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On Biot-like Models and Micro-Macrotransitions for Poroelastic Materials

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Abstract

The paper is devoted to the thermodynamic derivation of a two-component poroelastic model with balance equation of porosity as a prototype of Biot's model. It is shown that a constitutive dependence on the porosity gradient yields the possibility of the construction of the linear Biot's model of poroelastic materials provided one negelcts the relaxation of porosity. A procedure of micro-macrotransition for a homogeneous microstructure yields Gassmann relations as an approximation of the thermodynamic model. Simultaneously it is shown that the model with the porosity balance equation can be applied to porous materials with a rather rigid skeleton. In the case of soft granular materials there exists a correction to porosity changes which follows from an appropriate modification of the porosity source.

1 Introduction

This work is a continuation of my research on the status of Biot's model of linear poroelastic materials. As in the previous works (e.g. [28] where the other references are quoted) we begin with a fully nonlinear formulation of a model which describes isothermal processes in a two-component fully saturated poroelastic material. Changes of porosity are assumed to be described by the porosity balance equation.

The aim is to show that the linear Biot's model follows as a special case of such a general thermodynamic model. This question has been already risen and discussed by R. M. Bowen [13] who introduced a set of evolution equations for the so-called volume fractions describing the microstructure of porous media. In addition to thermodynamic considerations we check if a micro-macrotransition procedure supports general considerations. We aim to show not only that under well defined very restrictive conditions the classical Gassmann relations follow but we check as well the range of compatibility of both models and the limitations needed to apply the procedure of micro-macrotransition. This concerns particularly the assumption of homogeneity of the microstructure.

We expose particularly the role of microvariables and macrovariables. This type of an approach to the derivation of Gassmann relations has been indicated in the literature (e.g. [12]) but it seems to be not systematically investigated. In the case of extensions of the classical microstructure to incorporate, for instance, mass exchange between components, such a clear formulation is very adventageous.

We do not present here any review of a rather extensive literature on this subject. This can be found in numerous papers of Berryman (e.g. [17],[24]), Mavko (e.g. [19]), etc.

In section 2.1. we present a set of general thermodynamic admissibility conditions for a thermodynamic model of a poroelastic material with the equation for porosity. In section 2.2. we specify the form of these conditions under two additional assumptions: isotropy and linear deviation from the thermodynamic equilibrium.

Section 3 is devoted to the linear model of poroelastic materials. We compare equations of the thermodynamic model with the classical Biot's model.

In sections 4.1.-3. we present the procedure of micro-macrotransition for compressibility properties of both models. We discuss in section 4.4. some limitations on the porosity balance equation of the thermodynamic model which are related to this transition. Sections 4.5.-6. are devoted to the discussion of three Gedankenexperiments initiated by the work of Biot and Willis [7].

Considerations of this work are supposed to prepare the extension of the classical models on the class of unsaturated porous materials.

Generalizations of Biot's model and corresponding Gassmann relations to cover viscous effects are usually done by means of an assumption similar to this which is characteristic for classical linear visco-elasticity. It is well-known that a class of linear visco-elastic models can be covered by the so-called Alfrey-Lee elastic-viscoelastic analogy [4], [6]. This analogy relies on the fact that for some linear visco-elastic operators the Laplace transformed equations of motion coincide formally with these of classical elasticity. Solely the elastic parameters are not constant in such a case but they depend on the frequency. This type of an approach seems to be working in many cases of practical importance (e.g. [15], [26]) but it has, of course, very little to do with the modern thermodynamic way of constructing models. However, we skip this extension of poroelsticity in this work and consider only the class of classical inviscid models.

2 Thermodynamic foundations

2.1 Thermodynamic admissibility

As indicated in the Introduction we construct a continuum model for poroelastic two-component media undergoing isothermal processes and described by the following fields

$$\mathcal{F} = \left\{ \rho^F, \acute{\mathbf{x}}^F, \acute{\mathbf{x}}^S, \mathbf{F}^S, n \right\}, \quad \mathcal{B}_0 \times \mathcal{T} \rightarrow \mathcal{F}, \tag{1}$$

where $\mathcal{B}_0$ is a set of material points of the skeleton in its reference configuration, $\mathcal{T}$ is the time domain, ρ^F is the partial mass density of the fluid in the reference configuration, $\acute{\mathbf{x}}^F, \acute{\mathbf{x}}^S$ are the velocity fields of the fluid and of the skeleton, respectively, $\mathbf{F}^S$ denotes the deformation gradient of the skeleton with the property $\mathbf{F}^S = \mathbf{1}$ in the reference configuration, and n is the porosity. The latter relates the partial mass densities and the so-called true mass densities

$$\rho_t^F := \rho^F J^{S-1} = n\rho^{FR}, \quad \rho_t^S := \rho^S J^{S-1} = (1-n)\,\rho^{SR}, \quad J^S := \det \mathbf{F}^S, \qquad (2)$$

where ρ_t^F, ρ_t^S are current partial mass densities of the fluid and of the skeleton, respectively, ρ^{FR}, ρ^{SR} are current true mass densities of the fluid and of the skeleton. As there is no mass exchange between components the partial mass density of the skeleton in the reference configuration ρ^S is constant for homogeneous materials.

The above fields must fulfil balance laws of mass, momentum and porosity as well as the compatibility condition for deformations of the skeleton. They have the following form in the Lagrangian description when referred to the reference configuration of the skeleton

$$\begin{gathered}
\mathcal{E}^\rho := \frac{\partial \rho^F}{\partial t} + \operatorname{Div}\left(\rho^F \acute{\mathbf{X}}^F\right) = 0, \quad \acute{\mathbf{X}}^F := \mathbf{F}^{S-1}\left(\acute{\mathbf{x}}^F - \acute{\mathbf{x}}^S\right), \\
\mathcal{E}^{v^S} := \rho^S \frac{\partial \acute{\mathbf{x}}^S}{\partial t} - \left[\operatorname{Div}\mathbf{P}^S + \hat{\mathbf{p}}\right] = 0, \\
\mathcal{E}^{v^F} := \rho^F \frac{\partial \acute{\mathbf{x}}^F}{\partial t} + \operatorname{Div}\left(\rho^F \acute{\mathbf{x}}^F \otimes \acute{\mathbf{X}}^F\right) - \left[\operatorname{Div}\mathbf{P}^S - \hat{\mathbf{p}}\right] = 0, \qquad (3) \\
\\
\mathcal{E}^n := \frac{\partial n}{\partial t} + \operatorname{Div}\mathbf{J} - \hat{n} = 0, \\
\mathcal{E}^F := \frac{\partial \mathbf{F}^S}{\partial t} - \operatorname{Grad}\acute{\mathbf{x}}^S = 0,
\end{gathered}$$

where $\mathbf{P}^F, \mathbf{P}^S$ are Piola-Kirchhoff partial stress tensors for the fluid and skeleton, respectively, $\mathbf{J}$ is the flux of porosity, $\hat{\mathbf{p}}$ denotes the momentum source and $\hat{n}$ is the source of porosity. In addition the following symmetry relation must be satisfied

$$\operatorname{Grad}\mathbf{F}^S = \left(\operatorname{Grad}\mathbf{F}^S\right)^{\overset{23}{T}}. \qquad (4)$$

We construct a model for poroelastic materials and assume that the constitutive quantities

$$\mathcal{C} := \left\{\mathbf{P}^F, \mathbf{P}^S, \mathbf{J}, \hat{\mathbf{p}}, \hat{n}\right\}, \qquad (5)$$

are sufficiently smooth functions of the following constitutive variables

$$\mathcal{V} := \left\{\rho^F, \acute{\mathbf{X}}^F, \mathbf{F}^S, n, \mathbf{N}\right\}, \quad \mathcal{C} = \mathcal{C}(\mathcal{V}), \quad \mathbf{N} := \operatorname{Grad} n. \qquad (6)$$

Note that in addition to standard constitutive variables of a two-component immiscible mixture we introduce a dependence on the porosity gradient $\mathbf{N}$. It has been shown [28] that the simple model without higher gradients as constitutive variables does not contain couplings required by the Biot's model. However, in contrast to the above quoted paper, we do not make the assumption that the porosity n is described by an equilibrium relation. We use rather the full balance law for porosity.

Substitution of constitutive relations (6) in the balance equations (3) yields the set of field equations for the fields (1). Their solutions are called thermodynamic processes in poroelastic saturated materials under isothermal conditions.

Thermodynamic admissibility of constitutive relations (6) requires that thermodynamic processes satisfy identically the following inequality (e.g. [28])

$$\mathcal{E}^{\psi} := \rho^S \frac{\partial \psi^S}{\partial t} + \rho^F \left(\frac{\partial \psi^F}{\partial t} + \acute{\mathbf{X}}^F \cdot \operatorname{Grad} \psi^F \right) -$$

$$-\mathbf{P}^S \cdot \frac{\partial \mathbf{F}^S}{\partial t} - \mathbf{P}^F \cdot \operatorname{Grad} \acute{\mathbf{x}}^F - \left(\mathbf{F}^{ST} \hat{\mathbf{p}} \right) \cdot \acute{\mathbf{X}}^F \leq 0$$

$$\psi^S = \psi^S \left(\mathcal{V} \right), \quad \psi^F = \psi^F \left(\mathcal{V} \right), \tag{7}$$

where ψ^S, ψ^F denote the partial Helmholtz free energies.

The exploitation of the above condition is standard and can be made by use of the so-called Lagrange multipliers. In this case we replace the above condition by the following one. The inequality

$$\mathcal{E}^{\psi} - \Lambda^{\rho} \mathcal{E}^{\rho} - \mathbf{\Lambda}^{v^F} \mathcal{E}^{v^F} - \mathbf{\Lambda}^{v^S} \mathcal{E}^{v^S} - \Lambda^n \mathcal{E}^n - \mathbf{\Lambda}^F \mathcal{E}^F \leq 0, \tag{8}$$

$$\begin{aligned} \Lambda^{\rho} &= \Lambda^{\rho} \left(\mathcal{V} \right), \quad \mathbf{\Lambda}^{v^F} = \mathbf{\Lambda}^{v^F} \left(\mathcal{V} \right), \quad \mathbf{\Lambda}^{v^S} = \mathbf{\Lambda}^{v^S} \left(\mathcal{V} \right), \\ \Lambda^n &= \Lambda^n \left(\mathcal{V} \right), \quad \mathbf{\Lambda}^F = \mathbf{\Lambda}^F \left(\mathcal{V} \right), \end{aligned}$$

must be satisfied for all fields (1) and not only for thermodynamic processes.

Substitution of constitutive relations yields the linearity of the above inequality with respect to the following derivatives

$$\left\{ \frac{\partial \rho^F}{\partial t}, \frac{\partial \mathbf{F}^S}{\partial t}, \frac{\partial \acute{\mathbf{x}}^S}{\partial t}, \frac{\partial \acute{\mathbf{x}}^F}{\partial t}, \frac{\partial n}{\partial t}, \frac{\partial \mathbf{N}}{\partial t} \right\},$$

$$\left\{ \operatorname{Grad} \rho^F, \operatorname{Grad} \mathbf{F}^S, \operatorname{Grad} \acute{\mathbf{x}}^S, \operatorname{Grad} \acute{\mathbf{x}}^F, \operatorname{Grad} \mathbf{N}, \right\} \tag{9}$$

and there remains a nonlinear part which contains sources $\hat{\mathbf{p}}, \hat{n}$ and the gradient $\mathbf{N}$.

Elementary manipulations yield the following form of those relations

1. relations for the multipliers

$$\Lambda^{\rho} = \frac{\partial}{\partial \rho^F} \left[\rho^S \psi^S + \rho^F \psi^F \right] - \psi^F, \quad \Lambda^n = \frac{\partial}{\partial n} \left[\rho^S \psi^S + \rho^F \psi^F \right],$$

$$\rho^S \mathbf{\Lambda}^{v^S} = -\mathbf{F}^{ST} \frac{\partial}{\partial \acute{\mathbf{X}}^F} \left[\rho^S \psi^S + \rho^F \psi^F \right] = -\rho^F \mathbf{\Lambda}^{v^F},$$

$$\begin{aligned}\boldsymbol{\Lambda}^F \;=\;& -\rho^F\Lambda^\rho\mathbf{F}^{S-T} - \Lambda^n\mathbf{F}^{S-T}\left(\frac{\partial\mathbf{J}}{\partial\acute{\mathbf{X}}^F}\right)^T + \rho^F\mathbf{F}^{S-T}\frac{\partial\psi^F}{\partial\acute{\mathbf{X}}^F}\otimes\acute{\mathbf{X}}^F + \\ &+\mathbf{F}^{S-T}\left(\frac{\partial\mathbf{P}^S}{\partial\acute{\mathbf{X}}^F}\right)^{\overset{13}{T}}\boldsymbol{\Lambda}^{v^S} + \mathbf{F}^{S-T}\left(\frac{\partial\mathbf{P}^F}{\partial\acute{\mathbf{X}}^F}\right)^{\overset{13}{T}}\boldsymbol{\Lambda}^{v^F},\end{aligned} \tag{10}$$

2. constitutive relations for partial stresses

$$\begin{aligned}\mathbf{P}^S + \boldsymbol{\Lambda}^F &= \frac{\partial}{\partial\mathbf{F}^S}\left[\rho^S\psi^S + \rho^F\psi^F\right], \\ \mathbf{P}^F &= \boldsymbol{\Lambda}^F - \rho^F\boldsymbol{\Lambda}^{v^F}\otimes\acute{\mathbf{X}}^F,\end{aligned} \tag{11}$$

3. identities restricting the flux of porosity

$$\Lambda^n\frac{\partial\mathbf{J}}{\partial\rho^F} = -\rho^S\frac{\partial\psi^S}{\partial\rho^F}\acute{\mathbf{X}}^F + \left(\frac{\partial\mathbf{P}^S}{\partial\rho^F}\right)^T\boldsymbol{\Lambda}^{v^S} + \left(\frac{\partial\mathbf{P}^F}{\partial\rho^F}\right)^T\boldsymbol{\Lambda}^{v^F},$$

$$\begin{aligned}\operatorname{sym}_{13}\left\{\Lambda^n\frac{\partial\mathbf{J}}{\partial\mathbf{F}^S}\right\} \;=\;& \operatorname{sym}_{13}\left\{\rho^F\acute{\mathbf{X}}^F\otimes\left(\Lambda^\rho\mathbf{F}^{S-T} + \frac{\partial\psi^F}{\partial\mathbf{F}^S}\right) + \right. \\ &\left. + \left[\left(\frac{\partial\mathbf{P}^S}{\partial\mathbf{F}^S}\right)^{\overset{14}{T}}\boldsymbol{\Lambda}^{v^S} + \left(\frac{\partial\mathbf{P}^F}{\partial\mathbf{F}^S}\right)^{\overset{14}{T}}\boldsymbol{\Lambda}^{v^F}\right]^{\overset{23}{T}}\right\},\end{aligned} \tag{12}$$

$$\operatorname{sym}\left\{\Lambda^n\frac{\partial\mathbf{J}}{\partial\mathbf{N}}\right\} = \operatorname{sym}\left\{\rho^F\acute{\mathbf{X}}^F\otimes\frac{\partial\psi^F}{\partial\mathbf{N}} + \left(\frac{\partial\mathbf{P}^S}{\partial\mathbf{N}}\right)^{\overset{13}{T}}\boldsymbol{\Lambda}^{v^S} + \left(\frac{\partial\mathbf{P}^F}{\partial\mathbf{N}}\right)^{\overset{13}{T}}\boldsymbol{\Lambda}^{v^F}\right\}.$$

There remains the so-called residual inequality which describes the density of dissipation in the system

$$\begin{aligned}&\left\{\rho^F\frac{\partial\psi^F}{\partial n}\acute{\mathbf{X}}^F - \Lambda^n\frac{\partial\mathbf{J}}{\partial n} + \frac{\partial\mathbf{P}^{ST}}{\partial n}\boldsymbol{\Lambda}^{v^S} + \frac{\partial\mathbf{P}^{FT}}{\partial n}\boldsymbol{\Lambda}^{v^F}\right\}\cdot\mathbf{N} - \\ &-\left\{\mathbf{F}^{ST}\acute{\mathbf{X}}^F - \Lambda^{v^S} + \Lambda^{v^F}\right\}\cdot\hat{\mathbf{p}} + \Lambda^n\hat{n} \leq 0.\end{aligned} \tag{13}$$

Probably one cannot proceed any further with the thermodynamic considerations in the most general case. General results are quoted here to indicate possibilities and difficulties if one would like to include the effect of tortuosity in the model. As indicated by O. Coussy [18] such effects must be related to the dependence of free energies ψ^S, ψ^F on the relative velocity $\acute{\mathbf{X}}^F$.

If we neglect this effect the above relations simplify considerably. In this work we consider solely such a case and therefore we make the assumption

$$\frac{\partial\psi^S}{\partial\acute{\mathbf{X}}^F} = \frac{\partial\psi^F}{\partial\acute{\mathbf{X}}^F} = 0. \tag{14}$$

Under this assumption we obtain for multipliers

$$\mathbf{\Lambda}^{v^S} = \mathbf{\Lambda}^{v^F} = 0, \quad \mathbf{\Lambda}^F = \mathbf{P}^F, \tag{15}$$

and the remaining relations are as follows

$$\begin{gathered}
\mathbf{P}^S + \mathbf{P}^F = \frac{\partial}{\partial \mathbf{F}^S}\left[\rho^S \psi^S + \rho^F \psi^F\right], \quad \mathbf{P}^F = -\rho^F \Lambda^\rho \mathbf{F}^{S-T} - \Lambda^n \mathbf{F}^{S-T}\left(\frac{\partial \mathbf{J}}{\partial \acute{\mathbf{X}}^F}\right)^T, \\
\Lambda^n \frac{\partial \mathbf{J}}{\partial \rho^F} = -\acute{\mathbf{X}}^F \rho^S \frac{\partial \psi^S}{\partial \rho^F}, \quad \Lambda^n \frac{\partial \mathbf{J}}{\partial \mathbf{N}} = \operatorname{sym}\left\{\rho^F \acute{\mathbf{X}}^F \otimes \frac{\partial \psi^F}{\partial \mathbf{N}}\right\}. \\
\Lambda^n \frac{\partial \mathbf{J}}{\partial \mathbf{F}^S} = \operatorname{sym}_{13}\left\{\rho^F \acute{\mathbf{X}}^F \otimes \left[\Lambda^\rho \mathbf{F}^{S-T} + \frac{\partial \psi^F}{\partial \mathbf{F}^S}\right]\right\}.
\end{gathered} \tag{16}$$

In addition we have the following residual inequality

$$\mathcal{D} := -\mathbf{N} \cdot \left\{\rho^F \frac{\partial \psi^F}{\partial n} \acute{\mathbf{X}}^F - \Lambda^n \frac{\partial \mathbf{J}}{\partial n}\right\} + \hat{\mathbf{p}} \cdot \mathbf{F}^S \acute{\mathbf{X}}^F - \Lambda^n \hat{n} \geq 0. \tag{17}$$

The quantity $\mathcal{D}$ describes the dissipation and it vanishes in the thermodynamic equilibrium. Such a state is defined by two conditions $\acute{\mathbf{X}}^F = 0, \hat{n} = 0$, the first one referring to the lack of diffusion and the second one to the lack of relaxation of the microstructure. We prove this statement in the sequel.

The most important conclusion of this inequality follows from the analysis of the last contribution. We expect the source of porosity $\hat{n}$ to describe a spontaneous relaxation of porosity to its equilibrium value n_E which is in turn the solution of the porosity equation in the thermodynamic equilibrium. Consequently for smooth constitutive relations and small deviations from the thermodynamic equilibrium the source $\hat{n}$ must begin with a contribution which is linear in the deviation $\Delta_n = n - n_E$, i.e. $\hat{n} = O\left(|\Delta_n|\right)$. However, this means that the dissipation inequality can be satisfied if the multiplier Λ^n is of the same order, i.e. $\Lambda^n = O\left(|\Delta_n|\right)$. Then the contribution due to the relaxation is quadratic and we satisfy the above residual inequality by an appropriate choice of the sign of the constitutive coefficient (the characteristic relaxation time of porosity). Consequently the definition of this multiplier indicates the following relation for the free energies

$$\rho^S \psi^S + \rho^F \psi^F = \rho^S \psi_0^S + \rho^F \psi_0^F + O\left(\left|\Delta_n^2\right|\right), \tag{18}$$

where $\rho^S \psi_0^S + \rho^F \psi_0^F$ is independent of Δ_n. This indicates a very peculiar structure of the constitutive dependence on the nonequilibrium porosity if we want to preserve a coupling between the free energy of the skeleton ψ^S and changes of the mass density of the skeleton ρ^F. As required by Biot's model we would have to have

$$\frac{\partial \psi^S}{\partial \rho^F} = O\left(|\Delta_n|\right), \tag{19}$$

which immediately follows from the relation $(16)_3$. This would mean that this coupling would have to vanish in the termodynamic equilibrium which is, of course, not the case in Biot's model. We conclude that Biot's model may follow within the above desribed thermodynamic scheme solely if we neglect entirely the relaxation of porosity in the porosity balance equation, i.e. the porosity is required in such a model to satisfy the equation

$$\frac{\partial n}{\partial t} + \operatorname{Div}\mathbf{J} = 0. \tag{20}$$

Otherwise the model contains only dynamical couplings related to changes of porosity. These were considered in a series of papers (e.g. [20, 22, 23]). As the main purpose of this work is to discuss Biot-like models we shall rely on the porosity equation (20) and neglect relaxation effects. A particular case of such an approach was analyzed in [28].

2.2 An isotropic model

We simplify the model by the assumptions on linearity with respect to vectorial variables $\acute{\mathbf{X}}^F, \mathbf{N}$ and on the isotropy. Let us note that these assumptions indicate the assumption (14) because the dependence on $\acute{\mathbf{X}}^F$ of scalar isotropic functions must be at least quadratic. Under these assumptions we obtain immediately

$$\begin{aligned}
\psi^S &= \psi^S\left(\rho^F, I, II, III, n\right), \quad \psi^F = \psi^F\left(\rho^F, I, II, III, n\right), \\
\mathbf{J} &= \Phi\acute{\mathbf{X}}^F + \Xi\mathbf{N}, \\
\Phi &= \Phi\left(\rho^F, I, II, III, n\right), \quad \Xi = \Xi\left(\rho^F, I, II, III, n\right), \\
\hat{\mathbf{p}} &= \pi\mathbf{F}^S\acute{\mathbf{X}}^F - \mathcal{N}\mathbf{F}^{S-1}\mathbf{N}, \\
\pi &= \pi\left(\rho^F, I, II, III, n\right), \quad \mathcal{N} = \mathcal{N}\left(\rho^F, I, II, III, n\right),
\end{aligned} \tag{21}$$

where

$$I := tr\mathbf{C}^S, \quad II := \frac{1}{2}\left(I^2 - tr\mathbf{C}^{S2}\right), \quad III := J^{S2} = \det\mathbf{C}^S, \quad \mathbf{C}^S := \mathbf{F}^{ST}\mathbf{F}^S,$$

are principal invariants of the Cauchy-Green deformation tensor of the skeleton $\mathbf{C}^S$. Bearing $(16)_4$ in mind we obtain immediately

$$\Xi = 0. \tag{22}$$

Constitutive relations can be now written in the form[1]

$$\mathbf{P}^S = 2\rho^S\left[\left(\frac{\partial\psi^S}{\partial I} + I\frac{\partial\psi^S}{\partial II}\right)\mathbf{B}^S - \frac{\partial\psi^S}{\partial II}\mathbf{B}^{S2} + III\frac{\partial\psi^S}{\partial III}\mathbf{1}\right]\mathbf{F}^{S-T} +$$

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[1] We use here the following relation

$$\frac{\partial\psi^S}{\partial\mathbf{F}^S} = 2\mathbf{F}^S\frac{\partial\psi^S}{\partial\mathbf{C}^S} = 2\left\{\frac{\partial\psi^S}{\partial I}\mathbf{F}^S + \frac{\partial\psi^S}{\partial II}\left(I\mathbf{F}^S - \mathbf{F}^S\mathbf{C}^S\right) + III\frac{\partial\psi^S}{\partial III}\mathbf{F}^{S-T}\right\},$$

and similarly for other scalars.

$$+2\Lambda^n\left[\left(\frac{\partial\Phi}{\partial I}+I\frac{\partial\Phi}{\partial II}\right)\mathbf{B}^S-\frac{\partial\Phi}{\partial II}\mathbf{B}^{S2}+\left(\frac{1}{2}\Phi+III\frac{\partial\Phi}{\partial III}\right)\mathbf{1}\right]\mathbf{F}^{S-T},\quad(23)$$

$$\mathbf{P}^F=-\left(\rho^F\Lambda^\rho+\Lambda^n\Phi\right)\mathbf{F}^{S-T},\quad \mathbf{B}^S:=\mathbf{F}^S\mathbf{F}^{ST},$$

$$\mathbf{J}=\Phi\acute{\mathbf{X}}^F,\quad \hat{\mathbf{p}}=\pi\mathbf{F}^S\acute{\mathbf{X}}^F-\mathcal{N}\mathbf{F}^{S-T}\mathbf{N},$$

and the following identities must be satisfied

$$\begin{aligned}\Lambda^n\frac{\partial\Phi}{\partial\rho^F} &= -\rho^S\frac{\partial\psi^S}{\partial\rho^F},\quad \Lambda^n\frac{\partial\Phi}{\partial I}=\rho^F\frac{\partial\psi^F}{\partial I},\\ \Lambda^n\frac{\partial\Phi}{\partial II} &= \rho^F\frac{\partial\psi^F}{\partial II},\quad 2\Lambda^n III\frac{\partial\Phi}{\partial III}=\rho^F\left(\Lambda^\rho+2III\frac{\partial\psi^F}{\partial III}\right).\end{aligned}\tag{24}$$

In addition we have the residual inequality

$$-\mathbf{N}\cdot\acute{\mathbf{X}}^F\left(\rho^F\frac{\partial\psi^F}{\partial n}-\Lambda^n\frac{\partial\Phi}{\partial n}+\mathcal{N}\right)+\pi\left(\mathbf{F}^S\acute{\mathbf{X}}^F\right)\cdot\left(\mathbf{F}^S\acute{\mathbf{X}}^F\right)\geq 0,$$

which yields due to the linearity with respect to $\mathbf{N}$

$$\mathcal{N}=\Lambda^n\frac{\partial\Phi}{\partial n}-\rho^F\frac{\partial\psi^F}{\partial n},\quad \pi\geq 0.\tag{25}$$

Bearing the above identities in mind we can write the constitutive relations for partial Cauchy stresses

$$\begin{aligned}\mathbf{T}^S:= &\quad J^{S-1}\mathbf{P}^S\mathbf{F}^{ST}=\\ = &\quad 2\rho_t^S\left[\left(\frac{\partial\psi^S}{\partial I}+I\frac{\partial\psi^S}{\partial II}\right)\mathbf{B}^S-\frac{\partial\psi^S}{\partial II}\mathbf{B}^{S2}+III\frac{\partial\psi^S}{\partial III}\mathbf{1}\right]+\\ &\quad +2\rho_t^F\left[\left(\frac{\partial\psi^F}{\partial I}+I\frac{\partial\psi^F}{\partial II}\right)\mathbf{B}^S-\frac{\partial\psi^F}{\partial II}\mathbf{B}^{S2}\right]+\\ &\quad +\left(J^{S-1}\Lambda^n\right)\left(\Phi+2III\frac{\partial\Phi}{\partial III}\right)\mathbf{1},\end{aligned}\tag{26}$$

$$\begin{aligned}\mathbf{T}^F:= &\quad J^{S-1}\mathbf{P}^F\mathbf{F}^{ST}=\\ = &\quad -\rho^{F2}J^{S-1}\frac{\partial\psi^F}{\partial\rho^F}\mathbf{1}-\left(J^{S-1}\Lambda^n\right)\left(\Phi-\rho^F\frac{\partial\Phi}{\partial\rho^F}\right)\mathbf{1}.\end{aligned}$$

In these relations we have exposed the role of the flux of porosity Φ. We use it in order to introduce a coupling between stresses required by the linear Biot's model. We proceed under the following assumptions motivated by this condition. First of all we assume that the partial free energy for the fluid ψ^F is independent of the first two invariants I, II of deformation of the skeleton. According to relations $(24)_{2,3}$ we obtain under this assumption

$$\psi^F=\psi^F\left(J^S,\rho^F,n\right),\quad \Phi=\Phi\left(J^S,\rho^F,n\right).\tag{27}$$

It means that these two functions do not react on macroscopic shape changes (shearing) of the skeleton. In addition we assume

$$\Phi + 2III\frac{\partial\Phi}{\partial III} = 0 \quad \Rightarrow \quad \Phi = \Phi_0\left(\rho^F, n\right) J^{S-1}. \tag{28}$$

Then

$$\Lambda^n\left(\Phi - \rho^F\frac{\partial\Phi}{\partial\rho^F}\right) = -\Lambda^n\left(\rho^F\frac{\partial\Phi}{\partial\rho^F} + J^S\frac{\partial\Phi}{\partial J^S}\right) = -\rho^F\left(\rho^F\frac{\partial\psi^F}{\partial\rho^F} + J^S\frac{\partial\psi^F}{\partial J^S}\right). \tag{29}$$

It is convenient to change the variables of Lagrangian description to the set of Eulerian constitutive variables: $\left\{\rho_t^F, \mathbf{B}^S, n\right\}$ with I, II, J^{S2} being also the invariants of $\mathbf{B}^S$, i.e.

$$\psi^S = \check{\psi}^S\left(I, II, J^S, \rho_t^F, n\right), \quad \psi^F = \check{\psi}^F\left(J^S, \rho_t^F, n\right). \tag{30}$$

Then the relations for partial Cauchy stresses become

$$\mathbf{T}^S = 2\rho_t^S\left[\left(\frac{\partial\check{\psi}^S}{\partial I} + I\frac{\partial\check{\psi}^S}{\partial II}\right)\mathbf{B}^S - \frac{\partial\check{\psi}^S}{\partial II}\mathbf{B}^{S2}\right] + \rho_t^S\left(J^S\frac{\partial\check{\psi}^S}{\partial J^S} - \underline{\rho_t^F\frac{\partial\check{\psi}^S}{\partial\rho_t^F}}\right)\mathbf{1},$$

$$\mathbf{T}^F = -\rho_t^{F2}\frac{\partial\check{\psi}^F}{\partial\rho_t^F}\mathbf{1} + \rho_t^F\underline{J^S\frac{\partial\check{\psi}^F}{\partial J^S}}\mathbf{1}. \tag{31}$$

In order to obtain the required symmetry it is now sufficient to assume

$$\frac{\partial\left(\rho_t^F\check{\psi}^F\right)}{\partial J^S} = \frac{\partial\Gamma}{\partial J^S}, \quad \frac{\partial\left(\rho_t^S\check{\psi}^S\right)}{\partial\rho_t^F} = \frac{\partial\Gamma}{\partial\rho_t^F}, \quad \Gamma = \Gamma\left(J^S, \rho_t^F, n\right). \tag{32}$$

The second relation requires a splitting of ψ^S into a part dependent on I, II, J^S, n and another part dependent on J^S, ρ_t^F, n. This separation property shall not be used in this work.

Since the symmetry of interactions should yield equal contributions to both partial stresses we have (compare the underlined contributions in (31))

$$J^S\frac{\partial\Gamma}{\partial J^S} + \rho_t^F\frac{\partial\Gamma}{\partial\rho_t^F} = 0 \quad \Rightarrow \quad \Gamma = \Gamma\left(\rho_t^F J^{S-1}, n\right). \tag{33}$$

The relations for stresses have now the form

$$\mathbf{T}^S = 2\rho_t^S\left[\left(\frac{\partial\check{\psi}^S}{\partial I} + I\frac{\partial\check{\psi}^S}{\partial II}\right)\mathbf{B}^S - \frac{\partial\check{\psi}^S}{\partial II}\mathbf{B}^{S2}\right] + \rho_t^S J^S\frac{\partial\check{\psi}^S}{\partial J^S} - \rho_t^F J^{S-1}\Gamma'\mathbf{1},$$

$$\mathbf{T}^F = -\rho_t^{F2}\frac{\partial\check{\psi}^F}{\partial\rho_t^F}\mathbf{1} - \rho_t^F J^{S-1}\Gamma'\mathbf{1}, \tag{34}$$

where Γ' denotes the derivative with respect to the combined variable $\rho_t^F J^{S-1}$.

In contrast to models without a dependence on the gradient of porosity the above constitutive relations leave a high degree of freedom in constructing special models.

First of all the partial free energy for the skeleton ψ^S can depend on the current mass density of the fluid ρ_t^F and the partial energy of the fluid ψ^F can depend on the deformation of the skeleton, namely its volume changes described by J^S.

In addition the flux contribution Φ is related to all these couplings through the identities (24). This is different from the model considered in [28] where instead of the balance equation for porosity solely a dependence on the equilibrium porosity n_E was investigated. Consequently the linear Biot's model can be constructed from many different nonlinear models. We present in the next section such a linear model following from the present considerations.

Bearing the above results in mind we are in the position to reconsider the porosity balance (20). Accounting for the relation (23) for $\mathbf{J}$ we obtain

$$\frac{\partial n}{\partial t} + \Phi \operatorname{Div} \acute{\mathbf{X}}^F + \acute{\mathbf{X}}^F \cdot \operatorname{Grad} \Phi = 0. \tag{35}$$

Clearly the last contribution vanishes in the thermodynamic equilibrium. Hence we obtain

$$\frac{\partial n_E}{\partial t} + \Phi \operatorname{Div} \acute{\mathbf{X}}^F = 0 \quad \Rightarrow \quad \frac{\partial n_E}{\partial t} = \Phi \frac{\partial \ln \frac{\rho^F}{\rho_0^F}}{\partial t}, \tag{36}$$

where ρ_0^F is the reference value of the partial mass density of the fluid and the second relation follows from the mass conservation for the fluid. In a particular case important for linear models the coefficient Φ is constant and it follows

$$\frac{n_E - n_0}{n_0} = \gamma \ln \frac{\rho^F}{\rho_0^F} \equiv \gamma \ln \frac{\rho_t^F J^S}{\rho_0^F} \equiv \gamma \ln \frac{\rho_t^F}{\rho_0^F}\frac{\rho_0^S}{\rho_t^S}, \quad \gamma := \frac{\Phi}{n_0}. \tag{37}$$

For small changes of mass densities we can write this relation in the approximate form

$$\frac{n_E - n_0}{n_0} \approx \gamma \left(\frac{\rho_t^F}{\rho_0^F}\frac{\rho_0^S}{\rho_t^S} - 1 \right). \tag{38}$$

This relation was used in the paper [28] with the value $\gamma = 1$. It can be also written as follows

$$\frac{n_E - n_0}{n_0} = \gamma \left(e - \varepsilon \right), \tag{39}$$

where

$$e := J^S - 1 = \frac{\rho_0^S - \rho_t^S}{\rho_0^S}, \quad \varepsilon := \frac{\rho_0^F - \rho_t^F}{\rho_0^F}, \quad \max \left\{ |e|, |\varepsilon| \right\} \ll 1. \tag{40}$$

We discuss this relation further in this work.

3 Linear model

Now we proceed to construct the linear model for which the following condition is satisfied

$$\max \left\{ \left\| \mathbf{e}^S \right\|, |\varepsilon|, \left| \frac{n - n_0}{n_0} \right| \right\} \ll 1, \quad \mathbf{e}^S := \frac{1}{2} \left(\mathbf{1} - \mathbf{B}^{S-1} \right), \tag{41}$$

$$\left\| \mathbf{e}^S \right\| := \max \left\{ \left| \lambda_e^{(1)} \right|, \left| \lambda_e^{(2)} \right|, \left| \lambda_e^{(3)} \right| \right\}, \quad \varepsilon := \frac{\rho_0^F - \rho_t^F}{\rho_0^F}, \tag{42}$$

where $\lambda_e^{(1)}, \lambda_e^{(2)}, \lambda_e^{(3)}$ are eigenvalues of the deformation tensor $\mathbf{e}^S$.

The standard linearization procedure in (34) yields the following relations for partial stresses[2]

$$\begin{aligned} \mathbf{T}'^S &= \mathbf{T}_0^S + \lambda^S e \mathbf{1} + 2\mu^S \mathbf{e}^S + Q\varepsilon \mathbf{1}, \\ \mathbf{T}'^F &= -p'^F \mathbf{1}, \quad p'^F = p_0^F - \left(\rho_0^F \kappa \varepsilon + Qe \right), \end{aligned} \tag{43}$$

where $\mathbf{T}_0^S$ – initial partial stress in the skeleton, p_0^F – initial partial pressure in the fluid, and these quantities as well as the material parameters $\lambda^S, \mu^S, \kappa, Q$ are functions of an initial porosity n_0. We use here the prime to indicate a particular form of constitutive relations. These shall be modified in the sequel.

These relations can be also written in the form used customarily in soil mechanics for the total bulk stress $\mathbf{T}$ and for the pore pressure p'_f

$$\begin{aligned} \mathbf{T} := &\quad \mathbf{T}'^S + \mathbf{T}'^F = \\ = &\quad \mathbf{T}_0 + \left(\lambda^S + \rho_0^F \kappa + 2Q \right) e \mathbf{1} + 2\mu^S \mathbf{e}^S - \left(Q + \rho_0^F \kappa \right) \frac{1}{n_0} \zeta \mathbf{1}, \end{aligned}$$

$$p'_f := \frac{p'^F}{n_0} = p_f^0 - \left(Q + \rho_0^F \kappa \right) \frac{1}{n_0} e + \frac{\rho_0^F \kappa}{n_0^2} \zeta, \tag{44}$$

where the definition of the pore pressure contains already the approximation for small changes of porosity $(np'_f \approx n_0 p'_f)$ and

$$\zeta := n_0 \left(e - \varepsilon \right), \tag{45}$$

denotes the so-called increment of fluid content. Further we return to the interpretation of this variable.

The above relations should be now substituted in the partial momentum balance equations in order to obtain field equations for the velocity fields $\mathbf{v}^S, \mathbf{v}^F$. In the Eulerian description used in linear models they have the form

$$\begin{aligned} \rho_0^S \frac{\partial \mathbf{v}^S}{\partial t} &= \operatorname{div} \mathbf{T}^S + \hat{\mathbf{p}}, \quad \mathbf{v}^S = \acute{\mathbf{x}}^S (\mathbf{x}, t), \\ \rho_0^F \frac{\partial \mathbf{v}^F}{\partial t} &= \operatorname{div} \mathbf{T}^F - \hat{\mathbf{p}}, \quad \mathbf{v}^F = \acute{\mathbf{x}}^F (\mathbf{x}, t), \\ \hat{\mathbf{p}} &\approx \pi \left(\mathbf{v}^F - \mathbf{v}^S \right) - \mathcal{N} \operatorname{grad} n, \end{aligned} \tag{46}$$

---

[2] The first three contributions to $\mathbf{T}^S$ are linearized in the same way as in the nonlinear elasticity. The remaining contributions follow from expansions in which the following relations are used

$$\rho_t^S \approx \rho_0^S (1 - e), \quad \rho_t^F \approx \rho_0^F (1 - \varepsilon), \quad J^{S-1} \approx 1 - e, \quad \rho_t^F J^{S-1} \approx \rho_0^F (1 - e - \varepsilon).$$

where we have left out nonlinear contributions to the acceleration and to the source. In the linear model the material parameters $\pi, \mathcal{N}$ are constants depending solely on the initial porosity n_0. Due to this property the contribution of the gradient of porosity can be incorporated into the partial stresses. Namely if we define

$$\mathbf{T}^S := \mathbf{T}'^S - \mathcal{N}(n - n_0)\mathbf{1}, \quad \mathbf{T}^F := \mathbf{T}'^F + \mathcal{N}(n - n_0)\mathbf{1}, \tag{47}$$

then the source in the equations (46) contains solely the contribution of the diffusive force $\pi\left(\mathbf{v}^F - \mathbf{v}^S\right)$.

The complete set of field equations follows if we add the integrability condition for the deformation tensor of the skeleton $\mathbf{e}^S$

$$\frac{\partial \mathbf{e}^S}{\partial t} = \text{sym grad}\,\mathbf{v}^S, \tag{48}$$

and the equation for porosity which due to the constant value of Φ reduces to (36), i.e. to the solution for the equilibrium porosity (39).

In terms of the total stress and pore pressure we can write the momentum balance equations in the form

$$\begin{aligned} \rho_0 \frac{\partial \mathbf{v}}{\partial t} &= \operatorname{div}\mathbf{T}, \quad \rho_0 := \rho_0^S + \rho_0^F, \quad \rho_0 \mathbf{v} := \rho_0^S \mathbf{v}^S + \rho_0^F \mathbf{v}^F, \\ \rho_0^{FR} \frac{\partial \mathbf{v}^F}{\partial t} &= -\operatorname{grad} p_f + \frac{\pi \rho_0}{n_0 \rho_0^F} \operatorname{grad}\left(\mathbf{v} - \mathbf{v}^F\right), \quad \rho_0^{FR} := \frac{\rho_0^F}{n_0}, \end{aligned} \tag{49}$$

where

$$p_f := p_f' - \mathcal{N}\frac{n - n_0}{n_0}. \tag{50}$$

Let us comment in passing that the above form of equations of motion does not make much sense in truly dynamical situations. Namely the notion of the barycentric velocity $\mathbf{v}$ appearing in these equations is not related to any motion of a physical bearing. Consequently it may be used in formal relations local in time but it should never be used in practical applications. A displacement of a center of gravity of the two-component system which is coupled to $\mathbf{v}$ is of no practical interest.

Clearly the transformation (47) does not influence the constitutive relation $(44)_1$ for the total stress $\mathbf{T}$. In order to compare the above relations with those of the classical Biot's model we introduce the following identification of parameters

$$H := \lambda^S + 2\mu^S + \rho_0^F \kappa + 2Q, \quad C := \frac{1}{n_0}\left(Q + \rho_0^F \kappa\right), \quad M := \frac{\rho_0^F \kappa}{n_0^2}. \tag{51}$$

Then the constitutive relations have the form

$$\begin{aligned} \mathbf{T} &= \mathbf{T}_0 + \left(H - 2\mu^S\right) e\mathbf{1} + 2\mu^S \mathbf{e}^S - C\zeta\mathbf{1}, \\ p_f &= p_f^0 - Ce + M\zeta - \mathcal{N}\frac{n - n_0}{n_0}. \end{aligned} \tag{52}$$

The first relation is identical with this quoted by Stoll [16]. However, the second one coincides with the Biot's relation for pore pressure solely in two cases: a constant porosity or porosity changes described by the equilibrium relation (39). In the latter case we can write

$$p_f = p_f^0 - Ce + M_\gamma \zeta, \quad M_\gamma := M - \frac{\gamma}{n_0}\mathcal{N}. \tag{53}$$

The difference between material parameters M and M_γ creates problems with an identification of macroscopic parameters in terms of true material properties. In the next section we discuss the status of classical Gassmann relations in terms of the above difference between constants.

4 Gedankenexperiments for compressibilities

4.1 Preliminary remarks

In order to identify macroscopic (smeared-out) parameters appearing in the poroelastic model of granular materials attempts have been made to construct micro-macrotransitions from fundamental models in which real (true) properties of materials are utilized. Such transitions are related to various averaging procedures. For porous materials most of these procedures are based on the notion of the representative elementary volume (REV). REV is a domain of a real porous material whose volume is sufficiently small in comparison with macroscopic dimensions and sufficiently large in comparison with microscopic dimensions (e.g. [11], [25]). This rather vague definition means that microscopic properties averaged over REV represent local macroscopic properties which may vary from point to point of a two-component continuous model and, simultaneously one can use a continuum approach to describe microscopic properties. The procedure of micro-macrotransition is mostly based on volume averages but it is not always the case. For instance, macroscopic properties of granular gases are often calculated by means of the kinetic approach whose microscopic background is given by an appropriate Boltzmann-like kinetic equation for a distribution function.

Volume averaging procedures are usually based on mathematical homogenization techniques. We shall not enter this issue in the present work. We consider rather a very simple special case which seems to be sufficient for many important practical applications. We assume namely that processes are homogeneous within REV. In the case of granular materials this assumption has been employed in an implicit manner since the classical works of Gassmann [5] and Geertsma [9] concerning the compressibility properties. We present main features of this procedure in the following sections.

Much less has been done for shearing. Problems seem to be connected with a nonlinearity of the problem (e.g. connected to a dependence of the shear modulus on a confining pressure). Some rather elementary and not very convincing calculations of the shear modulus based on the solution of the contact Hertz problem are presented, for instance, by White [14] who follows the argument of Duffy and Mindlin [8]. We skip this problem in the present work and assume that the shear modulus μ^S is given.

The main purpose of this presentation is to prepare this procedure for the extension on such processes in which unsaturated media with effects of phase transformation, capillarity, etc. are included. This shall be the subject of a forthcoming work.

4.2 Geometrical compatibility relations

We begin with a few simple geometrical notions. On the *macroscopic* level changes of partial mass densities are described as follows

$$\begin{aligned} \rho_t^F &= \frac{\rho_0^F}{1+\varepsilon} \quad \Rightarrow \quad \varepsilon = \frac{\rho_0^F - \rho_t^F}{\rho_t^F} \approx \frac{\rho_0^F - \rho_t^F}{\rho_0^F}, \\ \rho_t^S &= \frac{\rho_0^S}{1+e} \quad \Rightarrow \quad e \equiv J^S - 1 = \frac{\rho_0^S - \rho_t^S}{\rho_t^S} \approx \frac{\rho_0^S - \rho_t^S}{\rho_0^S}, \end{aligned} \tag{54}$$

provided partial mass conservation laws are fulfilled, i.e. there is no mass exchange between components. The approximation holds in the case of small deformations appropriate for the linear model. Certainly, the quantities ε, e describe local macroscopic volume changes of components.

On the microscopic level we have

$$\begin{aligned} \rho^{FR} &= \frac{\rho_0^{FR}}{1+\varepsilon^R} \quad \Rightarrow \quad \varepsilon^R = \frac{\rho_0^{FR} - \rho^{FR}}{\rho_t^{FR}} \approx \frac{\rho_0^{FR} - \rho^{FR}}{\rho_0^{FR}}, \\ \rho^{SR} &= \frac{\rho_0^{SR}}{1+e^R} \quad \Rightarrow \quad e^R = \frac{\rho_0^{SR} - \rho^{SR}}{\rho^{SR}} \approx \frac{\rho_0^{SR} - \rho^{SR}}{\rho_0^{SR}}, \end{aligned} \tag{55}$$

where the index R refers to a real (true) quantity.

Between these two classes of quantities the following averaging relations hold in an arbitrary point $\mathbf{X}$ of the macroscopic continuum

$$\begin{aligned} \rho_t^F &= \frac{1}{V}\int_{REV} \rho^{FR} \chi_{\mathbf{X}}\left(\mathbf{Z},t\right) dV \overset{\text{hom}}{=} n\rho^{FR}, \quad n := \frac{1}{V}\int_{REV} \chi_{\mathbf{X}}\left(\mathbf{Z},t\right) dV, \\ \rho_t^S &= \frac{1}{V}\int_{REV} \rho^{SR}\left(1-\chi_{\mathbf{X}}\left(\mathbf{Z},t\right)\right) dV \overset{\text{hom}}{=} \left(1-n\right)\rho^{SR}, \end{aligned} \tag{56}$$

where $\chi_{\mathbf{X}}$ is the characteristic function (a window function) for the microdomain of the fluid component at the point $\mathbf{X}$ of macrocontinuum, V is the volume of REV and we have used the assumption on homogeneity in REV. Similar relations hold for the initial values of the mass densities and porosity.

Under the assumption of small deformations and small changes of porosity (41) the following *compatibility relations* follow after easy calculations

$$e = e^R + \frac{n - n_0}{1 - n_0}, \quad \varepsilon = \varepsilon^R - \frac{n - n_0}{n_0}. \tag{57}$$

Consequently microscopic and macroscopic volume changes coincide solely in the case of constant porosity.

Relations (57) form the basis for micro-macrotransitions related to volumetric effects.

Instead of the macroscopic volume change of the fluid ε a variable called the increment of the fluid content is used in soil mechanics. This variable is related to the fact that in contrast to the solid component the representative elementary volume is not material for the fluid component. It means that REV consists always of the same particles of the real skeleton but the fluid may open and leave this domain due to the diffusion. The instantaneous mass of the fluid per macroscopic volume dV_0 is given by $\rho_t^F \left(1 + e\right) dV_0$ because e is the macroscopic measure of volume changes. Simultaneously in the same volume material with respect to the skeleton the initial amount of fluid mass was $\rho_0^F dV_0$. Consequently the change of the fluid mass contained in this volume is given by the difference of these two quantities. We introduce the dimensionless measure of this quantity

$$\zeta := \frac{1}{\rho_0^{FR}} \left[\left(1 + e\right) \rho_t^F - \rho_0^F \right] \approx n_0 \left(e - \varepsilon\right). \tag{58}$$

It is called the *increment of the fluid content.* Bearing the relations (57) in mind we have

$$\zeta = n_0 \left(e^R - \varepsilon^R\right) + \frac{n - n_0}{1 - n_0}. \tag{59}$$

Hence if the porosity is changing it has an influence on the increment of the fluid content in terms of true (microscopic) quantities.

4.3 Constitutive relations for pressures

In the construction of Gedankenexperiments, from all components of stress tensors only pressures are appearing. Therefore we quote here the constitutive relations for all pressures needed in the construction of solutions. Let us begin with partial pressures. Relations (43) extended by the contribution of portosity have the form

$$
\begin{aligned}
p^S - p_0^S := & -\frac{1}{3} tr\left(\mathbf{T}^S - \mathbf{T}_0^S\right) = -\left(\lambda^S + \frac{2}{3}\mu^S\right) e - Q\varepsilon + \mathcal{N}\left(n - n_0\right) = \\
= & -\left(\lambda^S + \frac{2}{3}\mu^S + Q\right) e + \frac{1}{n_0} Q\zeta + \mathcal{N}\left(n - n_0\right),
\end{aligned}
$$

$$
\begin{aligned}
p^F - p_0^F := & -\frac{1}{3} tr\left(\mathbf{T}^F - \mathbf{T}_0^F\right) = -\rho_0^F \kappa\varepsilon - Qe - \mathcal{N}\left(n - n_0\right) = \\
= & \frac{1}{n_0} Q\zeta - \left(\rho_0^F \kappa + Q\right) e - \mathcal{N}\left(n - n_0\right).
\end{aligned} \tag{60}
$$

In terms of the bulk stresses and pore pressure we have (see: (52))

$$
\begin{aligned}
p - p_0 := -\frac{1}{3} tr\left(\mathbf{T} - \mathbf{T}_0\right) = -Ke + C\zeta, \quad K := H - \frac{4}{3}\mu^S, \\
p_f - p_f^0 = -Ce + M\zeta - \mathcal{N}\frac{n - n_0}{n_0}.
\end{aligned} \tag{61}
$$

Apart from these macroscopic relations we need microscopic constitutive laws for real pressures. These are assumed to have the form

$$
\begin{aligned}
p^{SR} - p_0^{SR} = -K_s e^R, \qquad & p^{SR} \approx \frac{p^S}{1 - n_0}, \quad p_0^{SR} = \frac{p_0^S}{1 - n_0}, \\
p^{FR} - p_0^{FR} = -K_f \varepsilon^R, \qquad & p^{FR} \approx \frac{p^F}{n_0} \equiv p_f, \quad p_0^{FR} = \frac{p_0^F}{n_0} \equiv p_f^0,
\end{aligned} \tag{62}
$$

where K_s, K_f are true (microscopic) compressibility moduli of the solid and of the fluid component, respectively. In contrast to macroscopic parameters they are independent of porosity. One of the main purposes of Gedankenexperiments is to establish relations between these microscopic and macroscopic parameters and the porosity n_0.

The relations bewteen microscopic and macroscopic partial pressures appearing in (62) can be called *dynamical compatibility conditions.*

4.4 Changes of porosity

The above considerations allow for the estimation of changes of porosity independently of the porosity balance equation described in Sect. 2. Bearing the geometrical compatibility relations (57), the dynamical compatibility relations and the microscopic constitutive laws (62) in mind we obtain

$$
\begin{aligned}
\left(1 - n_0\right) e &= -\frac{p^S - p_0^S}{K_s} + \left(n - n_0\right), \\
n_0 \varepsilon &= -\frac{p^F - p_0^F}{K_f} - \left(n - n_0\right).
\end{aligned} \tag{63}
$$

Hence it follows for porosity changes

$$(K_s - K_f)(n - n_0) = (1 - n_0) K_s e + n_0 K_f \varepsilon + (p - p_0). \tag{64}$$

Substitution of $(61)_1$ yields

$$\frac{n - n_0}{n_0} = \frac{C - K_f}{K_s - K_f}(e - \varepsilon) + \frac{(1 - n_0) K_s + n_0 K_f - K}{n_0 (K_s - K_f)} e. \tag{65}$$

This relation is independent of $\mathcal{N}$ and, consequently, it should hold for both models: the original Biot's model ($\mathcal{N} = 0$) as well as its thermodynamic correction presented in Sect. 2. Obviously the relation (65) does not coincide with the solution (39) of the porosity balance equation. We obtain the same result solely if the following two conditions are satisfied

$$\gamma = \frac{C - K_f}{K_s - K_f}, \quad K = (1 - n_0) K_s + n_0 K_f. \tag{66}$$

The second relation demonstrates limitations of the model with the porosity balance equation. It is identical with this proposed in the classical Voigt model. As shown by Hashin and Shtrikman [10] such a model is an upper bound for all models and it corresponds to a porous material with a very rigid skeleton. Consequently it seems to fit better the requirements for modeling rocks than granular materials.

In practical applications the relation (39) can be used as an approximation limited by the above observation.

In order to obtain an equilibrium solution of the porosity equation which agrees with the result (65) one would have to modify this equation. One of the possibilities is to extend the source of porosity $\hat{n}$ by contributions of volume changes of the skeleton. Such an extension is admissible from the microscopic point of view presented in [23]. These contributions would be proportional to the time derivative $\frac{\partial e}{\partial t}$ in the linear model. They would contribute to partial stresses in the order of magnitude $\frac{n - n_E}{n_0}$ which we neglected in the linear model. However, the equilibrium solution of the linear problem would contain independent contributions of volume changes of both components and not only their difference and, consequently, results from of the porosity balance equation would agree with the result (65). We return to this problem in the forthcoming paper.

Draw-backs of the porosity balance equation $(3)_4$ with the source term presented in this work in applications to problems of abrasion of granular materials have been pointed out in the Ph-D Thesis of Nina Kirchner [27].

4.5 Jacketed and unjacketed tests

Now we construct solutions for three homogeneous problems presented schematically in the two Figures below. We follow here the arguments used by Biot and Willis [7] and discussed in the present notation in the book of Stoll [16]. We consider a quasistatic process in which the *external pressure* has been increased on $\triangle p$. Then in all three tests the following equilibrium condition must be satisfied

$$p - p_0 = \triangle p. \tag{67}$$

1) The first and second Gedankenexperiment are shown in Figure 1. We consider first the problem with closed valve – *jacketed undrained test*. Then the increment of the fluid content must be zero and, consequently, we have to fulfil the following condition

$$\zeta = 0, \tag{68}$$

which defines this test. We have two macroscopic and two microscopic constitutive relations

– macroscopic

$$\triangle p = -Ke, \quad p_f - p_f^0 = -Ce - \mathcal{N}\frac{n - n_0}{n_0}, \tag{69}$$

– microscopic

$$\triangle p = -\left(1 - n_0\right) K_s e^R - n_0 \varepsilon^R, \quad p_f - p_f^0 = -K_f \varepsilon^R, \tag{70}$$

and the following geometrical compatibility relations

$$e^R = e - \frac{n - n_0}{1 - n_0}, \quad \varepsilon^R = e + \frac{n - n_0}{n_0}. \tag{71}$$

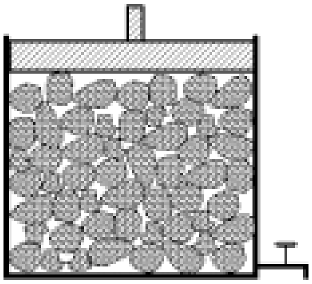

Fig 1.: *A scheme of the jacketed Gedankenexperiment*

Substitution of (71) in (70) yields four equations for three unknowns: $e, p_f - p_f^0, \frac{n-n_0}{n_0}$. Consequently we have to fulfil in addition a compatibility relation. The solution of this problem has the form

$$e = -\frac{\triangle p}{K}, \quad \frac{n - n_0}{n_0} = -\frac{C - K_f}{K\left(K_f - \mathcal{N}\right)}\triangle p, \tag{72}$$

$$p_f - p_f^0 = \left(\frac{C}{K} + \mathcal{N}\frac{C - K_f}{K\left(K_f - \mathcal{N}\right)}\right)\triangle p,$$

$$K = K_V - n_0\left(K_s - K_f\right)\frac{C - K_f}{K_f - \mathcal{N}}, \qquad K_V := \left(1 - n_0\right) K_s + n_0 K_f. \tag{73}$$

The last relation connects microscopic material parameters K_s, K_f and the macroscopic parameters $K, C, \mathcal{N}$ for any choice of the porosity n_0. The modulus K_V is analogous to the Voigt definition of the effective modulus of the classical crystallographic problem [2].

Let us remark that K should belong to the interval $0 < K < K_V$. We see later that $C - K_f > 0$. Consequently $\mathcal{N} < K_f$. Hence expected values of $\mathcal{N}$ are much smaller than typical values of compressibility coefficients of the solid component.

2) We proceed to discuss the second Gedankenexperiment – *the jacketed drained test*. Now the valve is open and we have the following condition defining the test

$$p_f - p_f^0 = 0, \tag{74}$$

i.e. the fluid flows freely from the vessel. Conditions (62) and (57) yield then

$$\varepsilon^R = 0 \quad \Rightarrow \quad \zeta = n_0 e + (n - n_0)\,. \tag{75}$$

We have two macroscopic and one microscopic constitutive relations at the disposal

– macroscopic

$$\begin{aligned} \triangle p &= -Ke + C\zeta, \\ 0 &= -Ce + M\zeta - \mathcal{N}\left(\frac{1}{n_0}\zeta - e\right), \end{aligned} \tag{76}$$

– microscopic

$$\begin{aligned} \triangle p &= -(1-n_0)\left(p^{SR} - p_0^{SR}\right) = -(1-n_0) K_s e^R = \\ &= -(1-n_0) K_s \left(e - \frac{n - n_0}{1 - n_0}\right). \end{aligned} \tag{77}$$

They serve the purpose to determine the quantities $e, \zeta, n - n_0$. It remains the relation $(75)_2$ which yields a compatibility relation for material parameters. After easy calculations we obtain

$$\begin{aligned} e &= -\frac{\triangle p}{K_b} - \frac{\mathcal{N}C}{K_b M} \frac{1 - (1-n_0)\frac{K_s}{K_b}}{\left[(1-n_0)\frac{\mathcal{N}C}{K_b M} - n_0\right] K_s} \triangle p, \\ \zeta &= -\frac{C}{K_b M}\left[1 + \frac{K\mathcal{N}}{CK_s} \frac{1 - (1-n_0)\frac{K_s}{K_b}}{(1-n_0)\frac{\mathcal{N}C}{K_b M} - n_0}\right] \triangle p, \\ \frac{n - n_0}{n_0} &= -\frac{1 - (1-n_0)\frac{K_s}{K_b}}{\left[(1-n_0)\frac{\mathcal{N}C}{K_b M} - n_0\right] K_s} \triangle p, \end{aligned} \tag{78}$$

where

$$K_b := K - \frac{C^2}{M}, \tag{79}$$

defines the *drained compressibility modulus* K_b.

There remains the geometrical compatibility condition $(75)_2$ to be satisfied. This yields

$$\frac{K_s}{K_b}\left(n_0-\frac{C}{M}\right)-\frac{1-(1-n_0)\frac{K_s}{K_b}}{(1-n_0)\frac{\mathcal{N}C}{K_bM}-n_0}\left[n_0-\frac{\mathcal{N}(K+C)}{K_bM}\right]=0. \tag{80}$$

3) The third Gedankenexperiment – *unjacketed test*, is shown schematically in Figure 2. It is a fully saturated granular heap immersed into a fluid which transfers the external loading. In this case the following condition must be fulfilled

$$\triangle p = p_f - p_f^0, \tag{81}$$

which is the equilibrium condition between the true fluid in the pores and the fluid outside the heap. This relation defines the test. Some limitations which are implicit in these conditions are discussed in the work [29].

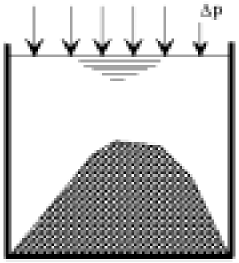

Fig 2.: *A scheme of the unjacketed Gedankenexperiment*

In order to find the solution for $e, \zeta, \frac{n-n_0}{n_0}$ we consider again macroscopic and microscopic constitutive relations:

– macroscopic

$$\begin{aligned}\triangle p &= -Ke + C\zeta, \\ \triangle p &= -Ce + M\zeta - \mathcal{N}\frac{n-n_0}{n_0},\end{aligned} \tag{82}$$

– microscopic

$$\begin{aligned}\triangle p &= -K_s e^R = -K_s\left(e-\frac{n-n_0}{1-n_0}\right), \\ \triangle p &= -K_f \varepsilon^R = -K_f\left(e-\frac{1}{n_0}\zeta+\frac{n-n_0}{n_0}\right).\end{aligned} \tag{83}$$

Combination of the last two equations yields

$$\frac{\triangle p}{K_W} = -e+\zeta, \quad \frac{1}{K_W} := \frac{1-n_0}{K_s}+\frac{n_0}{K_f}, \tag{84}$$

where the modulus K_W is identical with this introduced by Wood [1]. It is used in the description of polycrystals by Reuss [3]. From $(82)_1$ and (84) we obtain immediately

$$e = -\frac{\triangle p}{K}\frac{1-\frac{C}{K_W}}{1-\frac{C}{K}}, \quad \zeta = -\frac{\triangle p}{K}\frac{1-\frac{K}{K_W}}{1-\frac{C}{K}}. \tag{85}$$

The solution for the porosity follows from $(83)_1$. We have

$$n - n_0 = (1-n_0)\left(\frac{K}{K_s} - \frac{1-\frac{C}{K_W}}{1-\frac{C}{K}}\right)\frac{\triangle p}{K}. \tag{86}$$

It remains to use the relation $(82)_2$ to obtain the compatibility relation. It has the form

$$K = \frac{C - M + \frac{MK_b}{K_W}}{1-\frac{C}{K}} - \mathcal{N}\frac{1-n_0}{n_0}\left(\frac{K}{K_s} - \frac{1-\frac{C}{K_W}}{1-\frac{C}{K}}\right). \tag{87}$$

This completes the analysis of the Gedankenexperiments. The essential results are the compatibility relations (73), (80) and (87). In addition it is usually assumed that the drained modulus K_b can be measured in simple experimental tests and, for this reason, it should be considered as known in the same way as it is done for K_s and K_f. Consequently the relation (79) is also a relation for macroscopic material parameters. Hence we have four relations for four parameters: $K, C, M, \mathcal{N}$. We discuss these relations in the following sections.

4.6 Gassmann relations

We begin with the analysis of a special case which is an approximation for negligibly small values of the parameter $\mathcal{N}$. In the classical approach the jacketed experiments are ignored and solely the jacketed drained experiment is used as a motivation for the definition of the drained compressibility modulus K_b. In addition it is assumed that the porosity in the unjacketed experiment is constant. Details can be found, for instance, in the book of Stoll [16].

We proceed in a different way. We account for the jacketed undrained experiment and for the definition of the compressibility K_b but we leave out the compatibility relation (80).

In the particular case $\mathcal{N} = 0$ the above mentioned three relations have the form

$$K = K_V - n_0(K_s - K_f)\frac{C - K_f}{K_f}, \quad K_b = K - \frac{C^2}{M},$$

$$K\left(1 - \frac{C}{M}\right) = C - M + \frac{MK_b}{K_W}. \tag{88}$$

After easy manipulations we obtain the following relations for material parameters

$$K = \frac{(K_s - K_b)^2}{\frac{K_s^2}{K_W} - K_b} + K_b, \quad C = \frac{K_s (K_s - K_b)}{\frac{K_s^2}{K_W} - K_b}, \quad M = \frac{K_s^2}{\frac{K_s^2}{K_W} - K_b}. \tag{89}$$

These are exactly the same *Gassmann relations* as quoted by Stoll. Simultaneously the substitution of Gassmann relations in the formula (86) for changes of porosity shows that $n = n_0$, i.e. the porosity follows as a constant in this experiment and does not have to be assumed to be constant as claimed by Stoll.

It remains an open question what is the role of the ignored identity (80). We shall see further on a numerical example that the value of the compressibility K_b calculated by means of this identity is much bigger than a measured value of K_b (compare Fig. 4). Hence we have a contradiction. It may be explained in terms of simplifications of both the microscopic and macroscopic models used in the description of Gedankenexperiments. We do not account, for instance, for the cohesive forces appearing in the measurement of the drained compressibility modulus K_b. Results would be different if we had measured not only properties of drained but also fully dry samples. We leave out as well the influence of friction between grains. For all those reasons the jacketed drained Gedankenexperiment cannot be incorporated in the above scheme together with experimental values of K_b. Real measurements disqualify the drained jacketed test as a Gedankenexperiment.

The above conclusion must be extended on the model with the additional parameter $\mathcal{N}$. If we ignore (80) in the classical Biot's model we are not entitled to account for it in the thermodynamic model which is based on the same simplifications. We can still use the relation (66) for the constant γ. However, similarily to the second momentum source parameter – the permeability coefficient π, the parameter $\mathcal{N}$ cannot be specified from the above static Gedankenexperiments. One would have to construct an experiment with an influence of $\operatorname{grad} n$, the origin of constant $\mathcal{N}$, in order to find its relation to microscopic properties of the system. We can solely claim in general that this parameter is smaller than the true compressibility of the fluid K_f as indicated by the relation (66). In addition, as we indicated in earlier works, thermodynamic models with constraints yield this constant to be identical with the reaction pressure. This corresponds to the pore pressure in our case which is, certainly, much smaller than, say, K_f.

4.7 A numerical example

In order to see orders of magnitude of the above results we present in this section a few figures following from the experimental data published in the work of Cheng [21]. According to this work we choose the following values of parameters

$$K_s = 48.2 \times 10^9 Pa, \quad K_f = 3.3 \times 10^9 Pa, \quad K_b = 30.6 \times 10^9 Pa. \tag{90}$$

Then we obtain the following results. In Figure 3 we plot compressibility moduli K, C, M calculated according to the Gassmann relations for the porosity n_0 varying between 0.01 and 0.5. Consequently, the range of small porosities (app. $n_0 < 0.1$) seems to yield a singular behaviour and results may not be reliable.

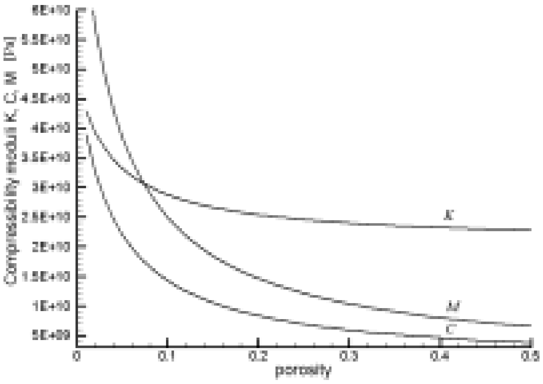

Fig 3.: *Compressibility moduli according to Gassmann relations*

Figure 4 shows the relation between the modulus K_b calculated from the compatibility relation (80) to the value given in (90). The relation is linear in n_0. As we see the jacketed drained test gives the values of K_b which are much bigger than these measured in real experiments.

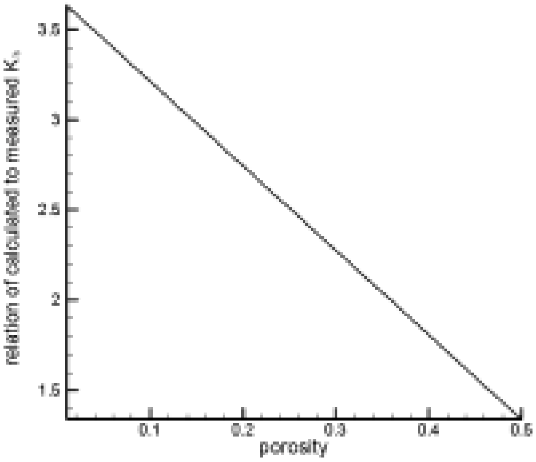

Fig 4.: *The relation of K_b calculated from the compatibility relation to the measured value.*

Finally in Figure 5 we show the values of the coefficient γ appearing in the flux of the equation of porosity. The value $\gamma = 1$ used in the earlier papers lies apparently near the present estimation solely in the range of very small porosities.

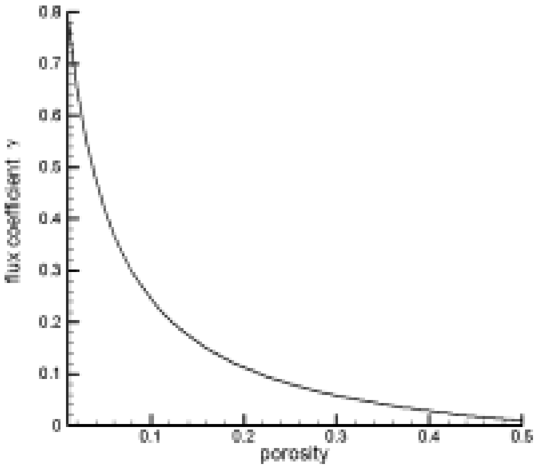

Fig 5.: *The flux coefficient γ as a function of porosity.*

These numerical results demonstrate clearly that we can put approximately $M_\gamma = M$ in the relation (53) and leave out the influence of $\mathcal{N}$ in compatibility relations. Hence the Gassmann relations apply to both the Biot's model and to the thermodynamic model.

5 Concluding remarks

The results presented in this work demonstrate the position of the classical Biot's model for linear poroelastic media among thermodynamically admissible models following as limit cases from a fully nonlinear model with a balance equation of porosity.

The most important conclusion is that the Biot's model may follow solely in the case when a constitutive dependence on the porosity gradient is incorporated. This extension of the classical two-component model of immiscible mixtures is necessary for the existence of coupling between partial stresses. In addition it is necessary to leave out relaxation properties of the porosity which may appear in the general thermodynamic model.

The analysis of micro-macrorelations for a homogeneous microstructure shows that the classical Gassmann relations for parameters of the Biot's model hold true provided one ignores one of the compatibility relations. This relation follows for the jacketed drained Gedankenexperiment and it contradicts results of real experiments for the drained compressibility modulus K_b. This may be a result of simplifications which are incorporated in Gedankenexperiments (homogeneity, lack of cohesive forces, lack of friction and abrasion, etc.).

Simultaneously the analysis of Gedankenexperiments reveals limitations for the applicability of the porosity equation of the thermodynamic model. We have shown that changes of porosity are properly described by the porosity balance equation solely for systems with a rather rigid skeleton. Hence the model is more appropriate for rocks than for granular materials. The analysis of this compatibility reveals as well that the additional material parameter $\mathcal{N}$ describing the contribution of porosity gradient to the momentum balance equations is small in comparison with other compressibility coefficients. This justifies the applicability of Gassmann relations of the Biot's model in the thermodynamic model with the porosity balance.

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