists a unique in law continuous $\mathcal{M}_{\mathrm{tem}}$–valued (in general time-inhomogeneous) Markov process*

$$(X,\,\mathbf{P}_{r,\mu}\,,\,r\ge 0,\,\mu\in\mathcal{M}_{\mathrm{tem}}) \tag{12}$$

in R^d with log-Laplace transition functional

$$-\log \mathbf{P}_{r,\mu}\,\mathrm{e}^{-\langle X_t,\varphi\rangle} \;=\; \langle \mu, v_r^t\rangle, \qquad 0\le r\le t, \quad \varphi\in\mathcal{C}_{\exp}^{+}. \tag{13}$$

Here, for t and φ fixed, $v^t = v^t[\varphi,\varrho] = \big\{v_r^t(x) : 0\le r\le t,\; x\in\mathsf{R}^d\big\}$ is the unique non-negative solution to the integral equation

$$v_r^t(x) \;=\; P_{r,x}\Big[\varphi(B_t) \;-\; 2\int_r^t L_{[\varrho,B]}(\mathrm{d}s)\,\big(v_s^t(B_s)\big)^2\Big], \quad (r,x)\in[0,t]\times\mathsf{R}^d. \tag{14}$$

This process $X = X[\varrho]$ is called the *continuous super-Brownian reactant with catalyst* ϱ.

Less formally, $v^t = v^t[\varphi,\varrho]$ can be understood as the mild solution of the (formal) partial differential equation

$$\begin{cases} -\dfrac{\partial}{\partial r}v_r^t(x) \;=\; \dfrac{1}{2}\Delta v_r^t(x) \;-\; 2\varrho_r(x)\,\big(v_r^t(x)\big)^2 \quad \text{on} \quad [0,t]\times\mathsf{R}^d \\ \qquad\qquad\qquad\qquad \text{with terminal condition } v_{t-}^t = \varphi. \end{cases} \tag{15}$$

Here, for r fixed, $\varrho_r(x)$ is the generalized(!) "density" at x of the measure $\varrho_r(\mathrm{d}x)$.

Example 5 (**Continuous super-Brownian reactants**). The well-studied examples of continuous super-Brownian reactants with catalyst ϱ listed in line with Example 3 are:

(i) Ordinary continuous super-Brownian motion X in R^d with branching rate c.
(ii) Single point-catalytic continuous super-Brownian reactant X in R (see [DFLG95] for a survey on it).
(iii) Continuous super-Brownian reactant X in R with stable catalyst ([DFM02] is a recent paper on this).
(iv) Continuous super-Brownian reactant X in R^d, $d \leq 3$, with continuous super-Brownian catalyst (see [FKX02] for a recent paper).
(v) Continuous super-Brownian reactant X in R^d with stable hyperplanes ([DF95]). ◇

1.4. **Part X^{f} of X in the catalyst-free region.** Inspired by [FK99], we first want to introduce the part X^{f} of X in the catalyst free region (where X is the continuous super-Brownian reactant with catalyst ϱ). Denote by $B_\delta(x)$ the open ball in R^d of radius δ centered at x. For $t>0$, write $F_t = F_t[\varrho]$ for the open set in R^d of all those $x \in \mathsf{R}^d$ such that there exists a $\delta = \delta(\varrho,t,x) \in (0,t)$ satisfying

$$\sup_{s\,\in\,[t-\delta,\,t+\delta]} \varrho_s\big(B_\delta(x)\big) \;=\; 0. \tag{16}$$

Then the *open* set

$$F = F[\varrho] \;:=\; \big\{(t,x):\; t>0,\;\; x\in F_t\big\} \tag{17}$$

in $\big(0,\infty)\times\mathsf{R}^d\big)$ is called the *catalyst-free region.* The random process $X^{\mathrm{f}} = X^{\mathrm{f}}[\varrho] = (X^{\mathrm{f}}_t)_{t>0}$ with

$$X^{\mathrm{f}}_t(\mathrm{d}x) \;:=\; 1_F(t,x)\, X_t(\mathrm{d}x), \qquad t>0, \tag{18}$$

is said to be the *part of X in the catalyst-free region.*

Example 6 (**Catalyst-free region**). For the uniform catalyst in R^d and the stable catalyst in R [as in Example 3(i) and (iii)] we have $F_t \equiv \emptyset = F$, whereas for the single point catalyst in R of (ii), $F_t \equiv \{x \neq 0\}$, $F = (0,\infty)\times\{x\neq 0\}$, and finally for the equidistant hyperplanes in R^d of (vi) one has $F_t \equiv \{x_1 \notin \mathsf{Z}\}$, $F = (0,\infty)\times\{x_1\notin\mathsf{Z}\}$. ◇

From Theorem 1 in [FK99] and the remark following it, we immediately get

Proposition 7 (**Heat flow in the catalyst-free region**). *Fix an admissible catalyst ϱ, a starting time $r\in\mathsf{R}_+$, and a measure $\mu\in\mathcal{M}_{\mathrm{tem}}$.*

(a) (Absolute continuity): $\mathbf{P}_{r,\mu}$*–almost surely, for all $t>r$, the measures X^{f}_t are absolutely continuous (with respect to Lebesgue measure ℓ).*

(b) (Smooth density field): *There is a version $\big(X^{\mathrm{f}}_t(x)\big)_{t>r,\;x\in\mathsf{R}^d}$ of the density field of X^{f} such that $\mathbf{P}_{r,\mu}$–almost surely the mapping $(t,x)\mapsto X^{\mathrm{f}}_t(x)$, $(t,x)\in F$, $t>r$, is of class $\mathcal{C}^\infty$ and solves the heat equation:*

$$\frac{\partial}{\partial t}X^{\mathrm{f}}_t(x) \;=\; \frac{1}{2}\Delta X^{\mathrm{f}}_t(x), \qquad (t,x)\in F, \quad t>r.$$

(c) (Local $L^2(\mathbf{P}_{r,\mu})$–Lipschitz continuity): *For each compact subset K of $F\cap\big((r,\infty)\times\mathsf{R}^d\big)$, there is a constant $c=c(\varrho,K)$ such that*

$$\big\|X^{\mathrm{f}}_{t_1}(x_1) - X^{\mathrm{f}}_{t_2}(x_2)\big\|_2 \;\leq\; c\,\big|(t_1,x_1)-(t_2,x_2)\big|\,, \qquad (t_1,x_1),(t_2,x_2)\in K.$$

This is a generalization of results on models with a single point catalyst in R due to Fleischmann and Le Gall [FLG95], with some time-independent catalysts in R^d by Delmas [Del96], as well as with a continuous super-Brownian catalyst in R^d $(d \leq 3)$ in Fleischmann and Klenke [FK99]. We mention also the recent paper Mörters and Vogt [MV02] on a continuous super-Brownian reactant in R^d with some time-independent catalysts of Lebesgue zero closed support and such that the related Revuz measure is absolutely continuous with respect to the catalytic measure. Then also a representation in the catalyst-free region is given as a generalization of [FLG95].

Remark 8 (Some moments of X). The $X_t^{\mathrm{f}}(x)$ belong to $\mathcal{L}^2 = \mathcal{L}^2(\mathbf{P}_{r,\mu})$, have expectation

$$\mathbf{P}_{r,\mu} X_t^{\mathrm{f}}(x) \;=\; \mu {*} \mathrm{p}_{t-r}\,(x), \qquad (t,x) \in F, \quad t > r, \tag{19}$$

and covariance

$$\begin{aligned} &\mathcal{C}\mathrm{ov}_{r,\mu}\left[X_{t_1}^{\mathrm{f}}(x_1), X_{t_2}^{\mathrm{f}}(x_2)\right] \\ &\quad= \; 4 \int_r^{t_1 \wedge t_2} \mathrm{d}s \; \Big\langle \varrho_s \,,\, (\mu {*} \mathrm{p}_{s-r})\,(\theta_{x_1}\mathrm{p}_{t_1-s})\,(\theta_{x_2}\mathrm{p}_{t_2-s}) \Big\rangle \;\geq\; 0, \end{aligned} \tag{20}$$

$(t_i, x_i) \in F,\ \ t_i > r,\ \ i = 1, 2.$ ◇

1.5. **Part X^{c} of X in the catalyst region.** Fix an admissible catalyst ϱ. Our next aim is to introduce the part X^{c} of X in the catalyst region. To this aim, we enrich the basic motion process B of the continuous super-Brownian reactant X with catalyst ϱ by its collision local time process. Indeed, recall the path spaces $\mathbb{C}^t$ and the related measure space $\mathcal{M}_{\mathrm{tem}}^r$, introduced in Subsection 1.2, and the stopped collision local time process $L_{[\varrho,B]}^t$ of Definition 1. For $r \geq 0$, $a = (x, f) \in \mathsf{R}^d \times \mathbb{C}^r$, and B under $P_{r,x}$, set

$$\tilde{B}_t \;=\; (\tilde{B}_t^1, \tilde{B}_t^2) \;:=\; \Big(B_t \,,\, f + L_{[\varrho,B]}^t\big([r,\cdot\,]\big)\Big), \qquad t \geq r. \tag{21}$$

Note that the second coordinate $\tilde{B}^2$ in the process $(\tilde{B},\, \tilde{P}_{r,a}\,,\, r \geq 0,\, a \in \mathsf{R}^d \times \mathbb{C}^r)$ is a path-valued process: $\tilde{B}$ lives in $\mathsf{R}^d \times \mathbb{C}$, with state at time t in $\mathsf{R}^d \times \mathbb{C}^t$. Replacing the basic motion process B in the definition of X (Proposition 4) by the continuous Markov process $\tilde{B}$, we get a (time-inhomogeneous) continuous superprocess denoted by

$$\big(\tilde{X},\, \tilde{\mathbf{P}}_{r,\nu}\,,\, r \geq 0,\, \nu \in \mathcal{M}_{\mathrm{tem}}^r\big). \tag{22}$$

We call $\tilde{X}$ the *continuous super-Brownian reactant with catalyst ϱ enriched with its collision local time history.* Clearly, under $\tilde{\mathbf{P}}_{r,\nu}\,,\ \nu \in \mathcal{M}_{\mathrm{tem}}^r\,,$ with the projection

$$t \;\mapsto\; \tilde{X}_t\big((\cdot) \times \mathbb{C}^t\big) \tag{23}$$

we gain back the continuous super-Brownian reactant X with catalyst ϱ, under the law $\mathbf{P}_{r,\mu}\,,$ where $\mu = \nu\big((\cdot) \times \mathbb{C}^r\big)$.

We want to use this additional coding to decide whether the reactant particle is in the catalyst region or not. Fix $t > 0$. We say, a function $f \in \mathbb{C}$ is *increasing in* $t-$, if $f(t) > f(t-\delta)$ for all $\delta \in (0,t)$. Write

$$C_t \;:=\; \Big\{ f \in \mathbb{C}^t \,:\, f \text{ is increasing in } t- \Big\} \tag{24}$$

for the set of all functions in $\mathbb{C}$ stopped at time t and with increase in $t-$.

Consider $\tilde{X}$ under the law $\tilde{\mathbf{P}}_{r,\mu\times\delta_0}$, $\mu\in\mathcal{M}_{\rm tem}$ (with δ_0 the delta measure at the zero function $0\in\mathbb{C}$), and X under $\mathbf{P}_{r,\mu}$. Then the *part* $X^{\rm c}=X^{\rm c}[\varrho]$ *of* X *in the catalytic region* is defined via

$$\langle X_t^{\rm c},\varphi\rangle \;:=\; \int_{\mathsf{R}^d\times C_t}\tilde{X}_t\big({\rm d}(x,f)\big)\,\varphi(x),\qquad t>r,\quad \varphi\in\mathcal{C}_{\rm exp}. \tag{25}$$

Remark 9 (**Warning for misinterpretation**). Note that $\tilde{B}_t^2\in C_t$ means that the reactant particle with path B was "significantly" in contact with the catalyst ϱ at time $t-$. Think of, for instance, the single point catalyst in R of Example 3(ii). Here, for example, in the moment of the first hitting of the catalyst, the reactant particle is certainly at the catalyst's position, but it does not yet have collected enough collision local time. This, of course, also means, that there is no branching in such moment. Moreover, in this example the reactant process X does not at all have mass at the catalysts position since it always has absolutely continuous states ([FLG95]). ◇

Remark 10 (**Relation between $X^{\rm f}$ and $X^{\rm c}$**). Note that $X_t^{\rm c}$ can take care only on those reactant "particles" at time t which had been significantly in contact with the catalyst ϱ at time $t-$. In other words, reactant "particles" which come significantly in contact with ϱ just at time $t+$ are neglected in $X_t^{\rm c}$. Nevertheless, the definition of $X^{\rm c}$ is very natural, since assertions on the future should not be involved. From this point of view one is obliged to replace the former definition of $X^{\rm f}$ by

$$\langle \hat{X}_t^{\rm f},\varphi\rangle \;:=\; \int_{\mathsf{R}^d\times(\mathbb{C}^t\setminus C_t)}\tilde{X}_t\big({\rm d}(x,f)\big)\,\varphi(x),\qquad t>r,\quad \varphi\in\mathcal{C}_{\rm exp}, \tag{26}$$

which implies

$$\langle \hat{X}_t^{\rm f}+X_t^{\rm c},\varphi\rangle \;=\; \int_{\mathsf{R}^d\times\mathbb{C}^t}\tilde{X}_t\big({\rm d}(x,f)\big)\,\varphi(x) \;=\; \langle X_t\,,\varphi\rangle. \tag{27}$$

But it is not at all clear how to get for $\hat{X}_t^{\rm f}$ the same properties as we have for $X^{\rm f}$ in Subsection 1.4. Actually we only know that

$$\hat{X}_t^{\rm f} \;\ge\; X_t^{\rm f}. \tag{28}$$

In fact, $X_t^{\rm f}\le X_t$ [recall (18)]. Moreover, $X_t^{\rm f}$ is determined by functions $\varphi\in\mathcal{C}_{\rm exp}$ with $\operatorname{supp}\varphi\subset F_t$ (introduced in the beginning of Subsection 1.4). But if x belongs to $\operatorname{supp}\varphi$ and for Brownian motion B we have $B_t=x$, then for the corresponding enriched process state $\tilde{B}_t=(x,f)$ we have $f\notin C_t$. Hence, for those φ,

$$\langle \hat{X}_t^{\rm f},\varphi\rangle \;=\; \int_{\mathsf{R}^d\times(\mathbb{C}^t\setminus C_t)}\tilde{X}_t\big({\rm d}(x,f)\big)\,\varphi(x) \;=\; \int_{\mathsf{R}^d\times\mathbb{C}^t}\tilde{X}_t\big({\rm d}(x,f)\big)\,\varphi(x) \;=\; \langle X_t\,,\varphi\rangle,$$

implying (28). However, this difference between $\hat{X}_t^{\rm f}$ and $X_t^{\rm f}$ is of no harm for us since we focus on $X^{\rm c}$ in the further procedure of the paper.

It might be worth to mention also the following *degenerate example:* Let ϱ be supported by a polar set of Brownian motion. Then $X^{\rm f}=0=X^{\rm c}$, and $\hat{X}^{\rm f}=X$ is the heat flow. ◇

The finite-dimensional distributions of $X^{\rm c}$ will be characterized in Lemma 16; for some moment formulae, see Remark 17.

Before we develop the Brownian snake representation of $X^{\rm c}$, in the following subsection first the concept of a catalytic Brownian snake and its excursion and some modified hitting measures will be sketched. For more details, see Section 3 below.

1.6. **A catalytic Brownian snake concept.** Recall that ϱ is a fixed admissible catalyst and the branching clock of an intrinsic reactant particle with path B under $P_{r,x}$ is governed by the collision local time

$$t \mapsto {}^{r}L(t) \;:=\; L_{[\varrho,B]}\bigl([r,t]\bigr), \qquad t \geq r. \tag{29}$$

Since we did not require that the collision local time $L = L_{[\varrho,B]}$ is (strictly) increasing, the inverse function

$${}^{r}L^{-1}(t) \;:=\; \inf\bigl\{s \geq r:\ {}^{r}L(s) \geq t\bigr\}, \qquad t \geq 0, \tag{30}$$

only belongs to $\mathcal{D}\bigl(\mathsf{R}_{+}, [r,\infty]\bigr)$ (càglàd functions on R_{+} with values in $[r,\infty]$). In particular, ${}^{r}L^{-1}(0) = r$. Introduce the random countable set

$$\mathsf{D} \;:=\; \bigl\{t \geq 0:\ {}^{r}L^{-1}(t+) \neq {}^{r}L^{-1}(t)\bigr\} \tag{31}$$

of all discontinuity points of ${}^{r}L^{-1}$ under $P_{r,x}$. We *require from now on* that these discontinuities do not occur at fixed times:

Hypothesis 11 (**Regularity of the catalyst**)**.** The admissible catalyst ϱ has to fulfil

$$P_{r,x}(t \in \mathsf{D}) \;=\; 0, \qquad t \geq 0, \tag{32}$$

for all $(r,x) \in \mathsf{R}_{+} \times \mathsf{R}^{d}$. $\diamond$

Example 12 (**Regular catalysts**)**.** Consider an *absolutely continuous catalyst* ϱ with bounded density function (recall Example 3) whose closed support $\operatorname{supp}\varrho$ is a regular domain in R^{d} (as a disc in R^{2}). Then, starting from inside $\operatorname{supp}\varrho$, the hitting time η, say, of the boundary of $\operatorname{supp}\varrho$ is a continuous random variable, hence the increment $\int_{0}^{\eta} \mathrm{d}s\, \varrho(B_{s})$ is a continuous variable. Thus, by the strong Markov property of Brownian motion, also the maximal positive increment is continuous. Moreover, starting directly from the boundary, which is regular by assumption, the interior is hit with certainty, and again by the strong Markov property we are back in the previous situation. Altogether, Hypothesis 11 is fulfilled.

Another example is the case of *hyperplanes weighted by a stable measure* [Example 3(v)]. Here, roughly speaking, $L_{[\varrho,B]}$ is the sum of Brownian local times over a dense set, implying that D is empty, almost surely. $\diamond$

Fix an admissible catalyst ϱ on R^{d} satisfying Hypothesis 11. Later we want to give a Brownian snake representation of the process $(X_{t}^{\mathrm{c}})_{t>r}$ under the law $\tilde{\mathbf{P}}_{r,\mu\times\delta_{0}}$, where, for the moment, $r \geq 0$ is fixed. Recall from (29) the collision local time process ${}^{r}L$ under $P_{r,x}$ and its inverse ${}^{r}L^{-1}$. Note that for $t > r$,

$${}^{r}L^{-1}\bigl({}^{r}L(t)\bigr) = t \quad \text{if and only if} \quad t \mapsto L(t) \text{ is increasing in } t-. \tag{33}$$

Note also that for $t > r$ an increase of ${}^{r}L = L_{[\varrho,B]}\bigl([r,\,\cdot\,]\bigr)$ in $t-$ just means that the reactant particle's path B is in the catalytic region at time $t-$.

Set $\mathsf{E} := \mathsf{R}_{+} \times (\mathsf{R}_{+} \times \mathsf{R}^{d})^{\bullet}$, where $(\mathsf{R}_{+} \times \mathsf{R}^{d})^{\bullet}$ denotes the one-point compactification of $\mathsf{R}_{+} \times \mathsf{R}^{d}$. The cemetery point $\dagger$ in $(\mathsf{R}_{+} \times \mathsf{R}^{d})^{\bullet}$ is also denoted by $(\infty,\dagger)$. Endowed with the product metric d_{E}, we get a Polish space $(\mathsf{E}, \mathrm{d}_{\mathsf{E}})$. A point $e \in \mathsf{E}$

is mostly written as (e^1,e^2,e^3) in the obvious meaning. We introduce a process $\xi=(\xi^1,\xi^2,\xi^3)$ in E by setting for $t\geq 0$,

$$\xi_t \;:=\; \begin{cases} \big(r'+t,\ {}^{r}L^{-1}(t),\ \dot B_{{}^{r}L^{-1}(t)}\big) & \text{if}\quad \xi_0=(r',r,x)\in\check{\mathsf{E}},\\ (r'+t,\infty,\dagger) & \text{if}\quad \xi_0=(r',\infty,\dagger),\end{cases} \tag{34}$$

where $\dot B$ is the extended Brownian motion [introduced before (8)] based on B under the law $P_{r,x}$, and we set

$$\check{\mathsf{E}} \;:=\; \big\{e=(e^1,e^2,e^3)\in\mathsf{E}:\ (e^2,e^3)\neq(\infty,\dagger)\big\}. \tag{35}$$

Lemma 13 (Underlying motion process). *Under Hypothesis* 11, $\xi=\xi[\varrho]$ *is a (time-homogeneous) weak Feller process* $(\xi,\,\Pi_e\,,\,e\in\mathsf{E})$ *with paths in the Skorohod space* $\mathcal{D}(\mathsf{R}_+,\mathsf{E})$ *of* E*–valued càglàd paths.*

Here with *weak Feller process* we mean that the semigroup $S=(S_t)_{t\geq 0}$ related to the Markov process ξ, acting on the Banach space $\mathcal{C}_{\mathrm{b}}(\mathsf{E})$ of bounded continuous functions $\varphi:\mathsf{E}\to\mathsf{R}$ (with the supremum norm),

$$S_t\varphi\,(e) \;:=\; \Pi_e\varphi(\xi_t),\qquad t\geq 0,\quad \varphi\in\mathcal{C}_{\mathrm{b}}(\mathsf{E}),\quad e\in\mathsf{E}, \tag{36}$$

is pointwise continuous; that is, $(t,e)\mapsto S_t\varphi\,(e)$ is continuous.

Proof. Since Brownian motion is strong Markov, ξ is a time-homogeneous Markov process. Fix $\varphi\in\mathcal{C}_{\mathrm{b}}(\mathsf{E})$. Let $(t_n,r_n',r_n,x_n)\to(t,r',r,x)$ in $\mathsf{R}_+\times\mathsf{E}$ as $n\uparrow\infty$. Then

$$\varphi\big(r_n'+t_n\,,\ {}^{r_n}L^{-1}(t_n),\ \dot B_{{}^{r_n}L^{-1}(t_n)}\big)\quad\text{under the law } P_{r_n,x_n} \tag{37}$$

equals in distribution to

$$\varphi\big(r_n'+t_n\,,\ {}^{r}L^{-1}(t_n-r_n+r),\ x_n-x+\dot B_{{}^{r}L^{-1}(t_n-r_n+r)}\big)\quad\text{under the law } P_{r,x} \tag{38}$$

which by our Hypothesis 11 converges in distribution to

$$\varphi\big(r'+t,\ {}^{r}L^{-1}(t),\ \dot B_{{}^{r}L^{-1}(t)}\big)\quad\text{under the law } P_{r,x}\,. \tag{39}$$

This gives the weak Feller property, finishing the proof. □

Let $\mathcal{C}_{\exp}(\mathsf{E})$ denote the space of all mappings $\psi:\mathsf{E}\to\mathsf{R}$ such that there is a function $\varphi=\varphi_\psi\in\mathcal{C}^+_{\exp}(\dot{\mathsf{R}}^d)$ with $|\psi|\,(e)\leq\varphi(e^3)$, $e=(e^1,e^2,e^3)\in\mathsf{E}$.

Modifying the approach of Section 4.1 in [BLGLJ97], we want to introduce the Brownian snake corresponding to the weak Feller motion process ξ. For this purpose, let $\mathcal{W}$ denote the collection of all stopped càglàd paths $w\in\bigcup_{s\geq 0}\mathcal{D}^s(\mathsf{R}_+,\mathsf{E})$. Set

$$\zeta(w) \;:=\; \inf\big\{s\geq 0:\ w(t)=w(s)\ \forall t\geq s\big\} \tag{40}$$

for the (finite) *lifetime* of the path $w\in\mathcal{W}$. Note that $\zeta(w)=s$ if and only if $w\in\mathcal{D}^s(\mathsf{R}_+,\mathsf{E})$ (recall that the first component of ξ is the identity function, except a shift). Recall also that d_{E} and $\mathrm{d}_{\mathcal{D}}$ (introduced in Subsection 1.2) are the metrics in E and $\mathcal{D}(\mathsf{R}_+,\mathsf{E})$, respectively. Introduce a metric $\mathrm{d}_{\mathcal{W}}$ in $\mathcal{W}$ by

$$\mathrm{d}_{\mathcal{W}}(w,\tilde w) \;:=\; \big|\zeta(w)-\zeta(\tilde w)\big| + \sup_{s\geq 0}\mathrm{d}_{\mathcal{D}}\big(w^{s\wedge\zeta(w)},\tilde w^{s\wedge\zeta(\tilde w)}\big),\qquad w,\tilde w\in\mathcal{W}, \tag{41}$$

with stopped paths $w^s=w\big(s\wedge(\cdot)\big)$. Note that $(\mathcal{W},\mathrm{d}_{\mathcal{W}})$ is a Polish space whose topology is *stronger* than the one used in [BLGLJ97]. In fact, $\mathrm{d}_{\mathcal{W}}$ controls in particular the *tip* $w\big(\zeta(w)\big)$ of w, that is, the fluctuations within the interval $\big(\zeta(w)\wedge\zeta(\tilde w),\,\zeta(w)\vee\zeta(\tilde w)\big)$. The price of this sharpening of the topology will be that the Brownian snake W we want to introduce will not anymore be a continuous process, but a càdlàg one, which is sufficient for our purpose.

For $e \in \check{\mathsf{E}}$ [recall (35)], set $\mathcal{W}_e := \{w \in \mathcal{W} : w(0) = e\}$. As we will see later (Proposition 19), the *Brownian snake* $W = W[\varrho]$ *with basic motion process* $\xi = \xi[\varrho]$ *and root* e is a certain $\mathcal{W}_e$–valued càdlàg (time-homogeneous) weak Feller and strong Markov process $(W, \mathbb{P}_w, w \in \mathcal{W}_e)$. Recalling notation (40) of the lifetime of a path in $\mathcal{W}$, put $\zeta_s := \zeta(W_s)$, $s \geq 0$. By construction, this *lifetime process* $\zeta = (\zeta_s)_{s\geq 0}$ of the snake W is a (standard) reflecting Brownian motion in R_+. Set

$$\sigma = \sigma(\zeta) := \inf\{s > 0 : \zeta_s = 0\}. \tag{42}$$

For $e \in \check{\mathsf{E}}$, denote by $\mathbf{e}$ the trivial path in $\mathcal{W}_e$ with life time 0. Since it is a regular point for the snake $(W, \mathbb{P}_w)$, *excursion measures* $\mathbb{N}_e = \mathbb{N}_e[\varrho]$ can be built. They are certain σ–finite measures on $\mathcal{C}(\mathsf{R}_+, \mathcal{W})$.

Fix $t \geq 0$. Put

$$D_t := \mathsf{R}_+ \times [0, t) \times \dot{\mathsf{R}}^d. \tag{43}$$

We always have

$$\Pi_e\big(\xi_s \in \mathsf{E}\backslash D_t \text{ for some } s \geq 0\big) = 1 \tag{44}$$

by our definition (34) of ξ based on ${}^r\!L^{-1}(s)$, which tends to infinity as $s \uparrow \infty$. Introduce the *exit time* τ_t of ξ from D_t :

$$\tau_t = \tau_t(\xi) := \inf\{s \geq 0 : \xi_s \notin D_t\}. \tag{45}$$

Remark 14 (Exit time formula). $\tau_t(\xi)$ is always finite and, in fact,

$$\tau_t(\xi) = {}^r\!L(t). \tag{46}$$

Indeed, from the definitions (45) of τ_t and (34) of ξ, under $\tau_t(\xi) < \infty$,

$${}^r\!L^{-1}\big(\tau_t(\xi) + \varepsilon\big) \geq t, \qquad \forall \varepsilon > 0. \tag{47}$$

Thus, from the continuity and monotonicity of $L_{[\varrho,B]}$,

$$\tau_t(\xi) + \varepsilon \geq {}^r\!L(t), \qquad \forall \varepsilon > 0, \tag{48}$$

resulting into $\tau_t(\xi) \geq {}^r\!L(t)$, which is clearly also true if $\tau_t(\xi) = \infty$. Assume now that

$$\tau_t(\xi) > {}^r\!L(t). \tag{49}$$

Take $s \in \big(0, \tau_t(\xi)\big)$. Then ${}^r\!L^{-1}(s) < t$, implying $s \leq {}^r\!L(t)$. Letting $s \uparrow \tau_t(\xi)$ leads to a contradiction to (49), giving claim (46). ◇

Recall that $r, t \geq 0$ are fixed. The *exit local time*

$$s \mapsto {}^r\!L_s^{D_t} := \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \int_0^s \mathrm{d}s' \, 1_{\{\tau_t(W_{s'}) < \zeta_{s'} < \tau_t(W_{s'}) + \varepsilon\}} \tag{50}$$

of W from D_t makes sense $\mathbb{P}_w$–a.s. and $\mathbb{N}_e$–a.e. (see also (86) below). Put

$$\hat{W}_s^t := \begin{cases} W_s(\zeta_s), & \text{if } \ W_s^2(\zeta_s) = t, \\ \dagger, & \text{otherwise}, \end{cases} \tag{51}$$

with $\dagger$ the cemetery point in $\dot{E}$. Note that $\hat{W}_s^t \neq \dagger$, that is $W_s^2(\zeta_s) = t$, can be interpreted as the *tip* $W_s(\zeta_s)$ of the snake state W_s at time s is in the catalytic region; recall the remark after (33).

The *modified hitting measure* $Z_t = Z_t^W$ of the boundary $\partial D_t = \mathsf{R}_+ \times \{t\} \times \dot{\mathsf{R}}^d$ of D_t by the snake W is now defined via

$$\langle Z_t\,,\psi\rangle \;:=\; \int_0^\sigma {}^r\!L_{\mathrm{d}s}^{D_t}\,\psi(\hat{W}_s^t), \qquad t\geq 0, \quad \psi\in\mathcal{C}_{\exp}^+(\mathsf{E}). \tag{52}$$

These modified hitting measures are a key tool for the intended snake representation of X^{c}.

1.7. **Brownian snake representation of X^{c}.** We are ready now to formulate the desired snake representation. Recall that ϱ is an admissible catalyst in R^d.

Theorem 15 (**Brownian snake representation of X^{c}**). *Fix $d\geq 1$, $r\geq 0$, and $\mu\in\mathcal{M}_{\mathrm{tem}}(\mathsf{R}^d)$. Impose Hypothesis 11. Consider the part X^{c} of X in the catalyst region as in Subsection 1.5, where X starts at time r from $X_r=\mu$. Furthermore, consider the Poisson point measure*

$$N(\mathrm{d}W) \;\; \textit{on} \;\; \mathcal{C}(\mathsf{R}_+,\mathcal{W}) \;\; \textit{with intensity measure} \;\; \int_{\check{\mathsf{E}}}(\delta_0\times\delta_r\times\mu)(\mathrm{d}e)\,\mathbb{N}_e(\mathrm{d}W). \tag{53}$$

Then the finite-dimensional distributions of the $\mathcal{M}_{\mathrm{tem}}(\mathsf{R}^d)$–valued processes $(X_t^{\mathrm{c}})_{t>r}$ and of

$$t \;\mapsto\; \int_{\mathcal{C}(\mathsf{R}_+,\mathcal{W})} N(\mathrm{d}W)\, Z_t^W\bigl(e^3\in(\cdot),\ e^3\neq\dagger\bigr), \quad t>r, \tag{54}$$

coincide.

Unfortunately, this theorem gives only an fdd representation, compared with other snake representations which work at all times. Nevertheless, we hope it can be used to deal with interface questions. It might be also helpful for dealing with properties of solutions to the singular equation (14/15).

The proof of Theorem 15 will be given in Subsection 3.5. It relies on log-Laplace representations of both processes on which we focus from now on. This requires, in particular, a detailed construction of Brownian snake tools in the present catalytic setting in Section 3.

2. Log-Laplace approach to X^{c}

Although the part X^{c} of X in the catalyst region is not a Markov process, we still use a log-Laplace equation to characterize its *finite-dimensional distributions.*

2.1. **Fdd characterization of X^{c}.** Recall notation ${}^rL$ from (29). Under $P_{r,x}$, $(r,x)\in\mathsf{R}_+\times\mathsf{R}^d$, introduce the following (left-continuous) *reduced Brownian motion* $\underline{B}=(\underline{B}_t)_{t>r}$:

$$t \;\mapsto\; \underline{B}_t \;:=\; \begin{cases} B_t & \text{if } t\mapsto {}^rL(t) \text{ is in creasing in } t-,\\ \dagger & \text{otherwise,}\end{cases} \tag{55}$$

which lives at time t only if it was in contact with the catalyst ϱ at time $t-$.

Lemma 16 (**Log-Laplace equation for X^{c}**). *Fix $\mu\in\mathcal{M}_{\mathrm{tem}}$, $m\geq 1$, $0\leq r_0=:$ $t_0<t_1<\cdots<t_m$, and $\varphi_1,\ldots,\varphi_m\in\mathcal{C}_{\exp}^+$. Then*

$$-\log\tilde{\mathbf{P}}_{r_0,\mu\times\delta_0}\,\exp\Bigl[-\sum_{j=1}^m\bigl\langle X_{t_j}^{\mathrm{c}}\,,\varphi_j\bigr\rangle\Bigr] \;=\; \langle\mu,v_{r_0}\rangle, \qquad 0\leq r_0<t_1\,, \tag{56}$$

where $v = v^{t_1,\ldots,t_m}[\varphi_1,\ldots,\varphi_m,\varrho]$ *is the unique non-negative solution of the following integral equation:*

$$v_r(x) \;=\; P_{r,x}\Big[\sum_{j=1}^{m} 1_{\{r\le t_j\}}\,\varphi_j(\underline{B}_{t_j}) \;-\; 2\int_r^\infty L_{[\varrho,B]}(\mathrm{d}s)\,v_s^2(B_s)\Big], \tag{57}$$

$(r,x)\in \mathsf{R}_+\times\mathsf{R}^d$.

Proof. Fix $r_0,\mu,t_j,\varphi_j\,,\ 1\le j\le m,$ as in the lemma. For $r\ge 0,\ (x,f)\in\mathsf{R}^d\times\mathbb{C}^r$, and $1\le i\le m,$ set

$$v_r(x) \;:=\; \begin{cases} -\log\tilde{\mathbf{P}}_{r,\delta_{(x,f)}}\exp\Big[-\sum_{j=i}^{m}\big\langle X^{\mathrm{c}}_{t_j},\varphi_j\big\rangle\Big] & \text{if}\quad t_{i-1}\le r<t_i\,,\\ 0 & \text{if}\quad r\ge t_m\,. \end{cases} \tag{58}$$

Note that the right hand side does not depend on f, justifying the notation on the left hand side. Obviously, this $v = v^{t_1,\ldots,t_m}[\varphi_1,\ldots,\varphi_m,\varrho]$ satisfies (57) on $[t_m\,,\infty)\times\mathsf{R}^d$. Assume now that $t_{i-1}\le r<t_i\,,\ 1\le i\le m.$

For $j=1,\ldots,m,$ put

$$\psi_j \;:=\; \varphi_j\otimes 1_{C_t}\quad\text{with } C_t \text{ defined in (24).} \tag{59}$$

Then,

$$\begin{aligned}\tilde{\mathbf{P}}_{r,\mu\times\delta_0}\exp\Big[-\sum_{j=i}^{m}\big\langle X^{\mathrm{c}}_{t_j},\varphi_j\big\rangle\Big] \;&=\; \tilde{\mathbf{P}}_{r,\mu\times\delta_0}\exp\Big[-\sum_{j=i}^{m}\big\langle \tilde{X}_{t_j},\psi_j\big\rangle\Big]\\ &=\; \exp\langle\mu\times\delta_0,-\tilde{v}_r\rangle,\end{aligned} \tag{60}$$

where, from standard facts on superprocesses, $\tilde{v} = \tilde{v}^{t_1,\ldots,t_m}[\psi_1,\ldots,\psi_m,\varrho]$ is the unique non-negative solution to the integral equation

$$\tilde{v}_r(a) \;=\; \tilde{P}_{r,a}\Big[\sum_{j=i}^{m} 1_{\{r\le t_j\}}\,\psi_j(\tilde{B}_{t_j}) \;-\; 2\int_r^\infty L_{[\varrho,B]}(\mathrm{d}s)\,\tilde{v}_s^2(\tilde{B}_s)\Big], \tag{61}$$

$r\ge t_{i-1}\,,\ a\in\mathsf{R}^d\times\mathbb{C}^r$. But by definitions of ψ_j in (59), $\tilde{B}$ in (21), C_{t_j} in (24), and $\underline{B}$ in (55),

$$\psi_j(\tilde{B}_{t_j}) \;=\; \varphi_j(B_{t_j})\,1_{C_{t_j}}\big({}^{r}L^{t_j}_{[\varrho,B]}\big) \;=\; \varphi_j(\underline{B}_{t_j}), \tag{62}$$

and by (60) and (58),

$$\tilde{v}_r(x,f) \;=\; v_r(x),\quad\text{hence}\quad \tilde{v}_s(\tilde{B}_s) \;=\; v_s(B_s). \tag{63}$$

Inserting (62) and (63) into (61) with $a=(s,f)$ shows that v as defined in (58) satisfies equation (57) on $[t_{i-1}\,,\infty)\times\mathsf{R}^d$, hence on $\mathsf{R}_+\times\mathsf{R}^d$, and that (56) holds.

The uniqueness statement follows from the uniqueness in the larger set of solutions to equation (61). □

2.2. **Example for a full interface.** For X^{c}, with any admissible catalyst ϱ, we have the following moment formulae, which can easily be derived from the log-Laplace equation (57). Denote by $P_{r,\mu}$ the "law" of Brownian motion in R^d started at time $r\ge 0$ with the measure $\mu\in\mathcal{M}_{\mathrm{tem}}$. Recall the reduced Brownian motion $\underline{B}$ introduced in (55).

Remark 17 (**Some moments of** X^{c}). Fix $r \geq 0$ and $\mu \in \mathcal{M}_{\mathrm{tem}}$. We have the following expectation and covariance formulae. For $0 \leq r < t_1 \leq t_2$ and $\varphi_1, \varphi_2 \in \mathcal{C}^2_{\mathrm{tem}}$,

$$\tilde{\mathbf{P}}_{r,\mu\times\delta_0} \langle X^{\mathrm{c}}_{t_1}, \varphi_1\rangle \;=\; P_{r,\mu}\, \varphi_1(\underline{B}_{t_1}), \tag{64}$$

$$\begin{aligned} &\tilde{\mathbf{C}}\mathrm{ov}_{r,\mu\times\delta_0}\big[\langle X^{\mathrm{c}}_{t_1}, \varphi_1\rangle, \langle X^{\mathrm{c}}_{t_2}, \varphi_2\rangle\big] \\ &\quad = 4\, P_{r,\mu} \int_r^{t_1} L_{[\varrho,B]}(\mathrm{d}s)\, (P_{s,B_s} \times P_{s,B_s})\, \varphi_1(\underline{B}'_{t_1})\, \varphi_2(\underline{B}''_{t_2}) \end{aligned} \tag{65}$$

with independent Brownian motions B, B', B''. ◇

As an application of these moment formulae, we mention the following example.

Example 18 (**Carrying Hausdorff dimension 2**). Suppose $d = 2$ and consider the catalyst with bounded density function of Example 3(v): $\varrho_t(\mathrm{d}x) = \varrho_t(x)\,\mathrm{d}x$. Start the reactant X at time $r \geq 0$ with a uniformly bounded density. Then all non-zero samples of X^{c}_t have carrying Hausdorff dimension two, for almost all $t \geq r$. ◇

It would be desirable, to verify such property for more general catalysts.

Now we give *details to Example* 18. According to Frostman's lemma (e.g., [Fal90, Theorem 4.13(a)]), it suffices to show that for fixed $0 \leq r < t$, absolutely continuous $\mu \in \mathcal{M}_{\mathrm{tem}}$ with bounded density function, as well as all $K \geq 1$ and $1 \leq \delta < 2$,

$$\tilde{\mathbf{P}}_{r,\mu\times\delta_0} \int_{|x|\leq K} X^{\mathrm{c}}_t(\mathrm{d}x) \int_{|y|\leq K} X^{\mathrm{c}}_t(\mathrm{d}y)\, |x-y|^{-\delta} \;<\; \infty. \tag{66}$$

By (65) and (64) this means that

$$P_{r,\mu} \int_r^{t_1} L_{[\varrho,B]}(\mathrm{d}s)\, (P_{s,B_s} \times P_{s,B_s})\, |\underline{B}'_t - \underline{B}''_t|^{-\delta}\, 1_{\{|\underline{B}'_t|, |\underline{B}''_t| \leq K\}} \;<\; \infty, \tag{67}$$

and

$$P_{r,\mu} \times P_{r,\mu}\, |\underline{B}_t - \underline{B}'_t|^{-\delta}\, 1_{\{|\underline{B}_t|, |\underline{B}'_t| \leq K\}} \;<\; \infty, \tag{68}$$

where we agree to read such integrand as 0 if a cemetery point is involved [recall notation (55)]. Using that μ has a bounded density function, the formula for $L_{[\varrho,B]}$ as written in Example 3(v), and passing from $\underline{B}$ to B, statement (68) follows from the estimates in the ordinary continuous super-Brownian motion case in dimension $d = 2$; see, for instance, [LG99, page 34.].

3. Catalytic Brownian snake

In this section we introduce the catalytic Brownian snake in more detail, culminating in the proof of the snake representation Theorem 15.

3.1. **Catalytic Brownian snake** W. Recall the weak Feller process $\xi = \xi[\varrho] = (\xi, \Pi_e, e \in \mathsf{E})$ from Lemma 13. For $e \in \check{\mathsf{E}}$, $w \in \mathcal{W}_e$, $a \in [0, \zeta(w)]$, and $b \geq a$, let $Q_{a,b}(w, \mathrm{d}\tilde{w})$ denote the unique probability law on $\mathcal{W}_e$ such that

- (i) $\zeta(\tilde{w}) = b$, $Q_{a,b}(w, \mathrm{d}\tilde{w})$–a.s.,
- (ii) $\tilde{w}(t) = w(t)$, $\forall t \in [0, a]$, $Q_{a,b}(w, \mathrm{d}\tilde{w})$–a.s., and
- (iii) the law of $t \mapsto \tilde{w}(a+t)$, $0 \leq t \leq b-a$, under $Q_{a,b}(w, \mathrm{d}\tilde{w})$ is equal to the law of $t \mapsto \xi_t$, $0 \leq t \leq b-a$, under $\Pi_{w(a)}$.

Let $\theta_s^{\zeta(w)}\big(\mathrm{d}(a,b)\big)$ denote the joint distribution of $(\inf_{0\le r\le s}\beta_r\,,\beta_s)\,,$ where β is a standard Brownian motion in R_+ reflecting at 0 and with initial state $\beta_0 = \zeta(w) \ge 0$. (For an explicit formula for $\theta_s^{\zeta(w)}\big(\mathrm{d}(a,b)\big)$, see [BLGLJ97], in front of Proposition 5.)

Proposition 19 (Brownian snake W with motion ξ). *Fix $e \in \check{\mathsf{E}}$. Impose Hypothesis* 11. *There is a $\mathcal{W}_e$–valued càdlàg (time-homogeneous) weakly Feller and strong Markov process $(W, \mathbb{P}_w\,,\, w \in \mathcal{W}_e)$ with transition probabilities*

$$Q_s(w, \mathrm{d}\tilde{w}) \;:=\; \int_{\mathsf{R}_+^2} \theta_s^{\zeta(w)}\big(\mathrm{d}(a,b)\big)\, Q_{a,b}(w, \mathrm{d}\tilde{w}). \tag{69}$$

Put $\zeta_s := \zeta(W_s),\; s \ge 0$. This lifetime process $\zeta = (\zeta_s)_{s\ge 0}$ of W is a standard reflecting Brownian motion in R_+. Furthermore, the conditional distribution of W given ζ is that of a time-inhomogeneous Markov process with transition function

$$R_{u,v}(w, \mathrm{d}\tilde{w}) \;:=\; Q_{\inf_{u\le r\le v}\zeta_r\,,\,\zeta_v}(w, \mathrm{d}\tilde{w}), \qquad 0 \le u \le v, \quad w \in \mathcal{W}_e\,. \tag{70}$$

This path-valued process $W = W[\varrho]$ is said to be the *Brownian snake with basic motion process $\xi = \xi[\varrho]$ and root e*. Note that the lifetime process ζ is strong Markov with respect to the natural filtration of W.

Proof of Proposition 19. This follows from [BLGLJ97, Proposition 5], based on the proof of Theorem 1.1 in [LG93] despite our setting of càglàd functions, and our stronger topology on $\mathcal{W}$. In fact, we only have to prove the weak Feller, càdlàg, and strong Markov property.

Fix $t \ge 0$ and $w \in \mathcal{W}_e$. Consider W_t under $\mathbb{P}_w$ and ξ under $\Pi_e\,$, and let ζ be a standard reflecting Brownian motion independent of ξ and starting from $\zeta_0 = \zeta(w)$. Then W_t and the stopped path $\xi^{\zeta_t} = \xi_{\zeta_t\wedge(\cdot)}$ are identically distributed. Moreover, the latter set of laws is included in the collection of all distributions of $\xi^s = \xi_{s\wedge(\cdot)}$ (stopped at any time $s \ge 0$). Now the latter set of laws on $\mathcal{W}$ is tight, hence, under $\mathbb{P}_w$, the family $\{W_t : t \ge 0\} \subset \mathcal{W}$ is tight in law. Thus, for each $n \ge 1$, we can choose a compact subset K_n of the Polish space $\mathcal{W}$ such that

$$\inf_{t\ge 0} \mathbb{P}_w\left(W_t \in K_n\right) \;\ge\; 1 - \frac{1}{n}\,. \tag{71}$$

We may additionally assume that $K_1 \subseteq K_2 \subseteq \cdots$. Let $K_\infty := \bigcup_{n\ge 1} K_n$. Then $\mathbb{P}_w(W_t \in K_\infty) = 1,\; t \ge 0$. Therefore, W is a Markov process on the σ–compact space K_∞. Denote by $\dot{K}_\infty := (K_\infty)^\bullet$ the one-point compactification of K_∞ (with additional point $\dagger$). Finally, denote by $\dot{W}$ the extension of W to $\dot{K}_\infty$ by declaring the cemetery point $\dagger$ of $\dot{K}_\infty$ as an absorbing point of $\dot{W}$.

Now we adapt arguments of [Wil79, § III.13] (which deals with Feller processes on locally compact spaces) to our situation.

Introduce the semigroup $T = (T_t)_{t\ge 0}$ related to the Markov process $\dot{W}$:

$$T_t\varphi\,(w) \;:=\; \mathbb{P}_w\varphi(\dot{W}_t), \qquad t \ge 0, \quad \varphi \in \mathcal{C}(\dot{K}_\infty), \quad w \in \dot{K}_\infty\,. \tag{72}$$

From the definition of $\dot{W}$, and since ξ is weakly Feller (recall Lemma 13), it is easy to see that T is a *weak Feller semigroup* acting on $\mathcal{C}(\dot{K}_\infty)$. This implies that also the related weak *resolvent* $R = (R_\lambda)_{\lambda>0}$ defined by

$$R_\lambda\varphi\,(w) \;=\; \int_0^\infty \mathrm{d}t\; \mathrm{e}^{-\lambda t}\, T_t\varphi\,(w), \qquad \lambda > 0, \quad \varphi \in \mathcal{C}(\dot{K}_\infty), \quad w \in \dot{K}_\infty\,, \tag{73}$$

is weakly continuous.

Fix $\varphi \in \mathcal{C}^+(\dot{K}_\infty)$, $\lambda > 0$. Then as in [Wil79, § III.13], $t \mapsto \mathrm{e}^{-\lambda t} R_\lambda \varphi(\dot{W}_t)$ is a supermartingale under $\mathbb{P}_w$. Hence, by Doob's regularity theorem ([Wil79, Theorem II.38]), the random process

$$t \mapsto R_\lambda \varphi(\dot{W}_t) \quad \text{has a càdlàg modification.} \tag{74}$$

Replace now (λ, φ) by a countable dense family of such (λ_m, φ_n), $m, n \geq 1$. Then $\{R_{\lambda_m}\varphi_n : m, n \geq 1\}$ is a countable family of functions on $\dot{K}_\infty$, separating points. Indeed, T is weakly continuous, and the φ_n separate points. Together with the càdlàg property (74) of the $t \mapsto R_\lambda \varphi(\dot{W}_t)$, we get the *càdlàg* modification of $\dot{W}$.

As the weak Feller process $\dot{W}$ has right-continuous paths, from the weak continuity of the corresponding resolvent and [GS75, Theorem 1.4.7] we get the *strong Markov* property of $\dot{W}$.

Since starting from K_∞, the process $\dot{W}$ coincides with W, we get the weak Feller, càdlàg and strong Markov property of W, finishing the proof. □

3.2. **Excursion measures** $\mathbb{N}_e$. Recall that $\mathbf{e}$ denotes the trivial path in $\mathcal{W}_e$ with life time 0. Since $\{s : W_s = \mathbf{e}\} = \{s : \zeta_s = 0\}$, $\mathbb{P}_{\mathbf{e}}$–a.s., $\mathbf{e}$ is a regular point for $(W, \mathbb{P}_w)$, and the local time of W at $\mathbf{e}$ can be described by the local time $\bigl(L_s^0(\zeta)\bigr)_{s\geq 0}$ of ζ at level 0. Therefore, we can built the Poisson point measure of excursion from e and the related *excursion measure* $\mathbb{N}_e = \mathbb{N}_e[\varrho]$. From [LG94, Proposition 2.2] we have

Lemma 20 (**Characterization of excursion measures** $\mathbb{N}_e$). *Fix* $e \in \check{\mathsf{E}}$. *The excursion measure* $\mathbb{N}_e$ *is characterized, up to a multiplicative constant, by the following properties:*

(i) *The lifetime process* ζ *of* W *under* $\mathbb{N}_e$ *is "distributed" according to Itô's measure* $n(\mathrm{d}f)$ *of positive excursions of linear Brownian motion.*
(ii) $W_0 = \mathbf{e}$, $\mathbb{N}_e$*–a.e.*
(iii) *The conditional law of* W *under* $\mathbb{N}_e$ *given* ζ, *denoted by* Θ_e^ζ, *can be described as in the end of Proposition* 19.

The excursion measures n and $\mathbb{N}_e$ are normalized as follows: For all $\varepsilon > 0$,

$$n\bigl(\sup_{s\geq 0} f(s) > \varepsilon\bigr) \;=\; \mathbb{N}_e\bigl(\sup_{s\geq 0} \zeta_s > \varepsilon\bigr) \;=\; \frac{1}{2\varepsilon}. \tag{75}$$

In order to formulate a strong Markov property of W under $\mathbb{N}_e$, we need some more notation. Recall notation $\sigma = \sigma(\zeta)$ introduced in (42). Write $\mathbb{P}_w^*$, $w \in \mathcal{W}_e$, for the law of the stopped process $W_{(\cdot)\wedge\sigma}$. Its transition function can be described similarly as those of W under $\mathbb{P}_w$. Here ζ becomes now a Brownian motion stopped when it hits 0 (instead of reflecting at 0).

Let T denote a stopping time of the filtration $(\mathcal{F}_{s+})_{s\geq 0}$ such that $T > 0$, $\mathbb{N}_e$–a.e. Also, let $F, G : \mathcal{C}(\mathsf{R}_+, \mathcal{W}) \to \mathsf{R}_+$ be measurable such that F is $\mathcal{F}_T$–measurable. Recall that θ_T denotes a time shift operator. Then the *strong Markov property* of W under $\mathbb{N}_e$ reads as follows:

$$\mathbb{N}_e\, 1_{\{T<\infty\}} F\, G \circ \theta_T \;=\; \mathbb{N}_e\, 1_{\{T<\infty\}} F\, \mathbb{P}_{W_T}^* G. \tag{76}$$

Fix $w \in \mathcal{W}$. Consider the Brownian snake W and its life time process $\zeta = \zeta[W]$, both under $\mathbb{P}_w^*$. Note that the life length $\sigma(\zeta)$ of ζ is finite $\mathbb{P}_w^*$–a.s. Introduce the minimum process $\check{\zeta}_r := \min\{\zeta_s : s \in [0, r]\}$ of ζ, and denote by (α_i, β_i), $i \in I$, the excursion intervals of $\zeta - \check{\zeta}$ away from 0 by time σ. For each $i \in I$, put

$$W^i_s(t) \ := \ W_{(\alpha_i+s)\wedge\beta_i}(\zeta_{\alpha_i}+t), \qquad s,t \geq 0. \tag{77}$$

From [LG94, Proposition 2.5] we have

Lemma 21 (Poisson point measure N of excursions). *Fix $w \in \mathcal{W}$. Under $\mathbb{P}^*_w$,*

$$N \ := \ \sum_{i\in I} \delta_{(\zeta_{\alpha_i}, W^i)} \tag{78}$$

is a Poisson point measure on $\mathsf{R}_+ \times \mathcal{C}(\mathsf{R}_+, \mathcal{W})$ with intensity measure

$$\mathbb{P}^*_w N\big(\mathrm{d}(t,\kappa)\big) \ = \ 2\,\ell_{[0,\zeta(w)]}(\mathrm{d}t)\,\mathbb{N}_{w(t)}(\mathrm{d}\kappa). \tag{79}$$

Proof. Follow the proof of [LG94, Proposition 2.5] up to the last equation array. Let $F(t,\kappa) = F_1(t)F_2(\kappa)$ be a non-negative measurable functional on $\mathsf{R}_+ \times \mathcal{C}(\mathsf{R}_+, \mathcal{W})$ such that $F_2(\kappa) = 0$ if $\sup_{s\geq 0} \zeta_s(\kappa) \leq \varepsilon$, for some fixed $\varepsilon > 0$. Then, conditioning on ζ and using the conditional independence of the W^i,

$$\begin{aligned} &\mathbb{P}^*_w \exp\Big[-\sum_{i\in I} F_1(\zeta_{\alpha_i})F_2(W^i)\Big] \\ &= \mathbb{P}^*_w \exp\Big[\sum_{i\in I} \log \mathbb{P}^*_w\Big\{\exp\big[-F_1(\zeta_{\alpha_i})F_2(W^i)\big]\,\Big|\,\zeta\Big\}\Big] \\ &= \mathbb{P}^*_w \exp\Big[\sum_{i\in I} \log \int_{\mathcal{W}} \Theta^{\zeta(W^i)}_{w(\zeta_{\alpha_i})}(\mathrm{d}\kappa)\ \exp\big[-F_1(\zeta_{\alpha_i})F_2(\kappa)\big]\Big] \end{aligned} \tag{80}$$

[recall notation Θ introduced in Lemma 20(iii)]. But under the law $\mathbb{P}^*_w$, the point measure $\sum_{i\in I} \delta_{(\zeta_{\alpha_i}, \zeta(W^i))}$ on $\mathsf{R}_+ \times \mathcal{C}(\mathsf{R}_+, \mathsf{R}_+)$ is Poissonian with (σ–finite) intensity measure $2\,\ell_{[0,\zeta(w)]}(\mathrm{d}t)\,n(\mathrm{d}f)$. Hence, by the Laplace functional formula for Poisson point measures, the latter chain of equations can be continued with

$$\begin{aligned} &= \exp\Big[-2\int_0^{\zeta(w)} \mathrm{d}t \int_{\mathcal{C}(\mathsf{R}_+,\mathsf{R}_+)} n(\mathrm{d}f)\Big(1 - \int_{\mathcal{W}} \Theta^f_{w(t)}(\mathrm{d}\kappa)\,\mathrm{e}^{-F_1(t)F_2(\kappa)}\Big)\Big] \\ &= \exp\Big[-2\int_0^{\zeta(w)} \mathrm{d}t \int_{\mathcal{C}(\mathsf{R}_+,\mathcal{W})} \mathbb{N}_{w(t)}(\mathrm{d}\kappa)\Big(1 - \mathrm{e}^{-F_1(t)F_2(\kappa)}\Big)\Big], \end{aligned} \tag{81}$$

since $\mathbb{N}_e(\mathrm{d}\kappa) = \int_{\mathcal{C}(\mathsf{R}_+,\mathsf{R}_+)} n(\mathrm{d}f)\,\Theta^f_e(\mathrm{d}\kappa)$ by a representation of the excursion measures $\mathbb{N}_e$ based on Lemma 20(iii). This gives the claim. □

3.3. **Exit local time and modified hitting measures Z_t.** Fix $e \in \check{\mathsf{E}}$. Let D denote an open connected subset of $\check{\mathsf{E}}$ such that $e \in D$. Assume

$$\Pi_e\big(\xi_t \in \mathsf{E}\backslash D \ \text{for some}\ t \geq 0\big) \ > \ 0, \tag{82}$$

and, in line with (45), we introduce the *exit time* of the left-continuous path ξ from D by

$$\tau^D(\xi) \ := \ \inf\{s \geq 0 :\ \xi_s \notin D\}. \tag{83}$$

Put

$$\gamma_s \ := \ \big(\zeta_s - \tau^D(W_s)\big)_+. \tag{84}$$

Lemma 22 (Continuity of γ). *The process $\gamma = (\gamma_s)_{s\geq 0}$ is continuous.*

Proof. Fix a sample of W and $s_0 \geq 0$. We want to prove continuity in s_0. Start with the case $\gamma_{s_0} > 0$. Then we find an $\varepsilon > 0$ such that $\zeta_{s_0} > \tau^D(W_{s_0}) + \varepsilon$. For all s close enough to s_0, we have $\zeta_s > \zeta_{s_0} - \varepsilon$. Hence, for these s, the W_s coincide on the time interval $[0, \zeta_{s_0} - \varepsilon]$. But $\tau^D(W_{s_0}) < \zeta_{s_0} - \varepsilon$, implying $\tau^D(W_s) = \tau^D(W_{s_0})$ for these s. Hence, $\gamma_s = \big(\zeta_s - \tau^D(W_{s_0})\big)_+$ which converges to γ_{s_0} as $s \to s_0$.

In the remaining case $\gamma_{s_0} = 0$, we have $\zeta_{s_0} \leq \tau^D(W_{s_0})$. For $\varepsilon > 0$, clearly $\zeta_{s_0} + \varepsilon > \zeta_s > \zeta_{s_0} - \varepsilon$ for all s sufficiently close to s_0 implying that the W_s coincide on the time interval $[0, \zeta_{s_0} - \varepsilon]$. Then we must have $\tau^D(W_s) \geq \zeta_{s_0} - \varepsilon$. Therefore,

$$\gamma_s \;=\; \big(\zeta_s - \tau^D(W_s)\big)_+ \;=\; \big(\zeta_s - \tau^D(W_s)\big)\, 1\{\zeta_s \geq \tau^D(W_s)\} \;\leq\; (\zeta_s - \zeta_{s_0} + \varepsilon) \;\leq\; 2\varepsilon$$

for all those s. Thus, $\gamma_s \to 0 = \gamma_{s_0}$ as $s \to s_0$, finishing the proof. □

Lemma 23 (Imbedded reflecting Brownian motion γ). *Fix $e \in \check{\mathsf{E}}$, $w \in \mathcal{W}_e$, consider the Brownian snake W under $\mathbb{P}_w$, and put*

$$A_r \;:=\; \inf\Big\{s \geq 0 : \int_0^s \mathrm{d}t\; 1_{\{\gamma_t > 0\}} > r\Big\}, \qquad r \geq 0. \tag{85}$$

Then, $A_r < \infty$, $\mathbb{P}_w$–a.s., and the process $r \mapsto \gamma_{A_r}$ is a one-dimensional (standard) reflecting Brownian motion starting from $\big(\zeta(w) - \tau^D(w)\big)_+$.

As in [LG94, Propositions 3.2], Lemma 23 implies that the limits

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} \int_0^s \mathrm{d}r\; 1_{\{\tau^D(W_r) < \zeta_r < \tau^D(W_r) + \varepsilon\}} \;:=\; L_s^D, \qquad s \geq 0, \tag{86}$$

make sense $\mathbb{P}_w$–a.s. and $\mathbb{N}_e$–a.e. In particular, definition (50) of the exit local times $s \mapsto L_s^{D_t}$ is justified, $\mathbb{P}_w$–a.s. and $\mathbb{N}_e$–a.e., for each $t \geq 0$. Moreover, as in [LG94, Propositions 3.3], (86) yields, for any non-negative measurable function F defined on $\mathcal{W}_e$,

$$\mathbb{N}_e \int_0^\sigma L^D_{\mathrm{d}s}\, F(W_s) \;=\; \Pi_e^D\, F(\xi), \tag{87}$$

with sub-probability law

$$\Pi_e^D \;:=\; \Pi_e\Big\{\big(\xi_{s\wedge\tau^D}\big)_{s\geq 0} \in (\cdot);\; \tau^D < \infty\Big\}. \tag{88}$$

Note that by definition,

$$\zeta(\xi) \;=\; \tau^D(\xi) \quad \text{under} \quad \Pi_e^D. \tag{89}$$

Fix for the moment $t \geq 0$ and $\psi \in \mathcal{C}^+_{\exp}(\mathsf{E})$. Returning to the case $D = D^t$ and rewriting (52) by using (51), that is,

$$\langle Z_t\,, \psi\rangle \;=\; \int_0^\sigma L^{D_t}_{\mathrm{d}s}\, \psi\big(W_s\,(\zeta_s)\big)\, 1_{\{W_s^2(\zeta_s) = t\}}, \tag{90}$$

from (87) and (89) we get

$$\mathbb{N}_e\, \langle Z_t\,, \psi\rangle \;=\; \Pi_e^{D_t}\, \psi(\xi_{\tau_t})\, 1_{\{\xi^2_{\tau_t} = t\}}. \tag{91}$$

As a counterpart to (51), we set

$$\hat{\xi}^t \;:=\; \begin{cases} \xi_{\tau_t} & \text{if } \;\xi^2_{\tau_t} = t, \\ \dagger & \text{otherwise.} \end{cases} \tag{92}$$

Then, instead of (91), we get the *first moment formula*

$$\mathbb{N}_e \langle Z_t , \psi \rangle \ = \ \Pi_e^{D_t} \psi(\hat{\xi}^t), \qquad e \in \check{\mathsf{E}}, \ t \geq 0, \ \psi \in \mathcal{C}_{\exp}^+(\mathsf{E}). \tag{93}$$

3.4. **Laplace equation for the modified hitting measure process Z.** In the special case $m = 1$, the following result is in formal analogy with [LG94, Proposition 4.2].

Proposition 24 (Laplace equation for Z). *Let $m \geq 1$, $0 \leq t_1 < \cdots < t_m$, and $\psi_1, \ldots, \psi_m \in \mathcal{C}_{\exp}^+(\mathsf{E})$. Put*

$$u(e) \ := \ \mathbb{N}_e \left(1 - \exp\Big[- \sum_{i=1}^m \langle Z_{t_i} , \psi_i \rangle \Big] \right), \qquad e \in \check{\mathsf{E}}. \tag{94}$$

Then $u = u^{t_1,\ldots,t_m}[\psi_1, \ldots, \psi_m, \varrho]$ is the unique non-negative solution to the integral equation

$$u(e) \ = \ \sum_{i=1}^m \Pi_e^{D_{t_i}} \ \psi_i(\hat{\xi}^{t_i}) \ - \ 2\,\Pi_e \int_{\mathsf{R}_+} \mathrm{d}s \ u^2(\xi_s), \qquad e \in \check{\mathsf{E}}. \tag{95}$$

Proof. For m, t_i, ψ_i, $1 \leq i \leq m$, as in the lemma, and $\lambda > 0$, put

$$u(\lambda, e) \ := \ \mathbb{N}_e \left(1 - \exp\Big[- \lambda \sum_{i=1}^m \langle Z_{t_i} , \psi_i \rangle \Big] \right), \qquad e \in \check{\mathsf{E}}, \tag{96}$$

and

$$f_i(\lambda, e) \ := \ \mathbb{N}_e \, \langle Z_{t_i} , \psi_i \rangle \exp\Big[- \lambda \sum_{j=1}^m \big\langle Z_{t_j} , \psi_j \big\rangle \Big], \qquad e \in \check{\mathsf{E}}. \tag{97}$$

By definition (52) of the modified hitting measures Z_{t_i},

$$f_i(\lambda, e) \ = \ \mathbb{N}_e \int_0^\sigma L_{\mathrm{d}s}^{D_{t_i}} \ \psi_i(\hat{W}_s^{t_i}) \exp\Big[- \lambda \sum_{j=1}^m \int_0^\sigma L_{\mathrm{d}r}^{D_{t_j}} \ \psi_j(\hat{W}_r^{t_j}) \Big]. \tag{98}$$

Let $0 =: s_{0,n} < s_{1,n} < \cdots$ be an equidistant decomposition of R_+ with mesh size $1/n$. Approximating the first Stieltjes integral in (98),

$$\begin{aligned} f_i(\lambda, e) \ &= \ \mathbb{N}_e \lim_{n \uparrow \infty} \sum_{k=1}^\infty \psi_i(\hat{W}_{s_{k,n} \wedge \sigma}^{t_i}) \, L_{[s_{k-1,n}, s_{k,n}] \wedge \sigma}^{D_{t_i}} \times \\ &\exp\Big[- \lambda \sum_{j=1}^m \int_0^{s_{k,n} \wedge \sigma} L_{\mathrm{d}r}^{D_{t_j}} \ \psi_j(\hat{W}_r^{t_j}) \Big] \exp\Big[- \lambda \sum_{j=1}^m \int_{s_{k,n} \wedge \sigma}^\sigma L_{\mathrm{d}r}^{D_{t_j}} \ \psi_j(\hat{W}_r^{t_j}) \Big]. \end{aligned} \tag{99}$$

But $L_{[s_{k-1.n}, s_{k,n}] \wedge \sigma}^{D_{t_i}} \leq L_\sigma^{D_{t_i}}$, and, by (87), $\mathbb{N}_e L_\sigma^{D_{t_i}} = \Pi_e^{D_{t_i}} 1 \leq 1$. Thus, by dominated convergence, we can interchange the integration with $\mathbb{N}_e$ and the limit procedure. Then, for k, n fixed, we apply the strong Markov property (76) of W under $\mathbb{N}_e$ to the stopping time $s_{k,n} \wedge \sigma$ to get (in an intuitive way)

$$\mathbb{N}_e \left\{ \exp\Big[- \lambda \sum_{j=1}^m \int_{s_{k,n} \wedge \sigma}^\sigma L_{\mathrm{d}r}^{D_{t_j}} \ \psi_j(\hat{W}_r^{t_j}) \Big] \,\Big|\, \mathcal{F}_{s_{k,n} \wedge \sigma} \right\} \ = \ F(W_{s_{k,n} \wedge \sigma}), \tag{100}$$

where

$$F(w) \ := \ \mathbb{P}_w^* \exp\Big[- \lambda \sum_{j=1}^m \int_0^\sigma L_{\mathrm{d}r}^{D_{t_j}} \ \psi_j(\hat{W}_r^{t_j}) \Big] \ = \ \mathbb{P}_w^* \exp\Big[- \lambda \sum_{j=1}^m \big\langle Z_{t_j} , \psi_j \big\rangle \Big], \tag{101}$$

$w \in \mathcal{W}$. Inserting (100) into (99), or more precisely, by a direct application of (76), and interchanging the limits again, as well as passing back to the Stieltjes integral, gives

$$f_i(\lambda,e) \;=\; \mathbb{N}_e \int_0^\sigma L_{\mathrm{d}s}^{D_{t_i}}\, \psi_i(\hat{W}_s^{t_i})\, F(W_s) \exp\Big[-\lambda \sum_{j=1}^m \int_0^s L_{\mathrm{d}r}^{D_{t_j}}\, \psi_j(\hat{W}_r^{t_j})\Big]. \tag{102}$$

Substituting $s \to \sigma - s$ yields

$$f_i(\lambda,e) \;=\; \mathbb{N}_e \int_0^\sigma L_{\sigma-\mathrm{d}s}^{D_{t_i}}\, \psi_i(\hat{W}_{\sigma-s}^{t_i})\, F(W_{\sigma-s}) \exp\Big[-\lambda \sum_{j=1}^m \int_s^\sigma L_{\sigma-\mathrm{d}r}^{D_{t_j}}\, \psi_j(\hat{W}_{\sigma-r}^{t_j})\Big].$$

But W, hence $\hat{W}^{t_i}$ from (51) and the measures $L^{D_{t_i}}$ are time-reversible with respect to σ under $\mathbb{N}_e$. Therefore,

$$f_i(\lambda,e) \;=\; \mathbb{N}_e \int_0^\sigma L_{\mathrm{d}s}^{D_{t_i}}\, \psi_i(\hat{W}_s^{t_i})\, F(W_s) \exp\Big[-\lambda \sum_{j=1}^m \int_s^\sigma L_{\mathrm{d}r}^{D_{t_j}}\, \psi_j(\hat{W}_r^{t_j})\Big]. \tag{103}$$

Applying again the strong Markov property as in (100), we find

$$f_i(\lambda,e) \;=\; \mathbb{N}_e \int_0^\sigma L_{\mathrm{d}s}^{D_{t_i}}\, \psi_i(\hat{W}_s^{t_i})\, F^2(W_s) \tag{104}$$

[with F from (101)].

Under $\mathbb{P}_w^*$ introduced after (75), for the modified hitting measure Z_{t_j} from (52),

$$\lambda \sum_{j=1}^m \big\langle Z_{t_j}, \psi_j \big\rangle \;=\; \int_{\mathsf{R}_+ \times \mathcal{C}(\mathsf{R}_+,\mathcal{W})} N\big(\mathrm{d}(r,\kappa)\big)\, \lambda \sum_{j=1}^m \big\langle Z_{t_j}^\kappa, \psi_j \big\rangle, \tag{105}$$

where $N\big(\mathrm{d}(r,\kappa)\big)$ is the Poisson point measure from Lemma 21 with intensity measure

$$\mathbb{P}_w^* N\big(\mathrm{d}(r,\kappa)\big) \;=\; 2\,\ell_{[0,\zeta(w)]}(\mathrm{d}r)\, \mathbb{N}_{w(r)}(\mathrm{d}\kappa). \tag{106}$$

Inserting (105) into (101) and calculating the Laplace functional gives

$$F(w) \;=\; \exp\Bigg[-2\int_0^{\zeta(w)} \mathrm{d}r \int_{\mathcal{C}(\mathsf{R}_+,\mathcal{W})} \mathbb{N}_{w(r)}(\mathrm{d}\kappa) \bigg(1 - \exp\Big[-\lambda \sum_{j=1}^m \big\langle Z_{t_j}^\kappa, \psi_j \big\rangle\Big]\bigg)\Bigg], \tag{107}$$

$w \in \mathcal{W}$. Putting this into (104) yields

$$\begin{aligned} f_i(\lambda,e) \;=\; &\mathbb{N}_e \int_0^\sigma L_{\mathrm{d}s}^{D_{t_i}}\, \psi_i(\hat{W}_s^{t_i}) \times \\ &\exp\Bigg[-4\int_0^{\zeta(W_s)} \mathrm{d}r \int_{\mathcal{C}(\mathsf{R}_+,\mathcal{W})} \mathbb{N}_{W_s(r)}(\mathrm{d}\kappa) \bigg(1 - \exp\Big[-\lambda \sum_{j=1}^m \big\langle Z_{t_j}^\kappa, \psi_j \big\rangle\Big]\bigg)\Bigg]. \end{aligned} \tag{108}$$

Moreover, applying (87) [as we derived (93)] gives

$$\begin{aligned} f_i(\lambda,e) \;=\; &\Pi_e^{D_{t_i}}\, \psi_i(\hat{\xi}^{t_i}) \times \\ &\exp\Bigg[-4\int_0^{\tau_{t_i}(\xi)} \mathrm{d}r \int_{\mathcal{C}(\mathsf{R}_+,\mathcal{W})} \mathbb{N}_{\xi_r}(\mathrm{d}\kappa) \bigg(1 - \exp\Big[-\lambda \sum_{j=1}^m \big\langle Z_{t_j}^\kappa, \psi_j \big\rangle\Big]\bigg)\Bigg]. \end{aligned} \tag{109}$$

Using notation (96) this results into

$$f_i(\lambda,e) = \Pi_e^{D_{t_i}}\,\psi_i(\hat{\xi}^{t_i})\exp\Big[-4\int_0^{\tau_{t_i}(\xi)}\mathrm{d}r\;u(\lambda,\xi_r)\Big] = \Pi_e^{D_{t_i}}\,\psi_i(\hat{\xi}^{t_i}) - \Pi_e^{D_{t_i}}\,\psi_i(\hat{\xi}^{t_i})\Big(1-\exp\Big[-4\int_0^{\tau_{t_i}(\xi)}\mathrm{d}r\;u(\lambda,\xi_r)\Big]\Big). \tag{110}$$

But by the fundamental theorem of calculus,

$$1-\exp\Big[-4\int_0^{\tau_{t_i}(\xi)}\mathrm{d}r\;u(\lambda,\xi_r)\Big] = 4\int_0^{\tau_{t_i}(\xi)}\mathrm{d}s\;u(\lambda,\xi_s)\exp\Big[-4\int_s^{\tau_{t_i}(\xi)}\mathrm{d}r\;u(\lambda,\xi_r)\Big].$$

Hence, for the last term in (110),

$$\Pi_e^{D_{t_i}}\,\psi_i(\hat{\xi}^{t_i})\Big(1-\exp\Big[-4\int_0^{\tau_{t_i}(\xi)}\mathrm{d}r\;u(\lambda,\xi_r)\Big]\Big) = 4\int_0^\infty \mathrm{d}s\;\Pi_e\,1_{\{s<\tau_{t_i}<\infty,\;\xi^2_{\tau_{t_i}}=t_i\}}\,u(\lambda,\xi_s)\,\psi_i(\xi_{\tau_{t_i}})\exp\Big[-4\int_s^{\tau_{t_i}}\mathrm{d}r\;u(\lambda,\xi_r)\Big], \tag{111}$$

where we additionally interchanged the order of integration and used definition (88) of $\Pi_e^{D_{t_i}}$ and notation (92) of $\hat{\xi}^{t_i}$. Using the Markov property at time s, (92), (88), and the first identity of (110), the latter equation can be continued with

$$= 4\int_0^\infty \mathrm{d}s\;\Pi_e\,1_{\{s<\tau_{t_i}\}}\,u(\lambda,\xi_s)\,f_i(\lambda,\xi_s) = 4\,\Pi_e\int_0^{\tau_{t_i}}\mathrm{d}s\;u(\lambda,\xi_s)\,f_i(\lambda,\xi_s)$$

$$= 4\,\Pi_e\int_{\mathsf{R}_+}\mathrm{d}s\;u(\lambda,\xi_s)\,f_i(\lambda,\xi_s). \tag{112}$$

Here, in the last step we used the fact that

$$f_i(\lambda,\xi_s) = 0 \quad \text{if} \quad s>\tau_{t_i}\,. \tag{113}$$

Indeed, $s>\tau_{t_i}$ implies that $\xi_s\notin D_{t_i}$, by the definition (45) of the exit time. Moreover, $e\notin D_{t_i}$ yields $\tau_{t_i}=\infty$, Π_e–a.s., again by (45), giving $\Pi_e^{D_{t_i}}=0$. Thus, for such e, by the first identity in (110), $f_i(\lambda,e)=0$, and (113) is true.

Inserting (112) into the last line of (110) yields

$$f_i(\lambda,e) = \Pi_e^{D_{t_i}}\,\psi_i(\hat{\xi}^{t_i}) - 4\,\Pi_e\int_{\mathsf{R}_+}\mathrm{d}s\;u(\lambda,\xi_s)\,f_i(\lambda,\xi_s). \tag{114}$$

As

$$\frac{\partial}{\partial\lambda}u(\lambda,e) = \sum_{i=1}^m f_i(\lambda,e), \tag{115}$$

we have

$$u(e) = u(1,e) = \int_0^1 \mathrm{d}\lambda\sum_{i=1}^m f_i(\lambda,e) = \sum_{i=1}^m \Pi_e^{D_{t_i}}\,\psi_i(\hat{\xi}^{t_i}) - 4\,\Pi_e\int_0^\infty\mathrm{d}s\int_0^1\mathrm{d}\lambda\;u(\lambda,\xi_s)\,\frac{\partial}{\partial\lambda}u(\lambda,\xi_s), \tag{116}$$

where we used (114). This shows that u solves (95).

The uniqueness statement is easy to establish. □

3.5. **Brownian snake representation (proof of Theorem 15).** Let $r \geq 0$ and $\mu \in \mathcal{M}_{\text{tem}}$. Consider $m \geq 1$, $r =: t_0 < t_1 < \cdots < t_m$, and $\varphi_1, \ldots, \varphi_m \in \mathcal{C}^+_{\text{exp}}$. Recall the log-Laplace representation (56) involving v solving uniquely the log-Laplace equation (57). On the other hand, setting

$$\psi_i(e) \;:=\; \varphi_i(e^3), \qquad e = (e^1, e^2, e^3) \in \mathsf{E}, \tag{117}$$

by the log-Laplace formula for Poisson point measures,

$$\begin{aligned}
&-\log \mathcal{P} \exp\Big[-\sum_{i=1}^m \int_{\mathcal{C}(\mathsf{R}_+,\mathcal{W})} N(\mathrm{d}W) \int_{\mathsf{E}} Z^W_{t_i}(\mathrm{d}e)\, \varphi_i(e^3)\Big] \\
&= \; -\log \mathcal{P} \exp\Big[-\int_{\mathcal{C}(\mathsf{R}_+,\mathcal{W})} N(\mathrm{d}W) \sum_{i=1}^m \langle Z^W_{t_i}, \psi_i\rangle\Big] \\
&= \; \int_{\check{\mathsf{E}}} (\delta_0 \times \delta_r \times \mu)(\mathrm{d}e) \int_{\mathcal{C}(\mathsf{R}_+,\mathcal{W})} \mathbb{N}_e(\mathrm{d}W) \Big(1 - \exp\Big[-\sum_{i=1}^m \langle Z^W_{t_i}, \psi_i\rangle\Big]\Big) \\
&= \; \int_{\check{\mathsf{E}}} (\delta_0 \times \delta_r \times \mu)(\mathrm{d}e)\; u(e) \;=\; \int_{\mathsf{R}^d} \mu(\mathrm{d}x)\; u(0, r, x)
\end{aligned} \tag{118}$$

with u the unique non-negative solution to (95) according to Proposition 24.

To finish the proof, it suffices to show that

$$v_r(x) = u(r', r, x), \quad (r', r, x) \in \mathsf{R}_+ \times \mathsf{R}_+ \times \mathsf{R}^d, \quad \text{under (117)}. \tag{119}$$

By (88), (92), and (33),

$$\Pi^{D_{t_i}}_{(r',r,x)}\, \psi_i(\hat{\xi}^{t_i}) \;=\; \Pi_{(r',r,x)}\, 1_{\{\tau_{t_i}<\infty,\; \xi^2_{\tau_{t_i}}=t_i\}}\, \varphi_i(\xi^3_{\tau_{t_i}}) \;=\; P_{r,x}\, 1_{\{r \leq t_i\}}\, \varphi_i(\underline{B}_{t_i}). \tag{120}$$

Thus, the linear terms at the right hand side of (57) and (95) coincide. Using definition (34) of ξ, for the non-linear term in (95) we obtain

$$\begin{aligned}
\Pi_{(r',r,x)} \int_{\mathsf{R}_+} \mathrm{d}s\; u^2(\xi_s) \;&=\; P_{r,x} \int_{\mathsf{R}_+} \mathrm{d}s\; u^2\big(r' + s, {}^rL^{-1}(s), \dot{B}_{{}^rL^{-1}(s)}\big) \\
&=\; P_{r,x} \int_r^\infty L(\mathrm{d}s)\; u^2\big(r' + {}^rL(s), s, B_s\big).
\end{aligned} \tag{121}$$

Consequently, (95) can be rewritten as

$$u(r', r, x) \;=\; P_{r,x} \Big[\sum_{i=1}^m 1_{\{r \leq t_i\}}\, \varphi_i(\underline{B}_{t_i}) \;-\; 2 \int_r^\infty L(\mathrm{d}s)\; u^2\big(r' + {}^rL(s), s, B_s\big)\Big]. \tag{122}$$

Recall that by Proposition 24, (122) has a unique non-negative solution. Now

$$\breve{u}(r', r, x) \;:=\; v_r(x), \qquad (r', r, x) \in \mathsf{R}_+ \times \mathsf{R}_+ \times \mathsf{R}^d, \tag{123}$$

with v from (57) also satisfies (122). By uniqueness, $\breve{u} = u$, and (119) follows, finishing the proof. □

Acknowledgment. Part of this work was accomplished during the first author's visit to the UTK in 2001 and to the U of A in 2003, and the second author's visit to the WIAS in 2002. Hospitality and financial support is gratefully acknowledged. We would like to thank Jean-François Le Gall for his interest in this work and for his critical remarks concerning an earlier version of the manuscript.

References

[BEP91] M.T. Barlow, S.N. Evans, and E.A. Perkins. Collision local times and measure-valued processes. *Canad. J. Math.*, 43(5):897–938, 1991.

[BLGLJ97] J. Bertoin, J.-F. Le Gall, and Y. Le Jan. Spatial branching processes and subordination. *Canad. J. Math.*, 49(1):24–54, 1997.

[DEF$^+$02] D.A. Dawson, A.M. Etheridge, K. Fleischmann, L. Mytnik, E.A. Perkins, and J. Xiong. Mutually catalytic branching in the plane: Finite measure states. *Ann. Probab.*, 30(4):1681–1762, 2002.

[Del96] J.-F. Delmas. Super-mouvement brownien avec catalyse. *Stochastics and Stochastics Reports*, 58:303–347, 1996.

[DF95] D.A. Dawson and K. Fleischmann. Super-Brownian motions in higher dimensions with absolutely continuous measure states. *Journ. Theoret. Probab.*, 8(1):179–206, 1995.

[DF97] D.A. Dawson and K. Fleischmann. A continuous super-Brownian motion in a super-Brownian medium. *Journ. Theoret. Probab.*, 10(1):213–276, 1997.

[DF02] D.A. Dawson and K. Fleischmann. Catalytic and mutually catalytic super-Brownian motions. In *Ascona 1999 Conference*, volume 52 of *Progress in Probability*, pages 89–110. Birkhäuser Verlag, 2002.

[DFLG95] D.A. Dawson, K. Fleischmann, and J.-F. Le Gall. Super-Brownian motions in catalytic media. In C.C. Heyde, editor, *Branching processes: Proceedings of the First World Congress*, volume 99 of *Lecture Notes in Statistics*, pages 122–134. Springer-Verlag, 1995.

[DFM02] D.A. Dawson, K. Fleischmann, and P. Mörters. Strong clumping of super-Brownian motion in a stable catalytic medium. *Ann. Probab.*, 30(4):1990–2045, 2002.

[EP94] S.N. Evans and E.A. Perkins. Measure-valued branching diffusions with singular interactions. *Canad. J. Math.*, 46(1):120–168, 1994.

[Fal90] K.J. Falconer. *Fractal geometry.* Wiley, New York, 1990.

[FK99] K. Fleischmann and A. Klenke. Smooth density field of catalytic super-Brownian motion. *Ann. Appl. Probab.*, 9(2):298–318, 1999.

[FKX02] K. Fleischmann, A. Klenke, and J. Xiong. Pathwise convergence of a rescaled super-Brownian catalyst reactant process. Universität zu Köln, Math. Instit., Preprint of June 4, 2002.

[FLG95] K. Fleischmann and J.-F. Le Gall. A new approach to the single point catalytic super-Brownian motion. *Probab. Theory Related Fields*, 102(1):63–82, 1995.

[GS75] I.I. Gihman and A.V. Skorohod. *Introduction to the Theory of Random Processes II*, volume 218 of *Grundlehren Math. Wiss.* Springer-Verlag, Berlin, 1975.

[Kle00] A. Klenke. A review on spatial catalytic branching. In Luis G. Gorostiza and B. Gail Ivanoff, editors, *Stochastic Models*, volume 26 of *CMS Conference Proceedings*, pages 245–263. Amer. Math. Soc., Providence, 2000.

[Kle03] A. Klenke. Catalytic branching and the Brownian snake. *Stoch. Proc. Appl.*, 103(2):211–235, 2003.

[LG93] J.-F. Le Gall. A class of path-valued Markov processes and its applications to superprocesses. *Probab. Theory Related Fields*, 95(1):25–46, 1993.

[LG94] J.-F. Le Gall. A path-valued Markov process and its connections with partial differential equations. In *First European Congress of Mathematics, Vol. II (Paris, 1992)*, pages 185–212. Birkhäuser, Basel, 1994.

[LG99] J.-F. Le Gall. *Spatial branching processes, random snakes and partial differential equations.* Birkhäuser Verlag, Basel, 1999.

[MV02] P. Mörters and P. Vogt. A construction of catalytic super-Brownian motion via collision local times. Preprint, University of Bath, 2002.

[Wil79] D. Williams. *Diffusions, Markov Processes, and Martingales; Volume1: Fondations.* Wiley Series in Probability and Mathematical Statistics. Wiley, Chichester, 1979.