Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Numerically Stable Computation of CreditRisk$^+$

Hermann Haaf[1], Oliver Reiß[2], John Schoenmakers[2]

submitted: 12.5.2003

[1] Commerzbank AG
60261 Frankfurt am Main
Germany
E-Mail: Hermann.Haaf@Commerzbank.com
URL: `http://www.commerzbank.com`

[2] Weierstrass Institute
for Applied Analysis and Stochastics
Mohrenstr. 39
10117 Berlin
Germany
E-Mail: reiss@wias-berlin.de,
schoenma@wias-berlin.de
URL: `http://www.wias-berlin.de`

No. 846
Berlin 2003

2000 *Mathematics Subject Classification.* 91B30, 60-08, 60E10.

Key words and phrases. Credit risk, Probability generating function, Computation of functions of power series, Numerical stability.

O. Reiß has been supported by the DFG Research Center “Mathematics for key technologies” (FZT 86) in Berlin.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

The CreditRisk$^+$ model launched by CSFB in 1997 is widely used by practitioners in the banking sector as a simple means for the quantification of credit risk, primarily of the loan book. We present an alternative numerical recursion scheme for CreditRisk$^+$, equivalent to an algorithm recently proposed by Giese, based on well-known expansions of the logarithm and the exponential of a power series. We show that it is advantageous to the Panjer recursion advocated in the original CreditRisk$^+$ document, in that it is numerically stable. The crucial stability arguments are explained in detail. Furthermore, the computational complexity of the resulting algorithm is stated.

1 Resume of the classical CreditRisk$^+$ model

We assume familiarity with the basic principles of CreditRisk$^+$, restrict ourselves to a concise resume and refer for a more detailed description to one of the following articles, e.g. the original CreditRisk$^+$ document [3], Lehrbass/Boland/Thierbach [7] or Bluhm/Overbeck/Wagner [1].

1.1 Notations

For a description of the CreditRisk$^+$ model based on probability generating functions in Section 1.2 we will use the following notations:

- Total loss of loan portfolio: X
- Basic loss unit: L_0
- Re-scaled total loss as multiple of basic loss units: $\tilde{X} = L_0^{-1} X$
- Exposure of i-th obligor: $L_0 \nu_i, \quad \nu_i \in \mathbb{N}_0$
- One period probability of default of i-th obligor: p_i
- Number of obligors: N
- Sector random variable: $\mathcal{S}_k$
- Volatility of $\mathcal{S}_k$: σ_k

- Number of sector variables $\mathcal{S}_k$: K
- Idiosyncratic risk affiliation of i-th obligor: $w_{0,i} \geq 0$
- Affiliation of i-th obligor to k-th sector: $w_{k,i}$ with $w_{k,i} \geq 0$ and $w_{0,i} + \sum_{k=1}^{K} w_{k,i} = 1$

Additional for this paper

- Highest polynomial degree of truncation: M

1.2 The elements of CreditRisk$^+$

The aggregate portfolio loss in CreditRisk$^+$ as a multiple of the basic loss unit L_0 is given by

$$\tilde{X} = \sum_{i=1}^{N} \nu_i \, Y_i \tag{1}$$

with ν_i denoting integer-valued multiplicities of L_0 corresponding the i-th obligor and Y_i being Poisson-distributed random variables with *stochastic* intensities

$$\mathcal{R}_i = p_i \left(w_{0,i} + \sum_{k=1}^{K} w_{k,i} \, \mathcal{S}_k \right), \quad (k = 1, \ldots, K; \; i = 1, \ldots, N)$$

conditional on independent Gamma distributed random variables

$$\mathcal{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_K)$$

with parameters $\mathbf{E}[\mathcal{S}_k] = 1$ and $\sigma_k^2 := \mathbf{Var}(\mathcal{S}_k)$, $(k = 1, \ldots, K)$. These sector variables $\mathcal{S}_k$ model the default behavior with respect to a number of meaningfully chosen sectors, corresponding to industry branches etc. Note that

$$\mathbf{E}[Y_i] = \mathbf{E}[\mathcal{R}_i] = p_i \quad \text{for} \quad i = 1, \ldots, N.$$

The probability generating function (PGF) of the CreditRisk$^+$ model $G_{\tilde{X}}(z) = \mathbf{E}[z^{\tilde{X}}]$ can be expressed in closed analytical form

$$G(z) = \exp\left(\sum_{i=1}^{N} w_{0,i} \, p_i \, (z^{\nu_i} - 1) - \sum_{k=1}^{K} \frac{1}{\sigma_k^2} \ln \left[1 - \sigma_k^2 \sum_{i=1}^{N} w_{k,i} \, p_i \, (z^{\nu_i} - 1) \right] \right), \tag{2}$$

with $G := G_{\tilde{X}}$ and z being a formal variable. On the other hand, from the definition of the PGF of a discrete, integer-valued random variable, we know that G may also be represented as

$$G(z) = \sum_{n=0}^{\infty} P[\tilde{X} = n] \, z^n. \tag{3}$$

The efficient and numerically stable computation of the probabilities $P[\tilde{X} = n]$ in (3) from (2) is the central problem in this paper.

2 Panjer Recursion

It is known that the algorithm advocated in the original CreditRisk$^+$ document in order to obtain the probabilities $p_n := P[\tilde{X} = n]$, the Panjer recursion scheme, is numerically unstable. The Panjer recursion is derived by requiring that the log-derivative of G is a rational function of the form $\frac{A(z)}{B(z)}$, with polynomials A and B. Its numerical instability arises from an accumulation of numerical roundoff errors, which is nicely explained in Gordy [5] and has got to do with the summation of numbers of similar magnitude but opposite sign, as both the polynomials A and B contain coefficients of both signs. In the Appendix we explain this issue in some more detail.

Several remedies have been offered in order to avoid the instability of the Panjer recursion. Amongst others we mention the saddlepoint approximations to the tail of the loss distribution proposed by Gordy [5] and Martin/Thompson/Browne [8], constituting an asymptotic result specific to the chosen quantile.

3 Numerically Stable Expansion of the PGF

We introduce the portfolio polynomial of the k-th sector to be

$$\mathcal{P}_k(z) := \sum_{i=1}^{N} w_{k,i}\, p_i\, z^{\nu_i}, \quad k \in \{0, \ldots, K\}.$$

For the further analysis, it is important to note that the coefficients of $\mathcal{P}_k$ are all non-negative. In terms of $\mathcal{P}_k$, G can be re-expressed as

$$G(z) = \exp\left[-\mathcal{P}_0(1) + \mathcal{P}_0(z) - \sum_{k=1}^{K} \frac{1}{\sigma_k^2} \ln\left(1 + \sigma_k^2\, \mathcal{P}_k(1) - \sigma_k^2\, \mathcal{P}_k(z)\right)\right]. \tag{4}$$

Observe that (3) can be interpreted as the power series representation of the analytical representation of G around $z = 0$, having a radius of convergence R strictly greater than 1, see Haaf/Tasche [6] for a more precise bound. Therefore, it is natural to calculate the coefficients, i.e. the probabilities p_n, directly, by applying standard algorithms for the logarithm and exponential of power series, which can be found in the analysis- and mathematical physics literature, see e.g. Brent/Kung [2] and the references therein. We systematically derive a method for calculating the coefficients of the power series expansion of (4) and present a two-step recursive scheme, where the sign structures of the coefficients involved are such that numerical stability of the two steps is ensured by two lemmas. For the convenience of the reader we provide detailed proofs of both lemmas. In fact, a basically equivalent recursion algorithm in this spirit was previously suggested by Giese [4]. However, in [4] the numerical stability is not analyzed.

Thus, we firstly look at the power series expansion of the logarithm of a power series[1]. Secondly, having gained information about the sign-structure of the coefficients of the resulting series, we investigate in a further step the power series expansion of its exponential.

We will show that the coefficients of the power series of $G(z)$ can be computed numerically stable by this method. In particular, by Lemma 1 and Lemma 2, it will be shown, that the stability follows from the particular sign structure of the polynomials under consideration. In fact, in the crucial operations of the recursion scheme, only non-negative terms are added up. The numerical stability of such summations is explained in the Appendix.

Lemma 1 *Expansion of the logarithm*
Consider a sequence $(a_k)_{k\geq 0}$ with $a_0 > 0$, $a_k \geq 0$ for all $k \geq 1$ and the function $g(z) := -\ln(a_0 - f(z))$, where $f(z) := \sum_{k=1}^{\infty} a_k z^k$. Let us assume that f has a positive convergence radius, so that g is analytic in a disc $\{z : |z| < R\}$ for some $R > 0$ and thus can be expanded as $g(z) =: \sum_{k=0}^{\infty} b_k z^k$ on this disc. Then, for the coefficients of g we have $b_k \geq 0$ for $k \geq 1$ and their computation by means of the following recursively defined sequence[2]

$$\begin{aligned} b_0 &= \ln(a_0), \\ b_k &= \frac{1}{a_0}\left[a_k + \frac{1}{k}\sum_{q=1}^{k-1} q\, b_q\, a_{k-q}\right] \qquad \text{for} \quad k \geq 1. \end{aligned} \tag{5}$$

is numerically stable.

Proof. Note that $g'(z) = f'(z)/(a_0 - f(z))$, hence

$$(a_0 - \sum_{k=1}^{\infty} a_k\, z^k) \sum_{k=0}^{\infty} (k+1) b_{k+1}\, z^k = \sum_{k=0}^{\infty} (k+1)\, a_{k+1}\, z^k.$$

Performing the Cauchy product of the power series on the L.H.S. of the preceding equation and comparing coefficients, it follows that $(b_k)_{k\geq 0}$ is given by (5) for $k \geq 1$. Substituting $z = 0$ gives $g(0) = \ln(a_0)$.
From the assumptions on the sequence (a_k) it follows by (5) that $b_k \geq 0$ for $k \geq 1$. So the recursive computation of $(b_k)_{k\geq 0}$ by (5) is numerically stable, as exclusively sums of non-negative terms are involved. ■

[1] We present this in a slightly more general context; for a mere application to the R.H.S. of (4) it would have been sufficient to consider the logarithm of a polynomial rather than of an (infinite) power series. However, if *stochastic* severities in the sense of Tasche [11] are introduced, arbitrary high exposure might be realized, leading naturally to the infinite power series formulation.

[2] As usual, an empty sum, if $k = 1$, is defined to be zero.

Lemma 2 *The exponential of a power series*
Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) := \exp(f(z)) = \sum_{n=0}^{\infty} b_n z^n$ in a disc $\{z : |z| < R\}$ for some $R > 0$. Then

$$\begin{aligned} b_0 &= \exp(a_0), \\ b_n &= \sum_{k=1}^{n} \frac{k}{n} b_{n-k}\, a_k \qquad \text{for} \quad n \geq 1. \end{aligned} \tag{6}$$

Moreover, the recursion (6) is numerically stable, if the coefficients of f satisfy $a_k \geq 0$ for $k \geq 1$.

Proof. The relation $b_0 = \exp(a_0)$ follows by substituting $z = 0$. For the j-th derivative we have

$$f^{(j)}(0) = j!\, a_j \qquad \text{and} \qquad g^{(j)}(0) = j!\, b_j. \tag{7}$$

On the other hand, for $n \geq 1$ one obtains

$$g^{(n)}(z) \;=\; \frac{d^n}{dz^n} \exp(f(z)) = \left(\frac{d}{dz}\right)^{n-1} [g(z) \cdot f'(z)].$$

Hence by Leibniz's rule for the higher derivative of a product

$$g^{(n)}(z) \;=\; \sum_{k=0}^{n-1} \binom{n-1}{k} f^{(k+1)}(z) g^{(n-(k+1))}(z) \tag{8}$$

holds. Then (6) follows straightforwardly by substituting $z = 0$ in (8) and using (7). Finally, the stability assertion is clear, since from $a_k \geq 0$ for $k \geq 1$ and $b_0 > 0$, it follows that $b_n \geq 0$ and so in (6) only positive terms are involved. ■

Remark: In fact, the results in Lemma 1 and Lemma 2 may be derived from one another. However, in order to clearly reveal the sign structures of the involved power series and their impact on numerical stability, we have chosen to treat them separately.

The Algorithm

Setting

$$
\begin{aligned}
a_0^{(k)} &:= 1+\sigma_k^2\,\mathcal{P}_k(1),\\
a_j^{(k)} &:= \sum_{i=1}^{N} w_{k,i}\,p_i\,\mathbf{1}_{\{\nu_i=j\}},\quad (j=1,\ldots,M),
\end{aligned}
$$

for $k=1,\ldots,K$, we compute with the procedure defined in Lemma 1 up to a pre-specified order[3] M, the M-th order expansion of

$$-\ln\left(1+\sigma_k^2\,\mathcal{P}_k(1)-\sigma_k^2\,\mathcal{P}_k(z)\right)$$

to obtain

$$\ln G(z)=\sum_{j=0}^{M}\beta_j\,z^j+\mathcal{O}(z^{M+1}).$$

Note, that Lemma 1 guarantees that $\beta_j\geq 0$ for $j\geq 1$.

In the next step we recursively compute the coefficients γ_n, $n=0,\ldots,M$, in the expansion

$$G(z)=\sum_{n=0}^{M}\gamma_n\,z^n+\mathcal{O}(z^{M+1})$$

from β_j, $j=0,..,M$, by applying Lemma 2.

The numerical stability of the Algorithm follows from Lemma 1 and Lemma 2, due to the sign structure of the coefficients $a_j^{(k)}$ and β_j, respectively. Note that the coefficients $\gamma_n=P[\tilde{X}=n]$, correspond to loss probabilities and are exact up to $n=M$.

4 Conclusion

We finally conclude that the calculation of the coefficients of the power series representation of G gives rise to a numerically stable algorithm. The computational complexity is obtained straightforwardly by counting the number of elementary operations to be

$$(K+1)M^2\,\mathrm{op}_\times+\frac{1}{2}(K+1)M^2\,\mathrm{op}_++\mathcal{O}(KN+KM)\max(\mathrm{op}_+,\mathrm{op}_\times),$$

where op_+ denotes the cost of an addition and $\mathrm{op}_\times$ the cost of a multiplication[4]. As a consequence, the loss distribution of CreditRisk$^+$ in the standard setting can

[3] A conservative upper bound for M, in the absence of multiple defaults, constitutes $\sum_{i=1}^{N}\nu_i$, corresponding to the case that each loan in the entire portfolio defaults. For practical purposes $M=\mathcal{O}(N)$ is a more meaningful choice.

[4] Note that on a modern PC the cost of a multiplication is roughly comparable with the one of an addition.

fast and reliably be determined. Therefore, the presented method for accurately determining the CreditRisk$^+$ -loss distribution or a pre-assigned quantile of it, is hard to beat by any other technique[5].
For generalizations of CreditRisk$^+$ -type models we refer to the work of Reiß [9], in which Fourier inversion techniques are consequently applied, allowing more freedom in the modelling. In addition, there is no need to introduce a basic loss unit L_0 anymore. In fact, for practical purposes we essentially yield by Fast Fourier Transformation(FFT) techniques the loss distribution on a continuous scale.
Of course, the Fourier inversion algorithm can also be applied to the standard CreditRisk$^+$ model. The computational effort of the Fourier inversion algorithm with given, pre-assigned numerical accuracy (i.e., in terms of the fineness of the discretization) and a fixed number of sectors, is of order $\mathcal{O}(N)$. On the other hand, the computational effort of the algorithm presented in this paper is of order $\mathcal{O}(N^2)$, since M ought to be chosen of order $\mathcal{O}(N)$. Hence the Fourier method is faster for very large portfolios. On the contrary, the presented series expansion of the PGF is computationally more advantageous for smaller portfolios.

Appendix: Propagation of Numerical Roundoff Errors

Recall that the relative error ε_{x+y} of the addition operation is given by

$$\varepsilon_{x+y} = \frac{x}{x+y}\,\varepsilon_x + \frac{y}{x+y}\,\varepsilon_y \quad \text{if} \quad x+y \neq 0, \tag{9}$$

in terms of the relative errors ε_x and ε_y of their arguments x and y, respectively.

If the summands x and y are of the same sign, we have that $|\varepsilon_{x+y}| \leq \max\{|\varepsilon_x|, |\varepsilon_y|\}$. On the other hand, if the arguments of the addition are of opposite sign, at least one of the terms $|\frac{x}{x+y}|$, $|\frac{y}{x+y}|$ is greater than 1 and hence at least one of the relative errors ε_x or ε_y gets amplified. This amplification becomes particularly big, if $x \approx y$ and hence a cancellation in the denominator term $x+y$ occurs, leading to an explosion of the relative error ε_{x+y}.

From the above it is clear that the error propagation of the addition of two numbers of equal sign can be considered as harmless, leading even under repeated application to no amplification of the original error terms.

On the other hand, if under repeated summation (e.g., in a recursive algorithm) there is only once the constellation that the summands are of similar magnitude, but opposite sign, cancellation effects will occur leading at least to spurious results, if not to a complete termination of the algorithm.

[5] Of course, the saddlepoint approximation [5, 8] still remains its importance with a view towards modifications of CreditRisk$^+$, particularly with regard to the original setting, where the default indicators are binomially distributed.

Furthermore, the relative error of a multiplication $x \cdot y$ is approximately given by

$$\varepsilon_{x \cdot y} \approx \varepsilon_x + \varepsilon_y, \tag{10}$$

i.e., the relative errors of the arguments simply add up.

Therefore, we conclude that a recursive algorithm relying exclusively on the summation and multiplication of non-negative numbers, can be considered as *numerically stable*.

We refer to standard text books on numerical analysis, e.g. Stoer/Bulirsch [10] for more details on the subject.

Acknowledgements

This paper was developed in co-operation with the bmbf project "Effiziente Methoden zur Bestimmung von Risikomaßen", which is supported by Bankgesellschaft Berlin AG. O.R. was supported by the DFG research center "Mathematics for key technologies" (FZT 86) in Berlin. The authors thank D. Tasche for helpful suggestions.

References

[1] C. Bluhm, L. Overbeck & C. Wagner: *An Introduction to Credit Risk Modeling*, second edition, Chapman & Hall/CRC *Financial Mathematics Series*, 2003.

[2] R.P. Brent & H.T. Kung: Fast algorithms for manipulating formal power series, *J. ACM,* Vol. **25,** 581-595, 1978.

[3] Credit Suisse First Boston: CreditRisk$^+$: A Credit Risk Management Framework, available at `http://www.csfb.com/creditrisk`, 1997.

[4] G. Giese: Enhancing CreditRisk$^+$, *Risk*, Vol. **16**, No. 4, 73-77, 2003.

[5] M. B. Gordy: Saddlepoint Approximation of CreditRisk$^+$, *Journal of Banking and Finance*, Vol. **26**, No. 7, 1337 – 1355, August 2002.

[6] H. Haaf & D. Tasche: Credit portfolio measurements, *GARP Risk Review*, Issue **07**, 43 – 47, July/August 2002.

[7] F. Lehrbass, I. Boland & R. Thierbach: Versicherungsmathematische Risikomessung für ein Kreditportfolio, *Blätter der Deutschen Gesellschaft für Versicherungsmathematik*, Vol. **XXV**, 285–308, 2002.

[8] R. J. Martin, K. E. Thompson & C. J. Browne: Taking to the saddle, *Risk*, Vol. **14**, No. 6, 91–94, 2001.

[9] O. Reiß: *Fourier inversion algorithms for generalized CreditRisk$^+$ models and an extension to incorporate market risk,* WIAS-Preprint No. 817, 2003.

[10] J. Stoer & R. Bulirsch: *Introduction to Numerical Analysis*, Springer-Verlag, 2002.

[11] D. Tasche: *Calculating and Allocating Capital for Credit Risk,* Risk Europe conference, Paris, April 2003, available at `http://www-m4.mathematik.tu-muenchen.de/m4/pers/tasche/paris.pdf`.