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An example of a resonant homoclinic loop of infinite cyclicity

Dmitry Turaev[1]

submitted:

[1] Weierstrass Institute for Applied Analysis and Stochastics
Mohrenstr. 39, D-10117 Berlin, Germany
E-Mail: turaev@wias-berlin.de

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

ABSTRACT. We describe a codimension-3 bifurcational surface in the space of C^r-smooth ($r \geq 3$) dynamical systems (with the dimension of the phase space equal to 4 or higher) which consists of systems which have an attractive two-dimensional invariant manifold with an infinite sequence of periodic orbits of alternating stability which converge to a homoclinic loop.

Introduction. Consider an $(n+1)$-dimensional C^r-smooth ($r \geq 3$) dynamical system with a saddle equilibrium state O. Let the stable manifold W^s of O be n-dimensional and the unstable manifold W^u be m-dimensional. The unstable manifold consists of the point O and two orbits, called separatrices, leaving O at $t = -\infty$ in opposite directions. Let one of the separatrices, Γ, tend to O as $t \to +\infty$ too, forming a homoclinic loop (i.e. Γ is an orbit of intersection of W^u and W^s).

Under certain assumptions which we formulate below, the system has a two-dimensional invariant manifold $\mathcal{M}$ which contains the equilibrium state O and the homoclinic loop Γ. This manifold persists for every C^r-close system, even when the homoclinic loop splits. It is an attractive manifold: every forward semiorbit which stays in a small neighborhood U of Γ tends to $\mathcal{M}$ as $t \to +\infty$ and every whole orbit which entirely lies in U must belong to $\mathcal{M}$.

The first statements of this kind can be found in [1, 2]; invariant manifold theorems for different classes of homoclinic loops can also be found in [3, 4, 5, 6, 7]. The significance of this result is obvious: it shows that the dynamics near our homoclinic loop Γ is essentially two-dimensional. We cannot expect chaotic dynamics, for example; and the only bifurcation we can expect here is a birth of a certain number of limit cycles (in the case when $\mathcal{M}$ is a Möbius band, one more bifurcation is possible — a formation of a double homoclinic loop). Therefore, the main questions which must be asked here are what is the number of limit cycles which can be born from Γ, can they coexist with Γ, etc..

For sufficiently smooth systems on a plane the answers are known due to the works of Dulac and Leontovich. Thus, it was shown in [8] that in the case of finite codimension (i.e. unless the system satisfies an infinite set of independent conditions of equality type) a homoclinic loop to a saddle on a plane is either stable (an ω-limit set) or unstable (an α-limit set). In [9] a sharp estimate on the number of limit cycles which can be born from the homoclinic loop on a plane was given (these results were rediscovered in [10]).

For a large class of heteroclinic cycles of sufficiently smooth systems on a plane, the finiteness of the number of periodic orbits which can be born from such heteroclinic cycles in the case of finite codimension was proven by Ilyashenko and Yakovenko [11] (see [12] for an overview).

The aim of the present paper is to demonstrate that in the case of planar systems obtained by reduction of a multidimensional system onto the two-dimensional invariant manifold the situation is quite different. Namely, we give an example of a codimension-3 homoclinic loop Γ for which the attractive two-dimensional invariant manifold $\mathcal{M}$ exists and, at the same time, on $\mathcal{M}$ there is a sequence of periodic orbits of alternating stability which has Γ as the limit.

As we mentioned, such situation is impossible in the case of sufficiently smooth planar systems. The main reason why this phenomenon happens in our case is that the smoothness of the non-local invariant manifold $\mathcal{M}$ is always very limited. In general, this manifold is only $C^{1+\varepsilon}$, with $\varepsilon < 1$. Another important ingredient of our construction is the presence of complex characteristic exponents of the saddle O. The corresponding two-dimensional invariant subspace of the system linearized at O is transverse to $\mathcal{M}$, so our example is at least four-dimensional.

Statement of the problem. Let us put the saddle O at the origin. The system near O is then written as

$$\dot{z} = Bz + o(z)$$

where the matrix B has n eigenvalues to the left of the imaginary axis and one eigenvalue to the right. The eigenvalues of B are called characteristic exponents; we denote them as $\lambda_1, \lambda_2, \dots, \lambda_n$ and γ, assuming that

$$\gamma > 0 > \mathrm{Re}\lambda_1 \geq \cdots \geq \mathrm{Re}\lambda_n.$$

We assume also that λ_1 is real and simple, so

A) $\gamma > 0 > \lambda_1 > \mathrm{Re}\lambda_2 \geq \cdots \geq \mathrm{Re}\lambda_n$.

One can introduce coordinates (x, y, w) (where $x \in R^1$, $y \in R^1$, $w \in R^{n-1}$) such that the system near O will take the form

$$\dot{y} = \gamma y + \dots, \quad \dot{x} = \lambda_1 x + \dots, \quad \dot{w} = Cw + \dots,$$

where the spectrum of the matrix C is $\lambda_2, \dots, \lambda_n$, and the dots stand for nonlinearities. In this case the unstable manifold W^u is tangent to the y-axis at O, and the (x, w)-space is the tangent to the stable manifold W^s at O.

In W^s there exists a uniquely defined $(n-1)$-dimensional smooth invariant manifold W^{ss} (the strong stable manifold) which is tangent at O to the w-space. The orbits which do not lie in W^{ss} tend to O along the leading direction (the x-axis) as $t \to +\infty$ (see more details about the strong-stable manifold, as well as about the hierarchy of various extended unstable manifolds mentioned below, in [7]). We will assume that the same holds true for the homoclinic orbit Γ, i.e.

B) $\Gamma \not\subset W^{ss}$.

The unstable manifold W^u lies within the so-called extended unstable manifold W^{ue} which is a two-dimensional C^1-smooth invariant manifold which is tangent at O to the eigenspace corresponding to the characteristic exponents λ_1, γ, i.e. to the (x, y). Since the orbit Γ is an intersection of W^u and W^s, it also lies in the intersection of W^{ue} and W^s. We make the following assumption:

C) the manifold W^{ue} is transverse to W^s at the points of the homoclinic orbit Γ.

Since W^{ue} is two-dimensional, and W^s is a manifold of codimension 1, they can indeed intersect transversely along a one-dimensional trajectory. Although the manifold W^{ue} is not defined uniquely, any of these manifolds contains W^u and all of them are tangent to each other at every point of W^u. In particular, all of them are tangent at every point of Γ, so the transversality condition above is well posed.

The conditions B,C are necessary and sufficient (see [5, 7]) for the existence of the two-dimensional attracting invariant C^1-manifold $\mathcal{M}$ which is transverse to W^{ss} at O and contains the homoclinic loop Γ. Moreover, it contains all orbits which stay in a small neighborhood of Γ for all times.

Conditions A,B,C are of inequality type, so the systems with a homoclinic loop satisfying this conditions form bifurcational surfaces of codimension 1 in the space of C^r-smooth systems. Now we impose two additional restrictions on the system which define a codimension 3 manifold within this surface. Namely, we assume that the saddle O is resonant, i.e.

D) the saddle value $\sigma = \lambda_1 + \gamma$ equals to zero.

We also assume

E) the separatrix value A, introduced below (see formula (13)), equals to 1.

Note that this condition is equivalent, as we will show below, to the vanishing of the integral $\int_{-\infty}^{+\infty} \mathrm{div} X(z(t))dt$, where $\{z(t)\}_{t\in(-\infty,+\infty)}$ denotes here the homoclinic solution Γ, and X denotes the vector field of the system on the two-dimensional invariant manifold $\mathcal{M}$.

When the saddle value $\sigma = \lambda_1 + \gamma$ is non-zero, bifurcations of the homoclinic loop under consideration were studied in [13] for systems on the plane, and in [14, 15] in the multidimensional case. Here, only one periodic orbit can be born from the loop. The finiteness of the number of limit cycles which can be born from the homoclinic loop with $\sigma = 0$ was established (in the cases of finite codimension) in [9] for systems on the plane, and in [16, 17] for three-dimensional systems. When $|A| \neq 1$ (i.e. condition E is violated) bifurcations of the resonant homoclinic loop on the plane were studied in [18]; for multidimensional systems the corresponding bifurcation diagrams was constructed in [19], with the final proof obtained in [20] in the three-dimensional case and in [21] in the general case. It follows from these works that no more than two limit cycles can be born from the resonant homoclinic loop if $|A| \neq 1$.

The main result of the present paper is the following theorem.

Theorem. *Let a C^r-smooth ($r \geq 3$) dynamical system in R^{n+1} ($n \geq 3$) have a homoclinic loop Γ, and let conditions A,B,C,D,E be satisfied. Let the next to λ_1 characteristic exponent be complex, i.e.* $0 > \lambda_1 > \mathrm{Re}\lambda_2 = \mathrm{Re}\lambda_3 > \mathrm{Re}\lambda_k$ ($3 < k \leq n$), $\mathrm{Im}\lambda_2 = -\mathrm{Im}\lambda_3 \neq 0$*; moreover, we assume*

$$\mathrm{Re}\lambda_2 > 2\lambda_1. \tag{1}$$

Then, provided conditions F,G of general position (formulated below) are satisfied, the homoclinic loop Γ is the limit of a sequence of isolated periodic orbits.

Note that the fact that the presence of complex characteristic exponents can lead to an infinite number of single-round periodic orbits near a homoclinic loop has been known since [22] where it was shown that the dynamics near Γ is chaotic if λ_1 is complex and $\gamma + \mathrm{Re}\lambda_1 > 0$. In our example the dynamics is simple — it is confined on the two-dimensional invariant manifold $\mathcal{M}$, but still we have infinitely many limit cycles. Note that condition (1) prevents the manifold $\mathcal{M}$ of being C^2 or of a higher smoothness.

Let us now formulate the remaining conditions F and G of the theorem. By our assumptions, the tangent space to W^s at O splits into two subspaces invariant with respect to the linearized system: one, the x-axis, corresponds to the characteristic exponent λ_1, and the second, the w-subspace corresponds to the characteristic exponents $\lambda_2, ..., \lambda_n$. We will write $w = (u, v)$ where $u \in R^2$ is the projection onto the invariant subspace corresponding the characteristic exponents λ_2 and λ_3, and v is the projection onto the invariant subspace corresponding to the rest of the characteristic exponents λ. We will show below that condition (1) guarantees that in the stable manifold W^s there is a uniquely defined $(n-3)$-dimensional C^{r-1}-smooth invariant submanifold W^{s0} which is tangent at O to the (x, v)-subspace. Recall that we assume $r \geq 3$, so the manifold W^{s0} is at least C^2. It is this smoothness condition which defines W^{s0} uniquely: we will see below that when (1) holds, every other manifold tangent to the (x, v)-space is only C^1. By condition B, the homoclinic loop Γ is tangent to the x-axis when it enters O at $t = +\infty$, so it is tangent to W^{s0} at O. We, however, assume that

F) $\Gamma \not\subset W^{s0}$.

Another invariant object we should mention is the invariant four-dimensional C^1-manifold W^{uee} which is tangent at O to the (x, y, u)-space. This manifold includes the unstable manifold W^u; the family of tangents N^{uee} to W^{uee} at the points of W^u is a uniquely defined continuous family of three-dimensional spaces, which is invariant with respect to the linearized flow and tends to the (x, y, u)-space when approaching the point O. The space N^{uee} contains a two-dimensional subspace N^{ue} which is the tangent to the manifold W^{ue} (the invariant two-dimensional manifold tangent at O to the (x, y)-space, see condition C). The family of subspaces N^{ue} is also invariant with respect to the linearized flow, continuous, and it is defined uniquely. We will show below that condition (1) guarantees the existence of another two-dimensional subspace N^{u0} of N^{uee} which is transverse to N^{ue} at every point of the unstable manifold W^u; the family of the subspaces N^{u0} is continuous, invariant with respect to the linearized flow, and it is uniquely defined by these conditions. Our last genericity assumption is that

G) the subspace N^{u0} is transverse to W^s at every point of Γ.

Proof of the theorem. Let us locally straighten the stable and unstable manifolds near the point O, i.e. we will make a C^r-transformation of the coordinates after which the equation of W^s_{loc} in a small neighborhood of O becomes $y = 0$ and the equation of W^u_{loc} becomes $(x, u, v) = 0$. Hence, in these coordinates the system near O is written as follows:

$$\frac{d}{dt}\begin{pmatrix} x \\ u \end{pmatrix} = (D_1 + f_{11}(x,u,v,y))\begin{pmatrix} x \\ u \end{pmatrix} + f_{12}(x,u,v,y)v,$$

(2) $$\dot{v} = D_2 v + f_{21}(x,u,v,y)\begin{pmatrix} x \\ u \end{pmatrix} + f_{22}(x,u,v,y)v,$$

$$\dot{y} = \gamma y\,(1 + g(x,u,v,y)).$$

Here

$$D_1 = \begin{pmatrix} -\lambda & 0 & 0 \\ 0 & -\rho & -\omega \\ 0 & \omega & -\rho \end{pmatrix},$$

where we denote $\lambda_1 = -\lambda$ and $\lambda_{2,3} = -\rho \pm i\omega$, so the spectrum of D_1 is $\{\lambda_1, \lambda_2, \lambda_3\}$; the spectrum of matrix D_2 consists of the characteristic exponents λ_k with $3 < k \le n$, so we may assume

(3) $$\|e^{D_2 t}\| = o(e^{-\beta t}) \text{ as } t \to +\infty$$

for some $\beta > \rho$. Recall that by our assumptions

(4) $$0 < \gamma = \lambda < \rho < 2\lambda$$

and $\omega \neq 0$. It will be also convenient for us to take $\beta < 2\lambda$.

The functions f_{ij} and g in (2) are C^{r-1}-functions vanishing at zero. By scaling the time we can always make

$$g \equiv 0,$$

which we will hereafter assume. What is also important, that the coordinates (x, u, v, y) can be chosen in such a way that the functions f_{ij} will satisfy the following identities:

(5) $$f_{i1}(x,u,v,0) \equiv 0, \qquad f_{1j}(0,0,0,y) \equiv 0.$$

The transformation which brings the system near O to the form (2) with the identities (5) satisfied is of class C^{r-1}. The existence and smoothness of this transformation is proven in [7] (following [23, 24]; note that the proof in [7] is conducted for the case where all the eigenvalues of the matrix D_1 have the same real parts, but it remains valid without any change in the case we consider here, when the spectrum of D_1 lies strictly in the strip $-\beta < \mathrm{Re}(\cdot) < -\beta'$ with $\beta' > 0$ and $\beta < \min(2\beta', \beta' + \gamma)$).

Note that the terms in functions f_{ij} which do not satisfy identities (5) are always non-resonant. So the possibility to achieve these identities means that these particular non-resonant terms can be killed by a single C^{r-1}-transformation of coordinates. Identities (5) have also certain geometrical meaning.

Thus, it is easy to see that the first of identities (5) imply that on the stable manifold (i.e. in the system obtained by plugging $y = 0$ in (2)) the evolution of the (x, u)-variables is independent on the v variables. Moreover, it is linear:

$$\dot{x} = -\lambda x, \quad \dot{u}_1 = -\rho u_1 - \omega u_2, \quad \dot{u}_2 = -\rho u_2 + \omega u_1$$

at $y = 0$. It is obvious from this equation, that the manifold $\{u = 0, y = 0\}$ is invariant; moreover, since $\rho \in (\lambda, 2\lambda)$, it is the only invariant manifold which is tangent to the $\{u = 0, y = 0\}$-space and which is C^2, at least. Thus, the invariant manifold W^{s0} from our condition F is given by the equation $\{u = 0, y = 0\}$ in our coordinates.

Analogously, the only invariant submanifold of W^s which is transverse to the x-axis is the manifold $\{x = 0, y = 0\}$; i.e. it is the manifold W^{ss} mentioned in condition B.

Let $(x = 0, u = 0, v = 0, y = y^\circ(t))$ be a trajectory in the unstable manifold. Taking into account the second of identities (5), we see that the linearization of system (2) (with $g \equiv 0$) is written for such trajectory as

$$\frac{d}{dt}\begin{pmatrix} X \\ U \end{pmatrix} = D_1 \begin{pmatrix} X \\ U \end{pmatrix} + f_{12}(0,0,0,y^\circ(t))V,$$

(6)
$$\dot{V} = (D_2 v + f_{22}(0,0,0,y^\circ(t)))V,$$

$$\dot{Y} = \gamma Y,$$

where we denote as (X, U, V, Y) the coordinates in the tangent space. One can see that the space $V = 0$ is invariant with respect to the linearized system. By uniqueness, $V = 0$ is the space N^{uee}, i.e. the tangent space to the invariant manifold W^{uee}. Within the space $V = 0$ the system (6) reduces to

$$\frac{d}{dt}\begin{pmatrix} X \\ U \end{pmatrix} = D_1 \begin{pmatrix} X \\ U \end{pmatrix};$$

this system has exactly two invariant subspaces: $X = 0$ and $U = 0$. The space $(U = 0, V = 0)$ is the space N^{ue} which is tangent to the invariant manifold W^{ue}; hence the space $(X = 0, V = 0)$ is the invariant space N^{u0} from our condition G.

We see that invariant manifolds and subspaces mentioned in the genericity conditions on our homoclinic loop have especially simple equations when identities (5) hold. In particular, the manifold W^{ue}_{loc} is tangent to the space $(u, v) = 0$ at every point of the local unstable manifold. The system on this invariant manifold can be written, hence, as

$$\dot{y} = \gamma y, \qquad \dot{x} = -\lambda x + p(x, y),$$

where the C^1-function p vanishes identically, along with its first derivative with respect to x, both at $y = 0$ and at $x = 0$ (see (2),(5)). Thus, the divergence of the vector field on W^{ue}_{loc} vanishes (recall that $\gamma = \lambda$ by assumption) at the points of W^{u}_{loc} and of $W^{ue}_{loc} \cap W^{s}_{loc}$. This means that the flow on W^{ue}_{loc}, when linearized at the points of any orbit from W^{u}_{loc} or $W^{ue}_{loc} \cap W^{s}_{loc}$, preserves the area (the manifold W^{ue}_{loc} is not uniquely defined, but this area-preservation property holds for any of them).

By assumption, the loop Γ coincides locally with one of the y-semiaxes when it leaves O at $t = -\infty$. Since $\Gamma \not\subset W^{ss}$, it enters O as $t \to +\infty$ along the x-axis. We assume that Γ adjoins O from the side of positive y and of positive x, as $t \to -\infty$ and $t \to +\infty$, respectively.

Take two cross-sections, S_0 and S_1 to the loop Γ. Namely, S_1 is $\{y = d\}$ and S_0 is $\{x = d\}$, for some small $d > 0$. Denote the coordinates on S_1 as $(x_1, u_{11}, u_{12}, v_1)$ and the coordinates on S_0 as $(y_0, u_{01}, u_{02}, v_0)$.

Since we have $g \equiv 0$ in (2), the last equation of (2) is easily integrated and gives

$$y(t) = e^{\gamma t} y_0. \tag{7}$$

Thus, the orbit of a point on S_0, when leaving the d-neighborhood of O, intersects the cross-section S_1 if and only if $y_0 > 0$; and the flight-time from S_0 to S_1 equals to

$$\tau = -\frac{1}{\gamma} \ln \frac{y_0}{d}. \tag{8}$$

The time-$\tau(y_0)$ map of the upper part $S_0^+ : \{y_0 > 0\}$ of S_0 into S_1 is called the local map T_0.

The flow outside a small neighborhood of O defines the global map $T_1 : S_1 \to S_0$ by the orbits close to Γ (the map T_1 takes any point from a small neighborhood of zero in S_1 into the first point of intersection of the forward orbit of this point with S_0). The composition $T = T_1 T_0$ is the Poincaré map near the homoclinic loop Γ; its fixed points correspond to periodic orbits of the flow. Thus, to prove the theorem we should show that the map $T : S_0^+ \to S_0$ has an infinite sequence of isolated fixed points converging to $y_0 = 0$.

Analogously to [24], one may show that the fulfillment of identities (5) implies the following estimates for the solution of the system starting at a point with the coordinates (x_0, u_0, v_0) at $t = 0$ and reaching $\{y = d\}$ at some $t = \tau$:

$$\begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} = e^{D_1 \tau} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} + \xi_2(x_0, u_0, v_0, \tau), \qquad v(\tau) = \xi_2(x_0, u_0, v_0, \tau) \tag{9}$$

where

$$\|\xi_{1,2}\|_{C^{r-2}} = o(e^{-\beta\tau}). \tag{10}$$

Here, $\beta > 0$ is the constant such that the spectrum of the matrix D_2 lies strictly to the left of the line $\mathrm{Re}(\cdot) = -\beta$ (i.e. (3) holds) and the spectrum of the matrix D_1 lies in the strip $\mathcal{R}_{\beta,\beta'} : -\beta < \mathrm{Re}(\cdot) < -\beta'$ with $\beta' > 0$ and $\beta < \min(2\beta', \beta' + \gamma)$ (i.e. $\beta \in (\rho, 2\lambda)$ in our case; see (4)). A detailed proof of estimates (9) can be found in [21] (formally, only the case where all the eigenvalues of D_1 have the same real parts is considered in [21]; however, the proof given there covers our case, where the real parts of the eigenvalues of D_1 are spread in the small strip $\mathcal{R}_{\beta,\beta'}$, without any change).

By plugging formula (8) for the flight time into (9), (10) we obtain the following estimate for the local map $T_0 : S_0^+ \to S_1$ (recall that $\gamma = \lambda$ by assumption and that $x_0 = d$ on S_0):

$$
\begin{aligned}
&x_1 = y_0 + \varphi_1(y_0, u_0, v_0),\\
&u_{11} = \left(\frac{y_0}{d}\right)^\nu (u_{01} \cos \Omega \ln \frac{y_0}{d} - u_{02} \sin \Omega \ln \frac{y_0}{d}) + \varphi_2(y_0, u_0, v_0),\\
&u_{12} = \left(\frac{y_0}{d}\right)^\nu (u_{01} \sin \Omega \ln \frac{y_0}{d} + u_{02} \cos \Omega \ln \frac{y_0}{d}) + \varphi_3(y_0, u_0, v_0),\\
&v_1 = \varphi_4(y_0, u_0, v_0),
\end{aligned}
\tag{11}
$$

where we denote $\Omega = \omega/\gamma$, $\nu = \rho/\gamma$, so $\Omega \neq 0$ and $1 < \nu < 2$; the functions φ_j, $j = 1, ..., 4$, satisfy the following estimates

$$
\begin{aligned}
&\varphi = o(y_0^\nu),\\
&\frac{\partial^{p+q}\varphi}{\partial(u_0, v_0)^p \partial y_0^q} = o(y_0^{\nu-q}) \qquad (p + q = 1, \cdots, r - 2).
\end{aligned}
\tag{12}
$$

The global map $T_1 : S_1 \to S_0$ is a diffeomorphism of a small neighborhood of the point $M^-(0, 0, 0, 0) = \Gamma \cap S_1$ into a small neighborhood of the point $M^+(0, u_1^+, u_2^+, v^+) = \Gamma \cap S_0$. Hence, it can be written as

$$
\begin{array}{ll}
y_0 & = a_{11}x_1 + a_{12}u_{01} + a_{13}u_{02} + a_{14}v_1 + \ldots,\\
u_{01} - u_1^+ & = a_{21}x_1 + a_{22}u_{01} + a_{23}u_{02} + a_{24}v_1 + \ldots,\\
u_{02} - u_2^+ & = a_{31}x_1 + a_{32}u_{01} + a_{33}u_{02} + a_{34}v_1 + \ldots,\\
v_0 - v^+ & = a_{41}x_1 + a_{42}u_{01} + a_{43}u_{02} + a_{44}v_1 + \ldots,
\end{array}
\tag{13}
$$

where a_{ij} are certain coefficients, and the dots stand for the quadratic and higher order terms.

The coefficient a_{11} is called the separatrix value (see [21]); it is exactly the value A from our condition E. One can show that in our case, where $\lambda = \gamma$, the value of a_{11} is invariant with respect to smooth coordinate transformations which keep the system in the form (2) with $g \equiv 0$ and with identities (5) satisfied. This can be verified by a direct computation. We choose to prove it in the following way.

Note that the two-dimensional invariant manifold $\mathcal{M}$ which contains the homoclinic loop Γ is transverse at O to the strong stable manifold W^{ss}; therefore, this manifold coincides with some local extended unstable manifold W^{ue} near O (see more details in [7]). We can choose (x, y) as the coordinates on $\mathcal{M}$ near O. For the flow on $\mathcal{M}$, the global map $T_1 : S_1 \cap \mathcal{M} \to S_0 \cap \mathcal{M}$ is written as

$$y_0 = a_{11}x_1 + o(x_1).$$

Thus, a_{11} is the coefficient the expansion (or contraction) of distances at the point M^- by the global map restricted on $\mathcal{M}$. In our coordinates, the phase velocity vectors $\dot{y} = \gamma y$ and $\dot{x} = -\lambda x$ for the flow on $\mathcal{M}$, taken at the points $M^-(y = d)$ and $M^+(x = d)$, respectively, have the same length (recall that $\gamma = \lambda$). Therefore, a_{11} is, as well, the coefficient of expansion/contraction of areas by the flow on $\mathcal{M}$, linearized at the points of the homoclinic orbit Γ on the segment from the point M^- to the point M^+. It follows that after a smooth coordinate transformation the coefficient a_{11} is multiplied to the factor $J(M^+)/J(M^-)$ where J is the Jacobian of the coordinate transformation on $\mathcal{M}$. Since $\mathcal{M}$ coincides locally with some manifold W^{ue}, the flow on $\mathcal{M}$ is divergence free at the points of W^u_{loc} and of $W^s_{loc} \cap \mathcal{M}$, provided $g = 0$ in (2) and identities (5) are satisfied. Hence, when linearized at the points of Γ, the flow on $\mathcal{M}$ is area-preserving near O. This implies that in our coordinates, the coefficient a_{11} is independent of the choice of the points M^+ and M^-, i.e. of the choice of the small constant d. Therefore, for the smooth coordinate transformations which keep the system in the form (2) with $g \equiv 0$ and with identities (5) satisfied, the factor $J(M^+)/J(M^-)$ have to be independent of d as well. By taking $d \to +0$, we get M^+ and M^- converging to the same point O, which gives $J(M^+)/J(M^-) \equiv 1$ for the coordinate transformations under consideration. Thus, a_{11} is an invariant of such transformations indeed.

By virtue of condition E, we have $a_{11} = 1$. By combining formulas (11) and (13) we obtain the following equation on the fixed points (y, u, v) (we dropped the index "0") of the Poincaré map $T = T_1 T_0$ on S_0^+:

$$
\begin{aligned}
y &= y + \left(\frac{y}{d}\right)^{\nu} |u| (a_{12} \cos(\Omega \ln \frac{y}{d} + \theta) + a_{13} \sin(\Omega \ln \frac{y}{d} + \theta)) + o(y^{\nu}), \\
u &= u^+ + O(y), \qquad v = v^+ + O(y),
\end{aligned}
\tag{14}
$$

where we denote $u = (|u| \cos\theta, |u| \sin\theta)$. For all small y, the last equations of this system can be resolved with respect to u and v, so that the system reduced to the following single equation on the y-variable:

$$
0 = y^{\nu} |u^+| (a_{12} \cos(\Omega \ln \frac{y}{d} + \theta) + a_{13} \sin(\Omega \ln \frac{y}{d} + \theta)) + o(y^{\nu}).
\tag{15}
$$

This equation has an infinite, converging to zero sequence of isolated positive roots

$$
y_m = d e^{-\frac{\pi}{\Omega} m} \, e^{(-\theta - \arctan \frac{a_{12}}{a_{13}})} (1 + o(1)),
$$

provided $u^+ \neq 0$ and $a_{12}^2 + a_{13}^2 \neq 0$. It remains to note that these two inequalities are, in fact, our conditions F and G, respectively. Indeed, by condition F, the point $M^+(0, u^+, v^+)$ does not belong to the manifold W^{s0}. The latter is given by the equation $(y = 0, u = 0)$ in our coordinates, so condition F reads as $u^+ \neq 0$ indeed. In turn, condition G reads in our coordinates as the transversality of the image of the plane $(x_0 = 0, v_0 = 0)$ from S_0 by the map T_1 to the space $y_1 = 0$ at the point M^+ in S_1. It follows immediately from (13) that this transversality condition is equivalent to $a_{12}^2 + a_{13}^2 \neq 0$.

Thus, we have proved the existence of an infinite sequence of the isolated fixed points of the Poincaré map, converging to the $W^s_{loc} \cap S_0$. The fixed points of the Poincaré map correspond to periodic orbits of the flow. End of the proof.

An interesting question is how the obtained family of limit cycles bifurcates. We deal here with a codimension-3 bifurcation, so we need at least three governing parameters. We denote them as (μ, δ, α). The parameter μ governs the splitting of the homoclinic loop Γ: we take it equal to the y-coordinate of the point $M^+ = T_1 M^-$ where $M^- = W^u_{loc} \cap S_1$. We also take $\delta = \lambda/\gamma - 1$, and $\alpha = a_{11} - 1$. Then, following the same lines as in the proof of the theorem, one may show that the fixed point of the Poincaré map satisfy the equation

$$y = \mu + (1+\alpha) y^{1+\delta} + K y^\nu \cos(\Omega \ln y + \theta) + o(y^\nu),$$

for some constants $K \neq 0$ and θ. This equation is bound to produce a rich bifurcation diagram. Thus, one can show that an infinite sequence of swallow tails exists in the parameter space.

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