Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Blue sky catastrophe in singularly-perturbed systems

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No. 841
Berlin 2003

2000 *Mathematics Subject Classification.* 37G15, 34E15, 37C27, 34C26.

Key words and phrases. saddle-node, global bifurcations, stability boundaries, slow-fast system, bursting oscillations, spikes, excitability.

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Abstract. We show that the blue sky catastrophe, which creates a stable periodic orbit whose length increases with no bound, is a typical phenomenon for singularly-perturbed (multi-scale) systems with at least two fast variables. Three distinct mechanisms of this bifurcation are described. We argue that it is behind a transition from periodic spiking to periodic bursting oscillations.

1. Stability boundaries for periodic orbits

Stable periodic orbits play a very special role in nonlinear dynamics. One of the basic questions here concerns the structure of the boundaries of their stability regions in the parameter space. Namely, suppose that a time-continuous dynamical system exhibits sustainable self-oscillations, i.e. has a stable periodic orbit. One may wonder how the periodic orbits evolves as the parameters of the system vary? In other words, consider a one-parameter family X_μ of dynamical systems with an exponentially stable periodic orbit at some μ. This periodic orbit will persist and remain stable within some interval of the parameter values. What is the boundary of this interval? Which type of bifurcation will it correspond to in a typical one-parameter family?

These questions gave an original impulse to the development of bifurcation theory in the pioneering works by Andronov and Leontovich [1] (see also [2]) who had discovered the following four codimension-1 boundaries of stability of limit cycles for systems of ODE's on a plane. The first one corresponds to the stable limit cycle bifurcating from/into a stable equilibrium state; on the second boundary the stable limit cycle coalesces with an unstable one and disappears; on the third boundary the periodic orbit transforms into a homoclinic loop of a simple saddle-node equilibrium state; the last, fourth boundary corresponds to the stable periodic orbit becoming a homoclinic orbit to a saddle equilibrium state with negative saddle value.

In the multi-dimensional case, generic one-parameter families have already seven such stability boundaries known as today. There are two kinds of them conditioned by whether the periodic orbit exists or not at the critical moment. In the former case the intersection of the periodic orbit with a local cross-section is the fixed point of the Poincaré map, so the problem reduces to the analysis of how the multipliers of the fixed point exit the unit circle. The first possibility is similar to the two-dimensional case: a single multiplier of the periodic orbit becomes equal to $(+1)$, this is the *saddle-node* bifurcation (see Fig.1). Two other codimension-1 bifurcations are the flip or period-doubling one and the birth of a torus. At the flip bifurcation there is a single multiplier equal to (-1). The periodic orbit itself does not disappear after this bifurcation (unlike the saddle-node case) but only loses stability. In the case where a pair of complex-conjugate multipliers crosses the unit circle outward the periodic orbit survives too but loses its skin — a stable two-dimensional invariant torus is born.

There are three stability boundaries of the second kind, as in the planar case. They correspond, to the birth of a periodic orbit off a stable equilibrium state

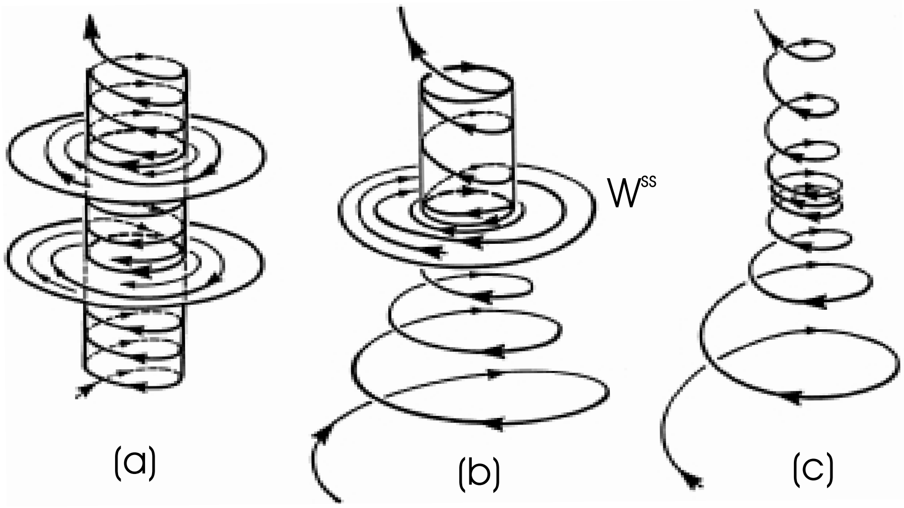

FIGURE 1. Saddle-node bifurcation: (a) $\mu < 0$, there are two periodic orbits: stable and saddle; (b) $\mu = 0$, the periodic orbits merge into a saddle-node one. Its strong stable manifold W^{ss} divides the neighborhood into the node region (below W^{ss} in the figure) and the saddle region (above W^{ss}). The unstable manifold is the part of the center manifold which lies in the saddle region. (c) $\mu > 0$, the saddle-node disappears; the drifting time throughout its neighborhood is estimated as $\sim 1/\sqrt{\mu}$.

(the Andronov–Hopf bifurcation), and to its flowing into a homoclinic loop of either a simple saddle-node equilibrium state or a hyperbolic equilibrium state with one-dimensional unstable manifold and with negative saddle value [3].

It can be shown that the list above gives *all* the main stability boundaries for the case where the *length* of the periodic orbit remains bounded at the bifurcation moment (although the period may tend to infinity if the orbit adheres to a homoclinic loop). One more (and, *conjecturally*, the last one) main boundary of stability that has no two-dimensional analogues and corresponds to the unbounded growth of the length of the periodic orbit was discovered in [4]. This is a codimension-1 bifurcation of smooth flows in at least three-dimensional phase space, such that for any one-parameter family X_μ of flows which crosses a corresponding bifurcational surface at, say, $\mu = 0$, for all small $\mu > 0$ (with the appropriate choice of the direction of increase of the parameter μ) the flow has a stable periodic orbit L_μ which stays in a bounded region of the phase space and is away from any equilibrium states; besides, it undergoes no bifurcations as $\mu \to +0$, whereas both its period and length increase without bound, and L_μ disappears at $\mu = 0$.

The likelihood of such type of bifurcation (called a "blue sky catastrophe") was a long-standing problem. In the construction suggested in [4] (see also [5, 6, 7]) the blue-sky stability boundary is an open subset of a codimension-1 bifurcational surface corresponding to the existence of a saddle-node periodic orbit. This open set is distinguished by some qualitative conditions that determine the geometry of the unstable manifold of the saddle-node (see Fig.2) as well as by a few quantitative restrictions (the Poincaré map introduced below must be a contraction). If everything is right, the stable periodic orbit L_μ whose period and length both tend to infinity when approaching the moment of bifurcation is born when the saddle-node orbit disappears.

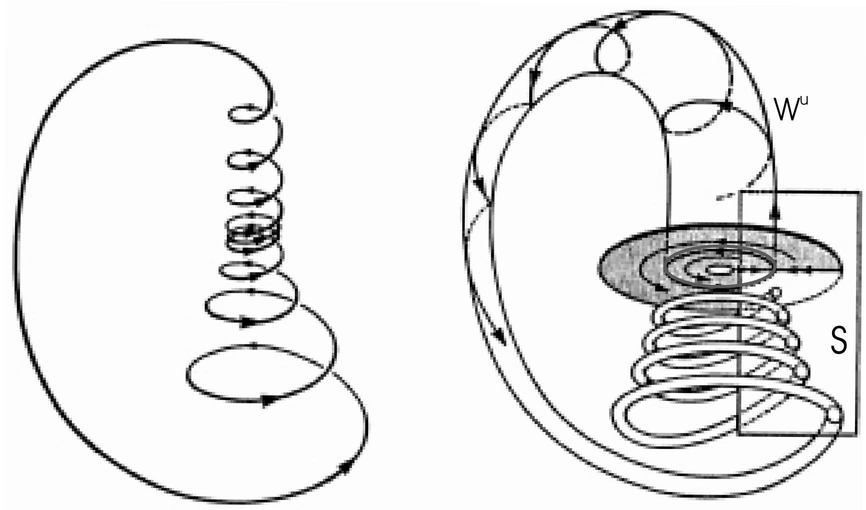

FIGURE 2. The global structure of the set W^u preset for the blue sky catastrophe. The intersection of W^u with the local cross-section S in the node region is a countable set of circles accumulating to $S \cap L$.

Although the global structure of the unstable set of the saddle-node for the blue sky catastrophe appears to be rather complex, and hence it may be unclear how this construction can be achieved plainly in dynamical systems of a natural origin, nevertheless the answer did not make one wait long. The first explicit example of a three-dimensional system of ODE's where the blue sky catastrophe occurs was constructed in [8] considering a global homoclinic Guckenheimer-Gavrilov bifurcation with an extra degeneracy. Another setup for the blue sky bifurcation was proposed in [7] where it was shown that this particular configuration of the unstable set of the saddle-node is, in fact, quite typical for the slow-fast (i.e. singularly perturbed) systems with at least two fast variables. In this paper we present and analyze specific scenarios which cause indeed the blue sky catastrophe in the singularly perturbed systems.

2. SLOW-FAST SYSTEMS.

A slow-fast system is a system of the form

$$
\begin{aligned}
\dot{x} &= g(x, y, \varepsilon), \\
\varepsilon \dot{y} &= h(x, y, \varepsilon),
\end{aligned}
\tag{1}
$$

where $\varepsilon > 0$ is a small parameter. This system may be regularized by rescaling the time $t = \varepsilon\tau$. With the new time τ system (1) becomes

$$x' = \varepsilon g(x, y, \varepsilon), \quad y' = h(x, y, \varepsilon), \tag{2}$$

where the prime denotes differentiating with respect to τ. Taking the limit $\varepsilon = 0$, we obtain

$$x' = 0, \quad y' = h(x, y, 0). \tag{3}$$

The second equation here is called the *fast* system. For simplicity, we assume that $x \in \mathbb{R}^1$. The variable x may be considered as a parameter which governs the motion of the fast y-variables; we assume $y \in \mathbb{R}^n$ with $n \geq 2$.

A trajectory of system (3) starting off any initial point (x, y) goes typically to an attractor of the fast system for the given value of x. The attractor may be a stable equilibrium, or a stable periodic orbit, or be of a less trivial structure — we would like to skip the discussions concerning the latter possibility for now. When the equilibrium state or the periodic orbit of the fast system is exponentially stable, it depends smoothly on x. Thus, we obtain a smooth attracting invariant manifold of system (3): equilibria of the fast system form curves M_{eq} in the (x, y)-space, while limit cycles form two-dimensional cylinders M_{po}, see Fig.3.

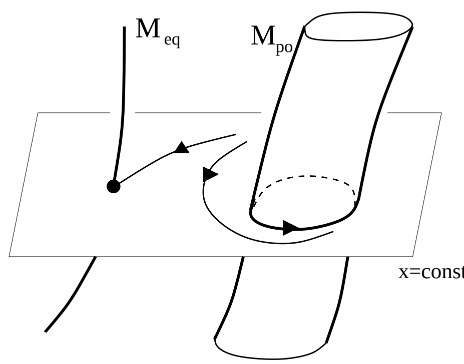

FIGURE 3. An orbit of the fast subsystem may tend to a stable equilibrium, or to a stable limit cycle.

Locally, near every exponentially stable equilibrium point, or a periodic orbit of the fast subsystem, such a manifold is a center manifold for system (3). Since the center manifold persists for any close system, it follows that the smooth attractive invariant manifolds $M_{eq}(\varepsilon)$ and $M_{po}(\varepsilon)$ will exist for all small ε in the whole system (2) (see [9, 10] for details).

Thus, a trajectory of the system (2) for small $\varepsilon > 0$ behaves in the following way: in a finite time it comes into a small neighborhood of one of the invariant manifolds M_{eq} or M_{po} so that its x-component stays nearly constant. Then, it begins drifting slowly along the chosen invariant manifold with the speed of change of x of order ε.

As for the original system (1) is concerned, one will observe, in contrast with the above development, an almost instant jump in the y-components towards the invariant manifold followed by a finite speed motion in the x-variable. Additionally, if this is the manifold M_{po}, then one observes a fast circular motion in the y-components, as depicted in Fig.4.

The equilibrium states of the fast system are found from the condition $h(x, y, 0) = 0$ yielding the algebraic equation for M_{eq}. If $y = y_{eq}(x)$ is a stable branch of M_{eq}, then the equation of motion of the x-component along it is given, up to the first order in ε, by

$$\dot{x} = g(x, y_{eq}(x), 0). \tag{4}$$

This is a one-dimensional system which may possess attracting and repelling equilibrium states corresponding to stable and saddle equilibrium states in the entire system (1) or (2). The evolution along M_{eq} is either limited to one of the stable points, or the trajectory hovers about M_{eq} onwards until it reaches a small neighborhood of a critical value of x. Recall that x is treated as a governing parameter for the fast system, hence its critical values correspond to bifurcations in the fast system. So, for instance, at such critical x^* two, stable and unstable, equilibrium states of the fast subsystem may collide, thereby forming a saddle-node. This will correspond to a maximum (or a minimum) of x on M_{eq}. The x-component of the trajectory may no further increase (resp. decrease) along the stable branch of M_{eq}. Instead, the orbit jumps towards another attractor, which is the ω-limit set of the outgoing separatrix of the saddle-node equilibrium state in the fast system at $x = x^*$, see Fig.5.

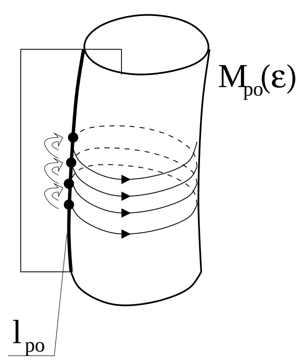

FIGURE 4. The fast circular motion on the cylinder $M_{po}(\varepsilon)$ defines the Poincaré map of the intersection curve $l_{po}(\varepsilon)$.

In order to determine the dynamics of the trajectory near the cylinder $M_{po}(\varepsilon)$ we must find firstly the equation $y = y_{po}(\tau; x)$ of the corresponding fast limit cycle for the given x; here y_{po} is a periodic function in τ of period $T(x)$. Then we substitute $y = y_{po}(t\varepsilon; x)$ into the right-hand side of the first equation in (1) and average it over the period $T(x)$. The resulting averaged system

$$\dot{x} = \phi(x) \equiv \frac{1}{T(x)} \int_0^{T(x)} g(x, y_{po}(\tau; x), 0) d\tau \tag{5}$$

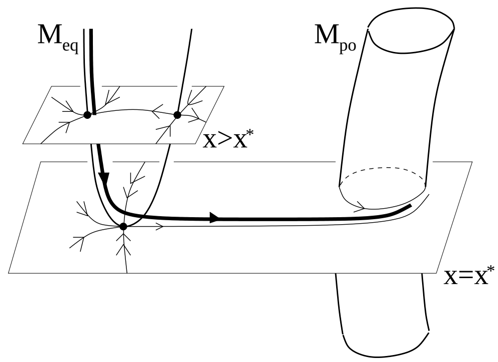

FIGURE 5. Fast jump of the trajectory taking off the fold towards the attracting cylindrical surface.

gives a first order approximation (see [11]) for the evolution of the x-component of the orbit near M_{po}.

By cutting the cylinder surface by a cross-section transverse to the fast motion (see Fig.4), one finds a Poincaré map defined on the intersection line $l_{po}(\varepsilon)$:

$$\bar{x} = x + \varepsilon\psi(x,\varepsilon) = x + \varepsilon\phi(x)T(x) + o(\varepsilon). \tag{6}$$

This one-dimensional map may have stable and unstable fixed points (sited at the zeros of $\psi(x)$). These points correspond to stable and saddle periodic orbits of the system (1). The iterates of any point on l_{po} either converge to one of the stable fixed points of the map, or continue to grow monotonically up to the critical value of x.

A critical value of x corresponds to a bifurcation in the fast system. We will next consider three types of such bifurcations. The first one (see Fig.6) corresponds to the case where the stable periodic orbit of the fast system collides with a saddle periodic orbit thereby forming a saddle-node one which later fades. After passing such critical value, orbits of the singularly-perturbed system (1) must follow orbits of the fast subsystem, i.e. they jump toward the ω-limit set of the unstable manifold of the saddle-node.

The second situation illustrated in Fig.7 corresponds to the case where the stable periodic orbit of the fast system shrinks to a focus. After passing through the critical value the phase point keeps on drifting along the corresponding branch of stable equilibria of the fast system.

The third situation (see Fig.8) corresponds to the case where the stable periodic orbit of the fast system becomes a homoclinic loop of a saddle equilibrium with one-dimensional unstable manifold. Thus, at this value of x the stable branch of M_{po} terminates by touching a saddle branch of M_{eq}.

At $\varepsilon = 0$ this branch of M_{eq} is comprised of saddle equilibria of the fast system. The union (over an interval of values of x) of their one-dimensional unstable manifolds gives a two-dimensional invariant manifold $W^u(M_{eq})$ and the union of their stable manifolds forms an n-dimensional invariant manifold $W^s(M_{eq})$. The manifold $W^u(M_{eq})$ is exponentially attracting, and the manifold $W^s(M_{eq})$ is exponentially repelling. Both are normally-hyperbolic invariant manifolds, and they, hence, persist for all sufficiently small ε [9]. The saddle branch $M_{eq}(\varepsilon)$ is the intersection of $W^u(M_{eq})$ and $W^s(M_{eq})$. The manifold $W^u(M_{eq})$ attracts orbits, so for every initial point close to M_{eq}, the orbit (may be after some drift along M_{eq}) leaves a small neighborhood of M_{eq} close to $W^u(M_{eq})$, i.e. it deserts M_{eq} at some x and follows one of the separatrices of the corresponding saddle of the fast subsystem.

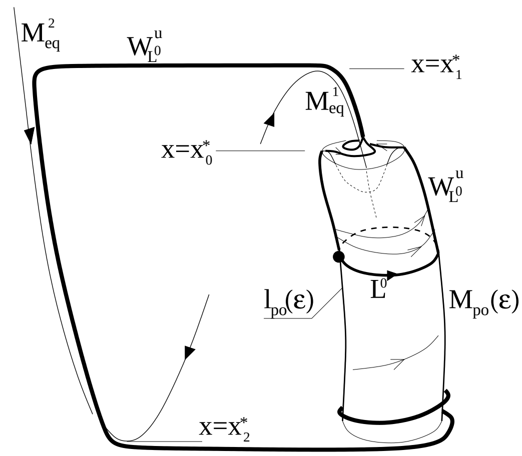

FIGURE 6. The fold on M_{po} (due to the saddle-node bifurcation in the fast system at $x = x_0^*$) triggers the fast jump towards the attracting slow motion surface M_{eq}^1 corresponding to equilibria of the fast subsystem. The unstable manifold of the saddle-node periodic orbit L^0 shrinks to a narrow tube after the jump.

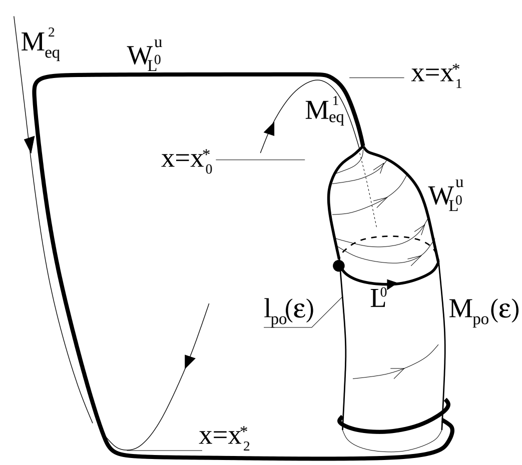

FIGURE 7. The surface M_{po} shrinks into M_{eq}^1 through the supercritical Andronov–Hopf bifurcation in the fast subsystem at $x = x_0^*$.

3. Blue sky catastrophe

Let us now suppose that there exists certain numbers $x_0^*, \ldots, x_k^*$ such that the following holds. Our singularly perturbed system has a number of branches $M_{eq}^1, \ldots, M_{eq}^k$

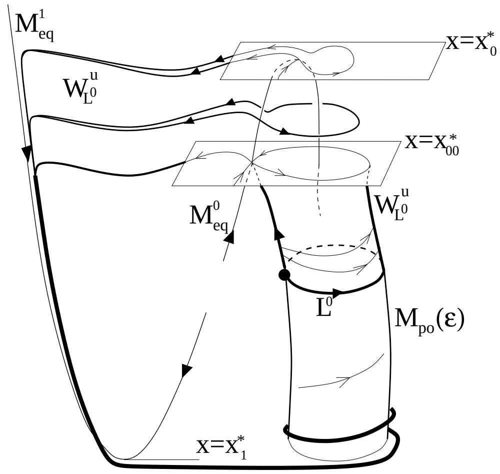

FIGURE 8. The surface M_{po} ends at $x = x^*_{00}$ which corresponds to a homoclinic loop in the fast subsystem. The saddle branch M^0_{eq} terminates at the fold at $x = x^*_0$. All the orbits starting near M^0_{eq} arrive eventually next to the stable branch M^1_{eq}.

composed of exponentially stable equilibria of the fast system at $\varepsilon = 0$. Each branch M^j_{eq} is given by an equation $y = y^j_{eq}(x)$ at $\varepsilon = 0$, where the function $y^j_{eq}(x)$ is defined on a certain interval of values of x, including the interval between x^*_{j-1} and x^*_j. The drift along the M^j_{eq} is directed from x^*_{j-1} towards x^*_j, i.e.

$$g(x, y^j_{eq}(x), 0) \neq 0 \text{ and } \operatorname{sign} g(x, y^j_{eq}(x), 0) = \operatorname{sign}(x^*_j - x^*_{j-1})$$

for all $x \in [x^*_{j-1}, x^*_j]$ (see (4)). At $x = x^*_j$ the branch M^j_{eq} ends up; namely, it collides with a certain saddle one so the fast system has a saddle-node equilibrium at $x = x^*_j$. The unstable manifold of this saddle-node tends to the exponentially stable equilibrium of the fast system on the branch M^{j+1}_{eq} at $j < k$. When $j = k$, the unstable manifold of the saddle-node tends to an exponentially stable periodic orbit of the fast system. The corresponding stable branch M_{po} extends in x until one of the three following events takes place.

(I) At $x = x^*_0$ the stable branch M_{po} meets that of saddle periodic orbits, i.e. the fast system has a saddle-node periodic orbit. The unstable manifold of this orbit in the fast system tends, as a whole, to the exponentially stable equilibrium on the branch M^1_{eq} (Fig.6).

(II) At $x = x^*_0$ the stable periodic orbit of the fast system shrinks to the equilibrium state lying in the branch M^1_{eq} (Fig.7).

(III) At some $x = x^*_{00}$ between x^*_k and x^*_0 the stable periodic orbit of the fast system adheres to a homoclinic loop of the saddle equilibrium of the fast system. The corresponding saddle branch M^0_{eq} extends in x until $x = x^*_0$ where it terminates at the fold representing a saddle-node equilibrium in the fast system; the direction of the shift in x on M^0_{eq} is from x^*_{00} towards x^*_0. For every x between x^*_{00} and x^*_0 both the one-dimensional separatrices of the saddle of the fast system tend to the stable equilibrium on the branch M^1_{eq} (at $x = x^*_{00}$, when one of the separatrices forms a homoclinic loop, the other tends to the equilibrium on M^1_{eq}). At $x = x^*_0$ the whole unstable set of the saddle-node of the fast system tends to the exponentially stable equilibrium on M_{eq}^1 (Fig.8).

In the latter case we need one more assumption. Let $\lambda^j(x)$ stand for maximum of the real part of the characteristic exponents (i.e. for the largest Lyapunov exponent) of the equilibrium state of the fast system for the fixed value of x on the branch M_{eq}^j at $\varepsilon = 0$. By construction, all $\lambda^1(x), \ldots, \lambda^k(x)$ are negative (for equilibria on the branches $M_{eq}^1, \ldots, M_{eq}^k$ are exponentially stable). Since the single branch M_{eq}^0 corresponds to a saddle equilibrium, it follows that $\lambda^0(x) > 0$. We assume that

$$
\begin{aligned}
&\sum_{j=2}^{k} \int_{x_{j-1}^*}^{x_j^*} \lambda^j(x) \frac{dx}{g(x, y_{eq}^j(x), 0)} + \\
(7) \qquad & \qquad + \max_x \left(\int_x^{x_1^*} \lambda^1(x) \frac{dx}{g(x, y_{eq}^1(x), 0)} + \int_{x_{00}^*}^{x} \lambda^0(x) \frac{dx}{g(x, y_{eq}^0(x), 0)} \right) < 0,
\end{aligned}
$$

where the maximum is taken over all $x \in [x_{00}^*, x_0^*]$.

Concerning the motion near the manifold M_{po}, for all three cases above we assume also that the function $\phi(x)$ that defines, to the first order, the direction of the drift along M_{po} (see (6)) has constant sign (the same as the sign of $x_0^* - x_k^*$) everywhere except for one point $x = x^{**}$ where ϕ vanishes. So, $\phi(x^{**}) = 0$, $\phi'(x^{**}) = 0$ and we may assume $\phi''(x^{**}) \neq 0$ (the functions g and h in (1) are required to be C^2 at least). Let us include our slow-fast system in a one-parameter family of systems (i.e. we assume that the functions g and h depend on some parameter μ varying near $\mu = 0$) such that $\phi(x^{**}) = 0$ at $\mu = 0$ and $\dfrac{\partial \phi}{\partial \mu}(x^{**}) > 0$.

It follows that there exists a smooth curve $\mu = \mu^*(\varepsilon)$, $\mu^*(0) = 0$, such that the function ψ from (6) has exactly two zeros at $\mu < \mu^*(\varepsilon)$ which collide at $\mu = \mu^*(\varepsilon)$, and at $\mu > \mu^*(\varepsilon)$ the function ψ is non-zero for all x between x_k^* and x_0^* (between x_k^* and x_{00}^* in case III). Zeros of ψ are the fixed points of the Poincaré map on the attracting invariant manifold $M_{po}(\varepsilon)$. Thus, if $\mu < \mu^*(\varepsilon)$ our system (2) (or (1) has two periodic orbits on $M_{po}(\varepsilon)$: a stable orbit L^+ and a saddle orbit L^- (Fig.9).

Let U be a small fixed neighborhood U enveloping the branches M_{po} and M_{eq}^j, as well as the orbits of the fast system which connect them. By construction, every orbit (excluding those in the stable manifold of L^-) within U tends to L^+ as time increases. Indeed, any orbit, which begins near M_{po} and reaches the threshold $x = x_0^*$ or $x = x_{00}^*$, will eventually jump next to the branch M_{eq}^1, then drift along it before making the next leap to the similar branch M_{eq}^2 and so forth until it will bound finally back to the initial branch M_{po} landing in the attraction basin of the periodic orbit L^+.

At $\mu = \mu^*(\varepsilon)$ the orbits L^+ and L^- unite into a saddle-node periodic orbit L^0. The manifold $M_{po}(\varepsilon)$ is a center manifold for this orbit; that part of $M_{po}(\varepsilon)$ where the orbits run away from L^0 as time increase is the unstable manifold of L^0. After approaching the critical value of x where the stable branch M_{po} finishes, all the orbits on the unstable manifold land closely next to the branch M^1_{eq} so that the manifold W^u_L will focus in a very narrow tube following around and bouncing amongst the slow motion branches M^j_{eq}, prior its final return to L^0 twirling around M_{po}. This gives exactly the configuration of the unstable manifold which was projected in Fig.2. Therefore, one should anticipate the blue sky catastrophe here, which is indeed justified by the following theorem.

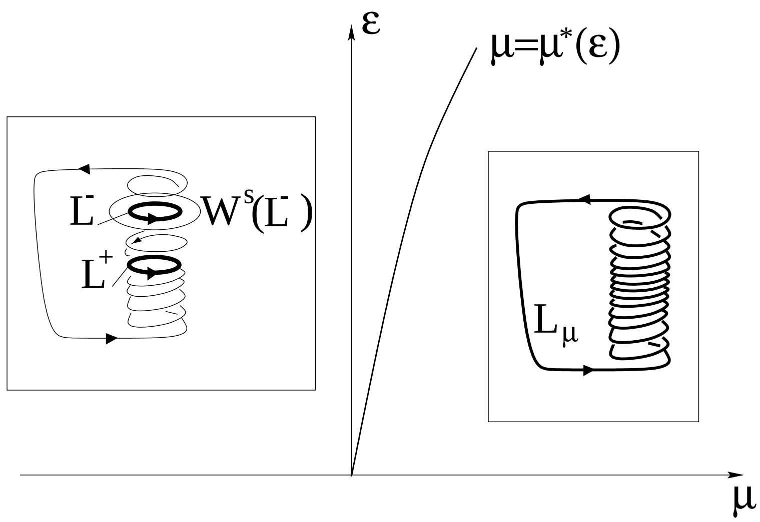

FIGURE 9. At $\mu < \mu^*(\varepsilon)$ the system has two periodic orbits: a stable orbit L^+ and a saddle orbit L^-. The orbits which do not lie in the stable manifold of L^- tend to L^+ as time increases. At $\mu > \mu^*(\varepsilon)$ the system has a single and attracting limit cycle L_μ whose length tends to infinity as $\mu \to \mu^*(\varepsilon) + 0$.

Theorem. *In any of the above cases I,II or III, for all sufficiently small $\varepsilon > 0$ and $\mu > \mu^*(\varepsilon)$, in the neighborhood U there exists a unique stable periodic orbit L_μ which attracts all orbits from U. Both the period and the length of L_μ tend to infinity as $\mu \to \mu^*(\varepsilon) + 0$.*

Proof. By assumption, at $\varepsilon = 0$ and $\mu = 0$ the fast system has a periodic orbit L^0 at $x = x^{**}$. Since L^0 is an exponentially stable periodic orbit of the fast subsystem the absolute value of every of its multipliers is less than 1. In the augmented slow-fast system (2) this orbit has, besides, an additional multiplier equal to $+1$ corresponding to the x-variable. Formally speaking, L^0 is a non-hyperbolic periodic orbit of (2) with the center variable x. It is well known that for such orbit there exist an invariant center manifold and an invariant strong-stable foliation which is transverse to the center manifold. Moreover, both ones persist for all close values of parameters. The center manifold coincides with the surface M_{po}, so it can be parameterized by the x-variable and by the angular variable $\varphi \in \mathbb{S}^1$ (which is indeed the phase on the periodic orbit in the fast system). The flow is uniformly exponentially contracting in the directions transverse to M_{po}. Let the coordinates in these contracting dimensions be denoted as $z \in \mathbb{R}^{n-1}$; we can always introduce the z-coordinates so that the center manifold becomes locally straightened, i.e. M_{po} near L^0 acquires the equation $z = 0$ for all small ε and μ.

The existence of the strong-stable invariant foliation implies that the variables (x, φ, z) in a small neighborhood of L^0 can be introduced in such a way that the evolution of the (x, φ)-variables will become independent of the z-variable for all small ε and μ (see [12] for details and proofs). Thus, the Poincaré map of an appropriate cross-section, say, $\varphi = 0$, is written near L^0 as

$$\bar{x} = x + \varepsilon\psi(x, \varepsilon, \mu), \quad \bar{z} = A(x, z, \varepsilon, \mu)z, \tag{8}$$

where ψ is the function from (6), and A is an $(n-1) \times (n-1)$-matrix such that $||A|| < 1$.

By assumption, when $\mu > \mu^*(\varepsilon)$ the function ψ vanishes nowhere; for definiteness we may assume $\psi > 0$ (in other words: $x_0^* > x_k^*$). Hence, by fixing $x^+ > x^{**}$, any trajectory beginning in a small neighborhood of $x = x^{**}$ will eventually hit the cross-section at a point (x, z) within the band $\Sigma^+ : x^+ \leq x < x^+ + \varepsilon\psi(x^+, \varepsilon, \mu)$. As time grows the orbit will further move towards the increase of x, then it will leap across onto a branch M_{eq}, etc., and as explained above, will finally return into a small neighborhood of $x = x^{**}$ on M_{po} from the side of $x < x^{**}$. Hence, given any $x = x^- < x^{**}$ fixed near $x = x^{**}$, the orbit will pierce the strip $\Sigma^- : x^- \leq x < x^- + \varepsilon\psi(x^-, \varepsilon, \mu)$ on the cross-section at some uniquely defined point. Thus, the flow outside the small neighborhood of L^0 determine the map: $\Sigma^+ \mapsto \Sigma^-$ which we will denote by T_1.

Analogously, the flow near M_{po} in the region $x^- \leq x < x^+ + \varepsilon\psi(x^+, \varepsilon, \mu)$ defines the map $T_0 : \Sigma^- \mapsto \Sigma^+$ for $\mu > \mu^*(\varepsilon)$. The composition $T_1 \circ T_0$ is a Poincaré map of Σ^-. We will show below that this map is a contraction, and hence has a single and stable fixed point attracting all other orbits. This fixed point corresponds to the sought periodic orbit L_μ of our slow-fast system. The number of iterations for the map (8) to take an orbit from Σ^- to Σ^+ tends to infinity as $\mu \to \mu^*(\varepsilon) + 0$; each iterate of the map corresponds to one complete revolution of the trajectory of the flow around M_{po}, i.e. to a non-zero length interval on L_μ. Consequently, the total length of L_μ increases without a bound as $\mu \to \mu^*(\varepsilon) + 0$. Thus, as soon as having the contraction of the map $T_1 \circ T_0$ evinced, we have the theorem proved.

Let us first show that the first derivative of the map T_0 is uniformly bounded from above. As we mentioned, the map is contracting in z, so we have solely to check the boundedness of the derivative of the map in the x-variable (which is independent of z). Take any $x_0 \in [x^-, x^- + \varepsilon\psi(x^-, \varepsilon, \mu))$ and let $\{x_1, ..., x_m\}$ be its orbit, i.e. $x_{j+1} = x_j + \varepsilon + \psi(x_j, \varepsilon, \mu)$, and $x_m \in [x^+, x^+ + \varepsilon\psi(x^-, \varepsilon, \mu))$. We need to prove the uniform boundedness of $\dfrac{dx_m}{dx_0}$ for arbitrarily large m.

Note that $\dfrac{dx_{j+1}}{dx_0} = (1+\varepsilon\psi'(x_j))\dfrac{dx_j}{dx_0}$ (we omit the dependence of ψ on ε and μ). Following [6], let us introduce $\xi_j = \ln \dfrac{1}{\psi(x_j)}\dfrac{dx_j}{dx_0}$. It is easy to see that

$$\xi_{j+1} = \xi_j + \ln \frac{(1+\varepsilon\psi'(x_j))\psi(x_j)}{\psi(x_j+\varepsilon\psi(x_j))} \le \xi_j + \ln \frac{1+\varepsilon\psi'(x_j)}{1+\varepsilon \min_{x\in[x_j,x_{j+1}]}\psi'(x)}.$$

It follows then that

$$\xi_{j+1} - \xi_j \le K\varepsilon(x_{j+1}-x_j),$$

where K is some constant. Hence,

$$\xi_m - \xi_0 \le K\varepsilon(x_m - x_0) \sim K\varepsilon(x^+ - x^-).$$

Thus, $\xi_m - \xi_0$ is uniformly bounded, which means that $\dfrac{\psi(x_0)}{\psi(x_m)}\dfrac{dx_m}{dx_0}$ is uniformly bounded too. Since x_0 is bounded away from x^{**}, the value of $\psi(x_0)$ is also bounded away from zero, and from here the required uniform boundedness of $\dfrac{dx_m}{dx_0}$ follows.

Next we will prove that the map T_1 for all μ is contracting with the contraction factor tending to zero as $\varepsilon \to +0$. Choose a point $M \in \Sigma^+$ and $\bar{M} = T_1 M \in \Sigma^-$. Since the phase velocity vectors $(\dot{x}, \dot{y})$ at both end points M and $\bar{M}$ are bounded away from zero and since the angle between these vectors and the cross-section is bounded away from zero as well for all small ε, it follows that in order to prove the strong contraction property for the map T_1 it is enough to show that the flow from M to $\bar{M}$ contracts strongly two-dimensional areas, for any initial point $M \in \Sigma^+$. To do so we split the flight from Σ^+ to Σ^- into a few stages such as the slow drift along M_{po}, then jumps towards and between the branches M_{eq}, the slow passages along ones, and the final leap back to M_{po} together with the drift along it until reaching Σ^-. Let us pick a sufficiently small $\delta > 0$. The interval of time (τ) needed for a trajectory of the system (2) to fly off the δ-neighborhood of one branch to the δ-neighborhood of the other branch is finite. Therefore, every such jump brings only a finite contribution into the contraction or expansion of areas. The number of such interbranch leaps is finite too, so altogether the jumps may contribute only a finite factor to the overall expansion/contraction of areas.

The first two Lyapunov exponents of the trajectory of the unperturbed system (2) at a point x on M^j_{eq} are 0 and $\lambda^j(x)$ when $\varepsilon = 0$ (the zero exponent corresponds to the x-variable, whereas λ^j is determined via the fast system). Therefore, when ε is nonzero and small, the time-$\Delta\tau$ shift ($\Delta\tau$ is small enough) set by the flow in the δ-neighborhood of the point x multiplies the areas by a factor bounded above by $e^{(\lambda^j(x)+O(\delta)+O(\varepsilon))\Delta\tau}$. It follows from here that the total coefficient of expansion or contraction of areas gained during the transport in the δ-neighborhood of the branch M^j_{eq} from a point x_1 to a point x_2 is bounded above by

$$C_1 \exp\left(\frac{1}{\varepsilon}\int_{x_1}^{x_2} (\lambda^j(x) + C_2\delta)\frac{dx}{g(x, y^j_{eq}(x), 0)}\right),$$

where $C_{1,2}$ are some constants independent of x, δ and ε. Recall that $\lambda^j < 0$ for $j = 1, ..., k$. Thus, if $\delta > 0$ is sufficiently small, it follows that in any of the cases I,II and III under consideration (in case III, inequality (7) is crucial), during that phase of motion between Σ^+ and Σ^- which corresponds to the drift in the δ-neighborhood of the branches M^j_{eq} and the interbranch jumps, the flow contracts areas with a factor $e^{-\alpha/\varepsilon}$ at least, where $\alpha > 0$ is a constant independent of δ and ε.

Now note that the flow in the δ-neighborhood of M_{po} but outside a small neighborhood of $x = x^{**}$ and the δ-neighborhood of the branches M_{eq} cannot produce a strong expansion of areas. Indeed, the first two Lyapunov exponents for the system (2) at $\varepsilon = 0$ are both zero on M_{po} (as earlier, the first one is due to the x-variable while the other one corresponds to the circular motion on the stable periodic orbit of the fast system). In fact, every complete revolution in the δ-neighborhood of M_{po} but outside the δ-neighborhood of the branches M_{eq} gives the rate of the area expansion estimated as $e^{O(\delta)}$. When $\varepsilon \neq 0$ is sufficiently small this becomes only slightly worse, i.e. this factor is modified to $e^{O(\delta)+O(\varepsilon)}$. The number of the turns that the orbit makes around M_{po} while travelling along the path from Σ^- towards Σ^+ (i.e. outside of a small neighborhood of x^{**}) is evaluated as $O(\varepsilon^{-1})$ (because the function ψ is bounded away from zero in this region, see (6)). Hence, the factor of possible expansion of areas accumulating during that phase of transport from Σ^+ to Σ^- which corresponds to the drift near M_{po} does not exceed some $e^{\left(C(1+\frac{\delta}{\varepsilon})\right)}$.

Thus, when ε is small enough, the areas are indeed strongly contracted during the flight from Σ^+ to Σ^-. Hence, the map T_1 is a strong contraction, so is the map $T_1 \circ T_0 : \Sigma^- \to \Sigma^-$. End of the proof.

4. Summary

To conclude we remark that the suggested mechanisms of the blue-sky catastrophe in slow-fast systems have indeed been reported in models of neuronal activity, for example, describing the dynamics of the leech heart interneurons, see [13]. The transition (illustrated in Fig.9) from one type of self-sustained oscillations (a round stable periodic orbit L^+) to the regime where the attractor is the "long" stable orbit L_μ can be interpreted as a transition from periodic tonic-spikes to periodic bursting oscillations. Here, each burst is constituted of the slow helix-like motion along M_{po} generating a large number of spikes, followed by the inter-burst "calm" phase due to the sluggish drive along M_{eq}.

Note as well that even before the transition to the bursting oscillations the spiking mode is in excitable state here: a perturbation which drives the initial point outside the saddle limit cycle L^- results in a long calm phase before the sustained spiking restores.

This work was supported by the grants No. 02-01-00273 and No. 01-01-00975 of RFBR, grant No. 2000-221 INTAS, scientific program "Russian Universities", project No. 1905. Leonid Shilnikov acknowledges the support of the Alexander von Humboldt foundation.

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