Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

Heat equation with strongly inhomogeneous noise

Henryk Zähle*[1]

submitted: 10. March 2003

[1] Weierstrass Institute
for Applied Analysis
and Stochastics
Mohrenstrasse 39
10117 Berlin, Germany
E-Mail: zaehle@wias-berlin.de

No. 824
Berlin 2003

2000 *Mathematics Subject Classification.* 60H15 35R60 60H20 60J80.

Key words and phrases. stochastic partial differential equation, martingale problem, singular measure, catalytic super-Brownian motion .

∗ Work was supported by Graduate College “Stochastic Processes and Probabilistic Analysis”, Technical University, Berlin.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

Abstract

We consider the stochastic heat equation in dimension one with singular drift and driven by an inhomogeneous space-time white noise whose quadratic variation measure is not absolutely continuous w.r.t. Lebesgue measure, neither in space nor in time. Under various assumptions we give statements on strong and weak existence as well as strong and weak uniqueness of continuous solutions.

1 Introduction

We consider the following stochastic partial differential equation (SPDE)

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) &= \frac{1}{2}\Delta u(t,x) + b(t,x,u(t,x))\frac{\sigma(dxdt)}{dxdt}(x,t) + a(t,x,u(t,x))\frac{\partial^2}{\partial x\partial t}w^{\varrho}(x,t)\\ u(0,x) &= \eta(x), \qquad x\in\mathbb{R}^1,\ t\geq 0 \end{aligned}\tag{1.1}$$

whose precise meaning will be given in Definition 2.1 below. Here $\Delta = \frac{\partial^2}{\partial x^2}$, a and b : $[0,\infty)\times\mathbb{R}^1\times\mathbb{R}\to\mathbb{R}$ as well as $\eta:\mathbb{R}^1\to\mathbb{R}$ are continuous functions, $\sigma(dxdt)$ and $\varrho(dxdt)$ are positive σ-finite Borel measures on $\mathbb{R}^1\times[0,\infty)$, $\frac{\sigma(dxdt)}{dxdt}$ is the density - possibly only existing as a generalized function (distribution) - of $\sigma(dxdt)$ and $w^{\varrho}:\mathbb{R}^1\times[0,\infty)\times\Omega\to\mathbb{R}$ is an inhomogeneous two-parameter Brownian motion on $\mathbb{R}^1\times[0,\infty)$ based on $\varrho(dxdt)$. The latter object is characterized by the relation $W^{\varrho}((x,x']\times(t,t']) = w^{\varrho}(x',t') - w^{\varrho}(x,t') - w^{\varrho}(x',t) + w^{\varrho}(x,t)$ where W^{ϱ} is a white noise "measure" based on $\varrho(dxdt)$, that is a real-valued random function on the algebra $\mathcal{A} = \cup_{n\geq 1}\mathcal{B}([-n,n]\times[0,n])\subset\mathcal{B}(\mathbb{R}^1\times[0,\infty))$ satisfying

- $W^{\varrho}(A)\sim N(0,\varrho(A))$ for all $A\in\mathcal{A}$,
- $W^{\varrho}(A), W^{\varrho}(A')$ indep. and $W^{\varrho}(A\cup A') = W^{\varrho}(A) + W^{\varrho}(A')$ for disjoint $A, A'\in\mathcal{A}$.

In [Wal86], p.269, W^{ϱ} is well-constructed as a Gaussian process on $\mathcal{A}$. Note that in a formal sense, w^{ϱ} is to be associated with the "distribution function" of W^{ϱ} and $\dot{w}^{\varrho}(x,t) = \frac{\partial^2}{\partial x\partial t}w^{\varrho}(x,t)$ with the "density" of W^{ϱ}. The latter is usually called white noise and coincides with W^{ϱ} in distribution sense. If $\varrho(dxdt) = \sigma(dxdt) = dxdt$, then $\frac{\sigma(dxdt)}{dxdt}(x,t)\equiv 1$ and w^{ϱ} becomes just the homogeneous two-parameter Brownian motion w on $\mathbb{R}^1\times[0,\infty)$. In this case equation (1.1) turns into

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) &= \frac{1}{2}\Delta u(t,x) + b(t,x,u(t,x)) + a(t,x,u(t,x))\frac{\partial^2}{\partial x\partial t}w(x,t)\\ u(0,x) &= \eta(x), \qquad x\in\mathbb{R}^1,\ t\geq 0\end{aligned}\tag{1.2}$$

and has been studied several times - under various assumptions - w.r.t. existence and uniqueness of continuous solutions ([Wal86], [Iwa87], [KS88], [Rei89], [MP92], [Shi94], [Myt98]). Also note that, in the sense of Definition 2.1 below,

$$\frac{\partial^2}{\partial x\partial t}w^{\varrho}(x,t) = \sqrt{\varrho(x,t)}\frac{\partial^2}{\partial x\partial t}w(x,t)\tag{1.3}$$

whenever $\varrho(dxdt)$ has a properly regular $dxdt$-density $\varrho(x,t)$.

One motivation to study equation (1.2) is the link to population systems. For example, if $b \equiv 0$ and $a(t,x,u(t,x)) = \sqrt{\varrho(x,t)u(t,x)}$, equation (1.2) describes the evolution of an infinitesimal system (high-density/short-lifetime limit) of critically branching Brownian particles where the branching intensity of an infinitesimal particle being at position x at time t is given by $\varrho(x,t)$, see [KS88], [Rei89] or [MRC88]+[MP92]. Here the medium $\varrho(.,.)$ was assumed to be a regular function. But media occurring in nature often have a more fractal shape, particularly they only can be modeled as a singular measure $\varrho(dxdt)$. For our example this means that the branching intensity is given by $\frac{\varrho(dxdt)}{dxdt}(x,t)$, which does not have a rigorous meaning any more, and that the particle system heuristically evolves according to equation (1.1) with $b \equiv 0$ and $a(t,x,u(t,x)) = \sqrt{u(t,x)}$, formally justified by (1.3). In case $\varrho(dxdt) = \varrho_t(dx)dt$, the corresponding infinitesimal particle system has been characterized ([DF91]) as a measure-valued process $(\bar{u}_t(dx))_{t\geq 0}$, the so-called catalytic super-Brownian motion. On the one hand, a continuous solution to SPDE (1.1) with mentioned a and b, provided it exists one, is the $dxdt$-density of $\bar{u}_t(dx)dt$. This can be checked using the characterization of $\bar{u}_t(dx)$ as unique solution to a certain martingale problem (cf. Proposition 2.6 of [Zäh02]). On the other hand, it was established in [DFLM95] that for $\varrho_t(dx) \equiv \delta_c(dx)$, $c \in \mathbb{R}^1$ fixed, $\bar{u}_t(dx)dt$ is singular w.r.t. $dxdt$. Consequently, we will not get a continuous solution to SPDE (1.1) with mentioned a and b for all measures $\varrho(dxdt) = \varrho_t(dx)dt$. However, as demonstrated in [Zäh02], there is a broad class of singular non-atomic measures $\varrho_t(dx)dt$ for which $\bar{u}_t(dx)dt$ possesses a continuous $dxdt$-density solving SPDE (1.1) with $b \equiv 0$ and $a(t,x,u(t,x)) = \sqrt{u(t,x)}$. Indeed, $\varrho_t(dx)$ only has to satisfy the following assumption on the concentration of local mass

$$\exists \alpha \in (0,1] \; \forall T>0 \; \exists c_T>0: \quad \sup_{t\leq T}\sup_{x\in\mathbb{R}^1} \varrho_t(B(x,r)) \leq c_T \; r^{\alpha} \qquad \forall r \in (0,1]$$

where $B(x,r)$ is the open ball with center x and radius r, that is $(x-r,x+r)$.

In the present paper we worry about solutions to SPDE (1.1) with more general coefficients (than $b \equiv 0$, $a(t,x,u(t,x)) = \sqrt{u(t,x)}$) under similar conditions on $\varrho(dxdt)$ and $\sigma(dxdt)$. For instance, assuming $\varrho(dxdt) \leq \varrho_1(dx)\varrho_2(dt)$ and $\sigma(dxdt) \leq \sigma_1(dx)\sigma_2(dt)$ ($\mu \leq \nu$ means $\mu(A) \leq \nu(A)$ for all Borel sets A) as well as

$$\exists \alpha_1,\alpha_2,\beta_1,\beta_2 \in [0,1] \text{ with } \frac{\alpha_1}{2}+\alpha_2>1,\; \frac{\beta_1}{2}+\beta_2>\frac{1}{2} \text{ such that } \forall T>0 \; \exists c_T>0:$$

$$\begin{aligned} &\sup_{x\in\mathbb{R}^1} \varrho_1(B(x,r)) \leq c\; r^{\alpha_1}, \qquad \sup_{t\leq T} \varrho_2(B(t,r)) \leq c_T\; r^{\alpha_2}, \qquad \forall r \in (0,1] \\ &\sup_{x\in\mathbb{R}^1} \sigma_1(B(x,r)) \leq c\; r^{\beta_1}, \qquad \sup_{t\leq T} \sigma_2(B(t,r)) \leq c_T\; r^{\beta_2}, \qquad \forall r \in (0,1] \end{aligned} \tag{1.4}$$

we can establish strong solutions and strong uniqueness for Lipschitz-continuous a,b and weak solutions for continuous a,b. For the exact condition see Definition 2.3 below.

We conclude this chapter with examples for measures $\varrho(dxdt)$ and $\sigma(dxdt)$ matching (1.4). It is easy to see that all $\varrho(dxdt)$ that can be bounded by a multiple of the Lebesgue measure $dxdt$, at least on compact time sets, fulfill (1.4). But $\varrho(dxdt)$ does not need to possess a Lebesgue density. Indeed, let $\mathcal{C}_\lambda(dx)$ be the "Cantor measure" on, say, $[0,1]$ with index $\lambda \in (0, \frac{1}{2})$. This measure is supported by an uncountable unification of single points (λ-Cantor set $C(\lambda)$, cf. [Mat95] 4.13) and possesses no atoms. In fact, $\mathcal{C}_\lambda(.) = \mathcal{H}^\gamma(C(\lambda) \cap .)$ where $\mathcal{H}^\gamma$ is the γ-dimensional Hausdorff measure and $\gamma = |\log 2|/|\log \lambda|$ the Hausdorff-dimension of $C(\lambda) = supp(\mathcal{C}_\lambda)$. $\mathcal{C}_\lambda(dx)$ has clearly no Lebesgue density. Furthermore, see e.g. Theorem 4.14 of [Mat95], there exist $0 < c < C$ such that for all $x \in supp(\mathcal{C}_\lambda)$,

$$c\, r^\gamma < \mathcal{C}_\lambda(B(x,r)) < C\, r^\gamma, \quad \forall r \in (0,1].$$

Therefore, $\varrho(dxdt) = \mathcal{C}_{\lambda_1}(dx)\mathcal{C}_{\lambda_2}(dt)$ satisfies (1.4) whenever $\lambda_1, \lambda_2 \in (0, \frac{1}{2})$ such that $|\log 2|/|2 \log \lambda_1| + |\log 2|/|\log \lambda_2| > 1$. Note that for $c > 0$, $\gamma \in (0,1)$ and a Borel measure $\mu(d\xi)$ on $\mathbb{R}$, $\sup_{\xi \in supp(\mu)} \mu(B(\xi,r)) \le c\, r^\gamma$ respectively $\inf_{\xi \in supp(\mu)} \mu(B(\xi,r)) \ge c\, r^\gamma$, $r \in (0,1]$, implies that the Hausdorff-dimension of the support $supp(\mu)$ of $\mu(d\xi)$ is at least respectively at most γ, see Theorem 5.7 of [Mat95]. The mentioned measures can also be taken as examples for $\sigma(dxdt)$. Moreover, we observe the remarkable fact that $\sigma(dxdt)$ may have spatial atoms ($\beta_1 = 0$). For instance, $\sigma(dxdt) = \delta_0(dx)dt$ and even $\sigma(dxdt) = \delta_0(dx)\mathcal{C}_{\lambda_2}(dt)$ with $\beta_2 := |\log 2|/|\log \lambda_2| > \frac{1}{2}$ fit into (1.4).

2 Preliminaries and main results

Let us give a precise meaning to SPDE (1.1). In case $\varrho(dxdt) = dxdt$, for instance, $w^\varrho(x,t)$ is known to be continuous in (x,t) but not differentiable in t and $\frac{\sigma(dxdt)}{dxdt}(x,t)$ does not need to be a function in t. So one might tend to regard (1.1) as a bunch of integral equations involving Itô-integrals. However, since $w^\varrho(x,t)$ is not differentiable in x and $\frac{\sigma(dxdt)}{dxdt}(x,t)$ does not need to be a function in x either, (1.1) will be understood in sense of Schwartz distributions. To do so one has to be able to integrate against $w^\varrho(dx,dt)$ which, by the known unbounded variation of $w^\varrho(x,t)$, is not a signed measure. The way out is a generalization of the Itô-integral for the space-time case, the so-called Walsh-integral. For a rigorous development of this theory see [Wal86]. The Walsh-integral is a stochastic integral having worthy martingale measures as integrators. We denote the quadratic variation measure of an orthogonal martingale measure M by $\langle M \rangle(dxdt)$ and the stochastic integral of a proper f against M by $f \bullet M$. We also set $\int_0^t \int_B f(r,y) M(dydr) = (f \bullet M)_t(B)$. Note that the white noise "measure" W^ϱ from Chapter 1 induces an orthogonal martingale measure, denoted by W^ϱ either, with $\langle W^\varrho \rangle(dxdt) = \varrho(dxdt)$. The probability domain of W^ϱ will be denoted by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \ge 0}, \mathbf{P})$ where $(\mathcal{F}_t)_{t \ge 0}$ is a filtration in $\mathcal{F}$ satisfying the usual conditions.

In the sequel, c will always refer to a finite positive constant that can vary from place to place. Possible subscripts of c stress the dependence of c on these subscripts. If we explicitely want to emphasize the possible difference of constants in front of different terms, then we write $\tilde{c}$ or $\bar{c}$ instead of c. Let $C(\mathbb{R}^1)$ be the space of real-valued continuous functions on $\mathbb{R}^1$ and C_b respectively C_c the subspace of bounded functions respectively functions with compact support. Superscripts $+$ and ∞ refer to the positive functions respectively functions having derivatives of any order. Furthermore, we introduce the subspaces of tempered functions as well as rapidly decreasing functions

$$\begin{aligned} C_{tem} &= \{\psi \in C(\mathbb{R}^1) : |\psi|_{(-\lambda)} < \infty \text{ for all } \lambda > 0\} \\ C_{rap} &= \{\psi \in C(\mathbb{R}^1) : |\psi|_{(\lambda)} < \infty \text{ for all } \lambda > 0\} \end{aligned}$$

where $|\psi|_{(\lambda)} = \|e^{\lambda|\cdot|}\psi\|_\infty$, $\lambda \in \mathbb{R}$, and $\|.\|_\infty$ is the usual supremum norm. C_{tem} and C_{rap} can be topologized by the metrics $d_{tem}(\phi, \psi) = \sum_{k=1}^\infty 2^{-k}(|\phi - \psi|_{(-1/k)} \wedge 1)$ and $d_{rap}(\phi, \psi) = \sum_{k=1}^\infty 2^{-k}(|\phi - \psi|_{(1/k)} \wedge 1)$, respectively. $\mathcal{S} := C_{rap}^\infty$ will play the rôle of the class of test functions. For proper functions ϕ and ψ on $\mathbb{R}^1$ set $\langle \phi, \psi \rangle = \int_{\mathbb{R}^1} \phi(x)\psi(x)dx$.

Definition 2.1 (strong and weak solution to SPDE) *A C_{tem}-valued continuous process $(u(t,.) : t \geq 0)$ is said to be a strong solution to SPDE (1.1) if, given the noise $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P}, W^\varrho)$, it is $(\mathcal{F}_t)$-adapted and*

$$\begin{aligned} \langle u(t,.), \psi \rangle &= \langle \eta, \psi \rangle + \int_0^t \langle u(r,.), \frac{1}{2}\Delta\psi \rangle dr \\ &+ \int_0^t \int_{\mathbb{R}^1} b(r,y,u(r,y))\psi(y)\sigma(dydr) + \int_0^t \int_{\mathbb{R}^1} a(r,y,u(r,y))\psi(y)W^\varrho(dydr) \end{aligned} \tag{2.5}$$

for all $t \geq 0$ and $\psi \in \mathcal{S}$, $\mathbf{P}$-almost surely. We say u is a weak solution to SPDE (1.1) if one can find any noise $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P}, W^\varrho)$ such that u is $(\mathcal{F}_t)$-adapted and (2.5) holds.

Definition 2.2 (strong and weak uniqueness of solutions) *A solution to SPDE (1.1) is said to be strongly unique if for any two solutions u and u' w.r.t. a given noise $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P}, W^\varrho)$, $u(t,x) = u'(t,x)$ for all $t \geq 0$ and $x \in \mathbb{R}^1$, $\mathbf{P}$-almost surely. We say a solution is weakly unique if any two solutions coincide in law.*

Next we are going to pose what sort of measures $\sigma(dxdt)$ and $\varrho(dxdt)$ will be considered. Let $\mathcal{M}(\mathbb{R}^1)$ be the space of positive Borel measures on $\mathbb{R}^1$ and equip

$$\mathcal{M}_{uni} := \{\mu \in \mathcal{M}(\mathbb{R}^1) : \sup_{x \in \mathbb{R}^1} \mu(B(x,1)) < \infty\}.$$

with the vague topology.

Definition 2.3 (classes (MD) and (MN)) *A positive Borel measure $\mu(dxdt) = \mu_1(t,dx)\mu_2(dt)$ on $\mathbb{R}^1 \times [0,\infty)$ is said to be of class* (MD) *(respectively* (MN)*) if μ_1 is a measurable kernel from $[0,\infty)$ into $\mathcal{M}_{uni}$, $\mu_2(dt)$ a positive Borel measure on $[0,\infty)$ and if there are $\alpha_1, \alpha_2 \in [0,1]$ such that for all $T > 0$,*

(i) $\sup_{t \leq T} \sup_{x \in \mathbb{R}^1} \mu_1(t, B(x,r)) \leq c_T\ r^{\alpha_1} \quad \forall r \in (0,1]$,

(ii) $\sup_{t \leq T} \mu_2(B(t,r)) \leq c_T\ r^{\alpha_2} \quad \forall r \in (0,1]$,

(iii) $\frac{\alpha_1}{2} + \alpha_2 > \frac{1}{2}$ *(respectively* $\frac{\alpha_1}{2} + \alpha_2 > 1$*).*

Note that (MN) $\subset$ (MD) and that (MN) requires $\alpha_1, \alpha_2 > 0$. For the sake of simplicity we henceforth only work rigorously with product measures $\mu(dxdt) = \mu_1(dx)\mu_2(dt)$, that is $\mu_1(.,dx) \equiv \mu_1(dx)$. However, all proofs in the sequel can trivially be extended to the general case. Let us now turn to our main results.

Theorem 2.4 (strong existence and uniqueness, Lipschitz case) *Assume a and b are continuous and satisfy*

$$\forall T > 0\ \exists c_T > 0 : \quad |a(t,x,u)| + |b(t,x,u)| \le c_T(1+|u|) \quad \forall t \le T, x \in \mathbb{R}^1, u \in \mathbb{R} \tag{2.6}$$

as well as

$$\forall T > 0\ \exists L_T > 0 : \quad \forall t \le T, x \in \mathbb{R}^1, u, u' \in \mathbb{R} :$$
$$|a(t,x,u) - a(t,x,u')| + |b(t,x,u) - b(t,x,u')| \le L_T |u - u'|. \tag{2.7}$$

Let $\varrho(dxdt)$ be of class (MN) *with α_1, α_2, $\sigma(dxdt)$ of class* (MD) *with β_1, β_2 and $\eta \in C_{tem}$. Then SPDE (1.1) has a strong solution which is strongly unique. Moreover, the solution is locally Hölder-γ-continuous for all $\gamma \in (0, \frac{\alpha}{2} \wedge \beta)$ where $\alpha := \frac{\alpha_1}{2} + \alpha_2 - 1$, $\beta := \frac{\beta_1}{2} + \beta_2 - 1/2$.*

Theorem 2.5 (non-negativity) *In the setting of Theorem 2.4 assume additionally $\eta \in C^+_{tem}$ and*

$$a(t,x,0) = 0,\ b(t,x,0) \ge 0 \quad \forall t \ge 0, x \in \mathbb{R}^1 \tag{2.8}$$

as well as

$$\exists \kappa > 0\ \forall T > 0\ \exists L_T > 0 : \quad \forall t \le T, x, x' \in \mathbb{R}^1, u, u' \in \mathbb{R} :$$
$$|a(t,x,u) - a(t,x',u')| + |b(t,x,u) - b(t,x',u')| \le L_T(|x - x'|^\kappa + |u - u'|) \tag{2.9}$$

instead of (2.7). Then we have for the unique solution $u(.,.)$ from Theorem 2.4,

$$u(t,x) \ge 0 \quad \forall t \ge 0, x \in \mathbb{R}^1, \quad \mathbf{P}\text{-almost surely.} \tag{2.10}$$

Theorem 2.6 (weak existence, non-Lipschitz case) *Consider the case $a(t,x,u) = a(u)$, $b(t,x,u) = b(u)$ continuous and assume condition (2.6), i.e.*

$$\exists c > 0 : \quad |a(u)| + |b(u)| \le c(1+|u|) \quad \forall u \in \mathbb{R}. \tag{2.11}$$

Furthermore, let $\varrho(dxdt)$ be of class (MN) *with α_1, α_2, $\sigma(dxdt)$ of class* (MD) *with β_1, β_2 and $\eta \in C_{tem}$. Then SPDE (1.1) has a weak solution being locally Hölder-γ-continuous for all $\gamma \in (0, \frac{\alpha}{2} \wedge \beta)$ where $\alpha := \frac{\alpha_1}{2} + \alpha_2 - 1$, $\beta := \frac{\beta_1}{2} + \beta_2 - 1/2$. If we additionally assume $\eta \in C^+_{tem}$ and condition (2.8), that is $a(0) = 0$ and $b(0) \ge 0$, then the solution is non-negative in sense of (2.10).*

Note that a and b in Theorem 2.6 could also depend on t and x. In that case just make sure that they can be uniformly approximated by functions that fit into Theorem 2.4.

In the Lipschitz case, strong uniqueness of solutions can be obtained comparatively easily. In the non-Lipschitz case, however, the question of uniqueness becomes much more delicate. While statements on strong uniqueness do not exist so far, weak uniqueness could be established, e.g., for equation (1.2) with $b \equiv 0$, $a(t,x,u) = u^\gamma$ and $\gamma = \frac{1}{2}$ ([RC86]) or $\gamma \in (\frac{1}{2}, 1)$ ([Myt98]). We now focus on equation (1.1) with $b \equiv 0$, $a(t,x,u) = \sqrt{u}$ and $\varrho(dxdt) = \varrho_1(t,dx)dt$ since here, as already mentioned in Chapter 1, the solution is the space-time Lebesgue density of the catalytic super-Brownian motion with catalyst ϱ.

Theorem 2.7 (weak uniqueness, catalytic super-Brownian motion) *Consider* $\eta \in C^+_{tem}$ *and* $\varrho(dxdt) = \varrho_1(t,dx)dt$ *of class* (MN). *Then the (non-negative) weak solution to SPDE (1.1) with* $b \equiv 0$ *and* $a(t,x,u) = \sqrt{u}$ *is weakly unique.*

The remainder of the paper is organized as follows. In the next chapter we give a series of technical lemmas. In Chapter 4 we shall establish the equivalence of SPDE (1.1) in sense of Definition 2.1 to a certain martingale problem and to a certain stochastic integral equation. Chapters 5, 6, 7 and 8 are devoted to the proofs of Theorems 2.4, 2.5, 2.6 and 2.7, respectively. Note that Theorems 2.4 and 2.6 were proved by Iwata [Iwa87] (under some stronger assumption on the coefficients) and Shiga [Shi94] (under the same assumptions on the coefficients) for SPDE (1.2). See also [MP92]. Shiga ([Shi94], Appendix) verified Theorem 2.5 for SPDE (1.2). The key for the proof of Theorem 2.7 is the method of duality.

3 Technical lemmas

Here we are going to provide basic tools for the proofs of our main results. We start with a result from [Iwa87], Lemma 5.4; see also [Shi94], Lemma 6.3. Note that $C([0,T],C_{tem})$ denotes the space of C_{tem}-valued continuous functions on $[0,T]$ and that a sequence of random elements is called tight if the sequence of their laws is tight.

Lemma 3.1 (Kolmogorov-type criterion) *(i) A stochastic process* $(u(t,x) : t \in [0,T], x \in \mathbb{R}^1)$ *has a* C_{tem}*-valued continuous modification if for every* $\lambda > 0$ *there are constants* $m, \epsilon, c_\lambda > 0$ *such that*

$$\mathbf{E}\Big[|u(t,x) - u(t',x')|^m\Big] \le c_\lambda\Big(|t-t'|^{2+\epsilon} + |x-x'|^{2+\epsilon}\Big)e^{\lambda|x|}$$

for all $t,t' \in [0,T]$ *and* $x,x' \in \mathbb{R}^1$ *with* $|x-x'| \le 1$. *In this case,* u *is locally jointly Hölder-*γ*-continuous for all* $\gamma \in (0,\epsilon/m)$.

(ii) A sequence $(u_n)_{n\ge1}$ *of* C_{tem}*-valued continuous processes* $(u_n(t,x) : t \in [0,T], x \in \mathbb{R}^1)$ *is tight in* $C([0,T],C_{tem})$ *if the sequence* $(u(0,.))_{n\ge1}$ *is tight in* C_{tem} *and for every* $\lambda > 0$ *there are constants* $m, \epsilon, c_\lambda > 0$ *such that*

$$\mathbf{E}\Big[|u_n(t,x) - u_n(t',x')|^m\Big] \le c_\lambda\Big(|t-t'|^{2+\epsilon} + |x-x'|^{2+\epsilon}\Big)e^{\lambda|x|}$$

for all $t,t' \in [0,T]$, $x,x' \in \mathbb{R}^1$ *with* $|x-x'| \le 1$ *and* $n \ge 1$. *In this case, any limit point* u *of* (u_n) *is locally jointly Hölder-*γ*-continuous for all* $\gamma \in (0,\epsilon/m)$.

The statements on the local Hölder-continuity are not explicitly given in [Iwa87] or [Shi94] but they can easily be verified using the classical Kolmogorov theorem, see e.g. Corollary 1.2 of [Wal86].

Lemma 3.2 (continuity of W^ϱ) *Let* $\varrho(dxdt)$ *be a positive Borel measure as in Definition 2.3 with* $\alpha_1, \alpha_2 > 0$ *instead of (iii) (that is for example of class* (MN)*). Then, the orthogonal martingale measure* W^ϱ *from Chapter 2 is a continuous one. Particularly, the stochastic integral* $f \bullet W^\varrho$ *is a continuous orthogonal martingale measure for every predictable* $f : [0,\infty) \times \mathbb{R}^1 \times \Omega \to \mathbb{R}$ *with* $\mathbf{E}[\int_0^t \int_{\mathbb{R}^1} f^2(r,y)\varrho(dydr)] < \infty\ \forall t \ge 0$.

Proof Consider a bounded Borel set B in $\mathbb{R}^1$, $T > 0$, $0 \le t \le t' \le T$ and recall $W^\varrho(A) \sim N(0, \varrho(A))$. Then, for $m \ge 1$,

$$\mathbf{E}\Big[\Big|W_t^\varrho(B) - W_{t'}^\varrho(B)\Big|^{2m}\Big] = \mathbf{E}\Big[\Big|W^\varrho(B \times (t,t'])\Big|^{2m}\Big] \le c_{B,T}|t-t'|^{2m\alpha_2}.$$

Hence, for m sufficiently large, Kolmogorov's theorem gives a continuous modification of $W_\cdot^\varrho(B)$ on $[0,T]$ for every $T > 0$. □

Lemma 3.3 (Burkholder-Davis-Gundy - type inequality) *Let $\varrho(dxdt)$ be a positive Borel measure as in Lemma 3.2. Then, for every $(\mathcal{F}_t)$-predictable $f : [0,\infty)\times\mathbb{R}^1\times\Omega \to \mathbb{R}$ with $\mathbf{E}[\int_0^t \int_{\mathbb{R}^1} f^2(r,y)\varrho(dydr)] < \infty$ $\forall t \ge 0$ and for every $m \ge 1$,*

$$\mathbf{E}\Big[\Big(\int_0^t \int_{\mathbb{R}^1} f(r,y) W^\varrho(dydr)\Big)^{2m}\Big] \le c_m \mathbf{E}\Big[\Big(\int_0^t \int_{\mathbb{R}^1} f^2(r,y)\varrho(dydr)\Big)^{m}\Big] \quad \forall t \ge 0.$$

Proof The statement is an immediate consequence of Lemma 3.2 and the Burkholder-Davis-Gundy inequality for continuous square-integrable martingales. □

The simple proof of the next lemma can be found, for instance, in [Mat95], p.15.

Lemma 3.4 (change of variable) *Let $\mu(dx)$ be a Borel measure on $\mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}_+$ Borel-measurable. Then,*

$$\int_{\mathbb{R}^1} g(x)\mu(dx) = \int_0^\infty \mu(x \in \mathbb{R} : g(x) \ge u)du.$$

Lemma 3.5 *Consider $\mu_1(dx) \in \mathcal{M}_{uni}$, $\alpha_1 \in (0,1]$, $x \in \mathbb{R}^1$, $\gamma > 0$ and*

(i) $\mu_1(B(x,r)) \le c\ r^{\alpha_1} \quad \forall r \in (0,1]$

(ii) $\int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{r}}\mu_1(dy) \le c\ r^{\alpha_1/2} \quad \forall r \in (0,1]$

(iii) $\int_{\mathbb{R}^1} |x-y|^\gamma e^{-\frac{(x-y)^2}{r}}\mu_1(dy) \le c_\gamma\ r^{\gamma/2+\alpha_1/2} \quad \forall r \in (0,1]$

(iv) $\sup_{x\in\mathbb{R}^1}\ e^{-\lambda|x|}\int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{r}} e^{+\lambda|y|}\mu_1(dy) \le c_\lambda\ r^{\alpha_1/2} \quad \forall r \in (0,1]$

(v) $\sup_{x\in\mathbb{R}^1}\sup_{x'\in\mathbb{R}^1}\ e^{-\lambda|x-x'|}e^{-\lambda|x|}\int_{\mathbb{R}^1} e^{-\frac{(x'-y)^2}{r}} e^{+\lambda|y|}\mu_1(dy) \le c_\lambda\ r^{\alpha_1/2} \quad \forall r \in (0,1]$.

Then, (i) ⇔ (ii) ⇒ (iii). Moreover, when (ii) holds uniformly in x on $\mathbb{R}^1$, (ii) ⇒ (iv), (v) for every $\lambda > 0$.

Proof (ii)⇒(i) Assuming *(ii)* we trivially get for all $r \in (0,1]$

$$e^{-1}\mu_1(B(x,\sqrt{r})) = \int_{\mathbb{R}^1} e^{-1}\mathbf{1}_{B(x,\sqrt{r})}(y)\mu_1(dy) \le \int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{r}}\mu_1(dy) \le c\ r^{\alpha_1/2}.$$

(i)⇒(ii) Using Lemma 3.4, one easily calculates

$$\int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{r}} \mu_1(dy) = \int_0^\infty \mu_1(y : e^{-\frac{(x-y)^2}{r}} \geq u) du = \int_0^1 \mu_1(B(x, (r \log \frac{1}{u})^{1/2})) du.$$

Substituting $s = \log \frac{1}{u}$, applying (ii) and noting that $\mu_2(dt)$ behaves globally as the Lebesgue measure dt, we get for $r \in (0,1]$

$$\int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{r}} \mu_1(dy) = \int_0^\infty e^{-s} \mu_1(B(x, \sqrt{sr})) ds$$

$$\leq \int_0^1 c\ (sr)^{\alpha_1/2} ds + \int_1^\infty e^{-s} c\sqrt{sr} ds \ \leq\ c\ r^{\alpha_1/2} + c\sqrt{r}\ \leq\ c\ r^{\alpha_1/2}.$$

(i)⇒(iii) For the sake of clarity assume $\gamma = 2$, the proof for general γ works analogously. Then the integral in (iii) equals

$$\int_0^\infty \mu_1\Big(y : |x-y|^2 e^{-\frac{(x-y)^2}{r}} \geq u\Big) du = \int_0^{r/e} \mu_1\Big(y : \frac{|x-y|^2}{r} e^{-\frac{(x-y)^2}{r}} \geq \frac{u}{r}\Big) du.$$

According to the elementary inequality $z^2 e^{-z^2} \leq e^{-z}$, $z \geq 0$, the latter term is bounded by

$$\int_0^{r/e} \mu_1\Big(y : e^{-\frac{|x-y|}{\sqrt{r}}} \geq \frac{u}{r}\Big) du$$

$$\leq\ \int_0^{r/e} \mu_1\Big(y : \sqrt{r} \log \frac{r}{u} \geq |x-y|\Big) du\ =\ \int_0^{r/e} \mu_1\Big(B(x, \sqrt{r} \log \frac{r}{u})\Big) du.$$

Substituting $s = \sqrt{r} \log \frac{r}{u}$, the inequality continues for $r \in (0,1]$

$$\leq\ \int_{\sqrt{r}}^\infty \mu_1(B(x,s)) \sqrt{r} e^{-s/\sqrt{r}} ds\ \leq\ \sqrt{r} \int_{\sqrt{r}}^1 c\ s^{\alpha_1} e^{-s/\sqrt{r}} ds + \sqrt{r} \int_1^\infty c\ s e^{-s/\sqrt{r}} ds$$

$$\leq\ c\ r^{1+\alpha_1/2} \int_0^\infty a^{\alpha_1} e^{-a} da + c\ r^{3/2} \int_0^\infty a e^{-a} da\ \leq\ c\ r^{1+\alpha_1/2}.$$

(ii)⇒(iv) Setting $\mu_1^x(dy) = \mu_1(x+dy)$ and using the assumption,

$$\int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{r}} e^{\lambda|y|} \mu_1(dy) = \int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{r}} e^{\lambda(|y|-|x|)} \mu_1(dy) e^{\lambda|x|}$$

$$\leq\ \int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{r}} e^{\lambda|x-y|} \mu_1(dy) e^{\lambda|x|} = \int_{\mathbb{R}^1} e^{-\frac{y^2}{r}} e^{\lambda|y|} \mu_1^x(dy) e^{\lambda|x|}$$

$$=\ \int_0^\infty e^{-\frac{(y-\frac{1}{2}r\lambda)^2}{r}} \mu_1^x(dy) e^{\frac{(\frac{1}{2}r\lambda)^2}{r}} e^{\lambda|x|} + \int_{-\infty}^0 e^{-\frac{(y+\frac{1}{2}r\lambda)^2}{r}} \mu_1^x(dy) e^{\frac{(\frac{1}{2}r\lambda)^2}{r}} e^{\lambda|x|}$$

$$\leq\ c_\lambda\ r^{\alpha_1/2}\ e^{\lambda|x|}.$$

(ii)⇒(v) First estimate

$$\int_{\mathbb{R}^1} e^{-\frac{(x'-y)^2}{r}} e^{\lambda|y|}\mu_1(dy) = \int_{\mathbb{R}^1} e^{-\frac{(x'-y)^2}{r}} e^{\lambda(|y|-|x|)}\mu_1(dy)e^{\lambda|x|}$$
$$\le \int_{\mathbb{R}^1} e^{-\frac{(x'-y)^2}{r}} e^{\lambda|x-y|}\mu_1(dy)e^{\lambda|x|} \le \int_{\mathbb{R}^1} e^{-\frac{(x'-y)^2}{r}} e^{\lambda(|x-x'|+|x'-y|)}\mu_1(dy)e^{\lambda|x|}$$
$$\le \int_{\mathbb{R}^1} e^{-\frac{(x'-y)^2}{r}} e^{\lambda|x'-y|}\mu_1(dy)e^{\lambda|x-x'|}e^{\lambda|x|},$$

then proceed as in the proof of $(ii) \Rightarrow (iv)$. □

Lemma 3.6 *Consider* $\alpha_2 \in (0,1]$ *and a Borel measure* $\mu_2(dt)$ *on* $[0,\infty)$ *with* $\mu_2([0,T]) < \infty$ *for all* $T > 0$. *Then,*

(o) $\sup_{t\in[0,T]} \mu_2(B(t,r)) \le c_T\ r^{\alpha_2} \quad \forall r \in (0,1]$

implies the following inequalities for all $0 \le t \le T$

(i) $\int_0^t r^{-\gamma}\mu_2(dr) \le c_T\ t^{\alpha_2-\gamma} \quad \forall\gamma \in [0,\alpha_2)$

(ii) $\int_u^t r^{-\gamma}\mu_2(dr) \le c_T\ u^{\alpha_2-\gamma} \quad \forall 0 < u \le t \quad \forall\gamma \in (\alpha_2,\infty)$

(iii) $\int_0^t \frac{r^\delta}{(t-r)^\gamma}\mu_2(dr) \le c_T\ t^{\delta+\alpha_2-\gamma}(\theta^\delta + (1-\theta)^{\alpha_2-\gamma}) \quad \forall\theta \in (0,1], \gamma \in [0,\alpha_2), \delta > 0$

(iv) $\int_0^T e^{-\gamma r}\mu_2(dr) \le c_T\ \gamma^{-\alpha_2} \quad \forall\gamma > 0.$

Proof **(i)** Using Lemma 3.4,

$$\int_0^t r^{-\gamma}\mu_2(dr) = \int_0^\infty \mu_2\Big(r : \mathbf{1}_{[0,t]}(r)r^{-\gamma} \ge u\Big)du \le \int_0^{t^{-\gamma}} \mu_2([0,t])du +$$
$$\int_{t^{-\gamma}}^\infty \mu_2\Big([0,t]\cap[0,u^{-1/\gamma}]\Big)du \ \le\ t^{-\gamma}c_T\ t^{\alpha_2} + \int_{t^{-\gamma}}^\infty \mu_2\Big([0,u^{-1/\gamma}]\Big)du.$$

A substitution $v = u^{-1/\gamma}$ yields the bound

$$c_T\ t^{\alpha_2-\gamma} + \int_0^t \mu_2\Big([0,v]\Big)\gamma v^{-\gamma-1}dv = c_T\ t^{\alpha_2-\gamma} + c_T\ \gamma\int_0^t v^{\alpha_2-\gamma-1}dv \le c_T\ t^{\alpha_2-\gamma}.$$

(ii) can be proved analogously to (i).
(iii) Setting $\mu_2^t(dr) = \mu_2(t-dr)$ one easily estimates

$$\int_0^t \frac{r^\delta}{(t-r)^\gamma}\mu_2(dr) \le \int_0^{\theta t} \frac{r^\delta}{(t-r)^\gamma}\mu_2(dr) + \int_{\theta t}^t \frac{r^\delta}{(t-r)^\gamma}\mu_2(dr)$$
$$\le (\theta t)^\delta\int_0^{\theta t} \frac{1}{(t-r)^\gamma}\mu_2(dr) + t^\delta\int_{\theta t}^t \frac{1}{(t-r)^\gamma}\mu_2(dr)$$
$$\le (\theta t)^\delta\int_{(1-\theta)t}^t \frac{1}{r^\gamma}\mu_2^t(dr) + t^\delta\int_0^{(1-\theta)t} \frac{1}{r^\gamma}\mu_2^t(dr).$$

Part (i) gives the bound $(\theta t)^\delta c_T t^{\alpha_2-\gamma} + t^\delta c_T((1-\theta)t)^{\alpha_2-\gamma} \le c_T t^{\delta+\alpha_2-\gamma}(\theta^\delta + (1-\theta)^{\alpha_2-\gamma})$.

(iv) With help of Lemma 3.4 one gets

$$\int_0^T e^{-\gamma r}\mu_2(dr) = \int_0^\infty \mu_2\Big(r : \mathbf{1}_{[0,T]}(r)e^{-\gamma r} \ge u\Big)du \le \int_0^1 \mu_2\Big([0, T\wedge \frac{1}{\gamma}\log\frac{1}{u}]\Big)du.$$

Substituting $v = \log\frac{1}{u}$, this can be bounded by

$$\begin{aligned}\int_0^\infty \mu_2\Big([0,T\wedge \frac{1}{\gamma}v]\Big)e^{-v}dv \;&=\; \int_0^{T\gamma} \mu_2\Big([0,T\wedge \frac{1}{\gamma}v]\Big)e^{-v}dv + \int_{T\gamma}^\infty \mu_2\Big([0,T\wedge \frac{1}{\gamma}v]\Big)e^{-v}dv \\ &\le\; \int_0^{T\gamma} c_T\Big(\frac{1}{\gamma}v\Big)^{\alpha_2}e^{-v}dv + c_T\,T^{\alpha_2}\int_{T\gamma}^\infty e^{-v}dv \;\le\; c_T\,\gamma^{-\alpha_2} + c_T\,T^{\alpha_2}e^{-T\gamma} \;\le\; c_{T,\alpha_2}\,\gamma^{-\alpha_2}\end{aligned}$$

where for the last inequality the estimate $e^{-T\gamma} \le c_{T,\alpha_2}\,\gamma^{-\alpha_2}\ \forall\gamma > 0$ was used. □

Before turning to the next lemma let us introduce the heat semigroup $(P_t)_{t\ge 0}$ corresponding to $\frac{1}{2}\Delta$. It is induced by $P_t\psi(x) = \int_{\mathbb{R}^1} p_t(x,y)\psi(y)dy$, $t > 0$, $x\in\mathbb{R}^1$ and $\psi\in C_b$, where p is the heat kernel given via $p_t(x,y) = (2\pi t)^{-1/2}e^{-\frac{(x-y)^2}{2t}}$, $t > 0$ and $x\in\mathbb{R}^1$. We set $p_t \equiv 0$, $t < 0$, and recall a known inequality. For every $\epsilon > 0$, $0 < t < t'$ and $x, y\in\mathbb{R}^1$,

$$|p_t(x,y) - p_{t'}(x,y)| \le c_\epsilon \int_t^{t'} \frac{1}{u}p_{(1+\epsilon)u}(x,y)du. \tag{3.12}$$

Lemma 3.7 *Let $\varrho(dxdt)$ be a Borel measure on $\mathbb{R}^1\times[0,\infty)$ of class* (MN) *with α_1,α_2 and $\sigma(dxdt)$ a Borel measure on $\mathbb{R}^1\times[0,\infty)$ of class* (MD) *with β_1,β_2. Set $\alpha := \alpha_1/2+\alpha_2-1$, $\beta := \beta_1+\beta_2-1/2$ and $p_u :\equiv 0$, $u < 0$. Then for all $(t,x),(t',x')\in[0,T]\times\mathbb{R}^1$ with $|t-t'|\le 1$, $|x-x'|\le 1$, w.l.o.g. $t\le t'$,*

$$\int_0^{t'}\int_{\mathbb{R}^1}\Big(p_{t-r}(x,y) - p_{t'-r}(x',y)\Big)^2\varrho(dydr) \le c_T\Big(|t-t'|^\alpha + |x-x'|^{2\alpha}\Big), \tag{3.13}$$

$$\int_0^{t'}\int_{\mathbb{R}^1}\Big|p_{t-r}(x,y) - p_{t'-r}(x',y)\Big|\sigma(dydr) \le c_T\Big(|t-t'|^\beta + |x-x'|^{2\beta}\Big). \tag{3.14}$$

Proof We only prove (3.13). (3.14) can be obtained analogously. The l.h.s. of (3.13) is bounded by

$$\begin{aligned}&\int_0^{t'}\int_{\mathbb{R}^1}\Big(p_{t'-r}(x',y) - p_{t'-r}(x,y)\Big)^2\varrho(dydr) \\ &+\; \int_0^{t}\int_{\mathbb{R}^1}\Big(p_{t'-r}(x,y) - p_{t-r}(x,y)\Big)^2\varrho(dydr) + \int_t^{t'}\int_{\mathbb{R}^1} p^2_{t'-r}(x,y)\varrho(dydr).\end{aligned} \tag{3.15}$$

The last term in (3.15) can be estimated, with help of Lemma 3.5 (ii), by

$$c\int_t^{t'}\frac{1}{t'-r}\int_{\mathbb{R}^1} e^{-\frac{(x-y)^2}{t'-r}}\varrho_1(dy)\varrho_2(dr) \le c\int_t^{t'}\frac{1}{t'-r}c\,(t'-r)^{\alpha_1/2}\varrho_2(dr)$$

which, by Lemma 3.6 (i), is dominated by $c_T|t-t'|^{\alpha_1/2+\alpha_2-1} = c_T|t-t'|^{\alpha}$.

The middle term in (3.15) equals

$$\int_0^{t-|t-t'|}\int_{\mathbb{R}^1}\Big(p_{t'-r}(x,y)-p_{t-r}(x,y)\Big)^2\varrho(dydr) \qquad (3.16)$$
$$+\int_{t-|t-t'|}^{t}\int_{\mathbb{R}^1}\Big(p_{t'-r}(x,y)-p_{t-r}(x,y)\Big)^2\varrho(dydr).$$

The second summand in (3.16) can be estimated by

$$c\int_{t-|t-t'|}^{t}\Big(\frac{1}{t-r}\int_{\mathbb{R}^1}e^{-\frac{(x-y)^2}{t-r}}\varrho_1(dy)$$
$$+\frac{1}{t'-r}\int_{\mathbb{R}^1}e^{-\frac{(x-y)^2}{t'-r}}\varrho_1(dy)+\frac{1}{t-r}\int_{\mathbb{R}^1}e^{-\frac{(x-y)^2}{2(t-r)}}\varrho_1(dy)\Big)\varrho_2(dr)$$

and so, again using Lemma 3.5 (ii) and Lemma 3.6 (i), by $c_T|t-t'|^{\alpha}$. Let $\epsilon>0$, then inequality (3.12) bounds the first summand in (3.16) by

$$c_\epsilon\int_0^{t-|t-t'|}\int_{\mathbb{R}^1}\Big(\int_{t-r}^{t'-r}\frac{1}{u}p_{(1+\epsilon)u}(x,y)du\Big)^2\varrho(dydr).$$

Once more exploiting Lemma 3.5 (ii) as well as Lemma 3.6 (ii), this can be estimated by

$$c\int_0^{t-|t-t'|}\int_{\mathbb{R}^1}\Big(\int_{t-r}^{t'-r}\frac{1}{u^{3/2}}e^{-\frac{(x-y)^2}{2(1+\epsilon)(t'-r)}}du\Big)^2\varrho_1(dy)\varrho_2(dr)$$
$$\leq c\int_0^{t-|t-t'|}\Big(\int_{t-r}^{t'-r}\frac{1}{u^{3/2}}du\Big)^2\int_{\mathbb{R}^1}e^{-\frac{(x-y)^2}{(1+\epsilon)(t'-r)}}\varrho_1(dy)\varrho_2(dr)$$
$$\leq c\int_0^{t-|t-t'|}\Big((t-r)^{-1/2}-(t'-r)^{-1/2}\Big)^2c_T(t'-r)^{\alpha_1/2}\varrho_2(dr)$$
$$\leq c_T\int_0^{t-|t-t'|}\frac{|t-t'|}{(t-r)(t'-r)}(t'-r)^{\alpha_1/2}\varrho_2(dr)$$
$$\leq c_T\,|t-t'|\int_0^{t-|t-t'|}\frac{1}{(t-r)^{2-\alpha_1/2}}\varrho_2(dr)\ \leq\ c_T|t-t'|^{\alpha}.$$

Therefore, (3.16) is dominated by $c_T|t-t'|^{\alpha}$.

The first term in (3.15) is smaller than

$$c\int_{t'-|x-x'|^2}^{t'}\frac{1}{t'-r}\int_{\mathbb{R}^1}\Big(e^{-\frac{(x'-y)^2}{t'-r}}+e^{-\frac{(x-y)^2}{t'-r}}\Big)\varrho(dydr) \qquad (3.17)$$
$$+\ c\int_0^{t'-|x-x'|^2}\frac{1}{t'-r}\int_{\mathbb{R}^1}\Big(e^{-\frac{(x'-y)^2}{2(t'-r)}}-e^{-\frac{(x-y)^2}{2(t'-r)}}\Big)^2\varrho(dydr).$$

Using Lemma 3.5 (ii) and Lemma 3.6 (i), the first summand in (3.17) can immediately be bounded by $c_T|x-x'|^{2\alpha}$. According to the mean value theorem for differentials, the second summand in (3.17) has the bound

$$c\int_0^{t'-|x-x'|^2}\frac{1}{t'-r}\int_{\mathbb{R}^1}\Big(|x-x'|\frac{|\bar{x}-y|}{t'-r}e^{-\frac{(\bar{x}-y)^2}{2(t'-r)}}\Big)^2\varrho(dydr)$$

for some $\bar{x}$ between x and x'. But by Lemma 3.5 (iii) and Lemma 3.6 (ii) this is dominated by

$$c_T|x-x'|^2\int_0^{t'-|x-x'|^2}\frac{1}{(t'-r)^3}(t'-r)^{1+\alpha_1/2}\varrho_2(dr)$$

$$\leq c_T\,|x-x'|^2\frac{|x-x'|^{\alpha_1}}{|x-x'|}\int_0^{t'-|x-x'|^2}\frac{1}{(t'-r)^{3/2}}\varrho_2(dr)\ \leq\ c_T|x-x'|^{2\alpha}.$$

Altogether we have the desired bound for (3.15), i.e. for the l.h.s. of (3.13). □

Lemma 3.8 (Gronwall-type lemma) *Consider Borel measures $\mu_2^i(dt)$ on $[0,\infty)$ satisfying $\mu_2^i([0,T])<\infty$ and (o) of Lemma 3.6 with $\alpha_2^i\in(0,1]$ for all $T>0$, $1\leq i\leq k$ and some $k\geq 1$. Moreover, let g and $g_n:[0,\infty)\to[0,\infty)$ be measurable functions $\forall n\geq 1$ with g,g_1 bounded and consider $\gamma^i\in[0,\alpha_2^i)$ $(1\leq i\leq k)$, $c_0\geq 0$ such that*

$$g_{n+1}(t)\leq c_T\Big(c_0+\sum_{i=1}^k\sup_{s\leq t}\int_0^s\frac{1}{(s-r)^{\gamma^i}}g_n(r)\mu_2^i(dr)\Big)\qquad\forall t\leq T,n\geq 1,$$

$$g(t)\leq c_T\Big(c_0+\sum_{i=1}^k\sup_{s\leq t}\int_0^s\frac{1}{(s-r)^{\gamma^i}}g(r)\mu_2^i(dr)\Big)\qquad\forall t\leq T.$$

Then for every $T>0$ there is a constant $q\in(0,1)$ such that

(i) $\sup_{t\leq T}g_n(t)\leq\tilde{c}_T\ c_0\ +\ \tilde{c}_T\ q^n\quad\forall n\geq 1,$

(ii) $\sup_{t\leq T}g(t)\leq\tilde{c}_T\ c_0.$

Note that the following proof also works (with the obvious changes) if one replaces $\sup_{s\leq t}\int_0^s\frac{1}{(s-r)^{\gamma^i}}\cdots$ by $\int_0^t\frac{1}{(t-r)^{\gamma^i}}\cdots$ in the assumptions of the lemma.

Proof Set $\delta=\min\{\alpha_2^i-\gamma^i:1\leq i\leq k\}>0$. By assumption $g_{n+1}(t)$ is bounded by

$$\sum_{i_1=1}^k c_T\Big(c_0+\sup_{t_1\in[0,t]}\int_0^{t_1}\frac{1}{(t_1-s_1)^{\gamma^{i_1}}}\sum_{i_2=1}^k c_T\Big(c_0+\sup_{t_2\in[0,s_1]}\int_0^{t_2}\frac{1}{(t_2-s_2)^{\gamma^{i_2}}}\cdots$$

$$\sum_{i_{n-1}=1}^k c_T\Big(c_0+\sup_{t_{n-1}\in[0,s_{n-2}]}\int_0^{t_{n-1}}\frac{1}{(t_{n-1}-s_{n-1})^{\gamma^{i_{n-1}}}}\tag{3.18}$$

$$\sum_{i_n=1}^k c_T\Big(c_0+\sup_{t_n\in[0,s_{n-1}]}\int_0^{t_n}\frac{1}{(t_n-s_n)^{\gamma^{i_n}}}g_1(s_n)\mu_2^{i_n}(ds_n)\Big)\cdots\Big)\mu_2^{i_1}(ds_1).$$

Since g_1 is bounded, Lemma 3.6 (i) bounds $\sup_{t_n\in[0,s_{n-1}]}\int_0^{t_n}\frac{1}{(t_n-s_n)^{\gamma^{i_n}}}g_1(s_n)\mu_2^{i_n}(ds_n)$ by $c\,s_{n-1}^{\alpha_2^{i_n}-\gamma^{i_n}}\leq\bar{c}_T s_{n-1}^{\delta}$. Hence, (3.18) can be estimated by

$$\bar{c}_T\sum_{i_1=1}^{k}c_T\Big(c_0+\sup_{t_1\in[0,t]}\int_0^{t_1}\frac{1}{(t_1-s_1)^{\gamma^{i_1}}}\sum_{i_2=1}^{k}c_T\Big(c_0+\sup_{t_2\in[0,s_1]}\int_0^{t_2}\frac{1}{(t_2-s_2)^{\gamma^{i_2}}}\cdots$$

$$\sum_{i_{n-2}=1}^{k}c_T\Big(c_0+\sup_{t_{n-2}\in[0,s_{n-3}]}\int_0^{t_{n-2}}\frac{1}{(t_{n-2}-s_{n-2})^{\gamma^{i_{n-2}}}}\sum_{i_{n-1}=1}^{k}c_T \tag{3.19}$$

$$\Big(c_0+\sup_{t_{n-1}\in[0,s_{n-2}]}\int_0^{t_{n-1}}\frac{1}{(t_{n-1}-s_{n-1})^{\gamma^{i_{n-1}}}}kc_T(c_0+s_{n-1}^{\delta})\mu_2^{i_{n-1}}(ds_{n-1})\Big)\cdots\Big)\mu_2^{i_1}(ds_1)$$

The expression in the most inner brackets is dominated by $[c_0+(kc_T)c_0s_{n-2}^{\delta}+(ckc_T)s_{n-2}^{2\delta}(\theta^{\delta}+(1-\theta)^{\delta})]$ for every $\theta\in(0,1)$. For the second summand we again used Lemma 3.6 (i), for the last summand we applied Lemma 3.6 (iii). The same steps bound the expression in the next brackets in (3.19), i.e. the expression in the brackets containing the $\int_0^{t_{n-2}}$-integral, by $[c_0+(ckc_T)c_0s_{n-3}^{\delta}+(ckc_T)^2c_0s_{n-3}^{2\delta}(\theta^{\delta}+(1-\theta)^{\delta})+(ckc_T)^2s_{n-3}^{3\delta}(\theta^{2\delta}+(1-\theta)^{\delta})(\theta^{\delta}+(1-\theta)^{\delta})]$. Going ahead recursively one estimates the term in (3.19) by

$$\begin{aligned}&\bar{c}_Tc_0\Big[(ckc_T)+(ckc_T)^2t^{\delta}+(ckc_T)^3t^{2\delta}\Big(\theta^{\delta}+(1-\theta)^{\delta}\Big)+(ckc_T)^4t^{3\delta}\Big(\theta^{2\delta}+(1-\theta)^{\delta}\Big)\\&\times\Big(\theta^{\delta}+(1-\theta)^{\delta}\Big)+\ldots+(ckc_T)^nt^{(n-1)\delta}\Big(\theta^{(n-1)\delta}+(1-\theta)^{\delta}\Big)\cdots\Big(\theta^{\delta}+(1-\theta)^{\delta}\Big)\Big]\\&+\bar{c}_T(ckc_T)^nt^{n\delta}\Big(\theta^{n\delta}+(1-\theta)^{\delta}\Big)\Big(\theta^{(n-1)\delta}+(1-\theta)^{\delta}\Big)\cdots\Big(\theta^{\delta}+(1-\theta)^{\delta}\Big).\end{aligned} \tag{3.20}$$

Now choose $\theta\in(0,1)$ sufficiently close to 1 and note that $\theta^{j\delta}$ is sufficiently close to 0 for all j greater than some sufficiently large j_0. Thus, for some $q\in(0,1)$, (3.20) can be estimated by

$$c_{T,j_0}c_0\Big[1+q+q^2+q^3+\ldots+q^{n-1}\Big]+c_{T,j_0}q^n\leq c_{T,j_0}c_0\sum_{i=0}^{\infty}q^i+q^n\leq\tilde{c}_Tc_0+\tilde{c}_Tq^n$$

for all $t\leq T$, which proves part (i). Assertion (ii) is an immediate consequence of (i). Indeed, set $g_n=g$ for all $n\geq 1$. □

4 SPDE (1.1) reformulated

In this chapter we establish the equivalence of SPDE (1.1) to a certain martingale problem (MP) and a certain stochastic integral equation (SIE).

Definition 4.1 (martingale problem) *The law on some* $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ *of an* $(\mathcal{F}_t)$*-adapted,* C_{tem}*-valued continuous process* $(u(t,.) : t \geq 0)$ *is said to be solution to the* (a, b, η)*-martingale problem if under this law*

$$M_t(\psi) = \langle u(t,.), \psi\rangle - \langle \eta, \psi\rangle - \int_0^t \langle u(r,.), \frac{1}{2}\Delta\psi\rangle dr - \int_0^t \int_{\mathbb{R}^1} b(r,y,u(r,y))\psi(y)\sigma(dydr),$$

$t \geq 0$*, is a square-integrable continuous* $(\mathcal{F}_t)$*-martingale having*

$$\langle M(\psi)\rangle_t = \int_0^t \int_{\mathbb{R}^1} a^2(r,y,u(r,y))\psi^2(y)\varrho(dydr),$$

$t \geq 0$*, as its quadratic variation process for all* $\psi \in \mathcal{S}$*. The solution is said to be unique if any two solutions coincide (in law).*

Recall that (P_t) was the heat semigroup defined before Lemma 3.7.

Definition 4.2 (stochastic integral equation) *Given the noise* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P}, W^\varrho)$*, an* $(\mathcal{F}_t)$*-adapted* C_{tem}*-valued continuous process* $(u(t,.) : t \geq 0)$ *satisfying* $\mathbf{P}$*-almost surely*

$$\begin{aligned} u(t,x) &= P_t\eta(x) + \int_0^t \int_{\mathbb{R}^1} p_{t-r}(x,y)b(r,y,u(r,y))\sigma(dydr) \\ &\quad + \int_0^t \int_{\mathbb{R}^1} p_{t-r}(x,y)a(r,y,u(r,y))W^\varrho(dydr) \quad \forall t \geq 0, x \in \mathbb{R}^1 \end{aligned} \tag{4.21}$$

is called solution to SIE (4.21). The solution is said to be unique if for any two solutions u *and* u' *w.r.t. a given noise* $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P}, W^\varrho)$*,* $u(t,x) = u'(t,x)$ *for all* $t \geq 0$ *and* $x \in \mathbb{R}^1$*,* $\mathbf{P}$*-almost surely.*

Proposition 4.3 (equivalence SPDE, MP) *Every weak solution to SPDE (1.1) in sense of Definition 2.1 is a solution to the* (a, b, η)*-martingale problem and vice versa.*

Proof Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P}, W^\varrho, u)$ be a weak solution to SPDE (1.1). Then, for all $\psi \in \mathcal{S}$,

$$\begin{aligned} M_t(\psi) &:= \int_0^t \int_{\mathbb{R}^1} a(r,y,u(r,y))\psi(y)W^\varrho(dydr) \\ &= \langle u(t,.), \psi\rangle - \langle \eta, \psi\rangle - \int_0^t \langle u(r,.), \frac{1}{2}\Delta\psi\rangle dr - \int_0^t \int_{\mathbb{R}^1} b(r,y,u(r,y))\psi(y)\sigma(dydr) \end{aligned}$$

provides a continuous square-integrable martingale with quadratic variation process

$$\langle M(\psi)\rangle_t = \int_0^t \int_{\mathbb{R}^1} a^2(r,y,u(r,y))\psi^2(y)\langle W^\varrho\rangle(dydr) = \int_0^t \int_{\mathbb{R}^1} a^2(r,y,u(r,y))\psi^2(y)\varrho(dydr).$$

Conversely, let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P}, u)$ be a solution of the (a, b, η)-martingale problem. The family $(M_t(\psi))$ of continuous martingales induces a continuous orthogonal martingale measure, say M. Then, for all $\psi \in \mathcal{S}$ and $t \geq 0$,

$$M_t(\psi) = \int_0^t \int_{\mathbb{R}^1} \psi(y)M(dydr) \quad \mathbf{P}\text{-a.s.} \tag{4.22}$$

Consequently, by the form of the quadratic variation process of $(M_t(\psi))_{t\geq 0}$,

$$\Big\langle \psi \bullet M \Big\rangle \Big(\mathbb{R}^1 \times (0,t] \Big) = \int_0^t \int_{\mathbb{R}^1} a^2(r,y,u(r,y))\psi^2(y)\varrho(dydr) \quad \mathbf{P}\text{-a.s.} \tag{4.23}$$

for all $\psi \in \mathcal{S}$ and $t \geq 0$. Note that the stochastic integral is well defined since $\langle M(\psi)\rangle_t \in L^1(\mathbf{P})$ $\forall t \geq 0$ which follows from the continuity and the square-integrability of $M_{\cdot}(\psi)$ as well as the Doob-Meyer decomposition theorem. Now pick an orthogonal martingale measure $\tilde{W}^\varrho$ being independent of M and having quadratic variation measure $\langle \tilde{W}^\varrho\rangle(dxdt) = \varrho(dxdt)$, if necessary on an enlargement of u's domain $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$. Set for all $\psi \in \mathcal{S}$ and $t \geq 0$

$$\begin{aligned} W_t^\varrho(\psi) = \int_0^t \int_{\mathbb{R}^1} \psi(y)\mathbf{1}_{a(r,y,u(r,y))\neq 0}\frac{1}{a(r,y,u(r,y))}M(dydr) \\ + \int_0^t \int_{\mathbb{R}^1} \psi(y)\mathbf{1}_{a(r,y,u(r,y))=0}\tilde{W}^\varrho(dydr). \end{aligned}$$

Then, using (4.23), it is easy to verify that W^ϱ provides an orthogonal martingale measure with $\langle W^\varrho\rangle(dxdt) = \varrho(dxdt)$ and satisfying

$$M_t(\psi) = \int_0^t \int_{\mathbb{R}^1} \psi(y)a(r,y,u(r,y))W^\varrho(dydr) \quad \mathbf{P}\text{-a.s.}$$

for all $\psi \in \mathcal{S}$ and $t \geq 0$. Since u is a solution of the (a,b,η)-martingale problem, it follows that u is a weak solution to SPDE (1.1). □

Proposition 4.4 (equivalence SPDE, SIE) *Assume (2.6) and for all $T \geq 0$ and $\mu \in \{\varrho, \sigma\}$, $\sup_{t\leq T}\sup_{x\in\mathbb{R}^1}\mu(B(x,1)\times B(t,1)) < \infty$. Then, every strong solution to SPDE (1.1) in sense of Definition 2.1 is a solution to SIE (4.21) and vice versa.*

The proof goes along the lines of the proof of Theorem 2.1 of [Shi94] with the obvious changes.

5 Proof of Theorem 2.4

We shall prove that SIE (4.21) has a unique solution and so, by Proposition 4.4, the same is true for SPDE (1.1). Given the noise $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P}, W^\varrho)$, let L_{tem} be the space of predictable functions $u : [0,\infty)\times\mathbb{R}^1\times\Omega \to \mathbb{R}$ with $\|u\|_{\lambda,T,m} < \infty$ for all $\lambda, T > 0$ and $m \geq 1$, where

$$\|u\|_{\lambda,T,m} := \Big(\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^1} e^{-\lambda|x|}\mathbf{E}\Big[|u(t,x)|^{2m}\Big]\Big)^{\frac{1}{2m}}.$$

For the sake of a Picard-Lindelöf iteration we introduce the functional

$$\begin{aligned} \Phi(u)(t,x) \;&:=\; P_t\eta(x) + \int_0^t\int_{\mathbb{R}^1} p_{t-r}(x,y)b(r,y,u(r,y))\sigma(dydr) \\ &\qquad + \int_0^t\int_{\mathbb{R}^1} p_{t-r}(x,y)a(r,y,u(r,y))W^\varrho(dydr) \\ &=: \; P_t\eta(x) \;+\; \Phi_2(u)(t,x) \;+\; \Phi_3(u)(t,x), \end{aligned}$$

and set $u_0 := P_{.}\eta(.)$ and $u_{n+1} := \Phi(u_n)$ for $n \geq 0$.

Step 1. First we prove that $\Phi(u)$ is a C_{tem}-valued continuous process whenever $u \in L_{tem}$. According to Lemma 3.1 of [Iwa87], $P_{.}\eta(.)$ is C_{tem}-valued continuous. Using Hölder's inequality ($\frac{2m-1}{2m} + \frac{1}{2m} = 1$), Lemma 3.5 (ii), (2.6), Lemma 3.6 (i), Lemma 3.5 (v) and Lemma 3.7, we get for all $(t,x),(t',x') \in [0,T] \times \mathbb{R}^1$ with $|t-t'| \leq 1$, $|x-x'| \leq 1$, w.l.o.g. $t \leq t'$,

$$
\begin{aligned}
&\mathbf{E}\Big[\Big|\Phi_2(u)(t,x) - \Phi_2(u)(t',x')\Big|^{2m}\Big] \\
&= \mathbf{E}\Big[\Big|\int_0^{t'}\int_{\mathbb{R}^1}\Big(p_{t-r}(x,y) - p_{t'-r}(x',y)\Big)b(r,y,u(r,y))\sigma(dydr)\Big|^{2m}\Big] \\
&\leq \Big(\int_0^{t'}\int_{\mathbb{R}^1}\Big|p_{t-r}(x,y) - p_{t'-r}(x',y)\Big|\sigma(dydr)\Big)^{2m-1} \\
&\qquad \times \int_0^{t'}\int_{\mathbb{R}^1}\Big|p_{t-r}(x,y) - p_{t'-r}(x',y)\Big|e^{\lambda|y|}e^{-\lambda|y|}\mathbf{E}\Big[(1+u(r,y))^{2m}\Big]\sigma_1(dy)\sigma_2(dr) \\
&\leq \Big(\int_0^{t'}\int_{\mathbb{R}^1}\Big|p_{t-r}(x,y) - p_{t'-r}(x',y)\Big|\sigma(dydr)\Big)^{2m-1} \\
&\qquad \times c\int_0^{t'}\Big(\int_{\mathbb{R}^1}e^{\lambda|y|}\Big(p_{t-r}(x,y) + p_{t'-r}(x',y)\Big)\sigma_1(dy)\Big)\Big(1 + \|u\|_{\lambda,r,m}^{2m}\Big)\sigma_2(dr) \\
&\leq \Big(\int_0^{t'}\int_{\mathbb{R}^1}\Big|p_{t-r}(x,y) - p_{t'-r}(x',y)\Big|\sigma(dydr)\Big)^{2m-1} \\
&\qquad \times c_T\int_0^{t'}\frac{1}{(t'-r)^{1/2-\beta_1/2}}e^{\lambda|x-x'|}e^{\lambda|x|}\Big(1 + \|u\|_{\lambda,r,m}^{2m}\Big)\sigma_2(dr) \\
&\leq c_T\Big(|t-t'|^{\beta} + |x-x'|^{2\beta}\Big)^{2m-1}e^{\lambda|x|}.
\end{aligned}
$$

Thus, for m sufficiently large, Lemma 3.1 (i) provides a C_{tem}-valued continuous modification of $\Phi_2(u)$. Using Lemma 3.3, Hölder's inequality ($\frac{m-1}{m} + \frac{1}{m} = 1$) and the same lemmas as above, we obtain analogously

$$
\begin{aligned}
&\mathbf{E}\Big[\Big|\Phi_3(u)(t,x) - \Phi_3(u)(t',x')\Big|^{2m}\Big] \\
&= c\,\mathbf{E}\Big[\Big|\int_0^{t'}\int_{\mathbb{R}^1}\Big(p_{t-r}(x,y) - p_{t'-r}(x',y)\Big)a(r,y,u(r,y))W^{\varrho}(dydr)\Big|^{2m}\Big] \\
&\leq c\,\mathbf{E}\Big[\Big|\int_0^{t'}\int_{\mathbb{R}^1}\Big(p_{t-r}(x,y) - p_{t'-r}(x',y)\Big)^2 a^2(r,y,u(r,y))\varrho(dydr)\Big|^{m}\Big] \\
&\leq c\,\Big(\int_0^{t'}\int_{\mathbb{R}^1}\Big(p_{t-r}(x,y) - p_{t'-r}(x',y)\Big)^2\varrho(dydr)\Big)^{m-1} \\
&\qquad \times \int_0^{t'}\int_{\mathbb{R}^1}\Big(p_{t-r}(x,y) - p_{t'-r}(x',y)\Big)^2 e^{\lambda|y|}e^{-\lambda|y|}\mathbf{E}\Big[(1+u(r,y))^{2m}\Big]\varrho_1(dy)\varrho_2(dr) \\
&\leq c\,\Big(\int_0^{t'}\int_{\mathbb{R}^1}\Big(p_{t-r}(x,y) - p_{t'-r}(x',y)\Big)^2\varrho(dydr)\Big)^{m-1}
\end{aligned}
$$

$$\times \int_0^{t'} \Big(\int_{\mathbb{R}^1} e^{\lambda|y|}\Big(p^2_{t-r}(x,y)+p^2_{t'-r}(x',y)\Big)\varrho_1(dy)\Big)\Big(1+\|u\|^{2m}_{\lambda,r,m}\Big)\varrho_2(dr)$$

$$\leq c_T\Big(\int_0^{t'}\int_{\mathbb{R}^1}\Big(p_{t-r}(x,y)-p_{t'-r}(x',y)\Big)^2\varrho(dydr)\Big)^{m-1}$$

$$\times \int_0^{t'} \frac{1}{(t'-r)^{1-\alpha_1/2}} e^{\lambda|x-x'|}e^{\lambda|x|}\Big(1+\|u\|^{2m}_{\lambda,r,m}\Big)\varrho_2(dr)$$

$$\leq c_T\Big(|t-t'|^{\alpha}+|x-x'|^{2\alpha}\Big)^{m-1}e^{\lambda|x|}$$

and so a C_{tem}-valued continuous modification of $\Phi_3(u)$. Altogether, $\Phi(u)$ has a C_{tem}-valued continuous modification. Due to the obtained estimates, $\Phi(u)$ also is locally Hölder-γ-continuous for all $\gamma \in (0, \frac{\alpha}{2}\wedge\beta)$. Note that the stochastic integral is well defined since by assumption $\|u\|_{\lambda,T,m}<\infty$ and $\varrho(dxdt)$ was of class (MN).

Step 2. Next we establish that u_n is in L_{tem} for every $n\geq 1$ whenever $\eta\in C_{tem}$ and, moreover, for all $\lambda, T>0$ and $m\geq 1$ even

$$\sup_{n\geq 1}\|u_n\|_{\lambda,T,m}\leq c_{\lambda,T,m}<\infty. \tag{5.24}$$

Since $\|u_0\|_{\lambda,T,m}\leq c_{\lambda,T,m}$, for (5.24) it is enough to show, by Lemma 3.8 (i), that for all $n\geq 0$

$$\|u_{n+1}\|^{2m}_{\lambda,T,m} \leq c_{\lambda,T,m}\Big\{1+sup_{t\in[0,T]}\int_0^t \frac{1}{(t-r)^{1-\alpha_1/2}}\|u_n\|^{2m}_{\lambda,r,m}\varrho_2(dr)$$
$$+sup_{t\in[0,T]}\int_0^t\frac{1}{(t-r)^{1/2-\beta_1/2}}\|u_n\|^{2m}_{\lambda,r,m}\sigma_2(dr)\Big\}. \tag{5.25}$$

Using Hölder's inequality ($\frac{2m-1}{2m}+\frac{1}{2m}=1$), Lemma 3.5 ($ii$), (2.6), Lemma 3.6 ($i$) and Lemma 3.5 ($iv$), we estimate

$$\|\Phi_2(u_n)\|^{2m}_{\lambda,T,m}\leq \sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^1} e^{-\lambda|x|}\mathbf{E}\Big[\Big|\int_0^t\int_{\mathbb{R}^1}p_{t-r}(x,y)b(r,y,u_n(r,y))\sigma(dydr)\Big|^{2m}\Big]$$

$$\leq \sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^1}e^{-\lambda|x|}\Big(\int_0^t\int_{\mathbb{R}^1}p_{t-r}(x,y)\sigma(dydr)\Big)^{2m-1}$$

$$\times\int_0^t\int_{\mathbb{R}^1}e^{\lambda|y|}p_{t-r}(x,y)e^{-\lambda|y|}\mathbf{E}\Big[(1+u_n(r,y))^{2m}\Big]\sigma_1(dy)\sigma_2(dr)$$

$$\leq c_{T,m}\Big\{1+\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^1}\int_0^t\Big(e^{-\lambda|x|}\int_{\mathbb{R}^1}e^{\lambda|y|}p_{t-r}(x,y)\sigma_1(dy)\Big)\|u_n\|^{2m}_{\lambda,r,m}\sigma_2(dr)\Big\}$$

$$\leq c_{T,m}\Big\{1+\sup_{t\in[0,T]}\int_0^t\frac{1}{(t-r)^{1/2-\beta_1/2}}\|u_n\|^{2m}_{\lambda,r,m}\sigma_2(dr)\Big\}. \tag{5.26}$$

Lemma 3.3 and again Hölder's inequality ($\frac{m-1}{m}+\frac{1}{m}=1$), Lemma 3.5 ($ii$), (2.6), Lemma 3.6 ($i$) and Lemma 3.5 ($iv$) give the following bound

$$\|\Phi_3(u_n)\|^{2m}_{\lambda,T,m}\leq\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^1}e^{-\lambda|x|}\mathbf{E}\Big[\Big|\int_0^t\int_{\mathbb{R}^1}p_{t-r}(x,y)a(r,y,u_n(r,y))W^{\varrho}(dydr)\Big|^{2m}\Big]$$

$$
\begin{aligned}
&\le c \sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^1} e^{-\lambda|x|}\mathbf{E}\Big[\Big|\int_0^t\int_{\mathbb{R}^1} p_{t-r}^2(x,y)a^2(r,y,u_n(r,y))\varrho(dydr)\Big|^m\Big]\\
&\le c \sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^1} e^{-\lambda|x|}\Big(\int_0^t\int_{\mathbb{R}^1} p_{t-r}^2(x,y)\varrho(dydr)\Big)^{m-1}\\
&\quad\times\int_0^t\int_{\mathbb{R}^1} e^{\lambda|y|}p_{t-r}^2(x,y)e^{-\lambda|y|}\mathbf{E}\Big[(1+u_n(r,y))^{2m}\Big]\varrho_1(dy)\varrho_2(dr)\\
&\le c_{T,m}\Big\{1+\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}^1}\int_0^t\Big(e^{-\lambda|x|}\int_{\mathbb{R}^1} e^{\lambda|y|}p_{t-r}^2(x,y)\varrho_1(dy)\Big)\|u_n\|_{\lambda,r,m}^{2m}\varrho_2(dr)\Big\}\\
&\le c_{T,m}\Big\{1+\sup_{t\in[0,T]}\int_0^t\frac{1}{(t-r)^{1-\alpha_1/2}}\|u_n\|_{\lambda,r,m}^{2m}\varrho_2(dr)\Big\}. \qquad (5.27)
\end{aligned}
$$

Thus, we reached (5.25). Furthermore, by Step 1, u_n is jointly continuous and so predictable for all $n\ge 1$. The well-definiteness of the stochastic integrals can be obtained successively using $u_0\in L_{tem}$ and the belonging of $\varrho(dxdt)$ to class (MN).

Step 3. Here we show $\|u_{n+1}-u_n\|_{\lambda,T,m}\to 0$ as $n\to\infty$ for all $\lambda,T>0$ and $m\ge 1$. By Lemma 3.8 (i) it suffices to verify,

$$
\begin{aligned}
\|u_{n+1}-u_n\|_{\lambda,T,m}^{2m} \;\le\; & c_{\lambda,T,m}\Big\{sup_{t\in[0,T]}\int_0^t\frac{1}{(t-r)^{1-\alpha_1/2}}\|u_n-u_{n-1}\|_{\lambda,r,m}^{2m}\varrho_2(dr)\\
&+sup_{t\in[0,T]}\int_0^t\frac{1}{(t-r)^{1/2-\beta_1/2}}\|u_n-u_{n-1}\|_{\lambda,r,m}^{2m}\sigma_2(dr)\Big\} \qquad (5.28)
\end{aligned}
$$

for all $n\ge 1$. But similar to getting the bounds (5.26) and (5.27) we obtain

$$
\|\Phi_2(u_n)-\Phi_2(u_{n-1})\|_{\lambda,T,m}^{2m}\le c_{T,m}\sup_{t\in[0,T]}\int_0^t\frac{1}{(t-r)^{1/2-\beta_1/2}}\|u_n-u_{n-1}\|_{\lambda,r,m}^{2m}\sigma_2(dr),
$$

$$
\|\Phi_3(u_n)-\Phi_3(u_{n-1})\|_{\lambda,T,m}^{2m}\le c_{T,m}\sup_{t\in[0,T]}\int_0^t\frac{1}{(t-r)^{1-\alpha_1/2}}\|u_n-u_{n-1}\|_{\lambda,r,m}^{2m}\varrho_2(dr)
$$

proving (5.28). Here we used the Lipschitz condition (2.7) instead of (2.6).

Step 4. According to Step 2 and Step 3, (u_n) is a Cauchy sequence in L_{tem}. Hence, there exists $u_\infty\in L_{tem}$ with $\|u_\infty-u_n\|_{\lambda,T,m}\to 0$ as $n\to\infty$ and, particularly, $u_\infty=\Phi(u_\infty)$ w.r.t. $\|.\|_{\lambda,T,m}$ for all $\lambda,T>0$ and $m\ge 1$. Thus, $u_\infty(t,x)=\Phi(u_\infty)(t,x)$ for $dxdt$-almost all (x,t), $\mathbf{P}$-almost surely. By Step 1, u_∞ is C_{tem}-valued continuous and so $u_\infty(t,x)=\Phi(u_\infty)(t,x)$ even holds for all (x,t), $\mathbf{P}$-almost surely. Consequently, u_∞ is a solution of SIE (4.21). Step 1 also gives the desired local Hölder-continuity.

Step 5. It remains to show strong uniqueness of solutions. Let u,u' be two solutions to SPDE (1.1) and so to $u=\Phi(u)$. Fix some $\lambda>0$ and define $\tau_K:=\inf\{t>0: |u(t,.)|_{(-\lambda/2)}\ge K \text{ or } |u'(t,.)|_{(-\lambda/2)}\ge K\}$ as well as $u_K(t,.):=\mathbf{1}_{t<\tau_K}u(t,.)$ and $u'_K(t,.):=\mathbf{1}_{t<\tau_K}u'(t,.)$ for each $K>0$. Arguments as in Step 3 yield

$$
\|u_K-u'_K\|_{\lambda,T,1}^2 \;=\; \|\Phi(u_K)-\Phi(u'_K)\|_{\lambda,T,1}^2
$$

$$\leq \; c_{\lambda,T,1}\Big\{sup_{t\in[0,T]}\int_0^{t\wedge\tau_K}\frac{1}{(t-r)^{1-\alpha_1/2}}\|u_K-u'_K\|^2_{\lambda,r,1}\varrho_2(dr)$$
$$+sup_{t\in[0,T]}\int_0^{t\wedge\tau_K}\frac{1}{(t-r)^{1/2-\beta_1/2}}\|u_K-u'_K\|^2_{\lambda,r,1}\sigma_2(dr)\Big\}$$

for all $T>0$. Note that $|u_K(t,x)-u'_K(t,x)|^2\leq 4K^2e^{\lambda|x|}$ for all (t,x). Thus, Lemma 3.8 (ii) gives $u_K(t,x)=u'_K(t,x)$ for $dxdt$-almost all (x,t), $\mathbf{P}$-almost surely. Hence, since u and u' are C_{tem}-valued continuous, and particularly $\tau_K\to\infty$ as $K\to\infty$, $u(t,x)=u'(t,x)$ holds for all (x,t), $\mathbf{P}$-almost surely. We are done.

6 Proof of Theorem 2.5

We follow an approach that was also used by Shiga ([Shi94], Appendix) in the homogeneous case. Define time measures $\sigma_x^\epsilon(dt):=\int_{\mathbb{R}^1}p_\epsilon(x,y)\sigma(dydt)$ and time white noises $W_x^\epsilon(dt):=\int_{\mathbb{R}^1}p_\epsilon(x,y)W_x^\varrho(dydt)$ (formally) for all $\epsilon>0$. Note that $\sigma_x^\epsilon(t):=\int_0^t\sigma_x^\epsilon(dr)$ provides an increasing process and $W_x^\epsilon(t)=\int_0^tW_x^\epsilon(dr):=\int_0^t\int_{\mathbb{R}^1}p_\epsilon(x,y)W_x^\varrho(dydt)$ a square-integrable martingale with quadratic variation process $\langle W_x^\epsilon\rangle(t)=\int_0^t\int_{\mathbb{R}^1}p_\epsilon^2(x,y)\varrho(dydr)$ for all $\epsilon>0$. Furthermore, introduce approximate Laplacians Δ_ϵ and corresponding semigroups (P_t^ϵ) via

$$\begin{aligned}
\Delta_\epsilon &:= \epsilon^{-1}(P_\epsilon-I)\\
P_t^\epsilon &:= e^{t\Delta_\epsilon}=e^{-t/\epsilon}e^{t/\epsilon P_\epsilon}=e^{-t/\epsilon}\sum_{n=0}^\infty\frac{(t/\epsilon)^n}{n!}P_{n\epsilon}=e^{-t/\epsilon}I+Q_t^\epsilon\\
Q_t^\epsilon f &:= \int_{\mathbb{R}^1}q_t^\epsilon(.,y)f(y)dy\\
q_t^\epsilon(x,y) &:= e^{-t/\epsilon}\sum_{n=1}^\infty\frac{(t/\epsilon)^n}{n!}p_{n\epsilon}(x,y)
\end{aligned}$$

for all $\epsilon>0$. Finally, let us equip with the following two technical lemmas yet.

Lemma 6.1 *For all $\lambda\geq 0$, $x\in\mathbb{R}^1$, $t\geq 0$ and $\epsilon\in(0,1]$,*

$$\int_{\mathbb{R}^1}\int_{\mathbb{R}^1}q_t^\epsilon(x,z)^2p_\epsilon(z,y)e^{\lambda|z|}dz\varrho_1(dy)\;\leq\;c\frac{1}{t^{1-\alpha_1/2}}e^{\lambda|x|},\tag{6.29}$$

$$\int_{\mathbb{R}^1}\int_{\mathbb{R}^1}q_t^\epsilon(x,z)p_\epsilon(z,y)e^{\lambda|z|}dz\sigma_1(dy)\;\leq\;c\frac{1}{t^{1/2-\beta_1/2}}e^{\lambda|x|}.\tag{6.30}$$

Proof First of all note that $e^{-h}\sum_{n=1}^\infty\frac{h^n}{n!}\frac{h^\gamma}{n^\gamma}\leq c$ for all $h\geq 0$ and $\gamma\in[0,1]$. We only prove (6.29), (6.30) can be checked analogously. The l.h.s. of (6.29) can be estimated by

$$\int_{\mathbb{R}^1}\int_{\mathbb{R}^1}e^{-2t/\epsilon}\Big(\sum_{n=1}^\infty\frac{(t/\epsilon)^n}{n!}p_{n\epsilon}(x,z)\Big)^2p_\epsilon(y,z)e^{\lambda|z|}dz\varrho_1(dy)\tag{6.31}$$

$$\leq \int_{\mathbb{R}^1}\int_{\mathbb{R}^1} e^{-t/\epsilon}\sum_{n=1}^{\infty}\frac{(t/\epsilon)^n}{n!}p_{n\epsilon}(x,z)p_\epsilon(z,y)e^{\lambda|z|}dz\varrho_1(dy)\ e^{-t/\epsilon}\sum_{n=1}^{\infty}\frac{(t/\epsilon)^n}{n!}\frac{c}{(n\epsilon)^{1/2}}$$

$$\leq e^{-t/\epsilon}\sum_{n=1}^{\infty}\frac{(t/\epsilon)^n}{n!}\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} p_{n\epsilon}(x,z)p_\epsilon(z,y)e^{\lambda|z|}dz\varrho_1(dy)\ c\frac{1}{t^{1/2}}.$$

Using Lemma 3.5 and the inequality

$$\int_{\mathbb{R}^1} p_s(x,z)p_t(z,y)e^{\lambda|z|}dz \ \leq\ p_{s+t}(x,y)e^{\lambda\left|y-\frac{t}{t+s}x\right|\frac{t+s}{t+s+t}}$$

it is not hard to verify that the r.h.s. of (6.31) is bounded by

$$e^{-t/\epsilon}\sum_{n=1}^{\infty}\frac{(t/\epsilon)^n}{n!}\frac{1}{((n+1)\epsilon)^{1/2-\alpha_1/2}}exp\Big[\frac{\epsilon+n\epsilon}{\epsilon+n\epsilon+\epsilon}\lambda|x|\Big]exp\Big[\frac{\epsilon}{\epsilon+n\epsilon+\epsilon}\lambda|x|\Big]\ c\frac{1}{t^{1/2}}$$

$$\leq\ c\frac{1}{t^{1-\alpha_1/2}}e^{\lambda|x|}\ e^{-t/\epsilon}\sum_{n=1}^{\infty}\frac{(t/\epsilon)^n}{n!}\frac{(t/\epsilon)^{1/2-\alpha_1/2}}{n^{1/2-\alpha_1/2}}\ \leq\ \tilde{c}\frac{1}{t^{1-\alpha_1/2}}e^{\lambda|x|}$$

giving the desired bound for the l.h.s. of (6.29). □

Lemma 6.2 *Fix $\delta>0$. Then for all $x,y\in\mathbb{R}^1$, $t>0$ and $\epsilon\in(0,1]$,*

$$\Big|\int_{\mathbb{R}^1} q_t^\epsilon(x,z)p_\epsilon(y,z)dz - p_t(x,y)\Big| \leq c_\delta\ \frac{1}{t^{1/2+\delta}}\ \epsilon^\delta. \tag{6.32}$$

The purely technical proof is omitted. Then the strategy is as follows. First (Step 1) we shall prove that, for fixed $\epsilon>0$, the family, with index $x\in\mathbb{R}^1$, of ordinary stochastic differential equations

$$\begin{aligned} u_\epsilon(t,x)\ &= \eta(x) \\ &+\int_0^t \frac{1}{2}\Delta_\epsilon u_\epsilon(r,x)dr + \int_0^t b(r,x,u_\epsilon(r,x))\sigma_x^\epsilon(dr) + \int_0^t a(r,x,u_\epsilon(r,x))W_x^\epsilon(dr) \end{aligned} \tag{6.33}$$

has a unique C_{tem}-valued continuous solution u_ϵ. Secondly, it will be established that this solution is non-negative (Step 2). In Step 3 we will approximate the unique solution u to SPDE (1.1) by u_ϵ as $\epsilon\to 0$, whereby the desired non-negativity of u will follow. The approximation of u by u_ϵ is not surprising since the equation family (6.33) is easily seen to be equivalent to the mollified version of SPDE (1.1)

$$\begin{aligned} \langle u_\epsilon(t,.),\psi\rangle\ &=\ \langle\eta,\psi\rangle + \int_0^t\langle u_\epsilon(r,.),\frac{1}{2}\Delta_\epsilon\psi\rangle dr \\ &+\int_0^t\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} b(r,z,u_\epsilon(r,z))\psi(z)p_\epsilon(y,z)dz\sigma(dydr) \\ &+\int_0^t\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} a(r,z,u_\epsilon(r,z))\psi(z)p_\epsilon(y,z)dzW^\varrho(dydr)\quad \forall t\geq 0,\psi\in\mathcal{S}; \end{aligned} \tag{6.34}$$

note $\langle P_\epsilon\phi,\psi\rangle = \langle \phi, P_\epsilon\psi\rangle$, i.e. particularly $\langle \Delta_\epsilon\phi,\psi\rangle = \langle\phi,\Delta_\epsilon\psi\rangle$, for all $\phi\in C_{tem}$, $\psi\in\mathcal{S}$.

Step 1. We establish a unique solution to (6.33). The crucial point is that (6.34), and so (6.33), is equivalent to the following mollified version of SIE (4.21)

$$\begin{aligned} u_\epsilon(t,x) &= P_t^\epsilon\eta(x) + \int_0^t\int_{\mathbb{R}^1} e^{-(t-r)/\epsilon} b(r,y,u_\epsilon(r,y))p_\epsilon(x,y)\sigma(dydr) \\ &\quad + \int_0^t\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} q_{t-r}^\epsilon(x,z)b(r,z,u_\epsilon(r,z))p_\epsilon(y,z)dz\sigma(dydr) \\ &\quad + \int_0^t\int_{\mathbb{R}^1} e^{-(t-r)/\epsilon} a(r,y,u_\epsilon(r,y))p_\epsilon(x,y)W^\varrho(dydr) \\ &\quad + \int_0^t\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} q_{t-r}^\epsilon(x,z)a(r,z,u_\epsilon(r,z))p_\epsilon(y,z)dzW^\varrho(dydr). \end{aligned} \tag{6.35}$$

The proof of the equivalence works analogously to the proof of Proposition 4.4; recall that Δ_ϵ was the generator of P_t^ϵ. Mimicking the proof of Theorem 2.4 we obtain a unique C_{tem}-valued continuous solution to SIE (6.35). This time one has to choose $\Phi^\epsilon(u) := P_\cdot^\epsilon\eta(.) + \Phi_{2,1}^\epsilon(u) + \Phi_{2,2}^\epsilon(u) + \Phi_{3,1}^\epsilon(u) + \Phi_{3,2}^\epsilon(u) :=$ r.h.s. of (6.35). Note that the essential technical tools are Lemma 3.5 and Lemma 3.6 as before, as well as Lemma 6.1. Particularly one obtains $\sup_{\epsilon\in(0,1]}\|u_\epsilon\|_{\lambda,T,m} < \infty$ for all $\lambda, T>0$ and $m\geq 1$. Having Lemma 6.1, one also gets an analogue to Lemma 3.7.

Step 2. Let us turn to the non-negativity of (6.33). Consider a sequence $x_n\uparrow 0$ with $x_0=-1$ and $\int_{x_{n-1}}^{x_n} x^{-2}dx = n$ $\forall n\geq 1$. Furthermore, pick functions $g_n\in C_c^\infty$ such that $supp(g_n)\subset(x_{n-1},x_n)$, $0\leq g_n(x)\leq \frac{2x^{-2}}{n}$ and $\int_{x_{n-1}}^{x_n} g_n(x)dx = 1$ for all $n\geq 1$. Set

$$f_n(x) = \begin{cases} \int_x^0\int_y^0 g_n(z)dzdy & , \quad x<0 \\ 0 & , \quad x\geq 0 \end{cases}.$$

Hence,

$$f_n'(x) = \begin{cases} -\int_x^0 g_n(y)dy & , \quad x<0 \\ 0 & , \quad x\geq 0 \end{cases} \qquad f_n''(x) = \begin{cases} g_n(x) & , \quad x<0 \\ 0 & , \quad x\geq 0 \end{cases}.$$

Note that f_n, f_n' and f_n'' approximate $f(x) = -\min\{0,x\}$, "$f'(x)$" $= -\mathbf{1}_{(-\infty,0]}(x)$ and "$f''(x)$" $=\delta_0(x)$, respectively. Applying Itô's formula to f_n and the semimartingale $u_\epsilon(t,x) = \eta(x) + V_x^\epsilon(t) + M_x^\epsilon(t) = \eta(x) + (V_x^{\epsilon,1}(t) + V_x^{\epsilon,2}(t)) + M_x^\epsilon(t) :=$ r.h.s. of (6.33) yields

$$\begin{aligned} f_n(u_\epsilon(t,x)) &= f_n(\eta(x)) \\ &\quad + \int_0^t f_n'(u_\epsilon(r,x))dV_x^\epsilon(r) + \int_0^t f_n'(u_\epsilon(r,x))dM_x^\epsilon(r) + \frac{1}{2}\int_0^t f_n''(u_\epsilon(r,x))d\langle M_x^\epsilon\rangle(r). \end{aligned}$$

Taking expectation as well as using $f_n''=g_n$, $b(r,y,u)\geq -L_T\,|u|$ $\forall(r,y,u)$, $f_n'(u)=0$ $\forall u\geq 0$, $-f_n'(u)\in[0,1]$ $\forall u$, $-u\leq f(u)$ $\forall u$ and $a(r,y,0)=0$, $b(r,y,0)\geq 0$ $\forall(r,y)$ (in what follows we should work with $u_\epsilon(t,x)e^{-|x|}$ instead of $u_\epsilon(t,x)$; but in order to avoid unnecessarily complicated expressions we assume boundedness of u_ϵ on time compacts; the general case causes no difficulties),

$$
\begin{aligned}
\mathbf{E}\Big[f_n(u_\epsilon(t,x))\Big] &= \mathbf{E}\Big[\int_0^t f_n'(u_\epsilon(r,x))dV_x^\epsilon(r)\Big] + \frac{1}{2}\mathbf{E}\Big[\int_0^t f_n''(u_\epsilon(r,x))d\langle M_x^\epsilon\rangle(r)\Big] \\
&= \mathbf{E}\Big[\int_0^t f_n'(u_\epsilon(r,x))\Delta_\epsilon u_\epsilon(r,x)dr\Big] \\
&\quad +\mathbf{E}\Big[\int_0^t\int_{\mathbb{R}^1} f_n'(u_\epsilon(r,x))b(r,x,u_\epsilon(r,x))p_\epsilon(x,y)\sigma(dydr)\Big] \\
&\quad +\frac{1}{2}\mathbf{E}\Big[\int_0^t\int_{\mathbb{R}^1} f_n''(u_\epsilon(r,x))a^2(r,x,u_\epsilon(r,x))p_\epsilon^2(x,y)\varrho(dydr)\Big] \\
&\le \mathbf{E}\Big[\frac{1}{\epsilon}\int_0^t f_n'(u_\epsilon(r,x))\int_{\mathbb{R}^1} p_\epsilon(x,y)u_\epsilon(r,y)dydr\Big] - \mathbf{E}\Big[\frac{1}{\epsilon}\int_0^t f_n'(u_\epsilon(r,x))u_\epsilon(r,x)dr\Big] \\
&\quad +\mathbf{E}\Big[\int_0^t\int_{\mathbb{R}^1}\Big(-f_n'(u_\epsilon(r,x))\;L_T|u_\epsilon(r,x)|\Big)p_\epsilon(x,y)\sigma(dydr)\Big] \\
&\quad +\frac{1}{2}\mathbf{E}\Big[\int_0^t\int_{\mathbb{R}^1}\frac{2|u_\epsilon(r,x)|^{-2}}{n}L_T^2|u_\epsilon(r,x)|^2p_\epsilon^2(x,y)\varrho(dydr)\Big] \\
&\le \mathbf{E}\Big[\frac{1}{\epsilon}\int_0^t\Big(-f_n'(u_\epsilon(r,x))\Big)\int_{\mathbb{R}^1} p_\epsilon(x,y)(-u_\epsilon(r,y))dydr\Big] \\
&\quad +c_{\epsilon,T}\mathbf{E}\Big[\int_0^t\Big(-f_n'(u_\epsilon(r,x))\Big)\Big(-u_\epsilon(r,x)\Big)\sigma_2(dr)\Big] + \frac{L_T^2}{n}\int_0^t\int_{\mathbb{R}^1} p_\epsilon^2(x,y)\varrho(dydr) \\
&\le c_\epsilon\int_0^t \sup_{y\in\mathbb{R}^1}\mathbf{E}\Big[f(u_\epsilon(r,y))\Big]dr + c_{\epsilon,T}\int_0^t\sup_{x\in\mathbb{R}^1}\mathbf{E}\Big[f_n'(u_\epsilon(r,x))u_\epsilon(r,x)\Big]\sigma_2(dr) + \frac{c_{\epsilon,T}}{n}
\end{aligned}
$$

for all $t\in[0,T]$ and $T>0$. Letting $n\to\infty$, we get by dominated convergence

$$
\Big\|\mathbf{E}\Big[f(u_\epsilon(t,.))\Big]\Big\|_\infty \le c_\epsilon\int_0^t\Big\|\mathbf{E}\Big[f(u_\epsilon(r,.))\Big]\Big\|_\infty dr + c_{\epsilon,T}\int_0^t\Big\|\mathbf{E}\Big[f(u_\epsilon(r,.))\Big]\Big\|_\infty\sigma_2(dr)
$$

for all $t\in[0,T]$ and $T>0$. An application of the Gronwall-type Lemma 3.8 (ii) leads to $\sup_{t\le T}\|\mathbf{E}[f(u_\epsilon(t,.))]\|_\infty = 0$ for all $T>0$. Thus, since $f\ge 0$, $f(u_\epsilon(t,x))=0$, $\mathbf{P}$-almost surely for all (t,x). Hence, $f(u_\epsilon(t,x))=0$ for almost all (t,x), $\mathbf{P}$-almost surely, and consequently, by the joint continuity of u_ϵ, $u_\epsilon(t,x)\ge 0$ for all (t,x), $\mathbf{P}$-almost surely.

Step 3. We approximate u by the u_ϵ. First note that

$$
\begin{aligned}
e^{-\lambda|x|}\;\mathbf{E}\Big[|u_\epsilon(t,x)-u(t,x)|^2\Big] &\le e^{-\lambda|x|}\Big\{\;|P_t^\epsilon\eta(x)-P_t\eta(x)|^2 \\
&\quad +\mathbf{E}\Big[\Big|\int_0^t\int_{\mathbb{R}^1} e^{-(t-r)/\epsilon}b(r,y,u_\epsilon(r,y))p_\epsilon(x,y)\sigma(dydr)\Big|^2\Big] \\
&\quad +\mathbf{E}\Big[\Big|\int_0^t\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} q_{t-r}^\epsilon(x,z)\Big(b(r,z,u_\epsilon(r,z))-b(r,z,u(r,z))\Big)p_\epsilon(y,z)dz\sigma(dydr)\Big|^2\Big] \\
&\quad +\mathbf{E}\Big[\Big|\int_0^t\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} q_{t-r}^\epsilon(x,z)\Big(b(r,z,u(r,z))-b(r,y,u(r,y))\Big)p_\epsilon(y,z)dz\sigma(dydr)\Big|^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbf{E}\Big[\Big|\int_0^t\int_{\mathbb{R}^1}\Big(\int_{\mathbb{R}^1} q^\epsilon_{t-r}(x,z)p_\epsilon(y,z)dz - p_{t-r}(x,y)\Big)b(r,y,u(r,y))\sigma(dydr)\Big|^2\Big]\\
&+\mathbf{E}\Big[\Big|\int_0^t\int_{\mathbb{R}^1} e^{-(t-r)/\epsilon}a(r,y,u_\epsilon(r,y))p_\epsilon(x,y)W^\varrho(dydr)\Big|^2\Big]\\
&+\mathbf{E}\Big[\Big|\int_0^t\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} q^\epsilon_{t-r}(x,z)\Big(a(r,z,u_\epsilon(r,z)) - a(r,z,u(r,z))\Big)p_\epsilon(y,z)dzW^\varrho(dydr)\Big|^2\Big]\\
&+\mathbf{E}\Big[\Big|\int_0^t\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} q^\epsilon_{t-r}(x,z)\Big(a(r,z,u(r,z)) - a(r,y,u(r,y))\Big)p_\epsilon(y,z)dzW^\varrho(dydr)\Big|^2\Big]\\
&+\mathbf{E}\Big[\Big|\int_0^t\int_{\mathbb{R}^1}\Big(\int_{\mathbb{R}^1} q^\epsilon_{t-r}(x,z)p_\epsilon(y,z)dz - p_{t-r}(x,y)\Big)a(r,y,u(r,y))W^\varrho(dydr)\Big|^2\Big]\ \Big\}\\
&=:\ e^{-\lambda|x|}\ \Big\{I_1^\epsilon(t,x)+\cdots+I_9^\epsilon(t,x)\Big\}.
\end{aligned}
$$

Then, using Lemmas 3.5, 3.6, 6.1, 3.3 and Hölder's inequality, we obtain for $t \le T$

$$
\begin{aligned}
&I_6^\epsilon(t,x)\\
&\le\ c\,\mathbf{E}\Big[\int_0^t\int_{\mathbb{R}^1} e^{-2\frac{t-r}{\epsilon}}a^2(r,y,u_\epsilon(r,y))p^2_\epsilon(x,y)\varrho(dydr)\Big]\\
&\le\ c\int_0^t e^{-2\frac{t-r}{\epsilon}}\int_{\mathbb{R}^1} p^2_\epsilon(x,y)e^{\lambda|y|}\varrho_1(dy)c(1+\|u_\epsilon\|_{\lambda,r,1})^2\varrho_2(dr)\\
&\le\ c_T\,\frac{1}{\epsilon^{1-\alpha_1/2}}\int_0^t e^{-2\frac{t-r}{\epsilon}}\varrho_2(dr)\ e^{\lambda|x|}\ =\ c_T\ \epsilon^{\alpha_1/2+\alpha_2-1}e^{\lambda|x|},
\end{aligned}
\tag{6.36}
$$

$$
\begin{aligned}
&I_7^\epsilon(t,x)\\
&\le\ c\,\mathbf{E}\Big[\int_0^t\int_{\mathbb{R}^1}\Big(\int_{\mathbb{R}^1} q^\epsilon_{t-r}(x,z)\Big(a(r,z,u_\epsilon(r,z)) - a(r,z,u(r,z))\Big)p_\epsilon(y,z)dz\Big)^2\varrho(dydr)\Big]\\
&\le\ c\,\mathbf{E}\Big[\int_0^t\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} q^\epsilon_{t-r}(x,z)^2e^{\lambda|z|}p_\epsilon(y,z)dz\ \times\\
&\qquad\qquad\int_{\mathbb{R}^1} p_\epsilon(y,z)e^{-\lambda|z|}|u_\epsilon(r,z)-u(r,z)|^2dz\ \ \varrho(dydr)\Big]\\
&\le\ c\int_0^t\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} q^\epsilon_{t-r}(x,z)^2e^{\lambda|z|}p_\epsilon(y,z)dz\varrho_1(dy)\ \|u_\epsilon-u\|^2_{\lambda,r,1}\ \varrho_2(dr)\\
&\le\ c_T\int_0^t\frac{1}{(t-r)^{1-\alpha_1/2}}\|u_\epsilon-u\|^2_{\lambda,r,1}\ \varrho_2(dr)\ e^{\lambda|x|}
\end{aligned}
\tag{6.37}
$$

and

$$
\begin{aligned}
&I_8^\epsilon(t,x)\\
&\le\ c\,\mathbf{E}\Big[\int_0^t\int_{\mathbb{R}^1}\Big(\int_{\mathbb{R}^1} q^\epsilon_{t-r}(x,z)\Big(a(r,z,u(r,z)) - a(r,y,u(r,y))\Big)p_\epsilon(y,z)dz\Big)^2\varrho(dydr)\Big]\\
&\le\ c\int_0^t\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} q^\epsilon_{t-r}(x,z)^2e^{\lambda|z|}p_\epsilon(y,z)dz\ \times
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^1} p_\epsilon(y,z)\Big(|z-y|^{2\kappa}+\mathbf{E}\Big[|u(r,z)-u(r,y)|^2\Big]\Big)e^{-\lambda|z|}dz\ \varrho(dydr)\\
\leq\ & c\int_0^t\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} q^\epsilon_{t-r}(x,z)^2e^{\lambda|z|}p_\epsilon(y,z)dz\ \times\\
&\int_{\mathbb{R}^1} p_\epsilon(y,z)\Big(|z-y|^{2\kappa}+c_T(|z-y|^{2(\alpha\wedge\beta)}+|z-y|)e^{\lambda|z-y|}e^{\lambda|z|}\Big)e^{-\lambda|z|}dz\ \varrho(dydr)\\
\leq\ & c_T\int_0^t\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} q^\epsilon_{t-r}(x,z)^2e^{\lambda|z|}p_\epsilon(y,z)dz\int_{\mathbb{R}^1}p_\epsilon(y,z)|z-y|^{2((\alpha\wedge\beta)\wedge\kappa)}e^{\lambda|z-y|}dz\ \varrho(dydr)\\
\leq\ & c_T\int_0^t\int_{\mathbb{R}^1}\int_{\mathbb{R}^1} q^\epsilon_{t-r}(x,z)^2e^{\lambda|z|}p_\epsilon(y,z)dz\varrho_1(dy)\varrho_2(dr)\ \epsilon^\delta\ \leq\ c_T\ e^{\lambda|x|}\ \epsilon^\delta \qquad (6.38)
\end{aligned}
$$

for some sufficiently small $\delta>0$. Here we used $\int_{\mathbb{R}^1}p_\epsilon(y,z)|z-y|^{2((\alpha\wedge\beta)\wedge\kappa)}e^{\lambda|z-y|}dz\leq c\,\epsilon^\delta$, which can be shown with help of Hölder's inequality ($\frac{l-1}{l}+\frac{1}{l}=1$, $(\alpha\wedge\beta)\wedge\kappa>\frac{1}{l}$) and $(iii),(iv)$ of Lemma 3.5, as well as $\mathbf{E}[|u(r,z)-u(r,y)|^2]\leq c_T(|z-y|^{2(\alpha\wedge\beta)}+|z-y|)e^{\lambda|z-y|}e^{\lambda|z|}$, $z,y\in\mathbb{R}^1$, which is implicitly included in Step 1 of the proof of Theorem 2.4. Using Lemma 6.2, we also obtain

$$
\begin{aligned}
&I^\epsilon_9(t,x)\\
\leq\ & c\,\mathbf{E}\Big[\int_0^t\int_{\mathbb{R}^1}\Big(\int_{\mathbb{R}^1}q^\epsilon_{t-r}(x,z)p_\epsilon(y,z)dz-p_{t-r}(x,y)\Big)^2a^2(r,y,u(r,y))\varrho(dydr)\Big]\\
\leq\ & c\int_0^t\int_{\mathbb{R}^1}\Big(\int_{\mathbb{R}^1}q^\epsilon_{t-r}(x,z)p_\epsilon(y,z)dz-p_{t-r}(x,y)\Big)^2e^{\lambda|y|}e^{-\lambda|y|}\mathbf{E}\Big[(1+u(r,y))^2\Big]\varrho(dydr)\\
\leq\ & c\int_0^t\int_{\mathbb{R}^1}\Big(\int_{\mathbb{R}^1}q^\epsilon_{t-r}(x,z)p_\epsilon(y,z)dz-p_{t-r}(x,y)\Big)^2e^{\lambda|y|}\varrho_1(dy)\|u\|^2_{\lambda,r,1}\ \varrho_2(dr)\\
\leq\ & c\int_0^t\int_{\mathbb{R}^1}\Big|\int_{\mathbb{R}^1}q^\epsilon_{t-r}(x,z)p_\epsilon(y,z)dz-p_{t-r}(x,y)\Big|\times\\
&\Big(\int_{\mathbb{R}^1}q^\epsilon_{t-r}(x,z)p_\epsilon(y,z)dz+p_{t-r}(x,y)\Big)e^{\lambda|y|}\ \varrho_1(dy)\varrho_2(dr)\\
\leq\ & c\int_0^t\int_{\mathbb{R}^1}c\frac{1}{(t-r)^{1/2+\delta}}\epsilon^\delta\times c_T\frac{1}{(t-r)^{1/2-\alpha_1/2}}e^{\lambda|x|}\varrho_2(dr)\leq\ c_T\ e^{\lambda|x|}\ \epsilon^\delta \qquad (6.39)
\end{aligned}
$$

where $\delta\in(0,\alpha_1/2+\alpha_2-1)$. Proceeding analogously we get estimates for $I^\epsilon_2(t,x)$, $I^\epsilon_3(t,x)$, $I^\epsilon_4(t,x)$ and $I^\epsilon_5(t,x)$ similar to (6.36), (6.37), (6.38) and (6.39). Altogether,

$$
\begin{aligned}
\|u_\epsilon-u\|^2_{\lambda,t,1}\ \leq\ & c_{\lambda,T}\Big\{\ \epsilon^\delta\ +\ sup_{s\in[0,t]}\int_0^s\frac{1}{(s-r)^{1-\alpha_1/2}}\|u_\epsilon-u\|^2_{\lambda,r,1}\varrho_2(dr)\\
&+sup_{s\in[0,t]}\int_0^s\frac{1}{(s-r)^{1/2-\beta_1/2}}\|u_\epsilon-u\|^2_{\lambda,r,1}\sigma_2(dr)\Big\}
\end{aligned}
$$

for all $t\leq T$, some $\lambda>0$ and some sufficiently small $\delta>0$. With help of the Gronwall-type Lemma 3.8 (ii) we deduce for every $T>0$, $\|u_\epsilon-u\|_{\lambda,T,1}\leq\tilde{c}_{\lambda,T}\ \epsilon^\delta\downarrow 0$ as $\epsilon\downarrow 0$. Since u_ϵ and u are jointly continuous and u_ϵ is non-negative for every $\epsilon>0$, u is non-negative, too. We are done.

7 Proof of Theorem 2.6

We go to exploit Theorem 2.4. There are sequences $(a_n(.))$ and $(b_n(.))$ of Lipschitz-continuous functions approximating a respectively b uniformly on $\mathbb{R}^1$ as $n \to \infty$ and satisfying (2.11) with a, b replaced by a_n, b_n for all $n \geq 1$. Then, according to Theorem 2.4, for every $n \geq 1$ there is a unique strong solution u_n to equation (1.1) with a, b replaced by a_n, b_n. Similarly to Step 1 of the proof of Theorem 2.4 one checks for every $T > 0$ and $m \geq 1$ that

$$\mathbf{E}\Big[|u_n(t,x) - u_n(t',x')|^{2m}\Big] \leq c_\lambda\Big(|t-t'|^{\alpha(m-1)\wedge\beta(2m-1)} + |x-x'|^{2(\alpha(m-1)\wedge\beta(2m-1))}\Big)e^{\lambda|x|}$$

for all $t, t' \in [0,T]$, $x, x' \in \mathbb{R}^1$ with $|x - x'| \leq 1$ and $n \geq 1$. Hence, Lemma 3.1 *(ii)* gives tightness of (u_n) in $C([0,T], C_{tem})$ and so, according to Prohorov's theorem, (u_n) is relative compact w.r.t. weak convergence. In order to complete the proof - note that by Lemma 3.1 any limit point is locally Hölder-γ-continuous for all $\gamma \in (0, \frac{\alpha}{2} \wedge \beta)$ - we only have to check that any limit point is a (weak) solution to equation (1.1). At this point we can take advantage of the equivalence of SPDE (1.1) and the martingale problem from Definition 4.1, recall Proposition 4.3. Let u denote any weak limit point of some subsequence $(u_k) \subset (u_n)$. Weak convergence of u_k towards u means

$$\mathbf{E}[f(u_k(.,.))] = \int f(\phi(.,.))\mathbf{P}_{u_k}(d\phi) \to \int f(\phi(.,.))\mathbf{P}_u(d\phi) = \mathbf{E}[f(u(.,.))] \qquad (7.40)$$

as $k \to \infty$ for all $f \in C_b(C([0,T], C_{tem}), \mathbb{R})$.

Note that $M_.(\psi)$ from the (a, b, η)-martingale problem is an $(\mathcal{F}_t^u)$-martingale ($\mathcal{F}_t^u := \sigma(u(r,.) : r \leq t)$) if and only if

$$\begin{aligned} 0 \;=\; & \mathbf{E}\Big[\Big(\langle u(t+s,.),\psi\rangle - \langle u(t,.),\psi\rangle - \int_t^{t+s}\langle u(r,.), \frac{1}{2}\Delta\psi\rangle dr \\ & - \int_t^{t+s}\int_{\mathbb{R}^1} b(u(r,y))\psi(y)\sigma(dydr)\Big)\prod_{i=1}^l h_i(u(t_i,.))\Big] \end{aligned} \qquad (7.41)$$

for all $0 \leq t_1 < \cdots < t_l \leq t$, $s \geq 0$, $l \geq 1$ and $h_1, \ldots, h_l \in C_b(C_{tem}, \mathbb{R})$, and that the mappings

$$\phi \mapsto \langle\phi(t,.),\psi\rangle\prod_{i=1}^l h_i(\phi(t_i,.)), \qquad \phi \mapsto \int_0^t\langle\phi(r,.), \tfrac{1}{2}\Delta\psi\rangle dr\prod_{i=1}^l h_i(\phi(t_i,.)),$$

$$\phi \mapsto \int_0^t\int_{\mathbb{R}^1} b(\phi(r,y))\psi(y)\sigma(dydr)\prod_{i=1}^l h_i(\phi(t_i,.))$$

are in $C_b(C([0,T], C_{tem}), \mathbb{R})$ when $\psi \in \mathcal{S}$, $t \in [0,T]$, $n \geq 1$ and the h_i, t_i are as above. Since u_k solves the (a_k, b_k, η)-martingale problem, (7.41) holds with u, a, b replaced by u_k, a_k, b_k. Also, according to (7.40), the r.h.s. of (7.41) with u, a, b replaced by u_k, a_k, b_k converges to the r.h.s. of (7.41) as $k \to \infty$. Consequently, the equation in (7.41) holds and $M_t(\psi) := \langle u(t,.),\psi\rangle - \langle\eta,\psi\rangle - \int_0^t\langle u(r,.), 2^{-1}\Delta\psi\rangle dr - \int_0^t\int_{\mathbb{R}^1} b(u(r,y))\psi(y)\sigma(dydr)$ is thus an $(\mathcal{F}_t^u)$-martingale. By the uniqueness of the Doob-Meyer decomposition, $M_t(\psi)$

has quadratic variation process $\langle M(\psi)\rangle_t = \int_0^t \int_{\mathbb{R}^1} a^2(r,y,u(r,y))\psi^2(y)\varrho(dydr)$ if and only if

$$0 \;=\; \mathbf{E}\Big[\Big(M^2_{t+s}(\psi) - M^2_t(\psi) - \int_t^{t+s}\int_{\mathbb{R}^1} a^2(u(r,y))\psi^2(y)\varrho(dydr)\Big)\prod_{i=1}^{l} h_i(u(t_i,.))\Big] \quad (7.42)$$

for all $0 \le t_1 < \cdots < t_l \le t$, $s \ge 0$, $l \ge 1$ and $h_1,\ldots,h_l \in C_b(C^+_{tem},\mathbb{R})$. Now, the analogue of (7.42) for u_k, a_k, b_k holds. The r.h.s. of this analogue converges to the r.h.s. of (7.42) as $k \to \infty$ which can be proved again using (7.40). Hence, (7.42) holds and so u is a solution to the (a,b,η)-martingale problem. Proposition 4.3 then gives the first claim. The claim on the non-negativity can easily be concluded with help of Theorem 2.5.

8 Proof of Theorem 2.7

Note that Theorem 2.7 is already included - in a slightly different setting - in Corollary 3.3 of [Zäh02]. However, the proof was omitted since it is more or less standard in the context of superprocesses. For the sake of completeness we here work rigorously. We go to apply the method of duality whose main idea is the following. Let E and E' be metric spaces. Then, two E-valued continuous processes U_1 and U_2 have the same one-dimensional distributions, and so the same law, if both are dual to the same E'-valued process V w.r.t. a sufficiently large class F of measurable functions $f : E \times E' \to \mathbb{R}$, that is

$$\mathbf{E}^x[f(U(t),x')] = \mathbf{E}^{x'}[f(x,V(t))] \qquad \forall f \in F, x \in E, x' \in E', t \ge 0 \quad (8.43)$$

for $U = U_1, U_2$. This statement holds since relation (8.43) trivially leads to

$$\int f(y,x')\mathbf{P}^x_{U_1(t)}(dy) = \int f(y,x')\mathbf{P}^x_{U_2(t)}(dy) \qquad \forall f \in F, x \in E, x' \in E', t \ge 0. \quad (8.44)$$

Particularly, F is "sufficiently large" if the set $\{f(.,x') : f \in F, x' \in E'\}$ is separating w.r.t. probability measures on E. For details see [EK86]. In our setting the rôle of E is played by C^+_{tem} and U_1, U_2 are to be associated with two weak solutions to SPDE (1.1) with $a(u) = \sqrt{u}$ and $b \equiv 0$. As already mentioned, solutions to the considered SPDE should describe the space-time density of the catalytic super-Brownian motion $\bar{u}_t(dx)$ with catalyst $\varrho_1(t,dx)$ - note that $\varrho_1(t,dx)$ is an admissible catalyst in sense of [DF91]. Recalling the Laplace functional representation of $\bar{u}_t(dx)$ from Theorem 2.5.1 of [DF91], we conjecture that every solution u to the considered SPDE satisfies

$$\mathbf{E}\Big[exp\Big(-\langle u(t,.),\psi\rangle\Big)\Big] = exp\Big(-\langle \eta, v_0(t,.)\rangle\Big) \qquad \forall \psi \in \mathcal{S}, \eta \in C^+_{tem} \quad (8.45)$$

where $(v_s(t,x) : 0 \le s \le t, x \in \mathbb{R}^1)$ is the unique non-negative solution to the integral equation

$$v_s(t,x) = P_{t-s}\psi(x) - \frac{1}{2}\int_s^t \int_{\mathbb{R}^1} p_{r-s}(x,y)v_r^2(t,y)\varrho_1(r,dy)dr, \quad (8.46)$$

that is heuristically to the formal backward cumulant equation

$$-\frac{\partial}{\partial s}v_s(t,x) = \frac{1}{2}\Delta v_s(t,x) - \frac{1}{2}\frac{"\varrho_1(s,dx)"}{dx}(x)v_s^2(t,x), \quad v_t(t,.) = \psi.$$

Hence, for any solution u and every $t \geq 0$, relation (8.45) would give duality of $(u(r,.) : r \leq t)$ to the deterministic process $(v_{t-r}(t,.) : r \leq t)$ w.r.t.

$$F = \Big\{ f : C_{tem}^+ \to \mathbb{R}\,; \quad f(.) = exp(-\langle ., \psi\rangle), \psi \in \mathcal{S} \Big\}$$

which is known to be separating w.r.t. probability measures on C_{tem}^+. We hence had weak uniqueness of solutions, justified by the above considerations. And these arguments really apply. We only have to prove (8.45) yet.

First of all note that equation (8.46) indeed has a unique non-negative measurable solution v, see Proposition 1(a) of [DF97]. Using techniques as in the proof of Theorem 3.4 (ii) of [Zäh02], one can even show that v is in $C([0,\infty), C_{rap}^+)$ (since ϱ was of class (MN)). Furthermore, for all $n \geq 1$, the backward cumulant equation

$$-\frac{\partial}{\partial s}v_s(t,x) = \frac{1}{2}\Delta v_s(t,x) - \frac{1}{2}\varrho_1^n(t,x)v_s^2(t,x), \quad v_t(t,.) = \psi \tag{8.47}$$

for the smoothed catalyst $\varrho_1^n(t,dx) = \varrho_1^n(t,x)dx := \int_{\mathbb{R}^1} p_{1/n}(x,y)\varrho_1^n(t,dy)dx$ possesses a unique solution v^n in the space $C^{1,2}([0,\infty), C_{rap}^+)$ of $C([0,\infty), C_{rap}^+)$-elements that are continuously differentiable once in time and twice in space. In fact, first one defines v^n as the (unique) solution to equation (8.46) with $\varrho_1(r,dx)$ replaced by $\varrho_1^n(r,dx)$, then, using standard arguments, one concludes that v^n is differentiable as desired and that it solves equation (8.47). Recalling the belonging of $\varrho(dxdt) = \varrho_1(t,dx)dt$ to class (MN) and exploiting the lemmas from Chapter 3, it is not hard to show that

$$d_{rap}^*(v, v^n) \to 0 \text{ as } n \to \infty \tag{8.48}$$

where

$$d_{rap}^*(v,v') := \sum_{k=1}^{\infty} 2^{-k}\Big[1 \wedge \sup_{t \leq k} d_{rap}\Big(v(t,.), v'(t,.)\Big)\Big]$$

is a metric on $C([0,\infty), C_{rap}^+)$.

Let u be any weak solution to the considered SPDE. As in the proof of Theorem 2.1 of [Shi94] (cf. (6.12) there) we get

$$\langle u_t, f_t\rangle = \langle \eta, f_0\rangle + \int_0^t \langle u_r, \frac{\partial}{\partial r}f_r + \frac{1}{2}\Delta f_r\rangle dr + \int_0^t \int_{\mathbb{R}^1} \sqrt{u(r,y)}f_r(y)W^{\varrho}(dydr) \tag{8.49}$$

for all $t \geq 0$ and $f \in C([0,\infty), C_{rap}^2)$ with $\frac{\partial}{\partial t}f_t(.) \in C([0,\infty), C_{rap})$. Setting $f_r(x) = v_r^n(t,x)$, for all $r \in [0,t]$, applying Itô's formula to the semimartingale $(\langle u(r,.), v_r^n(t,.)\rangle : r \in [0,t])$ and the function e^{-x}, as well as taking expectation yields

$$\mathbf{E}\Big[exp\Big(-\langle u(t,.), \psi\rangle\Big)\Big] - exp\Big(-\langle \eta, v_0^n(t,.)\rangle\Big)$$

$$
\begin{aligned}
&= \mathbf{E}\Big[\int_0^t\int_{\mathbb{R}^1} exp\Big(-\langle u(r,.),v_r^n(t,.)\rangle\Big)\frac{1}{2}v_r^n(t,y)^2u(r,y)\varrho_1(r,dy)dr\Big]\\
&\quad -\mathbf{E}\Big[\int_0^t exp\Big(-\langle u(r,.),v_r^n(t,.)\rangle\Big)\Big\langle u(r,.),\frac{1}{2}\Delta v_r^n(t,.)+\frac{\partial}{\partial r}v_r^n(t,.)\Big\rangle dr\Big]\\
&= \mathbf{E}\Big[\int_0^t\int_{\mathbb{R}^1} exp\Big(-\langle u(r,.),v_r^n(t,.)\rangle\Big)\frac{1}{2}v_r^n(t,y)^2u(r,y)\varrho_1(r,dy)dr\Big]\\
&\quad -\mathbf{E}\Big[\int_0^t\int_{\mathbb{R}^1} exp\Big(-\langle u(r,.),v_r^n(t,.)\rangle\Big)\frac{1}{2}v_r^n(t,y)^2u(r,y)\varrho_1^n(r,y)dydr\Big]\\
&\quad +\mathbf{E}\Big[\int_0^t\int_{\mathbb{R}^1} exp\Big(-\langle u(r,.),v_r^n(t,.)\rangle\Big)\times\\
&\qquad \Big(\frac{1}{2}\varrho_1^n(r,y)v_r^n(t,y)^2-\frac{1}{2}\Delta v_r^n(t,y)-\frac{\partial}{\partial r}v_r^n(t,y)\Big)dydr\Big].
\end{aligned}
$$

Noting that v^n was the solution to equation (8.47) and that it is dominated by a constant uniformly in n, recalling (8.48) and letting $n\to\infty$, we reach (8.45).

Acknowledgement. I am grateful to my supervisor, Sylvie Roelly, for many helpful conversations, for uncovering weaknesses of the original manuscript and for support in general.

References

[DF91] D.A. Dawson and K. Fleischmann. Critical branching in a highly fluctuating random medium. *Probab. Theory Related Fields*, 90(2):241–274, 1991.

[DF97] D.A. Dawson and K. Fleischmann. A continuous super-Brownian motion in a super-Brownian medium. *J. Theoret. Probab.*, 10(1):213–276, 1997.

[DFLM95] D.A. Dawson, K. Fleischmann, Y. Li, and C. Mueller. Singularity of super-Brownian local time at a point catalyst. *Ann. Probab.*, 23(1):37–55, 1995.

[EK86] S.N. Ethier and T.G. Kurtz. *Markov processes.* John Wiley & Sons Inc., New York, 1986. Characterization and convergence.

[Iwa87] K. Iwata. An infinite stochastic differential equation with state space C(R). *Probab. Theory Relat. Fields*, 74:141–159, 1987.

[KS88] N. Konno and T. Shiga. Stochastic partial differential equations for some measure-valued diffusions. *Probab. Theory Relat. Fields*, 79:201–225, 1988.

[Mat95] P. Mattila. *Geometry of Sets and Measures in Euclidean Spaces.* Cambridge University Press, 1995.

[MP92] C. Mueller and E.A. Perkins. The compact support property for solutions to the heat equation with noise. *Probab. Theory Related Fields*, 93(3):325–358, 1992.

[MRC88] S. Méléard and S. Roelly-Coppoletta. A generalized equation for a continuous measure branching process. *Lecture Notes Math.*, 1390:171–185, 1988.

[Myt98] L. Mytnik. Weak uniqueness for the heat equation with noise. *Ann. Probab.*, 26(3):968–984, 1998.

[RC86] S. Roelly-Coppoletta. A criterion of convergence of measure-valued processes: application to measure branching processes. *Stochastics*, 17:43–65, 1986.

[Rei89] M. Reimers. One dimensional stochastic partial differential equations and the branching measure diffusion. *Probab. Theory Relat. Fields*, 81:319–340, 1989.

[Shi94] T. Shiga. Two contrasting properties of solutions for one-dimensional stochastic partial differential equations. *Canad. J. Math.*, 46(2):415–437, 1994.

[Wal86] J.B. Walsh. An introduction to stochastic partial differential equations. In P.L. Hennequin, editor, *École d'Été de Probabilités de Saint-Flour XIV – 1984*, volume 1180 of *Lecture Notes Math.*, pages 266–439. Springer-Verlag Berlin, 1986.

[Zäh02] H. Zähle. Space-time regularity of catalytic super-Brownian motion. *WIAS Preprint*, 783, 2002.

Contents

1 Introduction **1**

2 Preliminaries and main results **3**

3 Technical lemmas **6**

4 SPDE (1.1) reformulated **13**

5 Proof of Theorem 2.4 **15**

6 Proof of Theorem 2.5 **19**

7 Proof of Theorem 2.6 **25**

8 Proof of Theorem 2.7 **26**