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Complete synchronization of symmetrically coupled autonomous systems

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Abstract

In this paper we derive conditions for complete synchronization of two symmetrically coupled identical systems of ordinary differential equations and differential-delay equations. Using Lyapunov function approach we give an estimate of the region of attraction of the synchronized solution. We also established that complete synchronization is robust with respect to small perturbations of the identical systems.

1 Introduction

We consider the problem of complete synchronization of identical dynamical systems which are symmetrically coupled. The dynamical systems under consideration are systems of autonomous ordinary differential equations (ODE systems)

$$\frac{dx}{dt} = f(x) \tag{1.1}$$

with $f : R^n \to R^n$, and systems of autonomous differential-delay equations (DDE systems)

$$\frac{dx}{dt} = f(x, x(t-1)) \tag{1.2}$$

with $f : R^n \times R^n \to R^n$.

In what follows we restrict ourselves to the case of two symmetrically coupled identical systems, that is, we study the non-autonomous ODE-system

$$\begin{aligned} \frac{dx}{dt} &= f(x) + g(t, x, y), \\ \frac{dy}{dt} &= f(y) + g(t, y, x) \end{aligned} \tag{1.3}$$

with $g : R^+ \times R^n \times R^n \to R^n$, as well as the non-autonomous DDE-system

$$\begin{aligned} \frac{dx}{dt} &= f(x, x(t-1)) + g(t, x, x(t-1), y, y(t-1)), \\ \frac{dy}{dt} &= f(y, y(t-1)) + g(t, y, y(t-1), x, x(t-1)) \end{aligned} \tag{1.4}$$

with $g : R^+ \times R^n \times R^n \times R^n \times R^n \to R^n$.

Our goal is to derive conditions on the function f and on the coupling term g in order to guarantee a complete synchronization.

The phenomenon of complete synchronization has been studied by many researchers in applied fields such as electrical engineering [1], laser physics [2], coupled semiconductor Josephson junctions [3], electro-chemical reactors [4] and others [5, 6]. Mathematical methods for studying this type of synchronization have been developed, in particular, in [7-10]. In [11, 12], the application of the uniform invariance principle to the synchronization problem was demonstrated. Rigorous results on coupled lattices of nonlinear oscillators are given in [13].

The paper is organized as follows. In section 2 we recall necessary definitions from the theory of dynamical systems. In section 3 we reconsider the general problem of complete synchronization for coupled identical systems of ordinary differential systems. We prove by applying the technique of Lyapunov functions that we can replace the usual Lipschitz condition on f (see, e.g. [14]) by a one-sided Lipschitz condition. We derive conditions for synchronization in a bounded region and give a lower estimate for a coupling constant to ensure synchronization of linearly coupled systems. At the same time we estimate the region of attraction of the synchronized solution. Section 4 is devoted to the problem of robustness of complete synchronization. We perturb the identical systems and estimate the synchronization error as a function of the perturbation. Section 5 generalizes the obtained results to the case of differential-delay systems. In section 6 we illustrate some of the obtained results by means of two modified Goodwin oscillators describing a control system for the production of an enzyme.

2 Preliminaries

Let $|\cdot|$ be the Euclidean norm in R^n. We define the distance $d(x, \mathcal{G})$ of a point $x \in R^n$ from a subset $\mathcal{G}$ of R^n by

$$d(x, \mathcal{G}) = \inf_{y \in \mathcal{G}} |x - y|.$$

A mapping $\varphi : \mathcal{G} \times R \to \mathcal{G}$ is called a *flow on* $\mathcal{G}$ if the following relations are satisfied:

(i). $\varphi(x, 0) = x \quad \forall x \in \mathcal{G}$.

(ii). $\varphi(\varphi(x, s), t) = \varphi(x, s + t) \quad \forall x \in \mathcal{G},\ \forall s, t \in R$.

(iii). $\varphi : \mathcal{G} \times R \to \mathcal{G}$ is continuous.

It is obvious that $\mathcal{G}$ is invariant under the flow φ, that is, it holds $\varphi(\mathcal{G}, t) \equiv \mathcal{G}$ for all $t \in R$. If we assume that f in (1.1) is such that to any initial value $x \in R^n$ there exists a unique solution $\varphi(x, t)$ defined for all t, then φ satisfies the relations (i)–(iii), and we say that f defines a flow on R^n.

A mapping $\varphi : \mathcal{G} \times R \to \mathcal{G}$ is called a *semiflow on* $\mathcal{G}$ if φ satisfies the properties (i) and (iii) above, and instead of (ii) the property (ii') $\varphi(\varphi(x,s),t) = \varphi(x, s+t) \quad \forall x \in \mathcal{G},\ \forall s,t \geq 0.$

We note that if φ is a semiflow on $\mathcal{G}$ then $\mathcal{G}$ is positively invariant with respect to φ, that is, we have $\varphi(\mathcal{G},t) \subset \mathcal{G}$ for all $t \in R^+$.

Let $\mathcal{A}$ and $\mathcal{B}$ be subsets of R^n. We say that the *set* $\mathcal{A}$ *attracts the set* $\mathcal{B}$ *under the semiflow* φ if

$$d(\varphi(x,t), \mathcal{A}) \to 0 \quad \text{as} \quad t \to \infty \quad \forall x \in \mathcal{B}.$$

A subset $\mathcal{A}$ of $\mathcal{G}$ is called an *attractor of the semiflow* φ *on* $\mathcal{G}$ if the following relations hold

(i). $\mathcal{A}$ is compact.

(ii) $\varphi(\mathcal{A},t) \equiv \mathcal{A} \quad \forall t \in R.$

(iii) There is a neighborhood $\mathcal{U}$ of $\mathcal{A}$ in $\mathcal{G}$ such that $\mathcal{A}$ attracts $\mathcal{U}$.

Let $z := (x,y) \in R^n \times R^n$. We denote by $\psi(z,t) = (\psi_1(z,t), \psi_2(z,t))$ a solution of (1.3) satisfying $\psi_1(z,0) = x$, $\psi_2(z,0) = y$.

Definition 2.1 *Let $\mathcal{W}$ be some subset of $R^n \times R^n$ such that for $z \in \mathcal{W}$ the solution $\psi(z,t)$ of (1.3) exists for $t \geq 0$. We say that two identical symmetrically coupled autonomous ODE-systems do completely synchronize for $z \in \mathcal{W}$ if it holds*

$$|\psi_1(z,t) - \psi_2(z,t)| \to 0 \quad as \quad t \to \infty. \tag{2.1}$$

Remark 2.2 *The problem of complete synchronization for $z \in \mathcal{W}$ consists in deriving conditions under which the components describing the behavior of the subsystems are asymptotically identical, that is, for $z \in \mathcal{W}$ the solution $\psi(z,t)$ of* (1.3) *satisfies*

$$d(\psi(z,t), \mathcal{P}) \to 0 \ as \ t \to \infty,$$

where $\mathcal{P}$ is defined by $\mathcal{P} := \{(x,y) \in R^n \times R^n : x = y\}$. It is easy to verify that $\mathcal{P}$ is invariant with respect to system (1.3).

In order to define the concept of complete synchronization for differential-delay systems we need the following notation.
Let $\mathcal{C}$ be the space of continuous functions mapping $[-1,0]$ into R^n. We denote by $\psi(\phi) : [-1,T] \to R^n \times R^n$ a solution of the DDE-system (1.4) defined on the interval $[-1,T]$ and satisfying $\psi(\phi)(0) = (\phi_1, \phi_2)$, where $\phi_1, \phi_2 \in \mathcal{C}$ represent initial functions.

Definition 2.3 *Let $\mathcal{V}$ be some subset of $\mathcal{C} \times \mathcal{C}$ such that for $\phi \in \mathcal{V}$ the corresponding solution $\psi(\phi)$ of* (1.4) *is defined on $[-1,\infty)$. We say that two symmetrically coupled autonomous DDE-systems do completely synchronize for $\phi \in \mathcal{V}$ if*

$$|\psi_1(\phi)(t) - \psi_2(\phi)(t)| \to 0 \quad as \ t \to \infty.$$

Remark 2.4 *As in the case of ODE systems, complete synchronization means that the components describing the behavior of the subsystems are asymptotically identical, i.e. we have*

$$d(\psi(\phi)(t), \mathcal{P}) \to 0 \ as \ t \to \infty \quad for \ all \quad \phi \in \mathcal{V},$$

3 Conditions for complete synchronization of autonomous ODE systems

We consider system (1.3) under the following assumptions

(A_1). The function f satisfies a global one-sided Lipschitz condition in $\mathcal{B} \subset R^n$, that is, there is a constant l (l can be negative!) such that

$$(f(x) - f(y))^T(x - y) \le l|x - y|^2 \quad \forall x, y \in \mathcal{B}, \tag{3.1}$$

where z^T means the transpose of the column vector z.

(A_2). Let $g : R^+ \times \mathcal{B} \times \mathcal{B} \to R^n$ be continuous. There are two constants $\beta_0 \in R$ and $\gamma_0 \in R$ such that

$$\begin{aligned} &(g(t, x_1, y) - g(t, x_2, y))^T(x_1 - x_2) \le \beta_0|x_1 - x_2|^2 \\ &\forall\ (t, y) \in R^+ \times \mathcal{B},\ \forall\ x_1, x_2 \in \mathcal{B}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &(g(t, x, y_1) - g(t, x, y_2))^T(y_1 - y_2) \ge \gamma_0|y_1 - y_2|^2 \\ &\forall\ (t, x) \in R^+ \times \mathcal{B},\ \forall\ y_1, y_2 \in \mathcal{B}. \end{aligned} \tag{3.3}$$

Inequality (3.2) says that $g(t, x, y)$ is uniformly one-sided Lipschitzian in x, from (3.3) it follows that g is uniformly strictly monotone in y in case of a positive γ_0.

Proposition 3.1 *Suppose the hypotheses (A_1) and (A_2) to be fulfilled with $\mathcal{B} = R^n$. Moreover, we assume that there is a subset $\mathcal{W} \in R^n \times R^n$ such that for $z \in \mathcal{W}$ the solution $\psi(z, t) = (\psi_1(x, y, t), \psi_2(x, y, t))$ of (1.3) exists for $t \ge 0$. Then, for $z \in \mathcal{W}$ the following estimate holds*

$$|\psi_1(x, y, t) - \psi_2(x, y, t)| \le e^{-\alpha t}|x - y|, \tag{3.4}$$

where

$$\alpha := \gamma_0 - \beta_0 - l. \tag{3.5}$$

Proof. We introduce the function $V : R^n \times R^n \to R^+$ by $V(x, y) := |x - y|^2$. Under our assumptions, for $z \in \mathcal{W}$ the derivative of $V(\psi(z, t))$ with respect to system (1.3) satisfies

$$\frac{dV(\psi(z,t))}{dt} = \frac{d}{dt}\left[(\psi_1(z,t) - \psi_2(z,t))^T(\psi_1(z,t) - \psi_2(z,t))\right]$$

$$= 2(\psi_1(z,t) - \psi_2(z,t))^T(f(\psi_1(z,t)) + g(t, \psi_1(z,t), \psi_2(z,t))$$

$$-f(\psi_2(z,t)) - g(t, \psi_2(z,t), \psi_1(z,t)))$$

$$= 2(\psi_1(z,t) - \psi_2(z,t))^T(f(\psi_1(z,t)) - f(\psi_2(z,t)))$$

$$+2(\psi_1(z,t) - \psi_2(z,t))^T(g(t, \psi_1(z,t), \psi_2(z,t)) - g(t, \psi_2(z,t), \psi_1(z,t)))$$

$$\le 2l|\psi_1(z,t) - \psi_2(z,t)|^2$$

$$+2(\psi_1(z,t)-\psi_2(z,t))^T(g(t,\psi_1(z,t),\psi_2(z,t))-g(t,\psi_2(z,t),\psi_2(z,t)))$$
$$+2(\psi_1(z,t)-\psi_2(z,t))^T(g(t,\psi_2(z,t),\psi_2(z,t))-g(t,\psi_2(z,t),\psi_1(z,t)))$$
$$\leq (2l+2\beta_0-2\gamma_0)|\psi_1(z,t)-\psi_2(z,t)|^2 = -2\alpha V(\psi(z,t)).$$

From the inequality

$$\frac{dV}{dt} \leq -2\alpha V \tag{3.6}$$

we obtain the estimate (3.4). ■

In view of the Proposition 3.1 we have the following result

Theorem 3.2 *Suppose the assumptions* (A_1) *and* (A_2) *to be fulfilled with* $\mathcal{B} = R^n$. *Moreover, we assume that for all* $z \in R^n \times R^n$ *the solution* $\psi(z,t) = (\psi_1(x,y,t), \psi_2(x,y,t))$ *of* (1.3) *exists for* $t \geq 0$. *Then, under the additional condition*

$$\alpha := \gamma_0 - \beta_0 - l > 0 \tag{3.7}$$

system (1.3) *synchronizes completely in* $R^n \times R^n$.

Proof. The proof follows immediately from (3.4), (3.7). ■

A special case of system (1.3) is the case of linearly diffusively coupled identical systems

$$\begin{aligned}\frac{dx}{dt} &= f(x) + K(y-x),\\ \frac{dy}{dt} &= f(y) + K(x-y)\end{aligned} \tag{3.8}$$

Concerning the vector field f and the coupling matrix K we assume:

(A_3). The function f satisfies a global Lipschitz condition in R^n, that is, there is a positive constant L such that

$$|f(x)-f(y)| \leq L|x-y| \quad \forall x,y \in R^n. \tag{3.9}$$

(A_4). The coupling matrix K has the form $K = kI$, where I is the identity in R^n and k is positive.

In [14] the following result about complete synchronization of (3.8) has been proved.

Theorem 3.3 *Suppose the hypotheses* (A_3) *and* (A_4) *to be valid. Then, under the condition*

$$k > L/2$$

system (3.8) *completely synchronizes in* $R^n \times R^n$.

If we put $g(t,x,y) \equiv K(y-x)$ in (1.3), then we get system (3.8). Thus, under the hypothesis (A_4) the relations

$$\beta_0 = -k,\ \gamma_0 = k$$

hold. Hence, we get from Theorem 3.2 the following result which generalizes Theorem 3.3.

Corollary 3.4 *Suppose f and K satisfy hypotheses (A_1) and (A_4) with $\mathcal{B} = R^n$. Furthermore, we assume that for all $z \in R^n \times R^n$ (3.8) has a solution $\psi(z,t) = (\psi_1(z,t), \psi_2(z,t))$ defined for $t \geq 0$. Then, under the condition*

$$k > l/2 \tag{3.10}$$

system (3.8) completely synchronizes in $R^n \times R^n$.

System (3.8) describes two autonomous systems which are bidirectionally coupled. If we consider the unidirectional coupling

$$\begin{aligned} \frac{dx}{dt} &= f(x) + K(y-x), \\ \frac{dy}{dt} &= f(y), \end{aligned} \tag{3.11}$$

then we can treat the problem of complete synchronization by the same approach. As a result we get

Theorem 3.5 *Suppose f and K satisfy hypotheses (A_1) and (A_4) with $\mathcal{B} = R^n$. Additionally we assume that for all $z \in R^n \times R^n$ there exists a unique solution $\psi(z,t)$ of (3.11) for $t \geq 0$. Then, under the condition $k > l$ system (3.11) completely synchronizes in $R^n \times R^n$.*

Theorem 3.5 can be formulated also in the following way.

Theorem 3.6 *Let $y^* : R^+ \to R^n$ be any half-trajectory of (1.1) starting for $t = 0$ at y. Then, under the assumptions of Theorem 3.5, there exists a linear feedback controller such that system (3.11) tracks this target trajectory, that is, it holds*

$$\lim_{t\to\infty} |\psi_1(x,y,t) - y^*(t)| = 0.$$

A coupling of identical systems of the type

$$\begin{aligned} \frac{dx}{dt} &= f(x) + g(t,x,y), \\ \frac{dy}{dt} &= f(y) - g(t,x,y), \end{aligned} \tag{3.12}$$

where g satisfies

$$g(t,x,x) \equiv 0 \quad \forall t \geq 0,\ x \in R^n, \tag{3.13}$$

can be considered as a generalization of the linear diffusive coupling. At the same time, (3.12) is a special case of system (1.3) when g fulfills

$$g(t, x, y) \equiv -g(t, y, x) \quad \forall t \geq 0,\ x, y \in R^n \tag{3.14}$$

which implies the validity of (3.13). It is easy to verify that a function $g(t, x, y)$ which satisfies a uniform one-sided Lipschitz condition with respect to x with the constant β_0 and fulfills relation (3.14), also satisfies a one-sided Lipschitz condition with respect to y and with the same Lipschitz constant. Thus, we get from Theorem 3.2 the result

Theorem 3.7 *Suppose f satisfies hypothesis (A_1) and g satisfies the inequality (3.2) and the relation (3.14). Moreover, we assume that that for all $z \in R^n \times R^n$ (3.12) has a unique solution $\psi(z, t) = (\psi_1(z, t), \psi_2(z, t))$ defined for $t \geq 0$. Then, under the condition*

$$-2\beta_0 - l > 0$$

system (3.12) synchronizes completely in R^n.

The assumptions of Theorem 3.2 do not guarantee that $\psi(z, t)$ stays in a bounded region for $t \geq 0$. In the sequel we consider identical systems with a generalized diffusive coupling and derive conditions to ensure that the complete synchronization takes place in a finite region.

Concerning the uncoupled system (1.1) we suppose

(A_5). There exist positive numbers R and γ and a differentiable function $V : |x| \geq R \to R^+$ such that

(i) $V(x) > 0$.
(ii) $V(x) \to \infty$ as $|x| \to \infty$.
(iii) $V'(x)f(x) \leq -\gamma$.
(iv) $V'(x)$ is uniformly continuous.

Assumption (A_5) implies that system (1.1) is dissipative. Thus, it has a global attractor $\mathcal{A}$.

With respect to the the coupling we consider a generalized diffusive coupling, that is, we study systems of type (3.12). We suppose

(A_6) g is continuous and satisfies (3.13). Additionally, to any ε there is a $\delta = \delta(\varepsilon)$ such that uniformly for $t \geq 0$ it holds

$$|g(t, x, y)| \leq \varepsilon \quad \text{for} \quad |x - y| \leq \delta.$$

Theorem 3.8 *Suppose the assumptions of Theorem 3.7 hold. Additionally, we assume that the hypotheses (A_5) and (A_6) are valid. Then system (3.12) completely synchronizes for $z \in R^n \times R^n$, where the synchronized state belongs to some bounded set.*

Proof. We define a δ-neighborhood of $\mathcal{P}$ by $\mathcal{P}_\delta := \{(x, y) \in R^n \times R^n : |x - y| \leq \delta\}$. Let ϵ_0 be some given positive number. According to hypothesis (A_6) there is a δ_0 such that uniformly for $t \geq 0$

$$|g(t, x, y)| \leq \epsilon_0 \quad \text{for} \quad |x - y| \leq \delta_0. \tag{3.15}$$

Under the hypotheses of Theorem 3.7, inequality (3.4) holds with $\alpha > 0$ such that any trajectory of (3.12) eventually enters the δ_0- neighborhood of $\mathcal{P}$. By assumption (A_5), $V'(x)$ is uniformly continuous for $|x| \geq R$. Hence, there is a δ_1, $\delta_1 \leq \delta_0$, such that

$$|V'(x) - V'(y)| \leq 1 \quad \text{for} \quad |x - y| \leq \delta_1.$$

We define the subregions $\mathcal{P}_{\delta_1}^R$ and $\mathcal{P}_{\delta_1}^s$ of $\mathcal{P}_{\delta_1}$, by

$$\mathcal{P}_{\delta_1}^R := \{(x, y) \in \mathcal{P}_{\delta_1} : |x| \geq R, |y| \geq R\}, \quad \mathcal{P}_{\delta_1}^s := \{(x, y) \in \mathcal{P}_{\delta_1} \setminus \mathcal{P}_{\delta_1}^R\}$$

In $\mathcal{P}_{\delta_1}^R$ we define the function W by

$$W(x, y) := V(x) + V(y).$$

For the function W it holds
(i): $W > 0$ for $|x| > R, |y| > R$.
(ii): $W(x, y) \to \infty$ as $|z| = |(x, y)| \to \infty$.
(iii): For $(x, y) \in \mathcal{P}_{\delta_1}^R$ we have by hypothesis (A_5)

$$\begin{aligned}
\left.\frac{dW}{dt}\right|_{(3.12)} &= \frac{\partial V(x)}{\partial x}(f(x) + g(t, x, y)) + \frac{\partial V(y)}{\partial y}(f(y) - g(t, x, y)) \\
&= \frac{\partial V(x)}{\partial x} f(x) + \frac{\partial V(y)}{\partial y} f(y) + \left(\frac{\partial V(x)}{\partial x} - \frac{\partial V(y)}{\partial y}\right) g(t, x, y) \\
&\leq -2\gamma + \left|\frac{\partial V(x)}{\partial x} - \frac{\partial V(y)}{\partial y}\right| |g(t, x, y)| \\
&\leq -2\gamma + \gamma = -\gamma < 0.
\end{aligned}$$

Thus, we can conclude that system (3.12) is dissipative and the synchronized state belongs to the region $\mathcal{P}_{\delta_1}^s$. ■

In the case of autonomous coupling

$$\begin{aligned}
\frac{dx}{dt} &= f(x) + g(x, y), \\
\frac{dy}{dt} &= f(y) - g(x, y),
\end{aligned} \tag{3.16}$$

where g satisfies

$$g(x, x) \equiv 0, \tag{3.17}$$

we can prove a more precise result.

Theorem 3.9 *Assume the hypotheses of Theorem* 3.2 *and assumption* (A_5) *to be fulfilled. Let* $g(x, y)$ *be uniformly continuous in a small neighborhood of* $\mathcal{P}$ *and satisfies* (3.17). *Then system* (3.16) *completely synchronizes in* $R^n \times R^n$*, where the omega-limit set* S *of the corresponding trajectory belongs to* $\mathcal{P} \cap \mathcal{A} \times \mathcal{A} = \{(x, y) : x = y \in \mathcal{A}\}$*, where* $\mathcal{A}$ *is the global attractor of* (1.1).

Proof. Under the assumptions of Theorem 3.2 system (3.16) generates a semiflow ψ on $R^n \times R^n$. We note that the hyperplane $\mathcal{P}$ is invariant under the semiflow ψ. According to (3.17), the dynamics of (3.16) on $\mathcal{P}$ is governed by system (1.1). Assumption (A_5) implies the dissipativity of (1.1) and the existence of a global attractor $\mathcal{A} \subset R^n$ of system (1.1).

Similarly to the proof of Theorem 3.8 we can show that the function $W(x,y) = V(x)+V(y)$ implies that the coupled system is dissipative. Therefore, there exists a global attractor $\mathcal{D} \subset R^n \times R^n$ of system (3.16). Since $\mathcal{P}$ is exponentially attracting for all orbits, we have $\mathcal{D} \subset \mathcal{P}$. Since $\mathcal{P}$ is invariant under the semiflow ψ and since ψ has a global attractor $\mathcal{P} \cap \mathcal{A} \times \mathcal{A}$ on $\mathcal{P}$, it follows $\mathcal{D} = \mathcal{P} \cap \mathcal{A} \times \mathcal{A}$. ■

In what follows we consider the linearly diffusively coupled system (3.8) and relax the assumption (A_5) for system (1.1) by the following two hypotheses.

$(A_{5.1})$ f is continuously differentiable $f \in C^1(\mathcal{G}, R^n)$ in some region $\mathcal{G} \subset R^n$. The corresponding system (1.1) generates a semiflow φ on G with a global attractor $\mathcal{A}$ in G. $(A_{5.2})$ There exists a convex set $U_{\mathcal{A}} \subset \mathcal{G}$ containing $\mathcal{A}$ with the properties
(i). $U_{\mathcal{A}}$ is positively invariant under the semiflow φ.
(ii). $\mathcal{A}$ attracts $U_{\mathcal{A}}$.

Theorem 3.10 *We assume the hypotheses (A_1), (A_4) with $\mathcal{B} = W_{\mathcal{A}}$ and $k > l/2$, and the assumptions $(A_{5.1})$ and $(A_{5.2})$ to be valid. Then system (3.8) completely synchronizes for $z \in W_{\mathcal{A}} = W_{\mathcal{A}} \times W_{\mathcal{A}}$ and the omega-limit set of the trajectory $\psi(z,t)$ belongs to the invariant set $\mathcal{S} = \mathcal{P} \cap (\mathcal{A} \times \mathcal{A})$.*

Proof. It follows from [15] that under the conditions $A_{5.1}$ and $A_{5.2}$, the region $W_{\mathcal{A}}$ is positively invariant with respect to (3.8). In analogy to Theorem 3.2 we can prove that under the additional hypotheses of Theorem 3.10, system (1.3) completely synchronizes for $z \in W_{\mathcal{A}}$. Therefore, we have only to prove that $\mathcal{S}$ attracts $W_{\mathcal{A}}$. But this follows immediately from the inequality (3.4), which has been used to establish the first part of the theorem, and the positive invariance of $W_{\mathcal{A}}$. ■

The property that f is continuously differentiable in $W_{\mathcal{A}}$, can be used to derive a condition ensuring that f satisfies a one-sided Lipschitz condition in $W_{\mathcal{A}}$. To this end we introduce the symmetric matrix $M(x)$ by

$$M(x) := f'(x) + (f'(x))^T, \tag{3.18}$$

where f' is the Jacobian matrix of f. If we denote by $\kappa(x)$ the maximum of all eigenvalues of $M(x)$ (they are real) and if we introduce the number

$$\bar{\kappa} = \sup_{x \in W_{\mathcal{A}}} \kappa(x), \tag{3.19}$$

then we have for all $x, y \in W_{\mathcal{A}}$

$$(f(x) - f(y))^T(x-y) = (f'(\bar{x})(x-y))^T(x-y) \leq \frac{\bar{\kappa}}{2}|x-y|^2. \tag{3.20}$$

Consequently, we have

Corollary 3.11 *We assume the hypotheses of Theorem* 3.10 *to be satisfied, except* (A_1). *Then, under the additional condition* $k > \bar{\kappa}/4$ *system* (3.8) *completely synchronizes for* $z \in W_{\mathcal{A}} = W_{\mathcal{A}} \times W_{\mathcal{A}}$ *where the closure of the synchronized state belongs to* $\mathcal{S} = \mathcal{P} \cap (\mathcal{A} \times \mathcal{A})$.

4 Robustness of synchronization

In the previous sections we studied symmetric coupling of two identical autonomous ODE systems. In this section we consider symmetric coupling of perturbed identical systems, that is, we consider the system

$$\begin{aligned} \frac{dx}{dt} &= f(x) + \varepsilon h_1(t,x,y) + g(t,x,y), \\ \frac{dy}{dt} &= f(y) + \varepsilon h_2(t,x,y) + g(t,y,x), \end{aligned} \tag{4.1}$$

where ε is a positive parameter. Concerning the perturbations h_1 and h_2 we assume

(A_8). For $i = 1, 2$, the functions $h_i : R^+ \times \mathcal{B} \times \mathcal{B} \to R^n$ are continuous and uniformly bounded in $R^+ \times \mathcal{B} \times \mathcal{B}$, that is

$$|h_i(t,x,y)| \leq m_0 \quad \forall \quad (t,x,y) \in R^+ \times \mathcal{B} \times \mathcal{B}.$$

Theorem 4.1 *Suppose the hypotheses* $(A_1), (A_2)$ *and* (A_8) *to be satisfied. Additionally, we assume that for* $z \in \mathcal{B} \times \mathcal{B}$ *the solution* $\psi(z,t) = (\psi_1(z,t), \psi_2(z,t))$ *of* (4.1) *is defined for all* $t \geq 0$. *Then it holds*

$$|\psi_1(x,y,t) - \psi_2(x,y,t)| \leq 2\varepsilon\frac{m_0}{\alpha} + e^{-\alpha t}\left(-2\varepsilon\frac{m_0}{\alpha} + |x-y|\right) \quad \forall z \in \mathcal{B}, t \geq 0 \tag{4.2}$$

where α *is defined by* $\alpha := \gamma_0 - \beta_0 - l$ *(see* (3.5)*).*

Proof. Let $V : \mathcal{B} \times \mathcal{B} \to R^+$ be defined by $V(x,y) = |x-y|^2$. The derivative of $V(\psi(z,t))$ along a solution $\psi(z,t)$ of (4.1) is

$$\begin{aligned} \frac{dV(\psi(z,t))}{dt} &= 2(\psi_1(x,y,t) - \psi_2(x,y,t))^T(f(\psi_1(x,y,t) + g(t,\psi_1(x,y,t),\psi_2(x,y,t)) \\ &+ \varepsilon h_1(t,\psi_1(x,y,t),\psi_2(x,y,t)) - f(\psi_2(x,y,t)) - g(t,y,\psi_1(x,y,t)) \\ &- \varepsilon h_2(t,\psi_1(x,y,t),\psi_2(x,y,t)) \\ &\leq -2\alpha|\psi_1(x,y,t) - \psi_2(x,y,t)|^2 + 4\varepsilon m_0|\psi_1(x,y,t) - \psi_2(x,y,t)| \\ &= -2\alpha V(\psi(z,t)) + 4\varepsilon m_0\sqrt{V(\psi(z,t))}. \end{aligned}$$

The solution of the initial value problem

$$\frac{d\bar{V}}{dt} = -2\alpha\bar{V} + 4\varepsilon m_0\sqrt{\bar{V}}, \quad \bar{V}(0) = V_0 > 0$$

is

$$\overline{V}(t) = \left[2\varepsilon\frac{m_0}{\alpha} + e^{-\alpha t}\left(-2\varepsilon\frac{m_0}{\alpha} + \sqrt{V_0}\right)\right]^2.$$

Hence, we have

$$V(\psi(z,t)) = |\psi_1(x,y,t) - \psi_2(x,y,t)|^2 \leq \left[2\varepsilon\frac{m_0}{\alpha} + e^{-\alpha t}\left(-2\varepsilon\frac{m_0}{\alpha} + |x-y|\right)\right]^2$$

for $t > 0$. This implies the validity of the relation (4.2). ■

From Theorem 4.1 we obtain

Corollary 4.2 *Under the hypotheses of Theorem* 4.1 *and under the additional assumption* $\alpha > 0$, *the solution* $\psi(z,t)$ *enters eventually a given small* δ*-neighborhood of the hyperplane* $\mathcal{P}$ *provided* ε *is sufficiently small.*

Under the conditions of Corollary 4.2 the slightly perturbed identical systems in (4.1) approach the synchronized state of the identical systems with an error characterized by $2\varepsilon m_0/\alpha$. Thus, Corollary 4.2 represents a robustness result for complete synchronization of symmetrically coupled identical systems.

5 Coupled systems with time delay

In this section we consider two identical differential-delay systems which are symmetrically coupled. More precisely, we investigate the system

$$\begin{aligned} \frac{dx(t)}{dt} &= f(x(t), x(t-1)) + g(t, x(t), x(t-1), y(t), y(t-1)), \\ \frac{dy(t)}{dt} &= f(y(t), y(t-1)) + g(t, y(t), y(t-1), x(t), x(t-1)) \end{aligned} \tag{5.1}$$

with $f : R^n \times R^n \to R^n$, $g : R^+ \times R^n \times R^n \times R^n \times R^n \to R^n$. Analogously to the case of ordinary differential equations, we assume that the function f satisfies a one-sided Lipschitz condition with respect to the first variable. With respect to the second variable f is assumed to obey a usual Lipschitz condition. Similar properties are assumed concerning the function g. Summarizing we suppose:

(A_9). There are constants $l_1, l_2, \beta_1, \beta_2, \gamma_1, \gamma_2$ such that for all $t \in R^+, x, y, x_i, y_i, \xi, \eta, \xi_i, \eta_i \in R^n$ the following inequalities hold

$$\begin{aligned} &(f(x_1,y) - f(x_2,y))^T(x_1 - x_2) \leq l_1|x_1 - x_2|^2, \\ &|f(x,y_1) - f(x,y_2)| \leq l_2|y_1 - y_2|, \\ &(g(t,x_1,y,\xi,\eta) - g(t,x_2,y,\xi,\eta))^T(x_1 - x_2) \leq \beta_1|x_1 - x_2|^2, \\ &|g(t,x,y_1,\xi,\eta) - g(t,x,y_2,\xi,\eta)| \leq \beta_2|y_1 - y_2|, \\ &(g(t,x,y,\xi_1,\eta) - g(t,x,y,\xi_2,\eta))^T(\xi_1 - \xi_2) \geq \gamma_1|\xi_1 - \xi_2|^2, \\ &|g(t,x,y,\xi,\eta_1) - g(t,x,y,\xi,\eta_2)| \leq \gamma_2|\eta_1 - \eta_2|, \end{aligned} \tag{5.2}$$

Theorem 5.1 *Suppose the hypothesis* (A_9) *to be satisfied. Let* $\mathcal{V}$ *be some subset of* $\mathcal{C} \times \mathcal{C}$ *such that for* $\phi \in \mathcal{V}$ (5.1) *has a unique solution* $\psi(\varphi)(t) = (\psi_1(\phi_1, \phi_2, t), \psi_2(\phi_1, \phi_2, t))$ *defined on* $[-1, \infty)$. *Then, under the additional condition*

$$l_1 + l_2 + \beta_1 + \beta_2 - \gamma_1 + \gamma_2 < 0$$

the following inequality holds

$$|\psi_1(\phi_1, \phi_2, t) - \psi_2(\phi_1, \phi_2, t)| \leq e^{-\alpha t}|\phi_1 - \phi_2|_C, \tag{5.3}$$

where $|\phi|_C = \sup_{-1 \leq t \leq 0} \phi(t)$, *and* $\alpha > 0$ *is uniquely determined from the equation*

$$2\alpha = \gamma_1 - \beta_1 - l_1 - (\gamma_2 + \beta_2 + l_2)e^{\alpha}. \tag{5.4}$$

Proof. As in the proof of Proposition 3.1, we use the function $V : R^n \times R^n \to R^+$ defined by $V(x, y) := |x - y|^2$. In order to simplify notation in the proof, we will write $\psi(\phi)(t) = \psi(t)$, since this will not create a misunderstanding in the context of the proof. The derivative of V with respect to a solution $\psi(\phi)(t)$ of (5.1) reads

$$\frac{1}{2}\frac{dV}{dt}(\psi((t)) = \frac{d}{dt}\left[(\psi_1(t) - \psi_2(t))^T(\psi_1(t) - \psi_2(t))\right]$$

$$\begin{aligned}
&= (\psi_1(t) - \psi_2(t))^T(f(\psi_1(t), \psi_1(t-1)) - f(\psi_2(t), \psi_2(t-1)) \\
&\quad + g(t, \psi_1(t), \psi_1(t-1), \psi_2(t), \psi_2(t-1)) \\
&\quad - g(t, \psi_2(t), \psi_2(t-1), \psi_1(t), \psi_1(t-1))) \\
&= (\psi_1(t) - \psi_2(t))^T[f(\psi_1(t), \psi_1(t-1)) - f(\psi_1(t), \psi_2(t-1)) \\
&\quad + f(\psi_1(t), \psi_2(t-1)) - f(\psi_2(t), \psi_2(t-1)) \\
&\quad + g(t, \psi_1(t), \psi_1(t-1), \psi_2(t), \psi_2(t-1)) \\
&\quad - g(t, \psi_1(t), \psi_2(t-1), \psi_2(t), \psi_2(t-1)) \\
&\quad + g(t, \psi_1(t), \psi_2(t-1), \psi_2(t), \psi_2(t-1)) \\
&\quad - g(t, \psi_2(t), \psi_2(t-1), \psi_2(t), \psi_2(t-1)) \\
&\quad + g(t, \psi_2(t), \psi_2(t-1), \psi_2(t), \psi_2(t-1)) \\
&\quad - g(t, \psi_2(t), \psi_2(t-1), \psi_1(t), \psi_2(t-1)) \\
&\quad + g(t, \psi_2(t), \psi_2(t-1), \psi_1(t), \psi_2(t-1)) \\
&\quad - g(t, \psi_2(t), \psi_2(t-1), \psi_1(t), \psi_1(t-1))].
\end{aligned}$$

Now using (A_9) we obtain

$$\frac{1}{2}\frac{dV}{dt} \leq l_2|\psi_1(t) - \psi_2(t)| \cdot |\psi_1(t-1) - \psi_2(t-1)|$$

$$+ l_1|\psi_1(t) - \psi_2(t)|^2 + \beta_2|\psi_1(t) - \psi_2(t)| \cdot |\psi_1(t-1) - \psi_2(t-1)|$$

$$
\begin{aligned}
&+\beta_1|\psi_1(t)-\psi_2(t)|^2-\gamma_1|\psi_1(t)-\psi_2(t)|^2\\
&+\gamma_2|\psi_1(t)-\psi_2(t)|\cdot|\psi_1(t-1)-\psi_2(t-1)|\\
&\le (l_1+\beta_1-\gamma_1)|\psi_1(t)-\psi_2(t)|^2\\
&+(l_2+\beta_2+\gamma_2)|\psi_1(t)-\psi_2(t)|\cdot|\psi_1(t-1)-\psi_2(t-1)|\\
&\le (l_1+\beta_1-\gamma_1)|\psi_1(t)-\psi_2(t)|^2\\
&+(l_2+\beta_2+\gamma_2)|\psi_1(t)-\psi_2(t)|\sup_{t-1\le\sigma\le t}|\psi_1(\sigma)-\psi_2(\sigma)|.
\end{aligned}
$$

Hence, we have shown that the function $W(t)=V(\psi(t))$ satisfies the following inequality

$$
\frac{dW(t)}{dt}\le\frac{1}{2}(l_1+\beta_1-\gamma_1)W(t)+\frac{1}{2}(l_2+\beta_2+\gamma_2)\sup_{t-1\le\sigma\le t}W(\sigma),\quad t\ge 0. \tag{5.5}
$$

Applying Halanay's inequality [16, 17] we get

$$
W(t)\le e^{-\alpha t}\sup_{-1\le\sigma\le 0}W(\sigma),\quad t\ge 0,
$$

where α is determined as in (5.4). This implies (5.3). ■

Remark 5.2 *Theorem* 5.1 *can be proved in the same manner for DDE systems with non-constant bounded delay* $\tau(t)$*. Our choice of fixed delay was made only to simplify notations.*

6 Application

We illustrate our results by means of a model describing the feedback control mechanism for the production of an enzyme (see [18], 145 ff.). It represents a slight generalization of a model proposed by Goodwin (Goodwin oscillator, see [19])

$$
\begin{aligned}
\frac{dx_1}{dt}&=\frac{1}{1+x_3^m}-x_1,\\
\frac{dx_2}{dt}&=x_1-x_2,\\
\frac{dx_3}{dt}&=x_2-0.5x_3.
\end{aligned} \tag{6.1}
$$

Here, x_1, x_1, x_3 represent the concentrations of the mRNA, the enzyme and the product, respectively, m is the Hill coefficient. It is known (see, e.g., [18]) that for $m\ge 8$ system (6.1) has a stable limit cycle Γ as a global attractor, otherwise it has a stable equilibrium point as a global attractor. It is easy to verify that the parallelepiped

$$
\mathcal{G}:=\{0\le x_1\le 1,\ 0\le x_2\le 1,\ 0\le x_3\le 2\} \tag{6.2}
$$

is a positively invariant set for (6.1). For the following we set $m = 20$. The symmetric matrix $M(x)$ introduced in (3.18) has the form

$$M(x) = \begin{pmatrix} -2 & 1 & -\dfrac{20x_3^{19}}{(1+x_3)^{20}} \\ 1 & -2 & 1 \\ -\dfrac{20x_3^{19}}{(1+x_3)^{20}} & 1 & -1 \end{pmatrix}$$

The maximal eigenvalue $\bar{\kappa}$ of $M(x)$ in $\mathcal{G}$ can be estimated by $\bar{\kappa} \leq 6$. Thus, applying Corollary 3.11, we get that two linearly diffusively coupled Goodwin oscillators completely synchronize for $k > 1.5$ and that the limit cycle Γ located in the plane $x = y$ attracts all points from the set $\mathcal{G} \times \mathcal{G}$ of the phase space. Figure 1 illustrates the limit cycle Γ located in the invariant manifold $\mathcal{P}$ and Figure 2 shows how the "synchronization error" $|\psi_1(z,t) - \psi_2(z,t)|$ tends to zero with the increasing of time for $k = 2.0$.

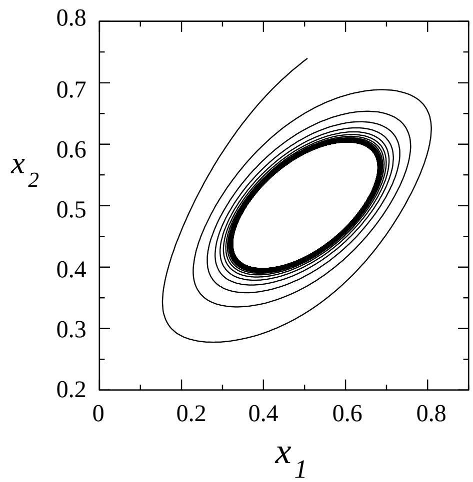

Figure 1: An orbit approaching asymptotically stable limit cycle of system (6.1)

In order to illustrate our result on the robustness of complete synchronization, we consider two non-identical Goodwin oscillators which are linearly diffusively coupled, that is, we consider the system

$$\frac{dx}{dt} = f(x) + \varepsilon g(t,x) + k(y-x), \quad \frac{dy}{dt} = f(y) + k(x-y), \tag{6.3}$$

where $x, y \in R^3$, f is determined by the right hand side of (6.1), $g(t,x) = (x_2 \sin t, 0, 0)^T$, k is the coupling constant, and ε is a perturbation parameter. For $k = 2$ we have

$$m_0 := \max_{t \in R, x \in \mathcal{G}} |g(t,x)| \leq 1, l \leq \bar{\kappa}/2 = 3, \gamma_0 = 2, \beta_0 = -2, \alpha = \gamma_0 - \beta_0 - l = 1$$

such that from Theorem 4.1 we obtain the following estimate for the synchronization error

$$\Delta(t) := |x(t, x_0, y_0) - y(t, x_0, y_0)| \leq 2\varepsilon + e^{-\alpha t}\big(-2\varepsilon + |x - y|\big).$$

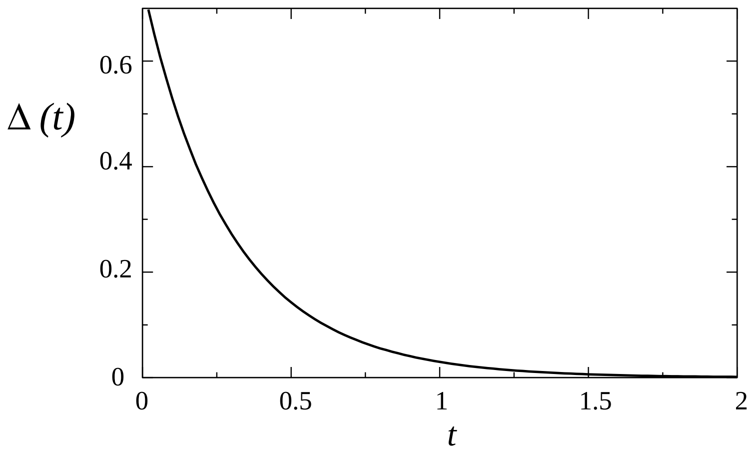

Figure 2: Behavior of the "synchronization error" $\Delta(t) := |\psi_1(x, y, t) - \psi_2(x, y, t)|$.

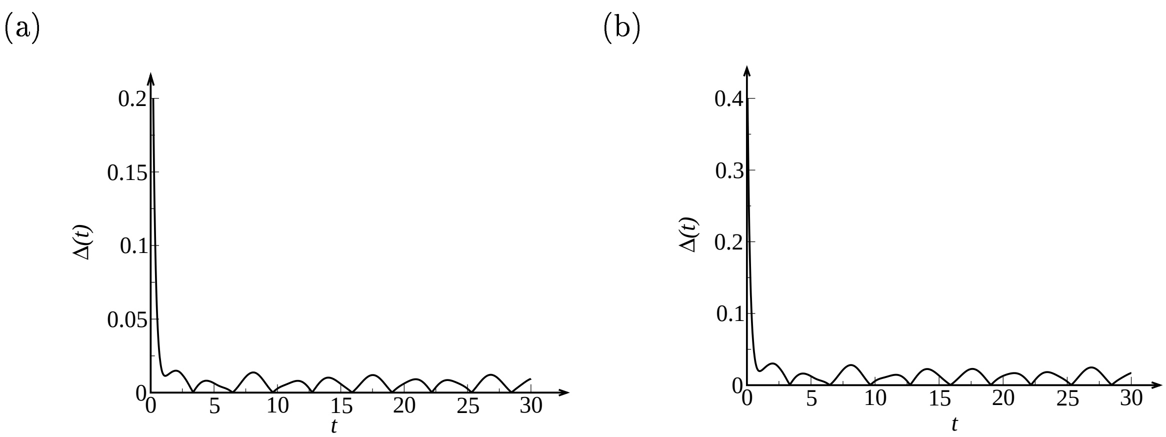

Figure 3: Behavior of the synchronization error $|\psi_1(x, y, t) - \psi_2(x, y, t)|$ for two nonidentical Goodwin oscillators (6.3) with $\varepsilon = 0.1$ (a) and $\varepsilon = 0.2$ (b).

Numerical calculations for two different values of the perturbation parameter are shown in Fig. 3 (note the different scales of the vertical axis).

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