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Compact interface property for symbiotic branching

Alison M. Etheridge*[1], Klaus Fleischmann**[2]

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[1] University of Oxford
Department of Statistics
1 South Parks Road
Oxford OX1 3TG
UK
E-Mail: etheridg@stats.ox.ac.uk

[2] Weierstrass-Institute
for Applied Analysis and Stochastics
Mohrenstr. 39
D-10117 Berlin
Germany
E-Mail: fleischmann@wias-berlin.de

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Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

ABSTRACT. A process which we call symbiotic branching, is suggested covering three well-known interacting models: mutually catalytic branching, the stepping stone model, and the Anderson model. Basic tools such as self-duality, particle system moment duality, measure case moment duality, and moment equations are still available in this generalized context. As an application, we show that in the setting of the one-dimensional continuum the compact interface property holds: starting from complementary Heaviside states, the interface is finite at all times almost surely.

CONTENTS

1. Introduction and main result

1.1. **Background.**

Consider the following system of stochastic partial differential equations

$$\frac{\partial}{\partial t} X_t^k(a) \;=\; \frac{\kappa^2}{2} \Delta X_t^k(a) \;+\; \sqrt{\gamma\, X_t^1(a) X_t^2(a)}\; \dot{W}_t^k(a), \tag{1}$$

$t > 0,\ \ a \in \mathbb{R},\ \ k = 1,2,$ starting from suitable $X_0^k \ge 0$. (Note that by a slight abuse of notation we always write the type $k = 1,2$ as an *upper* index, do not misunderstand it as a power.) Here $\kappa, \gamma > 0$ are constants, called the dispersion and collision rate, respectively. The one-dimensional Laplacian, Δ, acts on the real-valued variable a. Finally, $\dot{\mathbf{W}} = (\dot{W}^1, \dot{W}^2)$ is a *correlated* pair of standard white noises on $\mathbb{R}_+ \times \mathbb{R}$ with correlation constant $\varrho \in [-1,1]$:

$$E \dot{W}_{t^1}^1(a^1) \dot{W}_{t^2}^2(a^2) \;=\; \varrho\, \delta_0(t^1 - t^2)\, \delta_0(a^1 - a^2), \qquad t^1, t^2 \ge 0, \quad a^1, a^2 \in \mathbb{R}, \tag{2}$$

that is,

$$E W_{t^1}^1(\mathrm{d}a^1) W_{t^2}^2(\mathrm{d}a^2) \;=\; \varrho\, (t^1 \wedge t^2)\, \delta_0(a^1 - a^2)\, \mathrm{d}a^1 \mathrm{d}a^2, \tag{3}$$

$t^1, t^2 \ge 0,\ a^1, a^2 \in \mathbb{R},$ where δ_0 denotes the delta function at 0. Hence, (1) can be seen as a (vector-valued) stochastic partial differential equation with a "coloured noise". The three special cases $\varrho = 0$ and $\varrho = \pm 1$ already appear in the literature.

The first case, $\varrho = 0$, is the *mutually catalytic branching model* in $\mathbb{R}$ of Dawson/Perkins [DP98] and Mytnik [Myt98]. Roughly speaking, here $X_t^k(a)$ is interpreted as the density of mass of type k at time t at site a of a two-type population $\mathbf{X} = (X^1, X^2)$, where X^2 evolves as "catalytic super-Brownian motion" with time-space varying branching rate $X_t^1(a)$, with the analogous interpretation for X^1. Of course, the X^k are not classical superprocesses: even though X^1 and X^2 are uncorrelated, the branching property is violated.

The case $\varrho = -1$ with the additional requirement $X_0^1 + X_0^2 = 1$ corresponds to the *continuous space stepping stone model*

$$\frac{\partial}{\partial t} X_t^1(a) \;=\; \frac{\kappa^2}{2} \Delta X_t^1(a) \;+\; \sqrt{\gamma\, X_t^1(a) \big(1 - X_t^1(a)\big)}\; \dot{W}_t^1(a), \tag{4}$$

$t > 0,\ \ a \in \mathbb{R},$ of population genetics, see Shiga [Shi88, Shi94]. Indeed, here $\dot{W}^1 = -\dot{W}^2$, hence $X^1 + X^2$ solves the heat equation, implying $X_t^1 + X_t^2 \equiv 1$ in law.

Finally, in the case $\varrho = 1$ we have $\dot{W}^1 = \dot{W}^2 =: \dot{W}$. Consider the unique strong solution X of the *continuous space Anderson model*

$$\frac{\partial}{\partial t} X_t(a) \;=\; \frac{\kappa^2}{2} \Delta X_t(a) \;+\; \sqrt{\gamma}\, X_t(a)\, \dot{W}_t(a), \qquad t > 0,\ \ a \in \mathbb{R}; \tag{5}$$

see, for instance, Mueller [Mue91]. Then the pair (X, X) solves our system (1) (with $\varrho = 1$).

In mutually catalytic branching ($\varrho = 0$) each population only branches in the presence of the other one, but conditional on branching taking place, the number of offspring of each type is independent. In the stepping stone model (this needs $\varrho = -1$) too each population only branches in the presence of the other, but in that setting an increase in one population must be exactly matched by a decrease in the other in order to maintain a constant total population size. When $\varrho = 1$, the two populations increase or decrease together. The cases $|\varrho| < 1$ can be regarded as intermediate between these extremes: each population only branches in the presence of the other, but now the branching mechanisms are correlated, but not completely so.

To explain the *particle model* corresponding to $\mathbf{X}$, we describe the branching mechanism at a single point carrying n^1 type 1 particles and n^2 type 2 particles. Critical binary branching events happen at rate $2\gamma n^1 n^2$. Then with probability $(1-|\varrho|)/2$ only a type 1 particle branches, with probability $(1-|\varrho|)/2$ only a type 2 particle branches, and with probability $|\varrho|$ one particle of each type branches. In the last case, if $\varrho > 0$, then the two populations increase or decrease together, whereas if $\varrho < 0$ an increase of one is matched by a decrease of the other.

The novelty of the present model concerns the case $0 < |\varrho| < 1$. Once more, in contrast to the mutually catalytic branching model, here X^1 and X^2 are correlated, hence the fluctuation coefficient $\gamma X_t^1(a) X_t^2(a)$ in (1) gets smaller or larger in the mean depending on whether ϱ is negative or positive, respectively, (see formula (81) below).

For reasonable initial states $\mathbf{X}_0 \geq 0$ and $\varrho < 1$, the system has a unique (weak) solution $\mathbf{X} = (X^1, X^2) \geq 0$, which we call the *symbiotic*[1] *branching process* in $\mathbb{R}$ with correlation constant ϱ (and dispersion rate κ and collision rate γ), see Theorem 3 below. Existence of $\mathbf{X}$ with finite moments of all orders can be established by standard methods, as, for instance, in [FX01, Sections 3.1 and 3.2]. Under $|\varrho| < 1$, uniqueness in law and the strong Markov property follow from a self-duality, see Proposition 4 below, we skip any further details. If $\varrho = -1$, then $X^1 + X^2$ solves the heat equation, which gives a nice control of all moments of $\mathbf{X}$ under deterministic initial states $\mathbf{X}_0$; thus, uniqueness (in law) of $\mathbf{X}$ follows from a particle system moment duality, see Proposition 11 below. If $\varrho = 1$, then with symbiotic branching $\mathbf{X}$ we mean any solution to (1) (that is, without having established uniqueness).

1.2. **Compact interface property.** For a pair $\mathbf{x} = (x^1, x^2)$ of non-negative functions, the *interface* $\operatorname{Ifc}\mathbf{x}$ of $\mathbf{x}$ is defined by

$$\operatorname{Ifc}\mathbf{x} \;:=\; \operatorname{cl}\bigl\{a \in \mathbb{R} :\; x^1(a)\, x^2(a) > 0\bigr\} \tag{6}$$

(with $\operatorname{cl}A$ referring to the closure of a set A). To keep the setting relatively simple, we now restrict our attention to the "*Heaviside initial state*"

$$\mathbf{X}_0 \;=\; (1_{\mathbb{R}_-}, 1_{\mathbb{R}_+}) \tag{7}$$

with interface $\operatorname{Ifc}\mathbf{x} = \{0\}$. Our process $\mathbf{X}$ is said to have the *compact interface property,* if $\operatorname{Ifc}\mathbf{X}_t$ is compact for all $t > 0$, a.s.

In the case of the stepping stone model (4), the compact interface property holds, see Tribe [Tri95, Corollary 3.3] (although there only continuous initial states are considered).

The *main result* of our paper is that the compact interface property is true in *all* ϱ–cases. Actually, we show that the interface propagates at most with a linear speed, see Theorem 5 below.

Remark 1 (**Case** $\varrho = 1$)**.** At first sight, this theorem, specialized to $\varrho = 1$, seems to contradict the well-known result that the continuous space Anderson model X from (5) propagates instantaneously (see [Mue91]). The latter means, if X_0

[1] Ed Perkins told us that Joe Mc Kenna (Cornel University) suggested that it would be better to use the term "symbiotic" instead of "mutually catalytic". Since we want to have a name for our larger class of models covering the mutually catalytic case, we abuse this suggestion in this way.

is a compactly supported non-negative continuous function different from 0, then $X_t(a) > 0$ for all $t > 0$ and $a \in \mathbb{R}$, a.s. But recall that the continuous space Anderson model (5) leads to a solution of our system only in the case $X^1 = X^2$ implying $X_0^1 = X_0^2$, which contradicts the Heaviside initial state assumption (7) in our theorem. In our model, X^1, say, undergoes critical continuous-state branching with branching rate X^2. True, the Anderson model X can also be seen as having the feature of critical continuous-state branching, but only if this branching happens with rate X. In our model instead it might happen that $X_t^1(a) << X_t^2(a)$, say, implying that $X_t^1(a)$ is killed with very high probability, reducing the speed of propagation drastically. Altogether, concerning $\varrho = 1$ we have two different models, our $\mathbf{X}$ with Heaviside initial state $\mathbf{X}_0$, respectively (X, X) based on (5), each model with its own propagation property. ◇

As in [Tri95, Corollary 3.3], the theorem is proved by deriving a probability estimate on the supremum over a finite time interval of the position $R(X_t^1)$ of the "*rightmost individual*" in the X^1–population at time t. For the stepping stone model, Tribe shows moreover, that $t \mapsto R(X_t^1)$ is càdlàg and that, under diffusive rescaling, it converges to Brownian motion. In addition, in Mueller and Tribe [MT97] a limiting interface is shown to exist. But the corresponding questions remain *open* in our more general symbiotic branching model. Note however, that in the special case $\varrho = 0$ of the mutually catalytic branching model the behaviour of the two populations at the interface is highly irregular; recall the hot spots seen in simulations, or the explosions of densities everywhere at the interface in the $\mathbb{R}^2$–model of Dawson et al. [DEF$^+$02a, DEF$^+$02b, DFM$^+$02]. In addition, more precise moment calculations than those included in the present paper indicate the possibility that as ϱ varies there is a phase transition in the rate of growth of $t \mapsto R(X_t^1)$ as $t \uparrow \infty$. (Our estimate is uniform in ϱ, so it is "dictated" by the speed corresponding to $\varrho = 1$.)

1.3. **Outline.** The rest of the paper is laid out as follows. In Section 2, after introducing some notation, we reformulate equation (1) as a martingale problem and state in Theorem 3 the unique existence of the symbiotic branching model under $\varrho < 1$. Here uniqueness in the cases $|\varrho| < 1$ is based on self-duality (Proposition 4), and uniqueness under $\varrho = -1$ follows from the particle system moment duality (Proposition 11), whereas uniqueness for $\varrho = 1$ remains open. Our main result, the compact interface property is established in Theorem 5. We conclude Section 2 with a scaling property. The $\mathbb{Z}^d$–version of the symbiotic model is introduced in Section 3, together with two versions of a moment dual. The discrete version is needed to pass via a diffusion approximation in Section 4 to the corresponding moment duals in the $\mathbb{R}$–setting. The particle system moment dual is used in Subsection 4.2 to derive our basic higher moment estimate. Moment equations are contained in Subsection 4.4. After some preparations, in Section 5 Theorem 5 concerning the compact interface property is proved. The key is an estimation of the fluctuation term (Lemma 22). This is used in the proof of Proposition 23 to derive a probability estimate on the supremum over a finite time interval of the position of the rightmost individual in the X^1–population.

For background on superprocesses, we refer to Dawson [Daw93], [Eth00], [LG99], and [Per02], for instance, for background on stochastic partial differential equations to Walsh [Wal86], and for a recent survey on mutually catalytic branching to [DF02].

2. Symbiotic branching in $\mathbb{R}$

2.1. **Preliminaries: notation and spaces.** For $\lambda \in \mathbb{R}$, introduce the reference function

$$\phi_\lambda(a) := \mathrm{e}^{-\lambda|a|}, \qquad a \in \mathbb{R}^d. \tag{8}$$

For $f : \mathbb{R}^d \to \mathbb{R}$, put

$$|f|_\lambda := \|f/\phi_\lambda\|_\infty \tag{9}$$

where $\|\cdot\|_\infty$ is the supremum norm. Denote by $\mathcal{B}_\lambda$ the space of all measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ with $|f|_\lambda < \infty$ and such that $f(a)/\phi_\lambda(a)$ has a finite limit as $|a| \to \infty$. Introduce the spaces

$$\mathcal{B}_{\mathrm{rap}} = \mathcal{B}_{\mathrm{rap}}(\mathbb{R}^d) := \bigcap_{\lambda>0} \mathcal{B}_\lambda \quad \text{and} \quad \mathcal{B}_{\mathrm{tem}} = \mathcal{B}_{\mathrm{tem}}(\mathbb{R}^d) := \bigcap_{\lambda>0} \mathcal{B}_{-\lambda} \tag{10}$$

of *exponentially decreasing* and *tempered* measurable functions on $\mathbb{R}^d$, respectively. (Roughly speaking, the functions in $\mathcal{B}_{\mathrm{rap}}$ decay faster than exponentially, whereas the ones in $\mathcal{B}_{\mathrm{tem}}$ are allowed to have a subexponential growth.) Write $\mathcal{C}_\lambda$, $\mathcal{C}_{\mathrm{rap}}$, $\mathcal{C}_{\mathrm{tem}}$ for the respective subspaces of continuous functions. We also need the space $\mathcal{C}_{\mathrm{com}} = \mathcal{C}_{\mathrm{com}}(\mathbb{R}^d)$ of all continuous functions on $\mathbb{R}^d$ with compact (closed) support.

Write $\mathcal{C}_\lambda^{(m)} = \mathcal{C}_\lambda^{(m)}(\mathbb{R}^d)$, $\mathcal{C}_{\mathrm{rap}}^{(m)} = \mathcal{C}_{\mathrm{rap}}^{(m)}(\mathbb{R}^d)$, $\mathcal{C}_{\mathrm{tem}}^{(m)} = \mathcal{C}_{\mathrm{tem}}^{(m)}(\mathbb{R}^d)$, and $\mathcal{C}_{\mathrm{com}}^{(m)} = \mathcal{C}_{\mathrm{com}}^{(m)}(\mathbb{R}^d)$ if we additionally require that all partial derivatives up to the order $m \geq 1$ exist and belong to $\mathcal{C}_\lambda$, $\mathcal{C}_{\mathrm{rap}}$, $\mathcal{C}_{\mathrm{tem}}$, and $\mathcal{C}_{\mathrm{com}}$, respectively.

For $T > 0$ and $\lambda \in \mathbb{R}$, denote by $\mathcal{C}_{T,\lambda}^{(1,2)}$ the set of all real-valued functions ψ defined on $[0,T] \times \mathbb{R}$ such that $t \mapsto \psi_t$, $t \mapsto \frac{\partial}{\partial t}\psi_t$, and $t \mapsto \Delta\psi_t$ are continuous $\mathcal{C}_\lambda$–valued functions. Set $\mathcal{C}_{T,\mathrm{rap}}^{(1,2)} := \bigcap_{\lambda>0} \mathcal{C}_{T,\lambda}^{(1,2)}$ and $\mathcal{C}_{T,\mathrm{tem}}^{(1,2)} := \bigcap_{\lambda>0} \mathcal{C}_{T,-\lambda}^{(1,2)}$.

For each $\lambda \in \mathbb{R}$, the linear space $\mathcal{C}_\lambda$ equipped with the norm $|\cdot|_\lambda$ from (9) is a separable Banach space. On the other hand, the space $\mathcal{C}_{\mathrm{rap}}$ topologized by the metric

$$^{\mathcal{C}}\mathrm{d}_{\mathrm{rap}}(f,g) := \sum_{n=1}^{\infty} 2^{-n} \left(|f-g|_{-n} \wedge 1\right), \qquad f,g \in \mathcal{C}_{\mathrm{rap}}, \tag{11}$$

is a Polish space. $\mathcal{C}_{\mathrm{tem}}$ becomes a Polish space if we use the metric

$$^{\mathcal{C}}\mathrm{d}_{\mathrm{tem}}(f,g) := \sum_{n=1}^{\infty} 2^{-n} \left(|f-g|_{-1/n} \wedge 1\right), \qquad f,g \in \mathcal{C}_{\mathrm{tem}}, \tag{12}$$

instead. (For $\mathcal{C}_{\mathrm{com}}$ we will not need a topology.)

Let $\mathcal{M} = \mathcal{M}(\mathbb{R}^d)$ denote the set of all (non-negative) Radon measures μ on $\mathbb{R}^d$ and let d_0 be a complete metric on $\mathcal{M}$ which induces the vague topology. We identify μ with its density (if exists). We use the notation $\langle \mu, f\rangle$ for the integral of the function f with respect to the measure μ. We need the space $\mathcal{M}_{\mathrm{tem}} = \mathcal{M}_{\mathrm{tem}}(\mathbb{R}^d)$ of all measures μ in $\mathcal{M}$ such that $\langle \mu, \phi_\lambda\rangle < \infty$, for all $\lambda > 0$. We topologize this set $\mathcal{M}_{\mathrm{tem}}$ of *tempered* measures by the metric

$$^{\mathcal{M}}\mathrm{d}_{\mathrm{tem}}(\mu,\nu) := \mathrm{d}_0(\mu,\nu) + \sum_{n=1}^{\infty} 2^{-n} \left(|\mu-\nu|_{1/n} \wedge 1\right), \qquad \mu,\nu \in \mathcal{M}_{\mathrm{tem}}. \tag{13}$$

Here $|\mu-\nu|_\lambda$ is an abbreviation for $\left|\langle \mu, \phi_\lambda\rangle - \langle \nu, \phi_\lambda\rangle\right|$. Note that $\left(\mathcal{M}_{\mathrm{tem}}, {}^{\mathcal{M}}\mathrm{d}_{\mathrm{tem}}\right)$ is a Polish space.

Write $\mathfrak{C} := \mathcal{C}\big((0,\infty),(\mathcal{C}^+_{\rm tem})^2\big)$ for the set of all continuous paths $t \mapsto \mathbf{f}_t$ in $(\mathcal{C}^+_{\rm tem})^2$, where $\big((\mathcal{C}^+_{\rm tem})^2, {}^{\mathcal{C}}{\rm d}^2_{\rm tem}\big)$ is defined as the Cartesian product of $(\mathcal{C}^+_{\rm tem}, {}^{\mathcal{C}}{\rm d}_{\rm tem})$. When endowed with the metric

$$\text{(14)} \qquad {\rm d}_{\mathfrak{C}}(f_\cdot, \tilde{f}_\cdot) := \sum_{n=1}^{\infty} 2^{-n}\Big(\sup_{1/n \le t \le n} {}^{\mathcal{C}}{\rm d}^2_{\rm tem}(f_t, \tilde{f}_t) \wedge 1\Big), \qquad f_\cdot, \tilde{f}_\cdot \in \mathfrak{C},$$

$\mathfrak{C}$ is a Polish space. Let $\mathfrak{P}$ denote the set of all probability measures on $\mathfrak{C}$. Equipped with the Prohorov metric ${\rm d}_{\mathfrak{P}}$, $\mathfrak{P}$ is a Polish space, too ([EK86, Theorem 3.1.7]).

Analogously, $\mathcal{C}\big((0,\infty),(\mathcal{C}^+_{\rm rap})^2\big)$ is defined and handled.

Random objects are always thought of as being defined over a large enough stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_\cdot, \mathcal{P})$ satisfying the usual hypotheses. If $Y = \{Y_t : t \ge 0\}$ is a random process, then as a rule the law of Y is denoted by P^Y. We use $\mathcal{F}^Y_t$ to denote the completion of the σ–field $\bigcap_{\varepsilon>0} \sigma\{Y_s : s \le t+\varepsilon\}$, $t \ge 0$. Sometimes we write $\mathcal{L}(Y)$ and $\mathcal{L}(Y\,|\,\cdot)$ for the law and conditional law of Y, respectively.

For a constant $\kappa > 0$ let $p = p^\kappa$ denote the heat kernel in $\mathbb{R}^d$ related to $\frac{\kappa^2}{2}\Delta$:

$$\text{(15)} \qquad p_t(a) = p^\kappa_t(a) := (2\pi\kappa^2 t)^{-d/2} \exp\Big[-\frac{|a|^2}{2\kappa^2 t}\Big], \qquad t > 0, \quad a \in \mathbb{R}^d.$$

Write ξ for the related Brownian motion in $\mathbb{R}^d$, and $S = \{S_t : t \ge 0\}$ for its semigroup.

We denote by $c = c(q)$ a positive constant which (in the present case) may depend on the quantity q and whose value might change from place to place. Moreover, an index on c as $c_{(\#)}$ or $c_{\#}$ will indicate that this constant first occurred in formula line (#) or (for instance) Lemma #, respectively.

2.2. **Basic martingale problem.** It is convenient to reformulate the pair (1) of stochastic equations on $\mathbb{R}$ in terms of the following martingale problem. We *fix* the constants $\kappa, \gamma > 0$ for the remainder of this article. Let $\delta_{k,l}$ denote the Kronecker symbol. We use the abbreviation

$$\text{(16)} \qquad c_{k,l}(\varrho) := \big[\delta_{k,l} + (1-\delta_{k,l})\varrho\big], \qquad k,l = 1,2,$$

(where ϱ is our correlation constant).

Definition 2 (Martingale problem $\mathbf{MP}^\varrho_{\mathbf{x}}$). Fix $\varrho \in [-1,1]$ and $\mathbf{x} \in (\mathcal{B}^+_{\rm tem})^2$ [resp. $(\mathcal{B}^+_{\rm rap})^2$]. We say a stochastic process $\mathbf{X} = \{\mathbf{X}_t : t \ge 0\}$ with law $P_{\mathbf{x}}$ on the (restricted) path space $\mathcal{C}\big((0,\infty),(\mathcal{C}^+_{\rm tem})^2\big)$ [resp. $\mathcal{C}\big((0,\infty),(\mathcal{C}^+_{\rm rap})^2\big)$] is a *solution to the martingale problem* $\mathbf{MP}^\varrho_{\mathbf{x}}$, if for each test function $\varphi \in \mathcal{C}^{(2)}_{\rm rap}$ [resp. $\mathcal{C}^{(2)}_{\rm tem}$],

$$\text{(17)} \qquad M^k_t(\varphi) := \langle X^k_t, \varphi\rangle - \langle x^k, \varphi\rangle - \int_0^t {\rm d}s\, \Big\langle X^k_s, \frac{\kappa^2}{2}\Delta\varphi\Big\rangle,$$

$t \ge 0$, $k = 1,2$, are continuous square-integrable martingales $M^k(\varphi)$, $k = 1,2$, with $M^k_0(\varphi) \equiv 0$, and with bracket

$$\text{(18)} \qquad \big\langle\!\big\langle M^k(\varphi), M^l(\varphi)\big\rangle\!\big\rangle_t = \gamma\, c_{k,l}(\varrho) \int_0^t {\rm d}s\, \langle X^1_s X^2_s, \varphi^2\rangle,$$

$t \ge 0$, $k,l = 1,2$. ◇

Of course, $\mathcal{B}^+_{\rm rap} \subset \mathcal{B}^+_{\rm tem}$, but for $\mathbf{x} \in (\mathcal{B}^+_{\rm rap})^2$ as a rule, by $\mathbf{MP}^\varrho_{\mathbf{x}}$ we mean the martingale problem with the law $P_{\mathbf{x}}$ on the smaller space $\mathcal{C}\big((0,\infty),(\mathcal{C}^+_{\rm rap})^2\big)$.

Theorem 3 (**Unique existence of symbiotic branching $\mathbf{X}$ in $\mathbb{R}$**). *Let ϱ and $\mathbf{x}$ be as in Definition 2. If $\varrho < 1$, then there exists a unique (in law), hence strong Markov solution $\mathbf{X}$ to the martingale problem $\mathbf{MP}_{\mathbf{x}}^{\varrho}$.*

Clearly, the unique existence of a weak solution to the stochastic equation (1) (on an enlarged probability space) then follows from the standard martingale representation theorem ([Wal86]).

Recall that existence of $\mathbf{X}$ can be established by standard methods, as, for instance, in [FX01, Sections 3.1 and 3.2], and that in case $|\varrho| < 1$ uniqueness in law and the strong Markov property follow from self-duality.

2.3. **Self-duality.** Here we establish the self-duality relation which guarantees uniqueness in the martingale problem in the case $|\varrho| < 1$. We believe that under $\varrho > -1$ the moments of $\mathbf{X}$ grow too quickly for the moment problem to be well-posed and hence do not characterize the law of $\mathbf{X}$.

We start by introducing the *self-duality function* $\mathfrak{E}$. Fix $\varrho \in [-1,1]$. For $(\mathbf{x}, \tilde{\mathbf{x}}) \in (\mathcal{B}_{\text{tem}}^{+})^2 \times (\mathcal{B}_{\text{rap}}^{+})^2$, set

$$\mathfrak{E}(\mathbf{x}, \tilde{\mathbf{x}}) \;:=\; \mathfrak{E}\big(\langle y, \tilde{y}\rangle, \langle z, \tilde{z}\rangle\big) \;:=\; \exp\Big[-\sqrt{1-\varrho}\,\langle y, \tilde{y}\rangle + i\sqrt{1+\varrho}\,\langle z, \tilde{z}\rangle\Big], \tag{19}$$

where $i := \sqrt{-1}$ and

$$y := x^1 + x^2, \quad z := x^1 - x^2, \tag{20}$$

and $\tilde{y}, \tilde{z}$ are analogously defined.

Proposition 4 (**Self-duality**). *Fix $\varrho \in (-1,1)$. Let $(\mathbf{x}, \tilde{\mathbf{x}}) \in (\mathcal{B}_{\text{tem}}^{+})^2 \times (\mathcal{B}_{\text{rap}}^{+})^2$. If $(\mathbf{X}, P_{\mathbf{x}})$ and $(\tilde{\mathbf{X}}, P_{\tilde{\mathbf{x}}})$ solve the martingale problems $\mathbf{MP}_{\mathbf{x}}^{\varrho}$ and $\mathbf{MP}_{\tilde{\mathbf{x}}}^{\varrho}$, respectively, of Definition 2, then we have the self-duality relation*

$$E_{\mathbf{x}}\mathfrak{E}(\mathbf{X}_t, \tilde{\mathbf{x}}) \;=\; E_{\tilde{\mathbf{x}}}\mathfrak{E}(\mathbf{x}, \tilde{\mathbf{X}}_t), \qquad t \geq 0. \tag{21}$$

Of course, in the case $\varrho = 0$, we recover Mytnik's [Myt98] self-duality of the mutually catalytic branching model in $\mathbb{R}$. Note that we excluded $|\varrho| = 1$ since here one term in the exponent of (19) vanishes leading to a triviality.

Proof of Proposition 4. We apply the notation of (20) in an obvious way to introduce processes $Y, Z, \tilde{Y}, \tilde{Z}$. For $\varphi, \psi \in \mathcal{C}_{\text{rap}}^{(2)}$ we have from the martingale problem $\mathbf{MP}_{\mathbf{x}}^{\varrho}$ that

$$\begin{cases} \mathrm{d}\langle Y_t, \varphi\rangle \;=\; \Big\langle Y_t, \dfrac{\kappa^2}{2}\Delta\varphi\Big\rangle\,\mathrm{d}t \;+\; \mathrm{d}(\text{martingale}), \\[2ex] \mathrm{d}\langle Z_t, \psi\rangle \;=\; \Big\langle Z_t, \dfrac{\kappa^2}{2}\Delta\psi\Big\rangle\,\mathrm{d}t \;+\; \mathrm{d}(\text{martingale}), \end{cases} \tag{22}$$

and that

$$\begin{cases} \mathrm{d}\big\langle\!\big\langle \langle Y, \varphi\rangle \big\rangle\!\big\rangle_t \;=\; 2\gamma\,(1+\varrho)\,\big\langle X_t^1 X_t^2, \varphi^2\big\rangle\,\mathrm{d}t, \\ \mathrm{d}\big\langle\!\big\langle \langle Z, \psi\rangle \big\rangle\!\big\rangle_t \;=\; 2\gamma\,(1-\varrho)\,\big\langle X_t^1 X_t^2, \psi^2\big\rangle\,\mathrm{d}t, \\ \mathrm{d}\big\langle\!\big\langle \langle Y, \varphi\rangle, \langle Z, \psi\rangle \big\rangle\!\big\rangle_t \;=\; 0. \end{cases} \tag{23}$$

Note also that the trivial identity

$$X_t^1 X_t^2 \;=\; \frac{1}{4}\,(Y_t^2 - Z_t^2) \tag{24}$$

holds (with 2 in X_t^2 referring to the type, whereas 2 in Y_t^2 and Z_t^2 referring to a square). Next we replace φ and ψ formally by $\tilde{y}$ and $\tilde{z}$, respectively, although $\tilde{y}$ and $\tilde{z}$ do not meet the required smoothness. Thus, formally we get from Itô's formula,

$$\begin{aligned}(25)\quad \mathrm{d}\mathfrak{E}(\mathbf{X}_t,\tilde{\mathbf{x}}) &= \mathrm{d}\exp\Big[-\sqrt{1-\varrho}\,\langle Y_t,\tilde{y}\rangle + i\sqrt{1+\varrho}\,\langle Z_t,\tilde{z}\rangle\Big] \\ &= \mathfrak{E}(\mathbf{X}_t,\tilde{\mathbf{x}})\Big\{-\sqrt{1-\varrho}\,\mathrm{d}\langle Y_t,\tilde{y}\rangle + i\sqrt{1+\varrho}\,\mathrm{d}\langle Z_t,\tilde{z}\rangle \\ &\qquad + \frac{1}{2}(1-\varrho)\,\mathrm{d}\langle\!\langle\langle Y,\tilde{y}\rangle\rangle\!\rangle_t - \frac{1}{2}(1+\varrho)\,\mathrm{d}\langle\!\langle\langle Z,\tilde{z}\rangle\rangle\!\rangle_t \\ &\qquad - i\sqrt{1-\varrho^2}\,\mathrm{d}\langle\!\langle\langle Y,\tilde{y}\rangle,\langle Z,\tilde{z}\rangle\rangle\!\rangle_t\Big\}.\end{aligned}$$

Using (22) – (24), this amounts to

$$\begin{aligned}(26)\quad &\mathrm{d}\mathfrak{E}(\mathbf{X}_t,\tilde{\mathbf{x}}) \\ &= \mathfrak{E}(\mathbf{X}_t,\tilde{\mathbf{x}})\Big\{-\sqrt{1-\varrho}\,\Big\langle Y_t,\frac{\kappa^2}{2}\Delta\tilde{y}\Big\rangle + i\sqrt{1+\varrho}\,\Big\langle Z_t,\frac{\kappa^2}{2}\Delta\tilde{z}\Big\rangle \\ &\qquad + \frac{\gamma}{4}(1-\varrho^2)\,\big\langle Y_t^2 - Z_t^2,\tilde{y}^2-\tilde{z}^2\big\rangle\Big\}\,\mathrm{d}t + \mathrm{d}(\text{martingale}).\end{aligned}$$

Analogously,

$$\begin{aligned}(27)\quad &\mathrm{d}\mathfrak{E}(\mathbf{x},\tilde{\mathbf{X}}_t) \\ &= \mathfrak{E}(\mathbf{x},\tilde{\mathbf{X}}_t)\Big\{-\sqrt{1-\varrho}\,\Big\langle\frac{\kappa^2}{2}\Delta y,\tilde{Y}_t\Big\rangle + i\sqrt{1+\varrho}\,\Big\langle\frac{\kappa^2}{2}\Delta z,\tilde{Z}_t\Big\rangle \\ &\qquad + \frac{\gamma}{4}(1-\varrho^2)\,\big\langle y^2-z^2,\tilde{Y}_t^2-\tilde{Z}_t^2\big\rangle\Big\}\,\mathrm{d}t + \mathrm{d}(\text{martingale}),\end{aligned}$$

Comparing (26) and (27), the self-duality identity (21) follows by a standard procedure, compare, for instance, [Myt98]. Here in particular the symmetry of the Laplacian is exploited, and a regularization procedure using the heat kernel is needed to overcome the fact that the initial states $\mathbf{x},\tilde{\mathbf{x}}$ do not have the smoothness required for the test functions in the martingale problems $\mathbf{MP}_{\mathbf{x}}^{\varrho}$ and $\mathbf{MP}_{\tilde{\mathbf{x}}}^{\varrho}$, respectively. □

2.4. **Main result.** Recall definition (6) of the interface $\operatorname{Ifc}\mathbf{x}$ of a state $\mathbf{x}\in(\mathcal{B}_{\text{tem}}^{+})^2$. Our main result reads as follows.

Theorem 5 (Compact interface property). *Suppose $\mathbf{X}_0 = (1_{\mathbb{R}_-},1_{\mathbb{R}_+}) =: \mathbf{x}$. Then, there is a constant $c_5 = c_5(\gamma,\kappa)$ such that for each $\varrho\in[-1,1]$ and some random time T_0,*

$$(28)\qquad \bigcup_{t\le T}\operatorname{Ifc}\mathbf{X}_t \subseteq [-c_5T,\,c_5T] \quad \textit{for all } T\ge T_0,\ \ P_{\mathbf{x}}\text{-a.s.}$$

Consequently, the interface is compact and *propagates at most with a linear speed.* The proof of this theorem is postponed to Subsection 5.6.

2.5. **Scaling property.** The following scaling property will be a useful tool. The symbiotic branching process $\mathbf{X}$ in $\mathbb{R}$ evidently depends on the collision rate γ. When we want to make this dependence explicit we use the notation ${}^{\gamma}\mathbf{X}$.

Lemma 6 (Scaling of $\mathbf{X}$). *The symbiotic branching process* $\mathbf{X} = {}^{\gamma}\mathbf{X}$ *in* $\mathbb{R}$ *with collision rate* γ *and initial state* $\mathbf{X}_0 = \mathbf{x} \in \left(\mathcal{B}^{+}_{\mathrm{tem}}(\mathbb{R})\right)^2$ *has the following scaling property: for fixed constants* $K, c_0 > 0$, *the process* $\mathbf{Y} = (Y^1, Y^2)$ *defined by*

$$Y^k_t(a) \;:=\; c_0\, X^k_{Kt}(\sqrt{K}\, a), \qquad t \geq 0, \quad a \in \mathbb{R}, \quad k = 1, 2, \tag{29}$$

coincides in law with the symbiotic branching process ${}^{\sqrt{K}\gamma}\mathbf{X}$ *in* $\mathbb{R}$ *with collision rate* $\sqrt{K}\,\gamma$ *and initial state* $a \mapsto c_0\, \mathbf{x}(\sqrt{K}\, a)$.

Proof. The multiplication by the factor c_0 is trivial, so we will set $c_0 = 1$. For the remaining statement we only need observe that

$$\dot{V}^k_t(a) \;:=\; K^{3/4}\, \dot{W}^k_{Kt}(\sqrt{K}\, a), \qquad t \geq 0, \quad a \in \mathbb{R}, \quad k = 1, 2, \tag{30}$$

defines a correlated pair $\mathbf{V} = (V^1, V^2)$ of standard with noises on $\mathbb{R}_+ \times \mathbb{R}$ with the same correlation constant ϱ, [recall (2)], since for each fixed $a \in \mathbb{R}$,

$$\text{the generalized function} \quad b \mapsto K\, \delta_{Ka}(Kb) \quad \text{coincides with } \delta_a. \tag{31}$$

Hence, $\mathbf{Y}$ satisfies (1) with γ replaced by $\sqrt{K}\,\gamma$, finishing the proof. □

Note that for the Heaviside state $\mathbf{x} = (1_{\mathbb{R}_-, \mathbb{R}_+})$ or for $\mathbf{x} = (1, 1)$, we have $\mathbf{x}(\sqrt{K}\, a) \equiv \mathbf{x}(a)$, hence these initial states are invariant concerning the scaling procedure in the proposition provided that $c_0 = 1$.

3. Symbiotic branching in $\mathbb{Z}^d$

In this section, we introduce the $\mathbb{Z}^d$–version of symbiotic branching and develop two basic tools, the particle system moment dual (Proposition 7) and the measure case moment dual (Proposition 9).

3.1. The $\mathbb{Z}^d$–model.

The discrete space analogue of (1) is the following system of stochastic differential equations

$$\mathrm{d}X^k_t(a) \;=\; \frac{\kappa^2}{2}\Delta^{(1)} X^k_t(a)\, \mathrm{d}t \;+\; \sqrt{\gamma\, X^1_t(a) X^2_t(a)}\; \mathrm{d}W^k_t(a), \tag{32}$$

$t > 0$, $a \in \mathbb{Z}^d$, $k = 1, 2$, starting with suitable $X^k_0 \geq 0$. Here again $\kappa, \gamma > 0$ are constants, called the dispersion and collision rates, respectively, $\Delta^{(1)}$ is the discrete Laplacian in $\mathbb{Z}^d$,

$$\Delta^{(1)} f\,(a) \;:=\; \sum_{b \in \mathbb{Z}^d,\ |b-a|=1} \big[f(b) - f(a)\big], \qquad a \in \mathbb{Z}^d, \tag{33}$$

and

$$\big\{W^k(a) :\ a \in \mathbb{Z}^d,\ k = 1, 2\big\} \tag{34}$$

is a family of standard Brownian motions in $\mathbb{R}$ with *correlation*

$$E W^k_{t_1}(a_1) W^l_{t_2}(a_2) \;=\; \varrho\, (t_1 \wedge t_2)\, \delta_0(a_1 - a_2), \tag{35}$$

$t_1, t_2 \geq 0$, $a_1, a_2 \in \mathbb{Z}^d$, $k, l = 1, 2$. The correlation constant ϱ again belongs to $[-1, 1]$, and, in this discrete setting, δ_0 refers to the δ–measure at 0 (instead of the delta function). Under $\varrho < 1$, the unique solution to (32) is called the *symbiotic branching process* in $\mathbb{Z}^d$ with correlation constant ϱ (whereas for $\varrho = 1$ we call any solution to (32) symbiotic branching).

Just as in the $\mathbb{R}$–case, the special case $\varrho = 0$ is the mutually catalytic branching model in $\mathbb{Z}^d$ of [DP98], $\varrho = -1$ with $X^1_0 + X^2_0 = 1$ corresponds to the stepping stone model indexed by $\mathbb{Z}^d$, see Shiga [Shi80, Shi88], which goes back to Kimura [Kim53], and in the case $\varrho = 1$ the Anderson model X leads through the pair (X, X) to a special case of our model.

Existence of solutions with finite moments of all orders can be shown by standard methods, see, for instance, [DP98, Appendix], whereas uniqueness follows from the $\mathbb{Z}^d$–analogue of the self-duality of Proposition 4 if $|\varrho| < 1$, and from the particle system moment duality if $\varrho = -1$.

3.2. **Particle system moment dual N on $\mathbb{Z}^d$.** Moments of order $m \geq 1$ of the symbiotic process $\mathbf{X}$ in $\mathbb{Z}^d$ can be expressed in terms of a random system, $\mathbf{N}$, of m marked particles in $\mathbb{Z}^d$ with marks (types) in $\{1,2\}$. We shall call $\mathbf{N}$ the *particle system moment dual.* In $\mathbf{N}$, particles disperse according to independent random walks generated by $\frac{\kappa^2}{2}\Delta^{(1)}$. In addition, each pair of particles of the same type in the same site, a, at rate γ is replaced by a new pair of particles, one of each type.

More precisely, $\mathbf{N}$ is an $\mathcal{N}_{\mathrm{f}}$–valued Markov jump process with càdlàg paths, where $\mathcal{N}_{\mathrm{f}} = \mathcal{N}_{\mathrm{f}}\big(\mathbb{Z}^d \times \{1,2\}\big)$ is the set of finite counting measures on the marked space $\mathbb{Z}^d \times \{1,2\}$. Note that each $\mathbf{n} \in \mathcal{N}_{\mathrm{f}}$ has a representation $\sum_{i\in I} \delta_{(a(i),k(i))}$ as a finite sum of delta measures $\delta_{(a(i),k(i))}$ each interpreted as a particle at site $a(i)$ of type $k(i)$. We can also identify $\mathbf{n}$ with the pair (n^1, n^2) of "projections", where $n^k := \mathbf{n}\big((\cdot) \times \{k\}\big) \in \mathcal{N}_{\mathrm{f}}(\mathbb{Z}^d)$. For typographical simplification, we set $n^k(a) := n^k\big(\{a\}\big)$. The *generator* H of the process $\mathbf{N} = (N^1, N^2)$ is given by

$$
\begin{aligned}
(36)\quad Hf(\mathbf{n}) \;:=\; & \sum_{a\in\mathbb{Z}^d,\ k=1,2} n^k(a)\,\frac{\kappa^2}{2} \sum_{b\in\mathbb{Z}^d,\ |b-a|=1} \Big[f\big(\mathbf{n} - \delta_{(a,k)} + \delta_{(b,k)}\big) - f(\mathbf{n})\Big] \\
& + \gamma \sum_{a\in\mathbb{Z}^d,\ k=1,2} \binom{n^k(a)}{2} \big[f(\sigma_a^k\mathbf{n}) - f(\mathbf{n})\big], \quad \mathbf{n} = (n^1, n^2) \in \mathcal{N}_{\mathrm{f}}\,,
\end{aligned}
$$

where we use $\sigma_a^k\mathbf{n}$ to denote the element of $\mathcal{N}_{\mathrm{f}}$ obtained from $\mathbf{n}$ by *switching the type* of one of the particles of type k at position a [provided, of course, that $n^k(a) > 0$]. Consequently, besides the migration of the particles, each pair of particles of the same type and having the same position may experience a type jump with rate γ. Upon a jump, exactly one of the particles involved changes its type. Write $\mathbf{P}_{\mathbf{n}}$ for the law of $\mathbf{N}$ starting from $\mathbf{N}_0 = \mathbf{n} \in \mathcal{N}_{\mathrm{f}}$.

We need to introduce a *duality function* $\mathfrak{N}$. For $\mathbf{x} = (x^1, x^2) \in (\mathcal{B}^+_{\mathrm{tem}})^2$ and $\mathbf{n} = (n^1, n^2) \in \mathcal{N}_{\mathrm{f}}$, set

$$
(37)\qquad \mathfrak{N}(\mathbf{x}, \mathbf{n}) \;:=\; \mathbf{x}^{\mathbf{n}} \;:=\; \prod_{a\in\mathbb{Z}^d,\ k=1,2} \big(x^k(a)\big)^{n^k(a)}.
$$

Here the product is taken over those finitely many a where $n^k(a) > 0$ for some $k \in \{1,2\}$. In the duality relation (39) below we will also use the following notion: For $\mathbf{n} = (n^1, n^2) \in \mathcal{N}_{\mathrm{f}}$,

$$
(38)\qquad \|\mathbf{n}\|_{=} \;:=\; \sum_{a\in\mathbb{Z}^d,\ k=1,2} \binom{n^k(a)}{2}, \qquad \|\mathbf{n}\|_{\neq} \;:=\; \frac{1}{2}\sum_{a\in\mathbb{Z}^d} n^1(a)\,n^2(a).
$$

Thus, $\|\mathbf{n}\|_{=}$ is the number of pairs of particles in $\mathbf{n}$ having the same type and the same position, and $\|\mathbf{n}\|_{\neq}$ is just the total number of pairs of particles in $\mathbf{n}$ having the same position but different types.

Proposition 7 (**Particle system moment duality relation in $\mathbb{Z}^d$**). *Suppose* $\mathbf{x} = (x^1, x^2) \in (\mathcal{B}^+_{\mathrm{tem}})^2$ *and* $\mathbf{n} = (n^1, n^2) \in \mathcal{N}_{\mathrm{f}}$. *For* $\varrho \in [-1, 1]$, *consider the symbiotic process* $(\mathbf{X}, P_{\mathbf{x}})$ *and the particle system moment dual* $(\mathbf{N}, \mathbf{P}_{\mathbf{n}})$ *in* $\mathbb{Z}^d$. *Then, for all* $t \geq 0$,

$$E_{\mathbf{x}} \mathbf{X}^{\mathbf{n}}_t \;=\; \mathbf{E}_{\mathbf{n}}\, \mathbf{x}^{\mathbf{N}_t} \exp\Big[\gamma \int_0^t \mathrm{d}s\, \Big(\|\mathbf{N}_s\|_= \,+\, \varrho\, \|\mathbf{N}_s\|_{\neq}\Big)\Big]. \tag{39}$$

Remark 8 (**Stepping stone model**). It does not seem to be possible to derive from Proposition 7 Shiga's ([Shi80]) moment duality of the stepping stone model with coalescing random walks with delay. ◇

We mention, that for $\varrho = 0$ we get the particle system moment dual briefly indicated in [DP98, bottom of p.1091].

3.3. **Proof of the particle system moment duality relation.** By (32),

$$\mathrm{d}X^k_t(a) \;=\; \frac{\kappa^2}{2} \Delta^{(1)} X^k_t(a)\, \mathrm{d}t \,+\, \mathrm{d}(\text{martingale}), \tag{40}$$

and from (35),

$$E \mathrm{d}W^k_{t_1}(a) \mathrm{d}W^l_{t_2}(b) \;=\; \varrho\, \delta_0(a - b)\, \delta_0(t_1 - t_2)\, \mathrm{d}t_1 \mathrm{d}t_2 \tag{41}$$

(the first δ_0 refers to a delta measure since the argument $a - b \in \mathbb{Z}^d$ is discrete, but the second one to a delta function), implying

$$\mathrm{d}\,\big\langle\!\big\langle X^k(a), X^l(b) \big\rangle\!\big\rangle_t \;=\; \gamma\, c_{k,l}(\varrho)\, \delta_0(a - b)\, X^1_t(a) X^2_t(a)\, \mathrm{d}t, \tag{42}$$

where $c_{k,l}(\varrho)$ was defined in (16). This gives the generator $\mathcal{G}$ of $\mathbf{X}$ as

$$\begin{aligned} \mathcal{G} f(\mathbf{x}) \;=\; \sum_{a,k} \Big[&\frac{\kappa^2}{2} \Delta^{(1)} x^k(a)\, \frac{\partial}{\partial x^k(a)} \\ &+\, \frac{\gamma}{2}\, x^1(a) x^2(a) \sum_l c_{k,l}(\varrho)\, \frac{\partial^2}{\partial x^k(a)\, \partial x^l(a)} \Big] f(\mathbf{x}), \end{aligned} \tag{43}$$

where f is a function on $\mathbf{x} = (x^1, x^2) \in (\mathcal{B}^+_{\mathrm{tem}})^2$ depending on only finitely many components $x^k(a)$. Hence,

$$\begin{aligned} &\mathcal{G}\mathfrak{N}(\,\cdot\,, \mathbf{n})(\mathbf{x}) \\ &= \sum_{a,k} n^k(a)\, \frac{\kappa^2}{2} \Delta^{(1)} x^k(a)\, \mathbf{x}^{\mathbf{n} - \delta_{(a,k)}} \,+\, \gamma \sum_{a,k} \binom{n^k(a)}{2} \mathbf{x}^{\sigma^k_a \mathbf{n}} \,+\, \gamma\varrho\, \|\mathbf{n}\|_{\neq}\, \mathbf{x}^{\mathbf{n}} \end{aligned} \tag{44}$$

[with σ^k_a defined after (36)]. On the other hand, by (36),

$$\begin{aligned} H\mathfrak{N}(\mathbf{x}, \,\cdot\,)(\mathbf{n}) \;=\; &\sum_{a,k} n^k(a)\, \frac{\kappa^2}{2} \Delta^{(1)} x^k(a)\, \mathbf{x}^{\mathbf{n} - \delta_{(a,k)}} \\ &+\, \gamma \sum_{a,k} \binom{n^k(a)}{2} \mathbf{x}^{\sigma^k_a \mathbf{n}} \,-\, \gamma\, \|\mathbf{n}\|_=\, \mathbf{x}^{\mathbf{n}}. \end{aligned} \tag{45}$$

Therefore,

$$\mathcal{G}\mathfrak{N}(\,\cdot\,, \mathbf{n})(\mathbf{x}) \;=\; H\mathfrak{N}(\mathbf{x}, \,\cdot\,)(\mathbf{n}) \,+\, \gamma\, \Big[\|\mathbf{n}\|_= \,+\, \varrho\, \|\mathbf{n}\|_{\neq} \Big] \mathfrak{N}(\mathbf{x}, \mathbf{n}). \tag{46}$$

The claimed duality relation (39) now follows by standard arguments; see [EK86, Corollary 4.4.13]. □

3.4. **Measure case moment dual M on $\mathbb{Z}^d$.** In the case $\varrho \in [0,1]$ there is another way of expressing the moments of order $n \geq 1$ of $\mathbf{X}$ in $\mathbb{Z}^d$ in terms of a dual process. This will be a "measure-valued" dual process, that we denote by $\mathbf{M}$ (we said "measure-valued", since, later, in the analogous $\mathbb{R}$–case, its states have a measure-valued component). Such duality occurred in the case $\varrho = 0$ in [DEF+02a], and played a crucial role there for constructing the mutually catalytic branching process in $\mathbb{R}^2$ for finite measure states as a scaling limit of the $\mathbb{Z}^2$–model.

For fixed $n \geq 1$, the dual process is a $\mathcal{C}^+_{\text{tem}}(\mathbb{Z}^{dn}) \times \{1,2\}^n$–valued strong Markov process $t \mapsto \mathbf{M}_t = (\Phi_t, \mathbf{K}_t)$ with càdlàg paths. We interpret a state $(\phi, \mathbf{k})$ of $\mathbf{M}$ as the "law" of n marked particles described by $(\mathbf{a}, \mathbf{k}) = \big((a_1,k_1),\dots,(a_n,k_n)\big)$ in $(\mathbb{Z}^{dn}) \times \{1,2\}^n$, where the i^{th} particle is at site $a_i \in \mathbb{Z}^d$ and is of type $k_i \in \{1,2\}$. This process $\mathbf{M}$ can be described as follows. $\mathbf{K}$ changes according to the following "autonomous" rules. If $\mathbf{K}$ is in the state $\mathbf{k} = (k_1,\dots,k_n)$, then for $i,j = 1,\dots,n$ with $i \neq j$, with rate $\frac{\gamma}{2}\,\delta_{k_i,k_j}$ a jump occurs, in which the j^{th} particle changes its type k_j, whereas with rate $\frac{\gamma}{2}\,\varrho\,(1-\delta_{k_i,k_j})$ a jump occurs which will only effect Φ. Given all these jump events, Φ evolves deterministically. In fact, in-between these jump times, Φ changes according to the semigroup of n independent simple random walks in $\mathbb{Z}^d$ with dispersion rate κ, the related operator is denoted by $\frac{\kappa^2}{2}\boldsymbol{\Delta}^{(1)}$. But if t is a jump time point caused by the ordered pair (i,j) [that is, caused by the rate $\frac{\gamma}{2}\,\delta_{k_i,k_j} + \frac{\gamma}{2}\,\varrho\,(1-\delta_{k_i,k_j})$], then $\mathbf{a} \mapsto \Phi_{t-}(\mathbf{a})$ jumps to

$$\Phi_t(\mathbf{a}) \;:=\; \delta_0(a_i - a_j)\,\Phi_{t-}(\mathbf{a}), \qquad \mathbf{a} = (a_1,\dots,a_n) \in \mathbb{Z}^{dn}, \tag{47}$$

with δ–measure δ_0, that is, the components i and j are *linked* together [or the "law" $\Phi_{t-}(\mathbf{a})\,\mathrm{d}\mathbf{a}$ is localized to $\delta_0(a_i - a_j)\,\Phi_{t-}(\mathbf{a})\,\mathrm{d}\mathbf{a}$]. Note that the *generator* G of $\mathbf{M}$ is determined by

$$\begin{aligned} Gf\,(\phi,\mathbf{k}) \;=\; f\Big(\frac{\kappa^2}{2}\boldsymbol{\Delta}^{(1)}\phi,\mathbf{k}\Big) + \frac{\gamma}{2}\sum_{i\neq j}\bigg(&\delta_{k_i,k_j}\Big[f(\delta^{i,j}\phi,\sigma_j\mathbf{k}) - f(\phi,\mathbf{k})\Big] \\ &+ \varrho\,(1-\delta_{k_i,k_j})\Big[f(\delta^{i,j}\phi,\mathbf{k}) - f(\phi,\mathbf{k})\Big]\bigg), \end{aligned} \tag{48}$$

$\phi \in \mathcal{C}^+_{\text{tem}}(\mathbb{Z}^{dn})$, $\mathbf{k} \in \{1,2\}^n$. Here $\sigma_j\mathbf{k}$ means that the j^{th} coordinate in $\mathbf{k}$ is *flipped* to the opposite type, and for $i \neq j$,

$$\delta^{i,j}\phi\,(\mathbf{a}) \;:=\; \delta_0(a_i - a_j)\,\phi(\mathbf{a}), \qquad \mathbf{a} = (a_1,\dots,a_k) \in \mathbb{Z}^{dn}, \tag{49}$$

refers to the *linking procedure* as in (47).

Next we want to turn to the *duality function*, denoted by $\mathfrak{M}$. For each state $\mathbf{x} = (x^1,x^2) \in \big(\mathcal{B}^+_{\text{tem}}(\mathbb{Z}^d)\big)^2$ of $\mathbf{X}$ and marked point $(\mathbf{a},\mathbf{k}) = \big((a_1,\dots,a_n),(k_1,\dots,k_n)\big)$ in $\mathbb{Z}^{dn} \times \{1,2\}^n$, we introduce the *product brackets*

$$\big[\mathbf{x},(\mathbf{a},\mathbf{k})\big] \;:=\; x^{k_1}(a_1)\cdots x^{k_n}(a_n) \tag{50}$$

and, if additionally $\phi \in \mathcal{C}^+_{\text{tem}}(\mathbb{Z}^{dn})$, we set

$$\mathfrak{M}(\mathbf{x},\mathbf{m}) \;:=\; \mathfrak{M}\big(\mathbf{x},(\phi,\mathbf{k})\big) \;:=\; \sum_{\mathbf{a}}\phi(\mathbf{a})\,\big[\mathbf{x},(\mathbf{a},\mathbf{k})\big]. \tag{51}$$

Furthermore, in line with the notation in (38), for $\mathbf{k} = (k_1,\dots,k_n) \in \{1,2\}^n$ set

$$\|\mathbf{k}\|_{=} \;=\; \frac{1}{2}\sum_{i\neq j}\delta_{k_i,k_j} \quad\text{and}\quad \|\mathbf{k}\|_{\neq} \;=\; \frac{1}{2}\sum_{i\neq j}(1-\delta_{k_i,k_j}) \tag{52}$$

for the number of (non-ordered) pairs in $\mathbf{k}$ of the same and of different type, respectively.

Proposition 9 (Measure case moment duality in $\mathbb{Z}^d$). *Suppose* $\varrho \in [0,1]$, $\mathbf{x} = (x^1, x^2) \in \left(\mathcal{B}^+_{\text{tem}}(\mathbb{Z}^d)\right)^2$, *and* $\mathbf{m} = (\phi, \mathbf{k}) \in \mathcal{C}^+_{\text{tem}}(\mathbb{Z}^{dn}) \times \{1,2\}^n$. *Consider the symbiotic process* $(\mathbf{X}, P_{\mathbf{x}})$ *and the measure case moment dual* $\left(\mathbf{M} = (\Phi, \mathbf{K}), \mathbf{P_m}\right)$ *in* $\mathbb{Z}^d$. *Then, for* $t \geq 0$,

$$E_{\mathbf{x}} \mathfrak{M}(\mathbf{X}_t, \mathbf{m}) \; = \; \mathbf{E_m} \mathfrak{M}(\mathbf{x}, \mathbf{M}_t) \exp\left[\gamma \int_0^t \mathrm{d}s \left(\|\mathbf{K}_s\|_= + \varrho \, \|\mathbf{K}_s\|_{\neq}\right)\right]. \tag{53}$$

Remark 10 (The case $\varrho \in [-1,0)$). The restriction to $\varrho \in [0,1]$ in the previous proposition is based on the fact that ϱ enters as a factor in some jump rates of $\mathbf{M}$. Nevertheless, also for negative ϱ, one can work formally with this dual since this leads to useful formulae. An analogous approach to that of [AT00] should allow one to extend this duality relation to negative ϱ. ◇

3.5. **Proof of the measure case moment duality relation.** First of all, we apply the generator G of $\mathbf{M}$ from (48) to the duality function $\mathfrak{M}$ defined in (51) to obtain

$$G\mathfrak{M}(\mathbf{x}, \cdot\,)\,(\mathbf{m}) \; = \; G\mathfrak{M}(\mathbf{x}, \cdot\,)\,(\phi, \mathbf{k}) \; = \; \mathfrak{M}\Big(\mathbf{x}, \, \frac{\kappa^2}{2} \boldsymbol{\Delta}^{(1)} \phi, \mathbf{k}\Big) \tag{54a}$$

$$+ \; \frac{\gamma}{2} \sum_{i \neq j} \Big(\delta_{k_i, k_j} \left[\mathfrak{M}\big(\mathbf{x}, (\delta^{i,j} \phi, \sigma_j \mathbf{k})\big) - \mathfrak{M}\big(\mathbf{x}, (\phi, \mathbf{k})\big)\right] \tag{54b}$$

$$+ \; (1 - \delta_{k_i, k_j}) \varrho \Big[\mathfrak{M}\big(\mathbf{x}, (\delta^{i,j} \phi, \mathbf{k})\big) - \mathfrak{M}\big(\mathbf{x}, (\phi, \mathbf{k})\big)\Big]\Big). \tag{54c}$$

In the non-positive term in (54b) we obtain a factor

$$\frac{\gamma}{2} \sum_{i \neq j} \delta_{k_i, k_j} \; = \; \gamma \, \|\mathbf{k}\|_= \tag{55}$$

in front of $\mathfrak{M}\big(\mathbf{x}, (\phi, \mathbf{k})\big)$ [recall notation (52)]. Similarly, in the non-positive term in (54c) we obtain the factor $\gamma \varrho \, \|\mathbf{k}\|_{\neq}$. Moreover,

$$\mathfrak{M}\big(\mathbf{x}, (\delta^{i,j} \phi, \sigma_j \mathbf{k})\big) \; = \; \sum_{\mathbf{a}} \phi(\mathbf{a}) \left[\mathbf{x}, (\pi_{i,j} \mathbf{a}, \sigma_j \mathbf{k})\right] \tag{56}$$

where, for $i \neq j$, the n–tuple $\pi_{i,j} \mathbf{a}$ is created from $\mathbf{a}$ by *switching* the j^{th} coordinate from a_j to a_i (related to the linking procedure). Now, if $k_i = k_j$, since in (54) we are summing over $i \neq j$, the occurrence of $\pi_{i,j} \mathbf{a}$, allows us to replace $\sigma_j \mathbf{k}$ by $\sigma^{1,2}_{i,j} \mathbf{k}$ in the right hand side of (56). Here $\sigma^{1,2}_{i,j} \mathbf{k}$ is obtained from $\mathbf{k} = (k_1, \ldots, k_n)$ by replacing the pair (k_i, k_j) by $(1,2)$. Thus, in (54b),

$$\frac{\gamma}{2} \sum_{i \neq j} \delta_{k_i, k_j} \, \mathfrak{M}\big(\mathbf{x}, (\delta^{i,j} \phi, \sigma_j \mathbf{k})\big) \; = \; \sum_{\mathbf{a}} \phi(\mathbf{a}) \, \frac{\gamma}{2} \sum_{i \neq j} \delta_{k_i, k_j} \left[\mathbf{x}, (\pi_{i,j} \mathbf{a}, \sigma^{1,2}_{i,j} \mathbf{k})\right]. \tag{57}$$

Similarly, when $k_i \neq k_j$, dropping the σ_j in (56), we see that we may replace $\mathbf{k}$ by $\sigma_{i,j}^{1,2}\mathbf{k}$. Summarizing, recalling the notation $c_{k,l}(\varrho)$ from (16),

$$G\mathfrak{M}(\mathbf{x},\,\cdot\,)(\mathbf{m}) = \mathfrak{M}\Big(\mathbf{x},\, \frac{\kappa^2}{2}\boldsymbol{\Delta}^{(1)}\phi, \mathbf{k}\Big) - \gamma\,\big[\|\mathbf{k}\|_{=} + \varrho\,\|\mathbf{k}\|_{\neq}\big]\,\mathfrak{M}(\mathbf{x},\mathbf{m}) + \sum_{\mathbf{a}}\phi(\mathbf{a})\,\frac{\gamma}{2}\sum_{i\neq j} c_{k_i,k_j}(\varrho)\,\big[\mathbf{x},(\pi_{i,j}\mathbf{a},\sigma_{i,j}^{1,2}\mathbf{k})\big]. \tag{58}$$

On the other hand, for the generator $\mathcal{G}$ from (43),

$$\begin{aligned}\mathcal{G}\mathfrak{M}(\,\cdot\,,\mathbf{m})(\mathbf{x}) &= \sum_{\mathbf{a}}\phi(\mathbf{a})\,\mathcal{G}\big[\mathbf{x},(\mathbf{a},\mathbf{k})\big] \\ &= \sum_{\mathbf{a}}\phi(\mathbf{a})\Big(\frac{\kappa^2}{2}\boldsymbol{\Delta}^{(1)}\big[\mathbf{x},(\mathbf{a},\mathbf{k})\big] + \frac{\gamma}{2}\sum_{i\neq j} c_{k_i,k_j}(\varrho)\,\big[\mathbf{x},(\pi_{i,j}\mathbf{a},\sigma_{i,j}^{1,2}\mathbf{k})\big]\Big).\end{aligned} \tag{59}$$

By the symmetry of the discrete Laplacian, the first term on the right hand side of the second line of (59) coincides with the corresponding term in (58). Comparing now (59) and (58), we find

$$\mathcal{G}\mathfrak{M}(\,\cdot\,,\mathbf{m})(\mathbf{x}) = G\mathfrak{M}(\mathbf{x},\,\cdot\,)(\mathbf{m}) + \gamma\,\big[\|\mathbf{k}\|_{=} + \varrho\,\|\mathbf{k}\|_{\neq}\big]\,\mathfrak{M}(\mathbf{x},\mathbf{m}). \tag{60}$$

This gives the claim (53), again by standard methods. □

4. Moments for symbiotic branching in $\mathbb{R}$

Passing from $\mathbb{Z}$ to $\mathbb{R}$, in this section we establish the analogous moment duals (Propositions 11 and 14). From the first one, a higher moment estimate (Proposition 12) is derived. For a third tool, moment equations, see Proposition 15.

4.1. **Particle system moment duality on $\mathbb{R}$.**

Just as the symbiotic branching model in $\mathbb{R}$ from (1) can be obtained as a diffusion approximation to the symbiotic branching model in $\mathbb{Z}$ from (32) by rescaling space and mass by $\varepsilon > 0$ and time by $\varepsilon^{-1/2}$, and letting $\varepsilon \downarrow 0$, so also the particle system moment dual $\mathbf{N}$ on $\mathbb{Z}$ approaches a particle system moment dual on $\mathbb{R}$, that we denote by the same symbol $\mathbf{N}$. Moreover, the moment duality relation of Proposition 7 remains true. We abstain from all the painful details in these convergence procedures and only sketch the limiting process $\mathbf{N}$ and the limiting moment duality relation.

$\mathbf{N}$ is again an $\mathcal{N}_{\mathrm{f}}$–valued strong Markov process with càdlàg paths, where now $\mathcal{N}_{\mathrm{f}} = \mathcal{N}_{\mathrm{f}}\big(\mathbb{R}^d \times \{1,2\}\big)$. All particles move continuously, in fact according to independent Brownian motions with dispersion rate κ. The type of particles change according to the same rules. More precisely, a pair of particles with paths ξ and ξ' (independent Brownian motions) and common type experiences a type jump at position $a \in \mathbb{R}$ with a rate governed by the *collision local time*

$$L_{[\xi,\xi']}\big(\mathrm{d}(t,a)\big) = L^0_{\xi-\xi'}(\mathrm{d}t)\,\delta_a(\xi_t)\,\mathrm{d}a \tag{61}$$

of the pair (ξ,ξ'), where $t \mapsto L^0_\xi(t)$ denotes the famous continuous local time at level 0 of a Brownian motion ξ. Recall that upon a jump, exactly one of the involved particles changes its type. Note that the rule according to which the type-changing particle is selected is irrelevant since the pair of particles involved both have the same position and are not ordered.

As before, the duality relation involves a "correction factor" that depends on certain total pair collision local times that we now introduce. First of all, note that $\mathbf{N}=(N^1,N^2)$ can be represented with the help of δ–measures,

$$\mathbf{N}_t \;=\; \sum_{i=1}^{m} \delta_{(\xi_t(i),\varkappa_t(i))}\,, \qquad t\geq 0, \tag{62}$$

where the $t\mapsto\xi_t(i)$ are continuous paths (Brownian motion paths) and the $t\mapsto \varkappa_t(i)$ are piecewise constant càdlàg paths. For $k,l=1,2,$ introduce the *total pair collision local time* $t\mapsto \mathcal{L}_{[N^k,N^l]}(t)$ between N^k and N^l :

$$\mathcal{L}_{[N^k,N^l]}(\mathrm{d}t) \;:=\; \sum_{1\leq i<j\leq m} L^0_{\xi_t(i)-\xi_t(j)}(\mathrm{d}t)\,\delta_{\varkappa_t(i),k}\,\delta_{\varkappa_t(j),l} \tag{63}$$

(with Kronecker deltas here). The duality function $\mathfrak{N}$ is introduced just as in (37) (where the product is taken over those finitely many $a\in\mathbb{R}$ where $n^k(a)>0$). The $\mathbb{R}$–analogue of Proposition 7 reads as follows.

Proposition 11 (Particle system moment duality relation in $\mathbb{R}$). *Suppose $\mathbf{x}=(x^1,x^2)\in(\mathcal{B}^+_{\mathrm{tem}})^2$ and $\mathbf{n}=(n^1,n^2)\in\mathcal{N}_{\mathrm{f}}$. For $\varrho\in[-1,1]$, consider the symbiotic process $(\mathbf{X},P_{\mathbf{x}})$ and the particle system moment dual $(\mathbf{N},\mathbf{P_n})$ in $\mathbb{R}$. Then, for all $t\geq 0$,*

$$E_{\mathbf{x}}\mathbf{X}_t^{\mathbf{n}} \;=\; \mathbf{E_n}\mathbf{x}^{\mathbf{N}_t}\exp\Big[\gamma\left(\mathcal{L}_{[N^1,N^1]}+\mathcal{L}_{[N^2,N^2]}+\varrho\,\mathcal{L}_{[N^1,N^2]}\right)(t)\Big]. \tag{64}$$

As in the $\mathbb{Z}^d$–case, it does not seem to be clear how the usual moment duality of the continuous space stepping stone model should follow from this proposition.

4.2. **A higher moment estimate.** The following higher moment bound will help us later to control the unboundedness of states (remember the hot spots seen in simulations!). Recall that S denotes the heat flow semigroup.

Proposition 12 (Moment bound). *With $\mathbf{x}$ the Heaviside state $(1_{\mathbb{R}_-},1_{\mathbb{R}_+})$, for all $\varrho\in[-1,1]$ and $q\geq 1$,*

$$E_{\mathbf{x}}\Big[\big(X^1_t(a)X^2_t(a)\big)^q\Big] \;\leq\; 2\,\mathrm{e}^{4\gamma^2q^4t/\kappa^2}\sqrt{S_t1_{\mathbb{R}_-}(a)}, \qquad t\geq 0,\quad a\in\mathbb{R}. \tag{65}$$

Proof. We use the particle system moment duality of Proposition 11, with q type 1 particles and q type 2 particles, all at the position a. That is, $\mathbf{N}_0=(q\delta_a\,,q\delta_a)=:\mathbf{n}$. Then

$$E_{\mathbf{x}}\Big[\big(X^1_t(a)X^2_t(a)\big)^q\Big] \;=\; \mathbf{E_n}\mathbf{x}^{\mathbf{N}_t}\exp\Big[\gamma\left(\mathcal{L}_{[N^1,N^1]}+\mathcal{L}_{[N^2,N^2]}+\varrho\,\mathcal{L}_{[N^1,N^2]}\right)(t)\Big].$$

We bound this above by passing to $\varrho=1,$ and apply the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} E_{\mathbf{x}}\Big[\big(X^1_t(a)X^2_t(a)\big)^q\Big] \;&\leq\; \big(\mathbf{E_n}\,\mathbf{x}^{2\mathbf{N}_t}\big)^{1/2}\times \\ &\Big(\mathbf{E_n}\exp\Big[2\gamma\left(\mathcal{L}_{[N^1,N^1]}+\mathcal{L}_{[N^2,N^2]}+\mathcal{L}_{[N^1,N^2]}\right)(t)\Big]\Big)^{1/2}. \end{aligned} \tag{66}$$

Note that the system $\mathbf{N}_t$ must contain at least one type 1 particle. Recalling representation (62), we estimate $\mathbf{x}^{2\mathbf{N}_t}$ by $1_{\mathbb{R}_-}(\xi_t)$ (where ξ is Brownian motion with dispersion rate κ and starting from a). That is, for all but one type one particle we estimate x^k by the constant function one. This gives

$$\mathbf{E}_{\mathbf{n}} \mathbf{x}^{2\mathbf{N}_t} \;\le\; S_t 1_{\mathbb{R}_-}(a). \tag{67}$$

Let us assign labels from $I = \{1, \dots, 2q\}$ to the $2q$ particles of $\mathbf{N}$. Write $L_{i,j}$ for the local time at the origin of the difference in position between the i^{th} and j^{th} particle. Write also L_0 for the local time at the origin of a Brownian motion with dispersion rate $\sqrt{2}\,\kappa$ and starting from the origin. Notice that $L_{i,j} = L_0$ in law. Substituting into the expectation expression in (66) and applying Hölder's inequality, we obtain

$$\mathbf{E}_{\mathbf{n}} \exp\Big[2\gamma \left(\mathcal{L}_{[N^1,N^1]} + \mathcal{L}_{[N^2,N^2]} + \mathcal{L}_{[N^1,N^2]}\right)(t)\Big] \;=\; \mathcal{E} \prod_{1 \le i < j \le 2q} \exp\big[2\gamma L_{i,j}(t)\big]$$

$$\le \prod_{1 \le i < j \le 2q} \Big(\mathcal{E} \exp\big[2\gamma \tbinom{2q}{2} L_{i,j}(t)\big]\Big)^{1/\binom{2q}{2}} \;=\; \mathcal{E} \exp\big[2\gamma \tbinom{2q}{2} L_0(t)\big]. \tag{68}$$

(Here the expectation sign $\mathcal{E}$ refers to the underlying probability space $(\Omega, \mathcal{F}, \mathcal{P})$ on which the local times are defined.) Now $L_0(t)$ coincides in law with ${}^1L_0(t) \,/\, \sqrt{2}\,\kappa$, where ${}^1L_0$ refers to the local time at 0 with dispersion rate 1. However, ${}^1L_0(t)$ coincides in law with $|{}^1\xi_t|$ where ${}^1\xi$ is standard Brownian motion in $\mathbb{R}$ starting from 0. Hence, the last expectation expression in (68) can be estimated from above by

$$2\,\mathcal{E} \exp\Big[2\gamma \tbinom{2q}{2}\, {}^1\xi_t \,/\, \sqrt{2}\,\kappa\Big] \;=\; 2 \exp\Big[\Big(2\gamma \tbinom{2q}{2} / \sqrt{2}\,\kappa\Big)^2 t/2\Big], \tag{69}$$

from which (65) follows. □

For convenience, we add here the following simple heat flow estimate.

Lemma 13 (An elementary heat flow estimate). *There is a constant $c_{13} = c_{13}(\kappa)$ such that*

$$\sqrt{S_T 1_{\mathbb{R}_-}(A)} + \int_A^\infty \mathrm{d}b\, \sqrt{S_T 1_{\mathbb{R}_-}(b)} \;\le\; c_{13} \frac{T^2}{\sqrt{A}}\, p_{2T}(A), \qquad T, A \ge 1. \tag{70}$$

Proof. We start from the estimate

$$S_T 1_{\mathbb{R}_-}(A) \;=\; \int_{-\infty}^{-A} \mathrm{d}a\, p_T(a) \;\le\; \frac{\kappa^2 T}{A}\, p_T(A) \tag{71}$$

(which is valid for any $T, A > 0$). In the following, we write $c(\kappa)$ for a constant which may vary from place to place. Estimate (71) implies

$$\sqrt{S_T 1_{\mathbb{R}_-}(A)} \;\le\; c(\kappa) \frac{T^{3/4}}{\sqrt{A}}\, p_{2T}(A). \tag{72}$$

Hence,

$$\int_A^\infty \mathrm{d}b\, \sqrt{S_T 1_{\mathbb{R}_-}(b)} \;\le\; c(\kappa) \frac{T^{3/4}}{\sqrt{A}} \int_A^\infty \mathrm{d}b\, p_{2T}(b) \;\le\; c(\kappa) \frac{T^{3/4}}{\sqrt{A}} \frac{T}{A}\, p_{2T}(A), \tag{73}$$

where in the last step we used (71). Combining (72) and (73) gives the claim. □

4.3. **Measure case moment duality on $\mathbb{R}$.** By the diffusion approximation mentioned at the beginning of the previous subsection, the measure case moment duality Proposition 9 can also be transferred to the $\mathbb{R}$ case. We will now sketch what must be modified in the development of Subsection 3.4.

The *measure case moment dual* $\mathbf{M} = (\Phi, \mathbf{K})$ is now an $\mathcal{M}_{\text{tem}}(\mathbb{R}^n) \times \{1,2\}^n$–valued strong Markov process with càdlàg paths. The underlying jump mechanism for the $\mathbf{K}$–component is exactly the same. Given these jump events, Φ evolves deterministically. In-between the jump times, Φ changes according to the heat flow in $\mathbb{R}^n$ with dispersion rate κ, the related operator is denoted by $\frac{\kappa^2}{2}\boldsymbol{\Delta}$. Hence, for t in-between the jumps, Φ_t is an absolutely continuous measure, and, by a slight abuse of notation, we write $\Phi_t(\mathrm{d}\mathbf{a}) = \Phi_t(\mathbf{a})\,\mathrm{d}\mathbf{a}$ with $\mathbf{a} \mapsto \Phi_t(\mathbf{a})$ the related density function in $\mathcal{C}^+_{\text{tem}}(\mathbb{R}^n)$. If t is a jump time point caused by the (ordered) pair (i,j), then the absolutely continuous measure $\Phi_{t-}(\mathbf{a})\,\mathrm{d}\mathbf{a}$ jumps to the "slightly" singular, localized measure

$$\Phi_t(\mathrm{d}\mathbf{a}) \;=\; \delta_0(a_i - a_j)\,\Phi_{t-}(\mathbf{a})\,\mathrm{d}\mathbf{a}, \tag{74}$$

with δ–function δ_0 now, that is, the components i and j are *linked* together.

The *duality function*, once more denoted by $\mathfrak{M}$, is defined as follows. For each state $\mathbf{x} = (x^1, x^2) \in \big(\mathcal{B}^+_{\text{tem}}(\mathbb{R})\big)^2$ of $\mathbf{X}$ and $\mathbf{m} = (\phi, \mathbf{k}) \in \mathcal{C}^+_{\text{tem}}(\mathbb{R}^n) \times \{1,2\}^n$,

$$\mathfrak{M}(\mathbf{x}, \mathbf{m}) \;:=\; \int_{\mathbb{R}^n} \mathrm{d}\mathbf{a}\;\phi(\mathbf{a})\,\big[\mathbf{x}, (\mathbf{a}, \mathbf{k})\big], \tag{75}$$

[with the product brackets defined in (50)], whereas for $\mathbf{m} = (\phi, \mathbf{k}) \in \mathcal{M}_{\text{tem}}(\mathbb{R}^n) \times \{1,2\}^n$ which is different from the previous form, that is $\phi \in \mathcal{M}_{\text{tem}}(\mathbb{R}^n) \backslash \mathcal{C}^+_{\text{tem}}(\mathbb{R}^n)$, we set $\mathfrak{M}$ to 0. (Note that the following moment duality is formulated for a fixed t, hence in such t there will be almost surely no jump.) Recall the notation $\|\mathbf{k}\|_=$ and $\|\mathbf{k}\|_{\neq}$ from (52).

Proposition 14 (**Measure case moment duality in $\mathbb{R}$**). *Suppose $\varrho \in [0,1]$, $\mathbf{x} = (x^1, x^2) \in \big(\mathcal{B}^+_{\text{tem}}(\mathbb{R})\big)^2$, and $\mathbf{m} = (\phi, \mathbf{k}) \in \mathcal{C}^+_{\text{tem}}(\mathbb{R}^n) \times \{1,2\}^n$. Consider the symbiotic process $(\mathbf{X}, P_{\mathbf{x}})$ and the measure case moment dual $\big(\mathbf{M} = (\Phi, \mathbf{K}),\, \mathbf{P}_{\mathbf{m}}\big)$ in $\mathbb{R}$. Then, for $t > 0$,*

$$E_{\mathbf{x}}\mathfrak{M}(\mathbf{X}_t, \mathbf{m}) \;=\; \mathbf{E}_{\mathbf{m}}\mathfrak{M}(\mathbf{x}, \mathbf{M}_t)\exp\Big[\gamma\int_0^t \mathrm{d}s\,\Big(\|\mathbf{K}_s\|_= + \varrho\,\|\mathbf{K}_s\|_{\neq}\Big)\Big]. \tag{76}$$

4.4. **Moment equation system.** Fix an initial state $\mathbf{x} = (x^1, x^2) \in \big(\mathcal{B}^+_{\text{tem}}(\mathbb{R})\big)^2$ of the symbiotic branching process $\mathbf{X}$ in $\mathbb{R}$, and an integer $n \geq 1$. For $t > 0$, $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n$, and $\mathbf{k} = (k_1, \ldots, k_n)$ in $\{1,2\}^n$, introduce the n^{th} moment

$$m_t^{\mathbf{k}}(\mathbf{a}) \;:=\; E_{\mathbf{x}} X_t^{k_1}(a_1) \cdots X_t^{k_n}(a_n) \tag{77}$$

of $\mathbf{X}$. Recall the notation $\boldsymbol{\Delta}$ for the Laplacian in $\mathbb{R}^n$, $\sigma^{1,2}_{i,j}\mathbf{k}$ introduced after identity (56), and $c_{k,l}(\varrho)$ from (16).

Proposition 15 (**Moment equation system**). *Let $\varrho \in [-1,1]$. For fixed $n \geq 1$, the n^{th} moments of $\mathbf{X}$ satisfy the following closed system of heat equations on $(0,\infty) \times \mathbb{R}^n$ with singular coefficients in the creation term,*

$$\frac{\partial}{\partial t} m_t^{\mathbf{k}}(\mathbf{a}) \;=\; \boldsymbol{\Delta} m_t^{\mathbf{k}}(\mathbf{a}) \,+\, \gamma \sum_{1 \leq i < j \leq n} c_{k_i,k_j}(\varrho)\,\delta_0(a_i - a_j)\, m_t^{\sigma^{1,2}_{i,j}\mathbf{k}}(\mathbf{a}), \tag{78}$$

$t>0,\ \mathbf{a}\in\mathbb{R}^n$, *with initial condition* $m_0^{\mathbf{k}}(\mathbf{a}) := x^{k_1}(a_1)\cdots x^{k_n}(a_n)$.

Proof. For non-negative $\varphi_1,\ldots,\varphi_n\in\mathcal{C}_{\mathrm{com}}^{(2)}$, by Itô's formula,

$$\begin{aligned}\mathrm{d}\Big(\langle X_t^{k_1},\varphi_1\rangle\cdots\langle X_t^{k_n},\varphi_n\rangle\Big) \;&=\; \sum_i \mathrm{d}\langle X_t^{k_i},\varphi_i\rangle \prod_{j\neq i}\langle X_t^{k_j},\varphi_j\rangle \\ &\quad + \sum_{i<j}\mathrm{d}\left\langle\!\!\left\langle \langle X^{k_i},\varphi_i\rangle, \langle X^{k_j},\varphi_j\rangle\right\rangle\!\!\right\rangle_t \prod_{l\neq i,j}\langle X_t^{k_l},\varphi_l\rangle. \end{aligned} \tag{79}$$

The right hand side is

$$\begin{aligned}&\mathrm{d(martingale)} + \sum_i \Big\langle X_t^{k_i}, \frac{\kappa^2}{2}\Delta\varphi_i\Big\rangle \prod_{j\neq i}\langle X_t^{k_j},\varphi_j\rangle\,\mathrm{d}t \\ &\quad + \gamma\sum_{i<j} c_{k_i,k_j}(\varrho)\,\big\langle X^1X^2,\varphi_i\varphi_j\big\rangle\,\mathrm{d}t \prod_{l\neq i,j}\langle X_t^{k_l},\varphi_l\rangle.\end{aligned}\tag{80}$$

Taking expectations, using the symmetry of the Laplacian Δ, switching to $\boldsymbol{\Delta}$, and exploiting that the φ_i are arbitrary, we arrive at (78). □

Example 16 (Mixed 2^{nd} moment $m^{1,2}$). Solving (78) for $m^{1,2}$ via Feynman-Kac (or using the particle system moment duality of Proposition 11) gives the following formula for the "mixed" second moment of the symbiotic branching process $\mathbf{X}$ starting off from $\mathbf{X}_0=\mathbf{x}=(x^1,x^2)$:

$$m_t^{1,2}(\mathbf{a}) \;=\; \Pi_{\mathbf{a}}\, x^1(\xi_t^1)\, x^2(\xi_t^2)\exp\big[\gamma\varrho L^0_{\xi^1-\xi^2}(t)\big], \tag{81}$$

$t\geq 0,\ \mathbf{a}=(a_1,a_2)\in\mathbb{R}^2$. Here $\boldsymbol{\xi}=(\xi^1,\xi^2)$ refers to Brownian motion in $\mathbb{R}^2$ with dispersion rate κ and law denoted by $\Pi_{\mathbf{a}}$, that is, $\boldsymbol{\xi}$ starts in $\mathbf{a}$ at time 0, and L^0 denotes Brownian local time at level 0 as in (61). ◇

Clearly, Proposition 15 remains valid also for the $\mathbb{Z}^d$–model, where in (78) then $\boldsymbol{\Delta}$ has to be replaced by the discrete Laplacian $\boldsymbol{\Delta}^{(1)}$ in $(\mathbb{Z}^d)^n$, and δ_0 has to be read as the δ–measure at 0.

In case of the $\mathbb{Z}^2$–model with $\varrho=0$, scaled 4^{th} moment equations were the basic tool in [DEF+02b] to construct the mutually catalytic branching model in $\mathbb{R}^2$ in the infinite measure case.

5. Compact interface property

The purpose of this section is to prove Theorem 5. Here we follow closely [Tri95], which relates to the case $\varrho=-1$, but now we must overcome the additional complication that $\mathbf{X}$ is no longer bounded.

The structure of the proof is as follows. First we reformulate the martingale problem so that X^k is expressed as the heat semigroup acting on X_0^k plus a fluctuation term (Corollary 18). This is convenient for later estimates. We then identify the Laplace transform of the weighted occupation time of the set $[r,\infty)$ by the process X^1 in terms of a solution of a parabolic partial differential equation (Subsection 5.2). This reveals the action of X^2 as a 'counter force' to the X^1–population and suggests estimating separately the probability that the weighted occupation time is positive when $X_t^2(x)$ is bigger than some threshold (that we take to be $\frac12$) for all $x\geq\frac{r}{2}$, and the probability that $X_t^2(x)<\frac12$ for some $x\geq r/2$ [formula (138)]. The first part is then estimated by bounding the solution of the p.d.e. in terms of a singular boundary value problem (formula (98) implying Lemma

21). The second part is estimated by easy estimates for the heat semigroup and by bounding the fluctuation term in X^2 (Lemma 22). This calls on our moment bound from Proposition 12. Finally, we combine these bounds to estimate the probability that the rightmost point in the support of X_t^1 exceeds r for some $t \in [0,T]$ and for sufficiently large r (Subsection 5.5). An application of Borel-Cantelli finally completes the proof (Subsection 5.6).

5.1. **Extended martingale problem and Green function representation.** We now present the following consequence of the martingale problem $\mathbf{MP}_{\mathbf{x}}^{\varrho}$ of Definition 2.

Lemma 17 (Extension of the martingale problem $\mathbf{MP}_{\mathbf{x}}^{\varrho}$). *Let $\varrho \in [-1,1]$, and $\mathbf{x}, \mathbf{X}$ be as in Definition 2. Then for ψ^1, ψ^2 in $\mathcal{C}_{T,\mathrm{rap}}^{(1,2)}$ (resp. $\mathcal{C}_{T,\mathrm{tem}}^{(1,2)}$),*

$$\begin{aligned}\langle X_t^k, \psi_t^k \rangle \;=\; \langle x^k, \psi_0 \rangle \;+\; \int_0^t \mathrm{d}s\, \Big\langle X_s^k,\; \frac{\kappa^2}{2}\, \Delta\psi_s^k + \frac{\partial}{\partial s}\psi_s^k \Big\rangle \\ +\int_{[0,t]\times\mathbb{R}} M^k\big(\mathrm{d}(s,a)\big)\, \psi_s^k(a), \quad 0 \le t \le T,\; k=1,2,\end{aligned} \tag{82}$$

where $M^k\big(\mathrm{d}(s,a)\big)$ are (zero-mean) martingale measures with bracket

$$\begin{aligned}\Big\langle\!\!\Big\langle \int_{[0,\,\cdot\,]\times\mathbb{R}} M^k\big(\mathrm{d}(s,a)\big)\, f_s^k(a),\, \int_{[0,\,\cdot\,]\times\mathbb{R}} M^l\big(\mathrm{d}(s,a)\big)\, f_s^l(a) \Big\rangle\!\!\Big\rangle_t \\ = \gamma\, c_{k,l}(\varrho) \int_0^t \mathrm{d}s\, \langle X_s^1 X_s^2,\, f_s^k f_s^l \rangle, \qquad 0 \le t \le T, \quad k,l=1,2.\end{aligned} \tag{83}$$

Here f^1, f^2 belong to the set of predictable functions f defined on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ such that

$$E_{\mathbf{x}} \int_0^t \mathrm{d}s\, \Big\langle X_s^1 X_s^2,\, (f_s)^2 \Big\rangle \;<\; \infty, \qquad 0 \le t \le T. \tag{84}$$

Proof. The proof is standard, and we will outline it only in the case $\mathbf{x} \in (\mathcal{B}_{\mathrm{tem}}^+)^2$. We may fix a $\lambda > 0$ and note that S is a strongly continuous semigroup acting on the separable Banach space $\mathcal{C}_\lambda$, and that each S_t maps $\mathcal{C}_\lambda^{(2)}$ into itself. We then use Proposition 1.3.3 of [EK86] to bootstrap up to the domain of the generator of time-space Brownian motion on $C_0([0,T]\times\mathbb{R})$ (the space of continuous functions $[0,T]\times\mathbb{R}$ vanishing at infinity), and this domain contains $\mathcal{C}_{T,\mathrm{rap}}^{(1,2)}$. Approximate $\psi \in \mathcal{C}_{T,\mathrm{rap}}^{(1,2)}$ by an appropriate sequence of step functions in the time variable, and then proceed as in the proof of Proposition II.5.7 of [Per02]. □

By standard methods, the previous lemma gives the following result.

Corollary 18 (Green function representation of $\mathbf{MP}_{\mathbf{x}}^{\varrho}$). *Let $\varrho \in [-1,1]$, and $\mathbf{x}, \mathbf{X}$ be as in Definition 2. Then for φ in $\mathcal{C}_{\mathrm{rap}}$ (resp. $\mathcal{C}_{\mathrm{tem}}$), $k=1,2$, and $t \ge 0$,*

$$\langle X_t^k, \varphi \rangle \;=\; \langle x^k, S_t\varphi \rangle \;+\; \int_{[0,t]\times\mathbb{R}} M^k\big(\mathrm{d}(s,a)\big)\, S_{t-s}\varphi(a) \tag{85}$$

with the martingale measures M^1, M^2 from Lemma 17.

Again with standard methods, the previous corollary implies the following convolution form of equation (1):

Corollary 19 (**Convolution form**). *Let $\varrho \in [-1,1]$, and $\mathbf{x}, \mathbf{X}$ be as in Definition 2. Then, for $t \geq 0$, $a \in \mathbb{R}$, and $k = 1,2$ fixed,*

$$X_t^k(a) \;=\; S_t x^k(a) + \int_{[0,t]\times\mathbb{R}} M^k\big(\mathrm{d}(s,b)\big)\, p_{t-s}(b-a), \quad P_{\mathbf{x}}\text{-a.s.} \tag{86}$$

Of course, this in particular implies the *expectation formula*

$$E_{\mathbf{x}} X_t^k(a) \;=\; S_t x^k(a). \tag{87}$$

5.2. **A Laplace transform identity.** From the extended martingale problem in Lemma 17 we get the following identity.

Lemma 20 (**Laplace transform identity**). *Let $\varrho \in [-1,1]$. For $\mathbf{x} \in (\mathcal{B}_{\mathrm{tem}}^{+})^2$, $k = 1,2$, $t \geq 0$, non-negative $\psi \in \mathcal{C}_{T,\mathrm{rap}}^{(1,2)}$, $\varphi \in \mathcal{C}_{\mathrm{rap}}^{+}$, and each stopping time τ with respect to the filtration of $\mathbf{X}$,*

$$\begin{aligned} (88)\quad & E_{\mathbf{x}} \exp\Big[-\big\langle X_{t\wedge\tau}^k, \psi_{t\wedge\tau}\big\rangle - \int_0^{t\wedge\tau} \mathrm{d}s\; \big\langle X_s^k, \varphi\big\rangle \Big] \\ &\quad = \mathrm{e}^{-\langle x^k, \psi_0\rangle} + E_{\mathbf{x}} \int_0^{t\wedge\tau} \mathrm{d}s\, \exp\Big[-\big\langle X_s^k, \psi_s\big\rangle - \int_0^s \mathrm{d}r\; \big\langle X_r^k, \varphi\big\rangle \Big] \\ &\qquad\qquad \times \Big(\Big\langle X_s^k, -\frac{\partial}{\partial s}\psi_s - \frac{\kappa^2}{2}\Delta\psi_s - \varphi \Big\rangle + \frac{\gamma}{2}\big\langle X_s^1 X_s^2, \psi_s^2\big\rangle \Big). \end{aligned}$$

Proof. By the extended martingale problem of Lemma 17,

$$\mathrm{d}\big\langle X_t^k, \psi_t\big\rangle \;=\; \Big\langle X_t^k,\ \frac{\kappa^2}{2}\,\Delta\psi_t + \frac{\partial}{\partial t}\psi_t \Big\rangle\, \mathrm{d}t \;+\; \mathrm{d}(\text{martingale}) \tag{89}$$

and

$$\mathrm{d}\big\langle\!\big\langle X_\cdot^k, \psi_\cdot \big\rangle\!\big\rangle_t \;=\; \gamma\,\big\langle X_t^1 X_t^2, \psi_t^2\big\rangle\, \mathrm{d}t \;+\; \mathrm{d}(\text{martingale}). \tag{90}$$

Thus,

$$\begin{aligned} \mathrm{d}\exp\Big[-\big\langle X_t^k, \psi_t\big\rangle - \int_0^t \mathrm{d}s\; \big\langle X_s^k, \varphi\big\rangle \Big] \;=\; \exp\Big[-\big\langle X_t^k, \psi_t\big\rangle - \int_0^t \mathrm{d}s\; \big\langle X_s^k, \varphi\big\rangle \Big] \times \\ \Big(-\big\langle X_t^k, \varphi\big\rangle - \Big\langle X_t^k,\ \frac{\kappa^2}{2}\,\Delta\psi_t + \frac{\partial}{\partial t}\psi_t \Big\rangle + \frac{\gamma}{2}\big\langle X_t^1 X_t^2, \psi_t^2\big\rangle \Big)\, \mathrm{d}t \;+\; \mathrm{d}(\text{martingale}), \end{aligned}$$

and the claim follows. □

We now specialize Lemma 20 to the case when $\psi \in \mathcal{C}_{T,\mathrm{rap}}^{(1,2)}$ is the unique solution of the partial differential equation

$$\left\{ \begin{aligned} -\frac{\partial}{\partial t}\psi_t \;&=\; \frac{\kappa^2}{2}\,\Delta\psi_t - \alpha\psi_t^2 + \varphi \quad \text{on} \quad [0,T)\times\mathbb{R}, \\ &\text{with terminal condition} \quad \psi_{T-} = 0. \end{aligned} \right. \tag{91}$$

Here $\alpha > 0$ is a fixed constant, and $\varphi \in \mathcal{C}_{\mathrm{rap}}^{+}$. The expression in the last line of the array (88) then simplifies to

$$\Big\langle -\alpha X_s^k + \frac{\gamma}{2} X_s^1 X_s^2,\ \psi_s^2 \Big\rangle. \tag{92}$$

5.3. **A log-Laplace function estimate.** Note that (91) is the log-Laplace equation of the occupation time process of a continuous super-Brownian motion on $\mathbb{R}$. We need good estimates on ψ for some special φ. This was essentially done by Tribe [Tri95, formula (5)]. For completeness and ease of reference, we include here some details on this.

Let $f:\mathbb{R}\to[0,1]$ belong to $\mathcal{C}^{(2)}_{\mathrm{rap}}$, have support $\mathbb{R}_+$, and total mass $\langle f,1\rangle$ bounded by 1. For fixed $r>0$, set

$$f^r(a) \;:=\; f(a-r), \qquad a\in\mathbb{R}. \tag{93}$$

In equation (91), specialize φ to λf^r, for a $\lambda>0$:

$$\begin{cases} -\dfrac{\partial}{\partial t}\psi_t \;=\; \dfrac{\kappa^2}{2}\,\Delta\psi_t-\alpha\psi_t^2+\lambda f^r \quad \text{on} \quad [0,T)\times\mathbb{R}, \\ \qquad\qquad \text{with terminal condition} \quad \psi_{T-}=0. \end{cases} \tag{94}$$

Lemma 21 (**Log-Laplace function estimate**). *There exists a constant $c_{21}=c_{21}(\kappa)$ such that for all such f and for all $r,\lambda>0$, the solution $\psi=\psi^{\lambda,r}$ to equation (94) satisfies*

$$\psi^{\lambda,r}(a) \;\le\; \frac{c_{21}}{\alpha\sqrt{T}}\,p_{4T}(r-a), \qquad 0\le t\le T, \quad a\le r-2\sqrt{T}. \tag{95}$$

This estimate will enter in the proof of Proposition 23 below.

Proof. First note that

$$h(a) \;=\; h^r(a) \;:=\; \frac{3\kappa^2}{\alpha}\,(r-a)^{-2}, \qquad a<r. \tag{96}$$

solves

$$\frac{\kappa^2}{2}\Delta h-\alpha h^2 \;=\; 0 \quad \text{on} \quad (-\infty,r). \tag{97}$$

Arguing as in the proof of the maximum principle, we obtain for the solution $\psi=\psi^{\lambda,r}\ge 0$ to (94) and $h=h^r>0$ from (96)/(97), the following *comparison* result:

$$\psi_t(a) \;\le\; h(a), \qquad a<r, \quad 0\le t\le T. \tag{98}$$

In fact, $h-\psi=:u$ satisfies

$$\begin{cases} -\dfrac{\partial}{\partial t}u \;=\; \dfrac{\kappa^2}{2}\Delta u-\wp\, u \quad \text{on} \quad [0,T)\times(-\infty,r), \\ \qquad\quad \text{with terminal condition} \quad u_{T-}=h>0, \end{cases} \tag{99}$$

where we introduced the function $\wp:=\alpha(\psi+h)>0$. Moreover, $\lim_{x\downarrow-\infty}u_t(x)=0$ for all $t\le T$, whereas $u_t(r+)\equiv+\infty$. Assume that (98) is not true. Then there is a $(t_0,a_0)\in[0,T)\times(-\infty,r)$ such that $u_{t_0}(a_0)=:-\varepsilon<0$. Since $u_{t_0}(r+)=+\infty$, there is an $a_1\in[a_0\,,r)$ such that u_{t_0} takes on its minimum $\le-\varepsilon$ at a_1. Then

$$\frac{\kappa^2}{2}\Delta u_{t_0}(a_1)\ge 0 \quad \text{and} \quad \wp_{t_0}(a_1)\,u_{t_0}(a_1)<0, \tag{100}$$

implying by (99) that $\frac{\partial}{\partial t}u_{t_0}(a_1)<0$. Therefore there is a $t_1\in(t_0,T)$ such that $u_{t_1}(a_1)\le-\varepsilon$. Again, there is an $a_2\in[a_1\,,r)$ such that u_{t_1} takes on its minimum $\le-\varepsilon$ at a_2. Then $\frac{\partial}{\partial t}u_{t_1}(a_2)<0$. Continuing this way, one gets a sequence $(t_n,a_n)\in[0,T)\times[a_0\,,r)$ where always $u_{t_n}(a_n)\le-\varepsilon$. Take a subsequence of the (t_n,a_n), converging in $[0,T]\times[a_0\,,r]$ to some (t_∞,a_∞). Since everywhere $u_{t_n}(a_n)\le$

$-\varepsilon$, we conclude that actually $(t_\infty, a_\infty) \in [0,T) \times [a_0\,, r)$, and $u_{t_\infty}(a_\infty) \leq -\varepsilon$. Thus, we can continue our construction. Hence, we must reach T, and get a contradiction, proving (98).

On the other hand, the Feynman-Kac representation of (94) reads as

$$(101)\quad \psi_t(a) \;=\; \Pi_{t,a} \int_t^T \mathrm{d}s\ \lambda f^r(\xi_s) \exp\Big[-\int_s^T \mathrm{d}s'\ \alpha\psi_{s'}(\xi_{s'})\Big], \qquad (t,a) \in [0,T) \times \mathbb{R},$$

where $(\xi, \Pi_{t,a})$ denotes Brownian motion in $\mathbb{R}$ with dispersion rate κ and starting from a at time t. Consider $a < b < r$, and let η_b denote the hitting time of b by Brownian motion ξ. Notice that if $\eta_b > T$, then ξ will not hit r by time T, and $f^r(\xi_s) \equiv 0$. Then from the strong Markov property at time η_b,

$$(102)\quad \psi_t(a) \;=\; \Pi_{t,a} 1_{\{\eta_b \leq T\}} \psi_{\eta_b}(b) \;\leq\; h(b)\, \Pi_{t,a}(\eta_b \leq T), \quad a < b < r,\ \ 0 \leq t \leq T,$$

where we used (98) (cf. [DIP89, formula array (3.2.24)]). Specializing to $b := r - \sqrt{T}$ and $a \leq r - 2\sqrt{T}$ gives

$$(103)\qquad \psi_t(a) \;\leq\; \frac{3\kappa^2}{\alpha T}\, \Pi_{t,a}(\eta_b \leq T) \;\leq\; \frac{3\kappa^2}{\alpha T}\, \Pi_a(\eta_b \leq T),$$

where $\Pi_a := \Pi_{0,a}$. Now by the reflection principle,

$$(104)\qquad \Pi_a(\eta_b \leq T) \;=\; 2\Pi_a(B_T \geq b).$$

Next we use (71) and $(b-a) \geq \frac{1}{2}(r-a) \geq \sqrt{T}$ to see that

$$(105)\qquad \Pi_a(B_T \geq b) \;\leq\; \kappa^2 \sqrt{T}\, p_T\big(\tfrac{1}{2}(r-a)\big).$$

Combining this with (104) and (103) we obtain the claim (95). This completes the proof. □

5.4. **Estimation of a fluctuation term.** Let $\mathbf{X}$ start from the Heaviside state $\mathbf{x} = (1_{\mathbb{R}_-}, 1_{\mathbb{R}_+})$. With the martingale measures from Lemma 17, we set

$$(106)\qquad N_t^k(a) \;:=\; \int_{[0,t]\times\mathbb{R}} M^k\big(\mathrm{d}(s,b)\big)\, p_{t-s}(b-a),$$

$t > 0$, $a \in \mathbb{R}$, and $N_0^k(a) :\equiv 0$. Recall that by (86),

$$(107)\qquad X_t^2(a) \;=\; S_t x^2\,(a) + N_t^2(a).$$

The next lemma (cf. [Tri95, Lemma 3.1]) will allow us to deduce that, for $T \geq 1$ fixed and sufficiently large A,

$$(108)\qquad \inf_{t\leq T,\ a\geq A} X_t^2(a) \;\geq\; \frac{1}{2} \quad \text{with high probability}$$

[see (151) below.] In fact, if $N_t^2(a)$ is small with high probability, and a is big enough that $S_t x^2\,(a)$ is close to one, then by (107), $X_t^2(a)$ cannot be small.

Lemma 22 (Fluctuation term estimate). *Let $\varrho \in [-1,1]$ and $\mathbf{x} = (1_{\mathbb{R}_-}, 1_{\mathbb{R}_+})$. There is a constant $c_{22} = c_{22}(\gamma,\kappa)$ such that for all $\varepsilon \in (0,1]$, $T, A \geq 1$, and $k = 1,2$,*

$$(109)\qquad P_{\mathbf{x}}\Big(\big|N_t^k(a)\big| \geq \varepsilon \ \ \textit{for some}\ \ t \leq T \ \ \textit{and}\ \ a \geq A\Big) \;\leq\; c_{22}\, \varepsilon^{-10}\, \mathrm{e}^{5^5\gamma^2 T/\kappa^2}\, \frac{T^{14}}{\sqrt{A}}\, p_{2T}(A).$$

Proof. The proof mirrors the proof of the modulus of continuity of Brownian motion. We follow [Tri95, proof of Lemma 3.1], but now his trivial estimate $X^k \leq 1$ for the stepping stone model must be replaced by our moment estimates (65), and so things are a little more complicated. First of all, by [Shi94, Lemma 6.2(i)],

$$\begin{aligned} \text{(110)} \qquad & \int_0^T \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}b \; \big[p_{t'-s}(b-a') - p_{t-s}(b-a)\big]^2 \\ & \leq \; c_{(110)} \; \Big(|t'-t|^{1/2} + |a'-a|\Big) \wedge T^{1/2}, \qquad 0 < t, t' \leq T, \quad a, a' \in \mathbb{R}, \end{aligned}$$

with a constant $c_{(110)} = c_{(110)}(\kappa)$, where we use the convention

$$\text{(111)} \qquad p_r := 0 \;\text{ if }\; r < 0.$$

1° (*Higher moment estimate*). Let $q \geq 2$. For $0 < t' \leq t \leq T$ and $a, a' \in \mathbb{R}$, consider

$$\begin{aligned} & E_{\mathbf{x}}\Big[\,\big|N_t^k(a) - N_{t'}^k(a')\big|^{2q}\,\Big] \\ \text{(112)} \quad & = E_{\mathbf{x}}\left[\left|\int_{[0,T]\times\mathbb{R}} W_{\mathrm{d}s}^k(\mathrm{d}b)\,[p_{t-s}(b-a) - p_{t'-s}(b-a')]\,\sqrt{\gamma\,X_s^1(b)X_s^2(b)}\,\right|^{2q}\right], \end{aligned}$$

For t, t', a, a' fixed, as a function in T, the stochastic integral in (112) is a martingale (the integrability follows from the moment bound in Proposition 12). Hence, by Burkholder's inequality, the moment expression (112) can be estimated from above by

$$\text{(113)} \qquad c(q)\,E_{\mathbf{x}}\left[\left|\int_0^T \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}b \; \big[p_{t-s}(b-a) - p_{t'-s}(b-a')\big]^2 \;\gamma\,X_s^1(b)X_s^2(b)\right|^q\right].$$

Writing $\big[p_{t-s}(b-a) - p_{t'-s}(b-a')\big]^2 \; X_s^1(b)X_s^2(b)$ as

$$\big[p_{t-s}(b-a) - p_{t'-s}(b-a')\big]^{2(q-1)/q} \; \Big(\big[p_{t-s}(b-a) - p_{t'-s}(b-a')\big]^{2/q} \; \big[X_s^1(b)X_s^2(b)\big]\Big),$$

by Hölder's inequality the $|\ldots|^q$–term can be estimated from above by

$$\begin{aligned} \text{(114)} \qquad & \gamma^q \left|\int_0^T \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}b \; \big[p_{t-s}(b-a) - p_{t'-s}(b-a')\big]^2 \;\right|^{q-1} \\ & \times \int_0^T \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}b \, \big[p_{t-s}(b-a) - p_{t'-s}(b-a')\big]^2 \; \big[X_s^1(b)X_s^2(b)\big]^q\,. \end{aligned}$$

For the first double integral, we use (110), whereas in the second one we replace the square of the difference by twice the sum of squares. Together with the moment bound (65), this gives for (113) the bound

$$\begin{aligned} \text{(115)} \qquad & c(q)\,\gamma^q \; \Big[c_{(110)}\,\big(\,|t'-t|^{1/2} + |a'-a|\,\big) \wedge T^{1/2}\Big]^{q-1} \\ & \times\; 4\,\mathrm{e}^{4\gamma^2 q^4 T/\kappa^2} \left[\int_0^T \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}b \; p_{t-s}^2(b-a)\,\sqrt{S_s 1_{\mathbb{R}_-}(b)} \right. \\ & \qquad\qquad \left. + \int_0^T \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}b \; p_{t'-s}^2(b-a')\,\sqrt{S_s 1_{\mathbb{R}_-}(b)}\,\right]. \end{aligned}$$

We specialize T to t. For the first double integral, estimate one of the p–factors by $c\ (t-s)^{-1/2}$. Then, by Jensen's inequality,

$$\int_{\mathbb{R}} \mathrm{d}b\, p_{t-s}(b-a) \sqrt{S_s 1_{\mathbb{R}_-}(b)} \;\le\; \Big(\int_{\mathbb{R}} \mathrm{d}b\, p_{t-s}(b-a)\, S_s 1_{\mathbb{R}_-}(b)\Big)^{1/2} = \sqrt{S_t 1_{\mathbb{R}_-}(a)}, \tag{116}$$

whereas

$$\int_0^t \mathrm{d}s\ (t-s)^{-1/2} \;=\; 2\,t^{1/2}. \tag{117}$$

Combining (112), (113), and (115)–(117), we obtain for $0<t'\le t$ and $a,a'\in\mathbb{R}$,

$$E_{\mathbf{x}} \left|N_t^k(a) - N_{t'}^k(a')\right|^{2q} \;\le\; c_{(118)} \Big[\big(|t'-t|^{1/2} + |a'-a|\big) \wedge t^{1/2}\Big]^{q-1} \times \mathrm{e}^{4\gamma^2 q^4 t/\kappa^2} \Big(\sqrt{t\, S_t 1_{\mathbb{R}_-}(a)} + \sqrt{t'\, S_{t'} 1_{\mathbb{R}_-}(a')}\Big) \tag{118}$$

for a constant $c_{(118)} = c_{(118)}(q,\gamma,\kappa)$.

2° (*Dyadic grid technique*). For each $n\ge 1$, we introduce the *dyadic grid*

$$G_n \;:=\; \Big\{(t_{n,i}\,, a_{n,j}):\ \ t_{n,i} := i2^{-n},\ \ a_{n,j} := j2^{-n},\ \ i,j\in\mathbb{Z}_+\Big\} \tag{119}$$

partitioning $\mathbb{R}_+^2$. Two points $g=(t,a)$ and $g'=(t',a')$ in the grid G_n are called *neighbouring* points, if $t=t'$ and $|a-a'|=1$, or vice verse. Fix $k=1,2$. For $\varepsilon_0>0$ and neighbouring points $g,g'\in G_n$ with $g\le g'$, introduce the event

$$\mathfrak{A}_{\varepsilon_0,n}^{g,g'} \;:=\; \Big\{\left|N_t^k(a) - N_{t'}^k(a')\right| \ge \varepsilon_0\, 2^{-n/10}\Big\} \tag{120}$$

By Markov's inequality and (118),

$$\begin{aligned} P_{\mathbf{x}}\big(\mathfrak{A}_{\varepsilon_0,n}^{g,g'}\big) \;&\le\; \varepsilon_0^{-2q}\, 2^{nq/5}\, E_{\mathbf{x}} \left|N_t^k(a) - N_{t'}^k(a')\right|^{2q} \\ &\le\; \varepsilon_0^{-2q}\, 2^{nq/5}\, c_{(118)} \big(2^{-n/2}\big)^{q-1}\, \mathrm{e}^{4\gamma^2 q^4 t/\kappa^2} \Big(\sqrt{t\, S_t 1_{\mathbb{R}_-}(a)} + \sqrt{t'\, S_{t'} 1_{\mathbb{R}_-}(a')}\Big). \end{aligned} \tag{121}$$

Fix $T\ge 1$ and set

$$\mathfrak{A}_{\varepsilon_0} \;:=\; \bigcup_{n\ge 1}\ \bigcup_{g,g'\,\in\, G_n\cap[0,T]\times[A,\infty)} \mathfrak{A}_{\varepsilon_0,n}^{g,g'}\,. \tag{122}$$

Then,

$$\begin{aligned} &P_{\mathbf{x}}(\mathfrak{A}_{\varepsilon_0}) \\ &\le\; 2\sum_{n\ge 1}\ \sum_{\substack{i:\ 0\le t_{n,i}\le T,\\ j:\ a_{n,j}\ge A}} \varepsilon_0^{-2q}\, 2^{nq/5}\, c_{(118)} \big(2^{-n/2}\big)^{q-1}\, \mathrm{e}^{4\gamma^2 q^4 T/\kappa^2} \sqrt{T} \sqrt{S_{t_{n,i}} 1_{\mathbb{R}_-}(a_{n,j})} \\ &=\; 2\,\varepsilon_0^{-2q}\, c_{(118)}\, \mathrm{e}^{4\gamma^2 q^4 T/\kappa^2} \sqrt{T} \sum_{n\ge 1}\ \sum_{i:\ 0\le t_{n,i}\le T} 2^{3n(1-q)/10} \times \\ &\qquad \sum_{j:\ a_{n,j}\ge A} 2^{-n} \sqrt{S_{t_{n,i}} 1_{\mathbb{R}_-}(a_{n,j})}. \end{aligned}$$

But $\sqrt{S_{t_{n,i}}1_{\mathbb{R}_-}(a_{n,j})}$ is non-increasing in $a_{n,j} \geq 0$. Hence, the internal sum can be estimated from above by

$$2^{-n}\sqrt{S_{t_{n,i}}1_{\mathbb{R}_-}(A)} + \int_A^\infty \mathrm{d}a \sqrt{S_{t_{n,i}}1_{\mathbb{R}_-}(a)}. \tag{123}$$

Note that this disappears if $t_{n,i} = 0$ since $A \geq 1$ by assumption. On the other hand, for $0 < t_{n,i} \leq T$,

$$\sqrt{S_{t_{n,i}}1_{\mathbb{R}_-}(a)} \leq \bigl(t_{n,i}/T\bigr)^{-1/4}\sqrt{S_T 1_{\mathbb{R}_-}(a)}. \tag{124}$$

Thus,

$$\begin{aligned}
&P_{\mathbf{x}}(\mathfrak{A}_{\varepsilon_0}) \\
&\leq 2\,\varepsilon_0^{-2q}\,c_{(118)}\,\mathrm{e}^{4\gamma^2q^4T/\kappa^2}\sqrt{T}\sum_{n\geq 1}\sum_{i:\,0<t_{n,i}\leq T} 2^{3n(1-q)/10}\,\times \\
&\quad\Bigl(2^{-n}\bigl(t_{n,i}/T\bigr)^{-1/4}\sqrt{S_T 1_{\mathbb{R}_-}(A)} + \int_A^\infty \mathrm{d}a\,\bigl(t_{n,i}/T\bigr)^{-1/4}\sqrt{S_T 1_{\mathbb{R}_-}(a)}\Bigr) \\
&= 2\,\varepsilon_0^{-2q}\,c_{(118)}\,\mathrm{e}^{4\gamma^2q^4T/\kappa^2}\sqrt{T}\sum_{n\geq 1} 2^{n(13-3q)/10}\,\times \\
&\quad\Bigl(2^{-n}\sqrt{S_T 1_{\mathbb{R}_-}(A)} + \int_A^\infty \mathrm{d}a\,\sqrt{S_T 1_{\mathbb{R}_-}(a)}\Bigr)\sum_{i:\,0<t_{n,i}\leq T} 2^{-n}\,\bigl(t_{n,i}/T\bigr)^{-1/4}.
\end{aligned}$$

However, the internal sum is bounded from above by

$$\int_0^T \mathrm{d}t\,(t/T)^{-1/4} = \frac{4}{3}T. \tag{125}$$

Therefore, taking $q = 5$, there is a constant $c_{(126)} = c_{(126)}(\gamma,\kappa)$ such that

$$\begin{aligned}
&P_{\mathbf{x}}(\mathfrak{A}_{\varepsilon_0}) \\
&\quad\leq c_{(126)}\,\varepsilon_0^{-10}\,\mathrm{e}^{5^5\gamma^2T/\kappa^2}\,T^{3/2}\Bigl(\sqrt{S_T 1_{\mathbb{R}_-}(A)} + \int_A^\infty \mathrm{d}a\,\sqrt{S_T 1_{\mathbb{R}_-}(a)}\Bigr).
\end{aligned} \tag{126}$$

On the other hand, on $\mathfrak{A}^{\mathrm{c}}_{\varepsilon_0}$ (where all the considered G_n–neighbouring increments are bounded by $\varepsilon_0 2^{-n/10}$), since $N^k_0(a) \equiv 0$, we can estimate $N^k_t(a)$ for $t \leq T$ and $a \geq A$ by a countable number of increments across neighbouring dyadic points along a path from (t,a) to the boundary part $\{0\}\times[A,\infty)$ of $[0,T]\times[A,\infty)$. This we now want to make precise.

First we choose a point $(t_{1,i}, a_{1,j}) =: g_{i,j} \in G_1$ closest to (t,a). For the fixed $a_{1,j}$, we pass along neighbouring points in G_1 to the $t = 0$ axis. For this we need at most $[T/\frac{1}{2}] + 1 \leq 3T$ steps. Hence, on $\mathfrak{A}^{\mathrm{c}}_{\varepsilon_0}$,

$$\bigl|N^k_{t_{1,i}}(a_{1,j})\bigr| \leq 3T\,\varepsilon_0 2^{-1/10}. \tag{127}$$

Additionally, to reach $g_{i,j}$ from (t,a), we need at most one time and space increment of length at most 2^{-n} each, for each $n > 1$. Thus, on $\mathfrak{A}^{\mathrm{c}}_{\varepsilon_0}$,

$$\Bigl|N^k_t(a) - N^k_{t_{1,i}}(a_{1,j})\Bigr| \leq 2\sum_{n>1}\varepsilon_0\,2^{-n/10}. \tag{128}$$

Adding up (127) and (128) gives

$$(129)\qquad \big|N_t^k(a)\big| \;\le\; 3T\,\varepsilon_0 \sum_{n\ge 1} 2^{-n/10} \;=\; c_{(129)} T\,\varepsilon_0, \qquad t\le T, \quad a\ge A,$$

for an absolute constant $c_{(129)}$. For $0<\varepsilon\le 1$, setting

$$(130)\qquad \varepsilon_0 \;:=\; \varepsilon / c_{(129)} T,$$

on $\mathfrak{A}^{\mathrm{c}}_{\varepsilon_0}$ we have

$$(131)\qquad \big|N_t^k(a)\big| \;\le\; \varepsilon, \qquad 0\le t\le T, \quad a\ge A.$$

Therefore,

$$(132)\qquad P_{\mathbf{x}}\Big(\,\big|N_t^k(a)\big|\ge\varepsilon \text{ for some } t\le T \text{ and } a\ge A\Big) \;\le\; P_{\mathbf{x}}(\mathfrak{A}_{\varepsilon_0}),$$

and with (126), (130), and the estimate (70) in Lemma 13, we finish the proof. □

5.5. **The rightmost point in the support of X_t^1.** As in [Tri95], we set

$$(133)\qquad R(y) \;:=\; \sup_{a\in\mathbb{R}} \big\{a:\; y(a)>0\big\}, \qquad y\in\mathcal{B}^+_{\mathrm{tem}}(\mathbb{R}),$$

for the *rightmost point* in the support of a state component y. Recall that we start $\mathbf{X}$ from the Heaviside state $\mathbf{x}=(1_{\mathbb{R}_-}, 1_{\mathbb{R}_+})$ with interface $\mathrm{Ifc}\,\mathbf{x}=\{0\}$. We estimate the probability that, for r large, the component X^1 hits $[r,\infty)$ by time T (cf. [Tri95, Proposition 3.2]).

Proposition 23 (**Rightmost point in the support of X_t^1**). *For some constant $c_{23}=c_{23}(\gamma,\kappa)$, we have*

$$(134)\qquad P_{\mathbf{x}}\Big(\sup_{t\le T} R(X_t^1) > r\Big) \;\le\; c_{23}\, T^{14}\, p_{16T}(r),$$

for $T\ge 1$, and $r>6^4(\gamma\vee\kappa\vee 1)T$. The symmetric result holds for the leftmost point, $L(X_t^2)$, say, in the support of X_t^2.

Proof. 1° (*Decomposition*). Introduce the first time τ_r that X^2 is less than the level $\frac12$ beyond the space point $r>0$:

$$(135)\qquad \tau_r \;:=\; \inf\Big\{t\ge 0:\; X_t^2(a)<\tfrac12 \text{ for some } a>r\Big\}.$$

Moreover, denote by σ_r the first time the rightmost point in the support of X_t^1 exceeds r :

$$(136)\qquad \sigma_r \;:=\; \inf\big\{t\ge 0:\; R(X_t^1)>r\big\} \;=\; \inf\big\{t\ge 0:\; \langle X_t^1, f^r\rangle >0\big\},$$

where f^r was introduced in (93). Fix $T\ge 1$. We wish to estimate

$$(137)\qquad P_{\mathbf{x}}\Big(\sup_{t\le T} R(X_t^1)>r\Big) \;=\; P_{\mathbf{x}}(\sigma_r\le T).$$

For $0<r'<r$,

$$(138)\qquad P_{\mathbf{x}}(\sigma_r\le T) \;\le\; P_{\mathbf{x}}(\sigma_r\le T\wedge\tau_{r'}) + P_{\mathbf{x}}(\tau_{r'}<T).$$

To see this, partition into $\tau_{r'}\ge T$ and $\tau_{r'}<T$, and note that in the first case, $T=T\wedge\tau_{r'}$. So we may estimate both terms on the right hand side of (138) separately.

2° (*First term*). By identity (88) in Lemma 20 with $\tau = \tau_{r'}$, $\varphi = \lambda f^r$, and $\psi = \psi^{\lambda,r}$, implying the simplification (92), and by choosing $\alpha := \frac{\gamma}{4}$ in the log-Laplace equation (94),

$$E_{\mathbf{x}}\Big(1 - \exp\Big[-\big\langle X^1_{t\wedge\tau_{r'}}, \psi_{t\wedge\tau_{r'}}\big\rangle - \lambda\int_0^{t\wedge\tau_{r'}} \mathrm{d}s\ \big\langle X^1_s, f^r\big\rangle\Big]\Big) \ = \ 1 - \mathrm{e}^{-\langle x^1, \psi_0\rangle}$$

$$+ \ \frac{\gamma}{2} E_{\mathbf{x}} \int_0^{t\wedge\tau_{r'}} \mathrm{d}s\ \exp\Big[-\big\langle X^1_s, \psi_s\big\rangle - \lambda\int_0^s \mathrm{d}r\ \big\langle X^1_r, f^r\big\rangle\Big]\big\langle \tfrac{1}{2} X^1_s - X^1_s X^2_s,\ \psi^2_s\big\rangle,$$

$t \ge 0$. In the last term on the right hand side, we decompose the integral over a into two parts: the integral over $(r', +\infty)$ and over $(-\infty, r']$. Note that

$$\tfrac{1}{2} X^1_s(a) - X^1_s(a)\, X^2_s(a) \ \le \ 0 \quad \text{for} \quad s < \tau_{r'} \quad \text{and} \quad a > r' \tag{139}$$

[with $\tau_{r'}$ from (135)]. Hence, we can drop the integral over $(r', +\infty)$ to obtain:

$$E_{\mathbf{x}}\Big(1 - \exp\Big[-\big\langle X^1_{t\wedge\tau_{r'}}, \psi_{t\wedge\tau_{r'}}\big\rangle - \lambda\int_0^{t\wedge\tau_{r'}} \mathrm{d}s\ \big\langle X^1_s, f^r\big\rangle\Big]\Big)$$

$$\le \ 1 - \mathrm{e}^{-\langle x^1, \psi_0\rangle} \ + \ \frac{\gamma}{4} E_{\mathbf{x}} \int_0^t \mathrm{d}s\ \big\langle X^1_s,\ 1_{(-\infty, r']}\,\psi^2_s\big\rangle$$

$$\le \ \langle 1_{\mathbb{R}_-}, \psi_0\rangle \ + \ \frac{\gamma}{4}\int_0^t \mathrm{d}s \int_{-\infty}^{r'} \mathrm{d}a \int_{-\infty}^0 \mathrm{d}b\ p_s(b-a)\,\psi^2_s(a), \tag{140}$$

where in the last step we applied the expectation formula (87). If we additionally assume that $r > 2\sqrt{T}$, then we may exploit Lemma 21 to obtain for some (changing) constant $c(\kappa)$,

$$\langle 1_{\mathbb{R}_-}, \psi_0\rangle \ \le \ \frac{c_{21}}{\gamma\sqrt{T}} \int_{-\infty}^0 \mathrm{d}a\ p_{4T}(r-a) \ \le \ \frac{c(\kappa)}{\gamma}\, p_{4T}(r), \tag{141}$$

where in the last step we used (71). In the second term of (140), we estimate $\int_{-\infty}^0 \mathrm{d}b\ p_s(b-a) \le 1$. Moreover, if $r' := \frac{r}{2}$ and moreover $r > 4\sqrt{T}$, then for the remainder of the second term in (140) we may exploit Lemma 21 once again to obtain the bound

$$\frac{\gamma}{4} T \int_{-\infty}^{r/2} \mathrm{d}a\ \frac{c^2_{21}}{T\gamma^2/16}\, p^2_{4T}(r-a) \ = \ \frac{c(\kappa)}{\gamma\sqrt{T}} \int_{-\infty}^{-r/2} \mathrm{d}a\ p_{2T}(a) \ \le \ \frac{c(\kappa)}{\gamma}\, p_{8T}(r), \tag{142}$$

using (71) in the last step. Combined with (141), in place of (140) we have the cruder estimate

$$E_{\mathbf{x}}\Big(1 - \exp\Big[-\big\langle X^1_{t\wedge\tau_{r/2}}, \psi_{t\wedge\tau_{r/2}}\big\rangle - \lambda\int_0^{t\wedge\tau_{r/2}} \mathrm{d}s\ \big\langle X^1_s, f^r\big\rangle\Big]\Big)$$

$$\le \ \frac{c(\kappa)}{\gamma}\, p_{8T}(r), \qquad 0 \le t \le T, \quad r > 4\sqrt{T}. \tag{143}$$

We now claim that

$$P_{\mathbf{x}}(\sigma_r \le T \wedge \tau_{r/2}) \ \le \ \frac{c(\kappa)}{\gamma}\, p_{8T}(r), \qquad r > 4\sqrt{T}. \tag{144}$$

Because of (143), it suffices to show that

$$P_{\mathbf{x}}(\sigma_r > T \wedge \tau_{r/2}) \tag{145}$$

$$\ge \ \lim_{\lambda\uparrow\infty} E_{\mathbf{x}} \exp\Big[-\big\langle X^1_{T\wedge\tau_{r/2}}, \psi_{T\wedge\tau_{r/2}}\big\rangle - \lambda\int_0^{T\wedge\tau_{r/2}} \mathrm{d}s\ \big\langle X^1_s, f^r\big\rangle\Big].$$

Here we take into account that the solution $\psi = \psi^{\lambda,r}$ to (94) is monotone in λ. To verify (145), restrict the expectation to $\sigma_r \le T \wedge \tau_{r/2}$. Then the d$s$–integral is positive, and the exponential expression will disappear as $\lambda \uparrow \infty$, implying that the whole restricted expectation will disappear in the limit. On the other hand, restricted to $\sigma_r > T \wedge \tau_{r/2}$, the exponential expression can be bounded by 1, implying (145).

3° (*Second term*). We are now going to estimate $P_{\mathbf{x}}(\tau_{r/2} < T)$. Using (107),

$$P_{\mathbf{x}}(\tau_{r/2} < T) \;=\; P_{\mathbf{x}}\Big(S_t 1_{\mathbb{R}_+}(a) + N_t^2(a) \le \tfrac12 \quad \text{for some } a \ge \tfrac{r}{2} \text{ and } t \le T\Big)$$

$$\le\; P_{\mathbf{x}}\Big(\big|N_t^2(a)\big| \ge S_t 1_{\mathbb{R}_+}(a) - \tfrac12 \quad \text{for some } a \ge \tfrac{r}{2} \text{ and } t \le T\Big). \tag{146}$$

But under $a \ge \frac{r}{2}$ and $t \le T$,

$$S_t 1_{\mathbb{R}_+}(a) \;\le\; S_t 1_{\mathbb{R}_+}(\tfrac{r}{2}) \;\le\; S_T 1_{\mathbb{R}_+}(\tfrac{r}{2}) \;=\; 1 - S_T 1_{\mathbb{R}_-}(\tfrac{r}{2}). \tag{147}$$

Hence, the chain of inequalities (146) can be continued with

$$\le\; P_{\mathbf{x}}\Big(\big|N_t^2(a)\big| \ge \tfrac12 - S_T 1_{\mathbb{R}_-}(\tfrac{r}{2}) \quad \text{for some } a \ge \tfrac{r}{2} \text{ and } t \le T\Big)$$

$$\le\; P_{\mathbf{x}}\Big(\big|N_t^2(a)\big| \ge \tfrac12 - \tfrac{2\kappa^2 T}{r}\, p_T(\tfrac{r}{2}) \quad \text{for some } a \ge \tfrac{r}{2} \text{ and } t \le T\Big), \tag{148}$$

where we have used (71). Now

$$\frac{2\kappa^2 T}{r}\, p_T(\tfrac{r}{2}) \;<\; \frac{\sqrt{2\kappa^2 T}}{r} \;\le\; \frac{1}{2\sqrt{2}} \quad \text{if} \quad r > 4\kappa\sqrt{T}, \tag{149}$$

and altogether we have

$$P_{\mathbf{x}}(\tau_{r/2} < T) \;\le\; P_{\mathbf{x}}\Big(\big|N_t^2(a)\big| \ge \varepsilon \quad \text{for some } a \ge \tfrac{r}{2} \text{ and } t \le T\Big), \tag{150}$$

provided that $r > 4\kappa\sqrt{T}$, where $\varepsilon := \frac12 - \frac{1}{2\sqrt{2}}$. Hence, by Lemma 22, for a constant $c(\gamma,\kappa)$,

$$P_{\mathbf{x}}(\tau_{r/2} < T) \;\le\; c_{22}\, \varepsilon^{-10}\, \mathrm{e}^{5^5 \gamma^2 T/\kappa^2}\, \frac{T^{14}}{\sqrt{r/2}}\, p_{2T}(r/2)$$

$$\le\; c(\gamma,\kappa)\, T^{14}\, p_{16T}(r), \tag{151}$$

provided that $r > (6 \cdot 5^{5/2}\gamma T) \vee 4\kappa\sqrt{T}$. Combined with our estimate (144), this gives the claim (134). By symmetry, the proof is finished. □

5.6. **Completion of the proof of Theorem 5.** Armed with Proposition 23, we can now easily complete the proof of Theorem 5. Choose

$$r = r(T) := c_{(152)} T \quad \text{where} \quad c_{(152)} = c_{(152)}(\gamma,\kappa) > 6^4(\gamma \vee \kappa \vee 1). \tag{152}$$

Then, by Proposition 23,

$$\sum_{T\ge1} P_{\mathbf{x}}\Big(\sup_{t\le T} R(X_t^1) > c_{(152)} T\Big) \;\le\; \sum_{T\ge1} c_{23}\, T^{14}\, p_{16T}(c_{(152)} T)$$

$$\le\; c \sum_{T\ge1} T^{14}\, \mathrm{e}^{-cT} \quad \text{with constants} \quad c = c(\gamma,\kappa). \tag{153}$$

But the latter sum is finite, hence, by Borel-Cantelli, the rightmost point in the support of X_t^1 exceeds $c_{(152)}T$ for some $t \leq T$ only for finitely many T. Thus,

$$\sup_{t \leq T} R(X_t^1) \; < \; c_{(152)}T \quad \text{for all } T \text{ sufficiently large, } P_{\mathbf{x}}\text{–a.s.} \tag{154}$$

Therefore, the rightmost point in the support of X_t^1 propagates at most with speed t, provided that t is sufficiently large.

Clearly, by symmetry, an analogous statement holds for the leftmost point $L(X_t^2)$ of X_t^2. Since

$$\operatorname{Ifc} \mathbf{X}_t \; \subseteq \; \left[L(X_t^2), R(X_t^1)\right], \tag{155}$$

the proof is finished. □

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University of Oxford, Department of Statistics, 1 South Parks Road, Oxford OX1 3TG, UK

E-mail address: `etheridg@stats.ox.ac.uk`

URL: `http://www.stats.ox.ac.uk/people/etheridge/index.htm`

Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstr. 39, D–10117 Berlin, Germany

E-mail address: `fleischm@wias-berlin.de`

URL: `http://www.wias-berlin.de/~fleischm`