# Weierstraß-Institut
# für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

## A super-stable motion with infinite mean branching*

Klaus Fleischmann[1] , Anja Sturm[**,1]

submitted: 05 March 2003

[1] Weierstrass-Institute
for Applied Analysis and Stochastics
Mohrenstr. 39
D-10117 Berlin,
Germany
E-Mail: fleischmann@wias-berlin.de
E-Mail: sturm@wias-berlin.de.de

No. 821
Berlin 2003

1991 *Mathematics Subject Classification.* 60J80 60K35 60G57 60F15.

*Key words and phrases.* Neveu's continuous state branching process, superprocess, branching process with infinite mean, non-Lipschitz non-linearity, immortal process, instantaneous mass propagation, locally countably infinite biodiversity .

*Work supported by DFG Priority Program 1033

** current address: Technische Universität Berlin, Fakultät II, Institut für Mathematik, Straße des 17. Juni 135, 10623 Berlin.

Edited by
Weierstraß-Institut für Angewandte Analysis und Stochastik (WIAS)
Mohrenstraße 39
D — 10117 Berlin
Germany

Fax: + 49 30 2044975
E-Mail: preprint@wias-berlin.de
World Wide Web: http://www.wias-berlin.de/

**Abstract**

Impressed by Neveu's (1992) continuous-state branching process we learned about from Bertoin and Le Gall (2000), a class of finite measure-valued càdlàg superprocesses $X$ with Neveu's branching mechanism is constructed. To this end, we start from certain supercritical $(\alpha, d, \beta)$-superprocesses $X^{(\beta)}$ with symmetric $\alpha$-stable motion and $(1+\beta)$-branching and let $\beta \downarrow 0$. The log-Laplace equation related to $X$ has the locally non-Lipschitz function $u \log u$ as non-linear term (instead of $u^{1+\beta}$ in the case of $X^{(\beta)}$) and is thus interesting in its own. $X$ has infinite expectations, is immortal in all finite times, propagates mass instantaneously everywhere in space, and has locally countably infinite biodiversity.

# Contents

# 1 Introduction

## 1.1 Motivation, background, and purpose

Bertoin and Le Gall (2000) established in [1] connection between a special continuous-state branching process $\bar{X} = (\bar{X}_t)_{t\geq 0}$ and a coalescent process investigated by Bolthausen and Sznitman (1998) in [2] and also by Pitman (1999) in [21]. This process $\bar{X}$ was actually introduced in connection with Ruelle's (1987) [22] probability cascades by Neveu (1992) in the preprint [19], so we call it henceforth *Neveu's process.* It is indeed a strange branching process: Its (individual) branching mechanism is given by the function $u \log u$, hence belongs to the domain of attraction of a stable law of index 1. On the other hand, the state at time $t > 0$ has a stable law of index $\mathrm{e}^{-t} < 1$ varying in time and tending to 0 as $t \uparrow \infty$. Trivially, this process has no finite expectations.

Fascinated by this process, we asked the question whether this model can be enriched by a spatial motion component. Indeed, imagine the "infinitesimally small parts" of Neveu's process move in $\mathbb{R}^d$ according to independent Brownian motions. Can this be made mathematically rigorous? In other words, *does a super-Brownian motion $X = (X_t)_{t\geq 0}$ exist with Neveu's branching mechanism*, and what properties does it have? Clearly, via log-Laplace functionals, such a superprocess $X$ would be related to the Cauchy problem

$$\left.\begin{array}{rcl}\dfrac{\partial}{\partial t}u_t(x) & = & \Delta u_t(x) - u_t(x)\log u_t(x) \ \text{ on } \ (0,\infty)\times\mathbb{R}^d \\ & & \text{with initial condition} \quad u_{0+} = \varphi \geq 0\end{array}\right\} \tag{1}$$

(where $\varphi$ is an appropriate test function). Note that this diffusion-reaction equation is interesting in itself since the reaction term does not satisfy a local Lipschitz condition (the derivative has a singularity at 0).

## 1.2 Approach, sketch of the main result

As Neveu's process $\bar{X}$ can be approximated by a family $(\bar{X}^{(\beta)})_{0<\beta\leq 1}$ of supercritical continuous-state branching processes $\bar{X}^{(\beta)}$ of index $1+\beta$ by letting $\beta \downarrow 0$, we try to approximate the desired process $X$ (with more general $\alpha$-stable motion component) by a family of superprocesses $X^{(\beta)}$ with $(1+\beta)$-branching mechanism. More precisely, we assume that $X^{(\beta)}$ is a supercritical super-stable motion related to the log-Laplace equation

$$\left.\begin{array}{rcl}\dfrac{\partial}{\partial t}u_t^{(\beta)}(x) & = & \Delta_\alpha u_t^{(\beta)}(x) - \frac{1}{\beta}\left(u_t^{(\beta)}(x)\right)^{1+\beta} + \frac{1}{\beta}\,u_t^{(\beta)}(x) \ \text{ on } \ (0,\infty)\times\mathbb{R}^d \\ & & \text{with initial condition} \quad u_{0+}^{(\beta)} = \varphi \geq 0.\end{array}\right\} \tag{2}$$

Here $\alpha \in (0,2]$, and $\Delta_\alpha$ is the fractional Laplacian $-(-\Delta)^{\alpha/2}$ on $\mathbb{R}^d$. Consequently, the underlying motion is a symmetric stable process of index $\alpha$, hence Brownian motion if $\alpha = 2$. Of course, the relation between $X^{(\beta)}$ and $u^{(\beta)}$ from

(2) is realized via log-Laplace functionals:

$$-\log \mathbb{E}_\mu \Big[\exp\langle X_t^{(\beta)}, -\varphi\rangle\Big] = \langle \mu, u_t^{(\beta)}\rangle \tag{3}$$

where $\langle \mu, f\rangle$ denotes the integral $\int_{\mathbb{R}^d} f(x)\mu(\mathrm{d}x)$, and the expectation symbol $\mathbb{E}_\mu$ refers to the law $\mathbb{P}_\mu$ of $X^{(\beta)}$ starting from the finite measure $X_0^{(\beta)} = \mu$. We note that

$$\tfrac{1}{\beta}(v^{1+\beta} - v) \underset{\beta\downarrow 0}{\longrightarrow} v\log v, \qquad v \geq 0, \tag{4}$$

therefore such set-up seems to be reasonable if $X$ exists non-trivially at all.

Our *purpose* is to verify that the family $(X^{(\beta)})_{0<\beta\leq 1}$ of superprocesses is tight in law as $\beta \downarrow 0$ on the Skorohod space of càdlàg measure-valued paths, and that each limit point $X$ is related to the unique process solving the log-Laplace equation (1) (with $\Delta$ replaced by $\Delta_\alpha$). This then gives convergence to the desired process $X$ (see Theorem 2 below) with total mass process $\bar{X} = X(\mathbb{R}^d)$.

Note that many of the standard tools are not available for this route, since the Lipschitz constants related to the non-linear term in the log-Laplace equation (2) blow up along (4), or -viewed in probabilistic terms- the expectations of $X^{(\beta)}$ become infinite as $\beta \downarrow 0$. For the well-posedness of equations as in (1), see Theorem 1 below.

## 1.3 First properties of $X$

As Neveu's process $\bar{X}$ has strange properties, one expects that also $X$ has interesting new properties compared with usual superprocesses. For instance, we suspect that $X$ has absolutely continuous states in *all* dimensions. Recall that the $(\alpha, d, \beta)$-superprocesses $X^{(\beta)}$ have absolutely continuous states at almost all times in dimensions $d < \alpha/\beta$ (see the Appendix of Fleischmann (1988) [10] for the case of critical $(\alpha, d, \beta)$-superprocesses starting from Lebesgue measures), and we let $\beta \downarrow 0$. In this paper, however, we will content ourself with more modest properties.

Because $\bar{X}_t$ has a stable distribution with index $\mathrm{e}^{-t}$, the total mass process $t \mapsto X_t(\mathbb{R}^d) = \bar{X}_t$ is immortal for all finite times. Moreover, the underlying $\alpha$-stable mass flow, that is, the semigroup with generator $\Delta_\alpha$ applied to measures, propagates instantaneously everywhere in space. Thus, our super-stable process $X$ is expected to be *immortal* and its mass should *propagate instantaneously* in space (see Proposition 14 below). Of course, this is in sharp contrast to the approximating supercritical $X^{(\beta)}$ processes, which in each given region and at each fixed positive time have no mass with positive probability.

As a further consequence of this, we obtain that $X$ has *locally countably infinite biodiversity*, a notion introduced in Fleischmann and Klenke (2000) [12]. Roughly speaking, this means that, for fixed $t > 0$, in the clustering representation (family structure) of the infinitely divisible random measure $X_t$, infinitely many clusters (families) contribute to each given region (see Corollary

16 below). Clearly, this contrasts with the (locally) finite biodiversity of the random states of the approximating $(\alpha, d, \beta)$-superprocesses $X^{(\beta)}$.

The further *layout* of the paper is as follows: In Section 2, we first introduce some notation in Section 2.1 before, in Section 2.2, we rigorously define the process $X$ and its approximations $X^{(\beta)}$. We also state Theorem 1 concerning the solutions $u$ of equations as in (1). The main results concerning existence of and convergence to $X$ are given in Theorems 2 and 3. The proof is worked out in the remaining parts of Section 2 after the concept is explained in 2.3. In Section 3 we are concerned with immortality and infinite biodiversity of the constructed process $X$. The appendix gives the proof of an almost sure limit on $\bar{X}_t$ as $t \to \infty$ (see Proposition 8). It follows a sketch of proof in Neveu's unpublished work [19], which uses ideas of Grey (1977) [13] concerning the Galton-Watson case.

For background on superprocesses we refer to Dawson (1993) [3], Dynkin (1994) [5], and Etheridge (2000) [6].

# 2 Construction

## 2.1 Preliminaries

For any Polish space $E$ let $D(\mathbb{R}_+, E)$ and $C(\mathbb{R}_+, E)$ denote the space of functions $\mathbb{R}_+ := [0, \infty) \to E$, which are càdlàg and continuous respectively. The former is endowed with the Skorohod topology, the latter with the topology of uniform convergence on compact sets. By $C(E)$ we denote the class of continuous real valued functions on $E$, and we use $C_{\mathrm{b}}(E)$ if they are furthermore bounded, and $C_{\mathrm{com}}(E)$ if they have compact support. The subspace of functions whose derivatives up to order $n$ are also in $C_{\mathrm{b}}(E)$ is denoted by $C_{\mathrm{b}}^n(E)$. The superscripts "+" and "++" indicate the respective subspaces of non-negative functions and functions with positive infimum. We write $M_{\mathrm{f}} := M_{\mathrm{f}}(\mathbb{R}^d)$ for the finite measures on $\mathbb{R}^d$ equipped with the weak topology. Throughout, $c$ denotes generic positive constants, whose dependencies we sometimes cite in parentheses. The arrow $\Rightarrow$ is used to indicate convergence in law.

Fix $\alpha \in (0, 2]$. The semigroup associated with the fractional Laplacian $\Delta_\alpha = -(-\Delta)^{\alpha/2}$ is denoted by $T^\alpha$,

$$T_t^\alpha \varphi\,(x) = \int_{\mathbb{R}^d} p_t^\alpha(x-y)\varphi(y)dy, \qquad t > 0, \quad x \in \mathbb{R}^d, \tag{5}$$

where $p^\alpha$ is the (jointly continuous) kernel on $(0, \infty) \times \mathbb{R}^d$ of the symmetric $\alpha$-stable motion in $\mathbb{R}^d$, see for example the appendix of Fleischmann and Gärtner (1986) [11]. Clearly, for $\alpha = 2$ we have the heat kernel:

$$p_t^2(x) = (4\pi t)^{-\frac{d}{2}} \exp\Big(-\frac{|x|^2}{4t}\Big), \qquad t > 0, \quad x \in \mathbb{R}^d. \tag{6}$$

If $q^\eta$ denotes the continuous transition density function of a stable process on $\mathbb{R}_+$ with index $\eta \in (0,1)$ so normalized that for the Laplace transform we have

$$\int_0^\infty q_t^\eta(s) \mathrm{e}^{-s\theta} ds = \exp(-t\theta^\eta), \qquad t > 0, \quad \theta \geq 0, \tag{7}$$

then in the case $\alpha < 2$ the subordination formula

$$p_t^\alpha(x) = \int_0^\infty q_t^{\alpha/2}(s) p_s^2(x) ds, \qquad t > 0, \quad x \in \mathbb{R}^d \tag{8}$$

is well-known. Note that $T^\alpha$ from (5) is a strongly continuous, positive and conservative contraction semigroup on $C_\mathrm{b}^+(\mathbb{R}^d)$, which follows via subordination (8) from the corresponding properties of $T^2$.

## 2.2 Main results

The construction of our process $X$ is based on the following well-posedness of the Cauchy problem as in (1) in the mild sense:

$$u_t(x) = T_t^\alpha \varphi\,(x) - \int_0^t T_{t-s}^\alpha \big(g(u_s)\big)\,(x)\, ds, \tag{9}$$

$t \geq 0, \;\; x \in \mathbb{R}^d, \;\; \varphi \in C_\mathrm{b}^+(\mathbb{R}^d)$. Here,

$$g(v) := \rho\, v \log v, \qquad v \geq 0, \tag{10}$$

is a continuous function on $\mathbb{R}_+$, and $\rho > 0$ is an additional constant (for eventual scaling purposes). For a plot of $g$ in the case $\rho = 1$, see Figure 1. In Section 2.5 we will prove the following result.

**Theorem 1 (Well-posedness of log-Laplace equation).**

**(a) (Unique existence in the Lipschitz region).** *To each test function $\varphi$ in $C_\mathrm{b}^{++}(\mathbb{R}^d)$, there is a unique solution $u = u(\varphi)$ in $C([0,\infty), C_\mathrm{b}^{++}(\mathbb{R}^d))$ to equation* (9)*. It satisfies*

$$\big(\inf_{y\in\mathbb{R}^d} \varphi(y) \wedge 1\big) \leq u_t(\varphi)(x) \leq \big(||\varphi||_\infty \vee 1\big), \qquad t \geq 0, \quad x \in \mathbb{R}^d. \tag{11}$$

*Furthermore, for $\varphi \in C_\mathrm{b}^{2,++}(\mathbb{R}^d)$, $u(\varphi)$ is even a strong solution (satisfying* (1) *with $\Delta$ replaced by $\Delta_\alpha$) in $C^1((0,\infty), C_\mathrm{b}^+(\mathbb{R}^d))$ with $u_t \in C_\mathrm{b}^2(\mathbb{R}^d)$ for every $t \geq 0$.*

**(b) (Extension to the non-Lipschitz region).** *If $\varphi_n \in C_\mathrm{b}^{++}(\mathbb{R}^d)$, $n \geq 1$, such that pointwise $\varphi_n \downarrow \varphi \in C_\mathrm{b}^+(\mathbb{R}^d)$ as $n \uparrow \infty$, then pointwise $u(\varphi_n) \downarrow$ some $u(\varphi) \in C([0,\infty), C_\mathrm{b}^+(\mathbb{R}^d))$ as $n \uparrow \infty$, and the limit $u = u(\varphi)$ solves equation* (9)*, satisfies* (11)*, and is independent of the choice of the sequence $(\varphi_n)_{n\geq 1}$ converging to $\varphi$.*

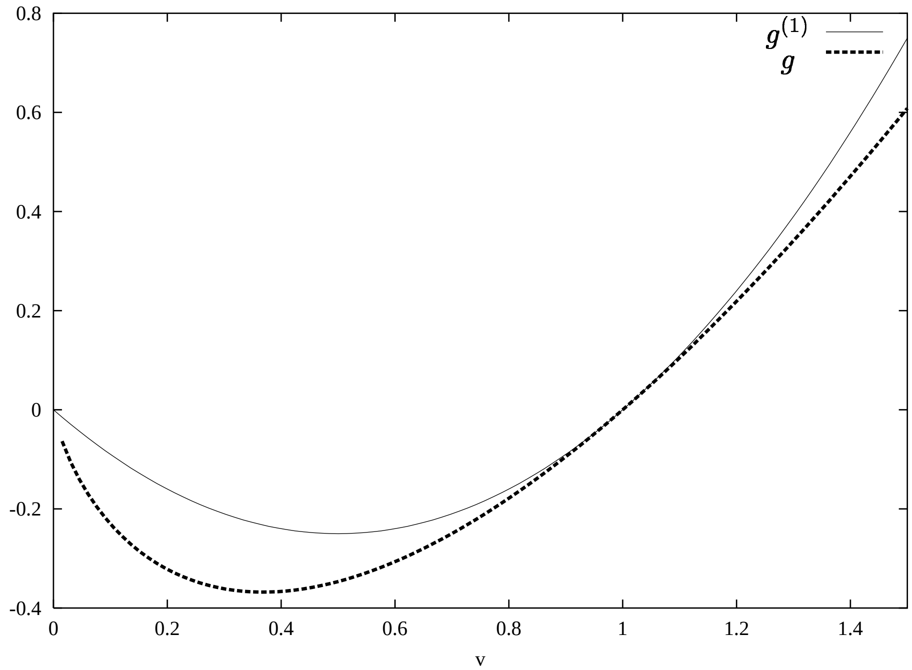


Figure 1: Branching mechanisms $g^{(1)}(v) = v^2 - v$ and $g(v) = v \log v$

We remark that the theorem implies the semigroup property for $u$, meaning that $u_{t+s}(\varphi) = u_t(u_s(\varphi))$ for $s, t \geq 0$ (see Dawson (1993) [3] p. 68). The semigroup property is tantamount to (17) below describing a time-homogeneous Markov process $X$.

The proof of this Theorem 11 will start from the well-posedness of the Cauchy problem as in (2) in the mild sense for $\beta \in (0, 1]$ fixed:

$$u_t^{(\beta)}(x) = T_t^{\alpha}\varphi\,(x) - \int_0^t T_{t-s}^{\alpha}\big(g^{(\beta)}(u_s^{(\beta)})\big)\,(x)\,ds, \tag{12}$$

$t \geq 0,\ x \in \mathbb{R}^d,\ \varphi \in C_{\mathrm{b}}^+(\mathbb{R}^d)$. Here,

$$g^{(\beta)}(v) := \tfrac{\rho}{\beta}\,(v^{1+\beta} - v), \qquad v \geq 0. \tag{13}$$

For a plot of $g^{(\beta)}$ in the case $\rho = 1$ and $\beta = 1$, see Figure 1. To each $\varphi$ in $C_{\mathrm{b}}^+(\mathbb{R}^d)$, there is a unique solution

$$u^{(\beta)} = u^{(\beta)}(\varphi) \in C\big([0, \infty), C_{\mathrm{b}}^+(\mathbb{R}^d)\big); \tag{14}$$

the function $\varphi \in C_{\mathrm{b}}^{++}(\mathbb{R}^d)$ is bounded away from 0 and $\infty$, which implies that also the solutions $u^{(\beta)}(\varphi)$ are bounded away from 0 and $\infty$, uniformly in $\beta$ (see Lemma 9 below). Therefore, passing to the limit as $\beta \downarrow 0$, we end up in a Lipschitz region of the function $g$ of (10). This idea is behind part (a) of Theorem 1. (We learned such trick from Watanabe (1968) [24] who worked in the simpler case of a compact phase space.) We remark that for the extension to (b) we partly use probabilistic arguments by arguing with the log-Laplace transform of $X_t$. Part (b) we need in the considerations of the properties of $X$.

As a starting point, for each $0 < \beta \leq 1$ we take the (unique) time-homogeneous càdlàg strong Markov process $(X^{(\beta)}, \mathbb{P}^{(\beta)}_{\mu^{(\beta)}}, \mu^{(\beta)} \in M_{\mathrm{f}})$ with log-Laplace functional

$$-\log \mathbb{E}_{\mu^{(\beta)}} \left[ \exp\langle X^{(\beta)}_t, -\varphi\rangle \right] = \langle \mu^{(\beta)}, u^{(\beta)}_t \rangle, \tag{15}$$

$t \geq 0$, $\varphi \in C^+_{\mathrm{b}}(\mathbb{R}^d)$, with $u^{(\beta)}$ the unique solution to (12). The construction of $X^{(\beta)}$ is nowadays standard; for references see, for instance, Iscoe (1986) [14], Fitzsimmons (1988 and 1991) [8, 9] and Chapter 4 of Dawson (1993) [3]. Note that $X^{(\beta)}$ is a supercritical $(\alpha, d, \beta)$-superprocess. Properties of $(\alpha, d, \beta)$-superprocesses have been widely studied in the critical case where $g^{(\beta)}$ in (12) is replaced by

$$g^{(\beta)}_{\mathrm{crit}}(v) := b\, v^{1+\beta}, \qquad v \geq 0, \tag{16}$$

with $b > 0$ a constant, see for example Iscoe (1986), Fleischmann (1988), Dawson and Vinogradov (1994) and Mytnik and Perkins (2003) [14, 10, 4, 18]. These processes have finite mean but infinite variance provided that $\beta < 1$. More precisely, $\mathbb{E}_{\mu^{(\beta)}}[\langle X^{(\beta)}_t, \varphi\rangle^\theta] < \infty$ for all $t \geq 0$, $\varphi \in C^+_{\mathrm{b}}(\mathbb{R}^d)$ with $\varphi \neq 0$, and $\mu^{(\beta)} \in M_{\mathrm{f}}$ with $\mu^{(\beta)} \neq 0$, if and only if $0 < \theta < 1 + \beta$ (see also Lemma 7). The case we are interested in corresponds to $\beta = 0$ in the sense that the branching mechanism is in the domain of attraction of a stable law of index 1.

Here is now our main result:

**Theorem 2 (Existence of $X$).** *For each $\mu \in M_{\mathrm{f}}$ there exists a unique time-homogeneous Markov process $X \in D(\mathbb{R}_+, M_{\mathrm{f}})$ with log-Laplace functional*

$$-\log \mathbb{E}_\mu \left[\exp\langle X_t, -\varphi\rangle\right] = \langle \mu, u_t\rangle, \quad t \geq 0, \quad \varphi \in C^+_{\mathrm{b}}(\mathbb{R}^d), \tag{17}$$

*with $u$ the unique solution to* (9) *according to Theorem* 1.

We call $X$ the *super-$\alpha$-stable motion with Neveu's branching mechanism* (and branching rate $\rho$).

Our proof of Theorem 2 actually yields the following limit theorem.

**Theorem 3 (Convergence theorem).** *Suppose that $X^{(\beta)}_0 \Rightarrow X_0$ in $M_{\mathrm{f}}$ as $\beta \downarrow 0$, as well as $\sup_{0<\beta\leq 1} \mathbb{E}[\langle X^{(\beta)}_0, 1\rangle^{\theta_0}] < \infty$, for some $0 < \theta_0 \leq 1$. Then in law on $D(\mathbb{R}_+, M_{\mathrm{f}})$,*

$$X^{(\beta)} \Rightarrow X \quad \textit{as} \quad \beta \downarrow 0. \tag{18}$$

*Furthermore, we have* $\mathbb{E}\left[\sup_{0\leq t\leq T} \langle X_t, 1\rangle^\theta\right] < \infty$ *for all* $0 < \theta < \theta_0 \mathrm{e}^{-\rho T}$.

The proof will proceed via tightness in law and convergence of subsequences combined with the uniqueness of the limit, which follows from the existence of log-Laplace solutions according to Theorem 1(a).

We remark finally that the "highly supercritical" process $X$ cannot be attained as the limit of critical ones. Observe that setting $\beta = 0$ for $g_{\rm crit}^{(\beta)}$ from (16) implies the linear log-Laplace equation

$$\frac{\partial}{\partial t} u_t^{(0,\mathrm{crit})} = \Delta_\alpha u_t^{(0,\mathrm{crit})} - b\, u_t^{(0,\mathrm{crit})}. \tag{19}$$

Hence, the corresponding measure-valued process is deterministic in this case. However, $X$ is also expected to be the *high density limit of supercritical branching particle systems* as the number of initial particles $N$ tends to infinity. Indeed, consider particles that move independently according to $\alpha-$stable motions in $\mathbb{R}^d$ leaving a random number of offspring after their exponentially distributed lifetime with mean $(\rho\,(1+\log N))^{-1}$. Let the number of offspring be sampled according to the probability generating function

$$\psi_N(r) := (1+\log N)^{-1}\bigl(\log N + r + (1-r)\log(1-r)\bigr), \qquad 0 \le r \le 1. \tag{20}$$

Since $N\rho(1+\log N)\bigl(\psi_N(1-\frac{v}{N}) - (1-\frac{v}{N})\bigr) = v\log v$ (cf. Chapter 3 of Le Gall (2000) [17] although there locally non-Lipschitz branching mechanisms are excluded), one then expects that, the empirical measures $\frac{1}{N}\sum_i \delta_{\xi_t^{\alpha,i}}$, where $\xi_t^{\alpha,i}$ are the positions of the particles alive at time $t$ and the sum is taken over all these particles, converge in law to $X_t$ as $N\uparrow\infty$ (provided that the initial states converge).

## 2.3 Concept of proof of Theorems 2 and 3

In preparation of the proofs, we consider in Section 2.4 properties of Neveu's continuous state branching process $\bar{X}$ and its approximations $\bar{X}^{(\beta)}$. We prove some (monotone) convergence of the related log-Laplace functions and their non-linear terms as well as of the processes, and show uniform boundedness of lower order moments, see Lemmas 6-7.

The log-Laplace equations given in (9) and (12) are studied in Section 2.5. We will deal with uniform convergences, comparisons, and solutions starting from "runaway" functions.

In order to show tightness in law of $X^{(\beta)}$ in $D(\mathbb{R}_+, M_{\rm f})$ we use Jakubowski's (1986) criterion (see Theorem 3.1 of [15]). Since $\{\langle\,\cdot\,,\varphi\rangle;\ \varphi\in C_{\rm b}^{++}(\mathbb{R}^d)\}$ is a family of continuous functions on $M_{\rm f}$ that separates points, Jakubowski's criterion states that the properties in the following claim are sufficient for tightness.

**Proposition 4 (Tightness of the $X^{(\beta)}$).** *Let $X_0^{(\beta)}, X_0$ be as in Theorem 3. Then the following statements hold:*

**(a) (Tightness of one-dimensional processes).** *For each $\varphi\in C_{\rm b}^{+}(\mathbb{R}^d)$, the family $(\langle X^{(\beta)},\varphi\rangle)_{0<\beta\le 1}$ is tight in law on $D(\mathbb{R}_+,\mathbb{R})$.*

**(b) (Compact containment).** *For any $T \geq 0$ and $\epsilon > 0$, there exists a compact set $K_{\epsilon,T}$ in $M_{\mathrm{f}}$ such that*

$$\inf_{0<\beta\leq 1} \mathbb{P}\left[X_t^{(\beta)} \in K_{\epsilon,T} \text{ for } 0 \leq t \leq T\right] \geq 1-\epsilon. \tag{21}$$

Part (a) is shown in Section 2.6. Compact containment (b) is verified in Section 2.7 followed by a completion of the proof of Theorems 2 and 3.

## 2.4 Neveu's continuous state branching process

We begin with studying the total mass $\bar{X}^{(\beta)} = X^{(\beta)}(\mathbb{R}^d)$ and $\bar{X} = X(\mathbb{R}^d)$ of the superprocesses that we are considering. Their log-Laplace functions $\bar{u}^{(\beta)}$ and $\bar{u}$, both independent of a spatial variable, can be calculated explicitly. Indeed, define for all $t \in \mathbb{R}$ and $\lambda \geq 0$,

$$\bar{u}_t^{(\beta)}(\lambda) := (\lambda^{-\beta}\mathrm{e}^{-\rho t} + 1 - \mathrm{e}^{-\rho t})^{-\frac{1}{\beta}}, \tag{22}$$

$$\bar{u}_t(\lambda) := \lambda^{(\mathrm{e}^{-\rho t})}, \tag{23}$$

reading the right-hand side as 0 for $\lambda = 0$. Then $\bar{u}_t^{(\beta)}(\lambda)$ and $\bar{u}_t(\lambda)$ restricted to $t \geq 0$ are the unique solutions of (9) and (12) for $\varphi \equiv \lambda$. The uniqueness follows in the former case by the local Lipschitz continuity of $g^{(\beta)}$. The latter case can equivalently be written as

$$\frac{\partial}{\partial t} w_t = -\rho w_t \log w_t = -g(w_t) \quad \text{on} \quad (0,\infty) \quad \text{with} \quad w_{0+} = \lambda \geq 0. \tag{24}$$

Although $g$ is not locally Lipschitz, (24) has a unique solution. In fact, for $\epsilon > 0$ fixed, the function $g$ is locally Lipschitz on $[\epsilon, \infty)$, and the unique solution $w$ with $w_{0+} = \lambda \geq \epsilon$ lives on $[\lambda \wedge 1, \lambda \vee 1]$. Therefore, (24) is uniquely solvable on $(0, \infty)$. Assume that $w$ is a non-zero solution to (24) with $w_{0+} = 0$. Then there is a $t > 0$ such that $w_t = \theta > 0$. But from the previously mentioned uniqueness, we necessarily obtain $w_s = \bar{u}_{-(t-s)}(\theta)$, $0 \leq s \leq t$. Thus, $w_0 > 0$, which is a contradiction.

We thus have for $t \geq 0$, and $\bar{X}_0^{(\beta)}, \bar{X}_0 \geq 0$,

$$\mathbb{E}\left[\exp(-\bar{X}_t^{(\beta)}\lambda)\right] = \mathbb{E}\left[\exp\left(-\bar{X}_0^{(\beta)}\bar{u}_t^{(\beta)}(\lambda)\right)\right], \tag{25}$$

$$\mathbb{E}\left[\exp(-\bar{X}_t\lambda)\right] = \mathbb{E}\left[\exp\left(-\bar{X}_0\bar{u}_t(\lambda)\right)\right]. \tag{26}$$

We can right away verify the following properties of $g^{(\beta)}$ and $g$ (introduced in (10) and (13)).

**Lemma 5 (Properties of the non-linear terms).** *For all $v \in \mathbb{R}_+$ we have $g^{(\beta)}(v) \downarrow g(v)$ as $1 \geq \beta \downarrow 0$. Furthermore, $g^{(\beta)}$ and $g$ are negative on $(0,1)$ and positive on $(1,\infty)$, with the only intersection points $g(v) = g^{(\beta)}(v) = 0$ for $v = 0$ and for $v = 1$.*

*Proof.* Let us start by showing that

$$\frac{\partial}{\partial \beta} g^{(\beta)}(v) = \rho \frac{v^{1+\beta}}{\beta^2} \left(\beta \log v - 1 + v^{-\beta}\right) \geq 0. \tag{27}$$

To see the non-negativity, we note that for $v = 0$ the derivative is zero. Otherwise we observe that $\beta \log v - 1 + v^{-\beta} \geq 0$ is equivalent to $1 + \log v^{-\beta} \leq v^{-\beta} = \exp(\log v^{-\beta})$, which is true. Thus, $g^{(\beta)}$ is monotonously non-increasing as $\beta \downarrow 0$. By Hôspital's Rule it is further easy to see that $g^{(\beta)}$ converges pointwise to $g$ as $\beta \downarrow 0$. In order to show that the only intersection points of $g^{(\beta)}$ and $g$ are 0 and 1, where both functions are zero, we observe that for $v \neq 0$, $g^{(\beta)}(v) = g(v)$ is equivalent to $\exp(v') = 1 + v'$ where $v' = v^{\beta} - 1$. The only solution is therefore $v' = 0$, which is equivalent to $v = 1$. Incidentally, these points are the only intersection points with the $v$-axis. To see that both functions are negative on $(0, 1)$ and positive on $(1, \infty)$, consider the derivatives of the two continuous functions,

$$\frac{\partial}{\partial v} g^{(\beta)}(v) = \tfrac{\rho}{\beta}\left((1+\beta)v^{\beta} - 1\right), \qquad \frac{\partial}{\partial v} g(v) = \rho\,(1 + \log v). \tag{28}$$

Thus, the derivative at $v = 0$ is $-\frac{1}{\beta}$ for $g^{(\beta)}$ and $-\infty$ in the limit. Likewise, at $v = 1$ the derivatives are all 1. □

>From the monotonicity of the non-linear terms we obtain the following monotone convergence result for the solutions to the corresponding ordinary differential equations.

**Lemma 6 (Monotone convergence of solutions).** *For all* $0 < \lambda \leq 1$ *and* $t \geq 0$ *we have* $\bar{u}_t^{(\beta)}(\lambda) \uparrow \bar{u}_t(\lambda)$ *as* $\beta \downarrow 0$.

*Proof.* For $0 < \lambda \leq 1$ and $t \geq 0$, we obtain $0 < \bar{u}_t^{(\beta)}(\lambda) \leq 1$, and so

$$\frac{\partial}{\partial \beta} \bar{u}_t^{(\beta)}(\lambda) = \log(\lambda)\lambda^{-\beta}\beta^{-1}\mathrm{e}^{-\rho t}(\bar{u}_t^{(\beta)}(\lambda))^{1+\beta} + \beta^{-2}\log(\bar{u}_t^{(\beta)}(\lambda))\bar{u}_t^{(\beta)}(\lambda) \leq 0.$$

Thus, $\bar{u}_t^{(\beta)}(\lambda)$ is non-decreasing as $\beta \downarrow 0$. We rewrite

$$\bar{u}_t^{(\beta)}(\lambda) = \left[\left(1 + \beta\big(\mathrm{e}^{-\rho t}\tfrac{1}{\beta}(\lambda^{-\beta} - 1)\big)\right)^{\frac{1}{\beta}}\right]^{-1}, \tag{29}$$

and since $\frac{1}{\beta}(\lambda^{-\beta} - 1) \to -\log(\lambda)$, it converges to $\mathrm{e}^{\mathrm{e}^{-\rho t}\log(\lambda)} = \bar{u}_t(\lambda)$. □

As an immediate consequence, for each $t \geq 0$ fixed, $\bar{X}_t^{(\beta)}$ converges in law to $\bar{X}_t$ as $\beta \downarrow 0$, provided that $\bar{X}_0^{(\beta)} \to \bar{X}_0$ in law. So armed we can prove the following uniform moment bound.

**Lemma 7 (Uniformly bounded lower order moments).** *Assume that* $\sup_{0<\beta\le 1}\mathbb{E}[(X_0^{(\beta)})^{\theta_0}] < \infty$ *for some* $0 < \theta_0 \le 1$. *Then, for all* $T > 0$ *and* $0 < \theta < \theta_0 \mathrm{e}^{-\rho T}$,

$$\sup_{0<\beta\le 1} \mathbb{E}\left[\sup_{t\le T}(\bar{X}_t^{(\beta)})^{\theta}\right] < \infty. \tag{30}$$

*Proof.* We use the following identity (see (2.1.11) of Zolotorev (1986) [26]),

$$x^{\theta}\theta^{-1}\Gamma(1-\theta) = \int_0^{\infty} \lambda^{-\theta-1}(1-\mathrm{e}^{-x\lambda})d\lambda. \tag{31}$$

which holds for any $x \ge 0$ and $0<\theta<1$. Therefore, for $0<\theta<\theta_0\mathrm{e}^{-\rho T}$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}(\bar{X}_t^{(\beta)})^{\theta}\right] \qquad (32)\\
&\le \frac{\theta}{\Gamma(1-\theta)}\int_0^{\infty}\lambda^{-\theta-1}\mathbb{E}\left[\sup_{0\le t\le T}(1-\mathrm{e}^{-\bar{X}_t^{(\beta)}\lambda})\right]d\lambda\\
&\le c(\theta)\left(\int_1^{\infty}\lambda^{-\theta-1}d\lambda + \int_0^1 \lambda^{-\theta-1}\mathbb{E}\left[\sup_{0\le t\le T}(1-\mathrm{e}^{-\bar{X}_t^{(\beta)}\lambda})\right]d\lambda\right)\\
&\le c(\theta)\Bigg(\theta^{-1}1^{-\theta} + \int_0^1 \lambda^{-\theta-1}\\
&\qquad \times \mathbb{E}\left[\sup_{0\le t\le T}\left(1-M_t^{(\beta)}(\lambda) - \int_0^t \exp(-\bar{X}_s^{(\beta)}\lambda)\bar{X}_s^{(\beta)}g^{(\beta)}(\lambda)\right)ds\right]d\lambda\Bigg),
\end{aligned}$$

where $t\mapsto M_t^{(\beta)}(\lambda) := \exp(-\bar{X}_t^{(\beta)}\lambda) - \int_0^t \exp(-\bar{X}_s^{(\beta)}\lambda)\bar{X}_s^{(\beta)}g^{(\beta)}(\lambda)ds$ is a martingale. We can therefore estimate by the martingale inequality that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left(1-M_t^{(\beta)}(\lambda)+\int_0^t \exp(-\bar{X}_s^{(\beta)}\lambda)\bar{X}_s^{(\beta)}g^{(\beta)}(\lambda)ds\right)\right] \qquad (33)\\
&\le \mathbb{E}\left[|1-M_T^{(\beta)}(\lambda))| + \int_0^T \exp(-\bar{X}_s^{(\beta)}\lambda)\bar{X}_s^{(\beta)}|g^{(\beta)}(\lambda)|ds\right]\\
&\le \mathbb{E}\left[|1-\exp(-\bar{X}_T^{(\beta)}\lambda)|\right] + 2\int_0^T |g^{(\beta)}(\lambda)|\mathbb{E}\left[\exp(-\bar{X}_s^{(\beta)}\lambda)\bar{X}_s^{(\beta)}\right]ds. \qquad (34)
\end{aligned}$$

Here, the first expectation is by the log-Laplace relation (25) equal to

$$\mathbb{E}\left[1-\exp(-\bar{X}_0^{(\beta)}\bar{u}_T^{(\beta)}(\lambda))\right] \le \mathbb{E}\left[1-\exp(-\bar{X}_0^{(\beta)}\bar{u}_T(\lambda))\right] \tag{35}$$

by Lemma 6. This implies that we have to estimate

$$\begin{aligned}
&\int_0^1 \lambda^{-\theta-1}\mathbb{E}\left[1-\exp(-\bar{X}_0^{(\beta)}\lambda^{(\mathrm{e}^{-\rho T})})\right]d\lambda \qquad (36)\\
&= \mathrm{e}^{\rho T}\int_0^1 \tilde{\lambda}^{-\mathrm{e}^{\rho T}\theta-1}\mathbb{E}\left[1-\exp(-\bar{X}_0^{(\beta)}\tilde{\lambda})\right]d\tilde{\lambda}\\
&\le \theta^{-1}\Gamma(1-\mathrm{e}^{\rho T}\theta)\mathbb{E}\left[(\bar{X}_0^{(\beta)})^{\mathrm{e}^{\rho T}\theta}\right],
\end{aligned}$$

where we have applied the transformation of variable $\tilde{\lambda} = \lambda^{(e^{\rho T})}$ and then used relation (31). Since $e^{\rho T}\theta < \theta_0$, this quantity is bounded uniformly over $0 < \beta \leq 1$ by assumption. We now turn to the second term of (34). The expectation in this term can be transformed with the help of the log-Laplace relation (25):

$$\begin{aligned}
&\mathbb{E}\left[\exp(-\bar{X}_s^{(\beta)}\lambda)\bar{X}_s^{(\beta)}\right] = -\frac{\partial}{\partial\lambda}\mathbb{E}\left[\exp(-\bar{X}_0^{(\beta)}\bar{u}_s^{(\beta)}(\lambda))\right] \qquad (37)\\
&= \mathbb{E}\left[\exp(-\bar{X}_0^{(\beta)}\bar{u}_s^{(\beta)}(\lambda))\bar{X}_0^{(\beta)}\right]\bar{u}_s^{(\beta)}(\lambda)^{1+\beta}\lambda^{-\beta-1}e^{-\rho s}\\
&= \mathbb{E}\left[(\bar{X}_0^{(\beta)})^{\theta_0}\exp(-\bar{X}_0^{(\beta)}\bar{u}_s^{(\beta)}(\lambda))(\bar{X}_0^{(\beta)}\bar{u}_s^{(\beta)}(\lambda))^{(1-\theta_0)}\right]\bar{u}_s^{(\beta)}(\lambda)^{\theta_0+\beta}\lambda^{-\beta-1}e^{-\rho s}.
\end{aligned}$$

Since $x \mapsto \exp(-x)x^{(1-\theta_0)}$ is a bounded function on $\mathbb{R}_+$, the expectation is bounded uniformly over $0 < \beta \leq 1$ by assumption. Going back to (32) it therefore remains to bound

$$\int_0^1\int_0^T \lambda^{-\theta-1}|g^{(\beta)}(\lambda)|\bar{u}_s^{(\beta)}(\lambda)^{\theta_0+\beta}\lambda^{-\beta-1}e^{-\rho s}ds d\lambda. \qquad (38)$$

Here we use that there exists a constant $c = c(T)$ so that for all $0 \leq s \leq T$ and for all $0 < \beta, \lambda \leq 1$,

$$|g^{(\beta)}(\lambda)|\bar{u}_s^{(\beta)}(\lambda)^{\beta}\lambda^{-\beta-1} \leq c(T)|\log\lambda|. \qquad (39)$$

To see this, we note that the inequality is equivalent to

$$(\lambda^{-\beta}e^{-\rho s} + 1 - e^{-\rho s})^{-1}(\lambda^{-\beta} - 1) \leq c(T)\log\lambda^{-\beta}, \qquad (40)$$

which is true for all $s \leq T$ if it is fulfilled for $s = T$. For further simplification we note that $\lambda^{-\beta} = 1 + a$ for some $a \in \mathbb{R}_+$ and hence

$$\frac{a}{(1 + e^{-\rho T}a)\log(1+a)} \leq c(T). \qquad (41)$$

Here the left hand side is a continuous function in $a \in \mathbb{R}_+$, which tends to 1 for $a \to 0$ and to 0 for $a \to \infty$, and is thus bounded. Using (39) and Lemma 6 we can now bound (38) by

$$\begin{aligned}
&c(T)\int_0^1\int_0^T \lambda^{-\theta-1}\lambda^{\theta_0(e^{-\rho s})}|\log\lambda|e^{-\rho s}ds d\lambda\\
&\leq c(T)\int_0^1 \lambda^{\theta_0 e^{-\rho T}-\theta-1}(1 + |\log\lambda|)d\lambda, \qquad (42)
\end{aligned}$$

which is finite as required since $\theta < \theta_0 e^{-\rho T}$ by assumption. □

Asymptotic properties as $t \uparrow \infty$ of the total mass process $\bar{X}$ have been explored in the Galton-Watson setting, amongst others by Grey (1977) [13]. This lead Neveu (1992) [19] to sketch the following proposition, whose proof is given in our appendix:

**Proposition 8 (Almost sure limit of total mass process).** *For all (deterministic) initial states $m > 0$, there exists an exponentially distributed random variable $V$ with mean $1/m$, so that as $t \uparrow \infty$,*

$$\mathrm{e}^{-\rho t} \log(\bar{X}_t) \to \log(\frac{1}{V}) \quad a.s. \tag{43}$$

An interesting *open problem* is the long-term behaviour of the spatial process $X$ constructed here.

## 2.5 Log-Laplace equations

In this section we construct solutions to (9) as the limit of solutions to (12), and investigate properties needed in the proof of Theorems 2 and 3, as well as in Section 3.

**Lemma 9 (Approximating solutions).** *For each $\varphi \in C_{\mathrm{b}}^{+}(\mathbb{R}^d)$ there is a unique solution $u^{(\beta)} \in C([0,\infty), C_{\mathrm{b}}^{+}(\mathbb{R}^d))$ to the integral equation (12). If additionally $\varphi \in C_{\mathrm{b}}^{2,+}(\mathbb{R}^d)$ (contained in the domain of $\Delta_\alpha$), it belongs to $C^1((0,\infty), C_{\mathrm{b}}^{+}(\mathbb{R}^d))$ with $u_t^{(\beta)} \in C_{\mathrm{b}}^{2}(\mathbb{R}^d)$ for every $t \geq 0$, and it solves the Cauchy problem (2). All solutions $u^{(\beta)}$ satisfy*

$$(\inf_{y \in \mathbb{R}^d} \varphi(y) \wedge 1) \leq u_t^{(\beta)}(\varphi)(x) \leq (||\varphi||_\infty \vee 1), \qquad t \geq 0, \quad x \in \mathbb{R}^d. \tag{44}$$

*Also, monotonicity in the initial conditions holds, meaning that for $\varphi_1, \varphi_2$ in $C_{\mathrm{b}}^{+}(\mathbb{R}^d)$,*

$$\varphi_1 \leq \varphi_2 \quad \textit{implies} \quad u_t^{(\beta)}(\varphi_1) \leq u_t^{(\beta)}(\varphi_2), \qquad t \geq 0. \tag{45}$$

*Furthermore,*

$$\lim_{\delta \downarrow 0} \sup_{0 < \beta \leq 1} ||u_\delta^{(\beta)}(\varphi) - \varphi||_\infty = 0. \tag{46}$$

*Proof.* Let us first observe that $g^{(\beta)}$ interpreted as a function $C_{\mathrm{b}}^{+}(\mathbb{R}^d) \to C_{\mathrm{b}}^{+}(\mathbb{R}^d)$, is Lipschitz continuous, indeed it is continuously differentiable. Since $T^\alpha$ is strongly continuous on $C_{\mathrm{b}}^{+}(\mathbb{R}^d)$, Theorem 6.1.4 of Pazy (1983) [20] then implies that for any $\varphi \in C_{\mathrm{b}}^{+}(\mathbb{R}^d)$ there exist unique mild solutions $u^{(\beta)} \in C([0,t_0), C_{\mathrm{b}}^{+}(\mathbb{R}^d))$ to (12) up to a possible "explosion time" $t_0 \leq \infty$. Because $g^{(\beta)}$ is continuously differentiable we may further apply Theorem 6.1.5 of Pazy (1983) [20] in order to conclude that if $\varphi \in C_{\mathrm{b}}^{2,+}(\mathbb{R}^d)$ additionally, $u^{(\beta)} \in C^1((0,t_0), C_{\mathrm{b}}^{+}(\mathbb{R}^d))$, and that all $u_t^{(\beta)}$ belong to $C^{2,+}(\mathbb{R}^d)$, and that $u^{(\beta)}$ solves (2) up to the explosion time $t_0$.

By a probabilistic argument, we show next the bound on the solutions $u^{(\beta)}$ as in (44) implying that $t_0 = \infty$. Here, we use the monotonicity in the initial condition stated in (45), which follows from the log-Laplace representation (12). Thus, we may estimate $u^{(\beta)}$ with the $\bar{u}^{(\beta)}$ given in (22), related to the total mass process. We obtain for all $x \in \mathbb{R}^d$ and $t \geq 0$,

$$\bar{u}_t^{(\beta)}(\inf_{y \in \mathbb{R}^d} \varphi(y)) \leq u_t^{(\beta)}(\varphi)(x) \leq \bar{u}_t^{(\beta)}(||\varphi||_\infty). \tag{47}$$

Since $\bar{u}_t^{(\beta)}(||\varphi||_\infty) \downarrow 1$ for $||\varphi||_\infty \geq 1$ and $\bar{u}_t^{(\beta)}(\inf_{y\in\mathbb{R}^d}\varphi(y)) \uparrow 1$ for $0 < \inf_{y\in\mathbb{R}^d}\varphi(y) \leq 1$ as $t \uparrow \infty$, the bounds on $u^{(\beta)}$ follow.

For the uniform convergence in $t$, we consider

$$\begin{aligned} ||u_\delta^{(\beta)}(\varphi) - \varphi||_\infty &\leq ||T_\delta^\alpha \varphi - \varphi||_\infty + ||\int_0^\delta T_{t-s}^\alpha g^{(\beta)}(u_s^{(\beta)}(\varphi))ds||_\infty \\ &\leq ||T_\delta^\alpha\varphi - \varphi||_\infty + C(\varphi)\delta, \end{aligned} \tag{48}$$

where the second term has been estimated by noting that $g^{(\beta)}(v)$ is bounded uniformly over all $0 < \beta \leq 1$ and $v \in [0, 1 \vee ||\varphi||_\infty]$. The result now follows since $||T_\delta^\alpha\varphi - \varphi||_\infty \to 0$ as $\delta \downarrow 0$, by the strong continuity of the semigroup $T^\alpha$. □

**Lemma 10 (Convergence to a limiting solution).** *Take $\varphi \in C_{\mathrm{b}}^{++}(\mathbb{R}^d)$. Then there exists a unique solution $u(\varphi) \in C([0,\infty), C_{\mathrm{b}}^{++}(\mathbb{R}^d))$ to (9), which satisfies for any $T > 0$,*

$$\lim_{\beta\downarrow 0} \sup_{0\leq t\leq T} ||u_t^{(\beta)}(\varphi) - u_t(\varphi)||_\infty = 0. \tag{49}$$

*For all $t \geq 0$, the solution $u$ fulfills*

$$0 < \inf_{y\in\mathbb{R}^d}\varphi(y) \wedge 1 \leq u_t(\varphi)(x) \leq ||\varphi||_\infty \vee 1, \qquad t \geq 0, \quad x \in \mathbb{R}^d, \tag{50}$$

*and is monotone in the initial condition (see (45)). Furthermore, for $\varphi$ in $C_{\mathrm{b}}^{2,++}(\mathbb{R}^d)$ they even solve (1) and $u \in C^1((0,\infty), C_{\mathrm{b}}^{+}(\mathbb{R}^d))$ with $u_t \in C_{\mathrm{b}}^2(\mathbb{R}^d)$ for every $t \geq 0$.*

*Proof.* Solutions to (12) with initial condition $\varphi \in C_{\mathrm{b}}^{++}(\mathbb{R}^d)$ are bounded below and above according to (50) of Lemma 9. We can therefore estimate for $0 < \beta_1 \leq \beta_2 \leq 1$,

$$\begin{aligned} |u_t^{(\beta_1)} - u_t^{(\beta_2)}| &= \left|\int_0^t T_{t-s}^\alpha \left(g^{(\beta_2)}(u_s^{(\beta_2)}) - g^{(\beta_1)}(u_s^{(\beta_1)})(x)\right) ds\right| \\ &\leq \int_0^t T_{t-s}^\alpha \left|g^{(\beta_2)}(u_s^{(\beta_2)}) - g^{(\beta_2)}(u_s^{(\beta_1)})\right|(x)ds \\ &\quad + \int_0^t T_{t-s}^\alpha \left|g^{(\beta_2)}(u_s^{(\beta_1)}) - g^{(\beta_1)}(u_s^{(\beta_1)})\right|(x)ds \\ &\leq C(\beta_1,\varphi)\int_0^t ||u_s^{(\beta_2)} - u_s^{(\beta_1)}||_\infty ds + \delta(\beta_1,\beta_2,\varphi)t. \end{aligned} \tag{51}$$

Here, we have set

$$C(\beta,\varphi) := \sup_{\inf_{y\in\mathbb{R}^d}\varphi(y)\wedge 1\leq v\leq ||\varphi||_\infty \vee 1} \frac{\partial g^{(\beta)}}{\partial v}(v) < \infty, \tag{52}$$

$$\delta(\beta_1,\beta_2,\varphi) := \sup_{\inf_{y\in\mathbb{R}^d}\varphi(y)\wedge 1\leq v\leq ||\varphi||_\infty \vee 1} |g^{(\beta_2)}(v) - g^{(\beta_1)}(v)| < \infty. \tag{53}$$

We now note that $\frac{\partial g^{(\beta)}}{\partial v}$ converges to $\frac{\partial g}{\partial v}$, uniformly on compact intervals in $(0,\infty)$ as $\beta \downarrow 0$, and hence $\sup_{0<\beta\leq 1} C(\beta,\varphi) < \infty$. Likewise, $g^{(\beta)}$ converges to $g$, uniformly on compacta in $(0,\infty)$ and thus $\sup_{\beta_1,\beta_2\leq\epsilon} \delta(\beta_1,\beta_2,\varphi) \to 0$ as $\epsilon \downarrow 0$. But by Gronwall's Inequality,

$$\sup_{t\leq T} ||u_t^{(\beta_2)} - u_t^{(\beta_1)}||_\infty \leq \delta(\beta_1,\beta_2,\varphi) T \exp(C(\beta_1,\varphi)T), \tag{54}$$

and so $(u^{(\beta_n)})_{n\geq 1}$ with $\beta_n \downarrow 0$ form a Cauchy sequence on $C([0,T], C_{\mathrm{b}}^{++}(\mathbb{R}^d))$. Of course, the limit, which we call $u$, also fulfills (50) as well as monotonicity in the initial condition (see (45)). We can therefore repeat essentially the same arguments as in (51) to show that

$$\lim_{\beta\downarrow 0} \sup_{t\leq T} \left\| \int_0^t T_{t-s}^\alpha \left| g(u_s) - g^{(\beta)}(u_s^{(\beta)}) \right| ds \right\|_\infty = 0. \tag{55}$$

Hence, $u$ satisfies (9). Because of the boundedness away from 0 we are securely in the Lipschitz region of $g$. Thus, the same arguments concerning further regularity for initial conditions $\varphi \in C_{\mathrm{b}}^{2,++}(\mathbb{R}^d)$ as detailed in the proof of Lemma 9 apply. This concludes the existence part of the lemma.

It remains to show uniqueness of solutions. We first note that for any solution $u(\varphi)$ to (9) with $\varphi \in C_{\mathrm{b}}^{++}(\mathbb{R}^d)$ there exists a $t_0 > 0$ so that $u_t(\varphi)(x) \geq \frac{1}{2}\inf_{y\in\mathbb{R}^d} \varphi(y) > 0$ for all $t \leq t_0$ and $x \in \mathbb{R}^d$. Indeed, $u$ is bounded above, $u_t(x) \leq ||\varphi||_\infty + t \sup_{v\in[0,||\phi||_\infty]} g(v) \leq C(T)$ for $t \leq T$, where we choose $C(T) > 1$. Thus, on $[0,T]$, we can bound $u$ from below, $u_t(x) \geq \inf_y \varphi(y) - g(C(T))t$ so that we can choose a positive $t_0 \leq (g(C(T)))^{-1}(\frac{1}{2}\inf_y \varphi(y)) \wedge T$.

The nonlinearity $g$ is Lipschitz continuous on compact intervals of $(0,\infty)$ so that uniqueness on $[0,t_0]$ follows by Gronwall's Inequality. Thus, the solution on $[0,t_0]$ must be the one that we constructed above, which is in fact bounded below by $\inf_{y\in\mathbb{R}^d}\varphi(y)$. Hence, we can reiterate the same argument to see that uniqueness must hold on any arbitrary time interval, and that $u \in C([0,\infty), C_{\mathrm{b}}^{++}(\mathbb{R}^d))$. □

**Lemma 11 (Comparison of solutions).** *Fix $0 < \beta_1 \leq \beta_2 \leq 1$, and $\varphi$ in $C_{\mathrm{b}}^{+}(\mathbb{R}^d)$ so that $\varphi(x) = 1$ for all $|x| > c$, where $c > 0$ is some constant. We obtain that $u_t^{(\beta_2)}(\varphi)(x) \leq u_t^{(\beta_1)}(\varphi)(x)$, $x \in \mathbb{R}^d$ and $t \geq 0$. In particular, for all $\varphi \in C_{\mathrm{b}}^{++}(\mathbb{R}^d)$, we have $\sup_{0<\beta\leq 1} u_t^{(\beta)}(\varphi)(x) \leq u_t(\varphi)(x)$, $x \in \mathbb{R}^d$.*

*Proof.* The proof is an adaptation of standard arguments, see for example Theorem 10.1 of Smoller (1983) [23]. Let us first assume that $\varphi \in C_{\mathrm{b}}^{2,+}(\mathbb{R}^d)$. We define (the possibly signed function) $v_t(x) := u_t^{(\beta_1)}(x) - u_t^{(\beta_2)}(x)$, which then satisfies according to Lemma 9,

$$\left.\begin{array}{rcl} \frac{\partial}{\partial t} v_t(x) &=& \Delta_\alpha v_t(x) - g^{(\beta_1)}(u_t^{(\beta_1)}(x)) + g^{(\beta_2)}(u_t^{(\beta_2)}(x)), \\ v_0(x) &=& 0. \end{array}\right\} \tag{56}$$

Let $f \in C([0,T], C_{\mathrm{b}}^{+}(\mathbb{R}^d))$ be defined by $f_t = -g^{(\beta_1)}(u_t^{(\beta_2)}) + g^{(\beta_2)}(u_t^{(\beta_2)})$ (recall that the $g^{(\beta)}$ are non-decreasing in $\beta$). Then, for some $\xi(t,x)$ between $u_t^{(\beta_1)}(x)$ and $u_t^{(\beta_2)}(x)$,

$$-g^{(\beta_1)}(u_t^{(\beta_1)}(x)) + g^{(\beta_2)}(u_t^{(\beta_2)}(x)) = -(g^{(\beta_1)})'(\xi(t,x))v_t(x) + f_t(x) \qquad (57)$$

(with $(g^{(\beta)})'$ denoting the derivative of $g^{(\beta)}$ ). Note that the following double supremum $\sup_{t\in\mathbb{R}_+}\sup_{x\in\mathbb{R}^d}(g^{(\beta_1)})'(\xi(t,x))$ is finite because $g^{(\beta_1)}$ is locally Lipschitz and $\xi$ is bounded uniformly over all $t$ and $x$. The latter follows from the boundedness of the solutions from above and below as stated in Lemma 9, which also implies the uniform boundedness of $v$. Thus, we can find some constant $R \in \mathbb{R}_+$ so that $-(g^{(\beta_1)})'(\xi(t,x))v_t(x) + R > 0$ for all $t$ and $x$. Therefore, $\tilde{v}_t := \mathrm{e}^{Rt}v_t$ satisfies

$$\left.\begin{aligned} \frac{\partial}{\partial t}v_t(x) &= \Delta_\alpha \tilde{v}_t(x) + (-(g^{(\beta_1)})'(\xi(t,x))\tilde{v}_t(x) + R) + f_t(x)\mathrm{e}^{Rt}, \\ \tilde{v}_0(x) &= 0. \end{aligned}\right\} \qquad (58)$$

Let $T > 0$ and suppose that $\tilde{v}_t(x) < 0$ for some $(t,x) \in [0,T]\times\mathbb{R}^d$. Then, $\tilde{v}$ must attain a negative minimum on $[0,T]\times\mathbb{R}^d$ in some point $(t_{\min}, x_{\min})$. This follows from the fact that for any $t\in[0,T]$, $\tilde{v}_t(x) \to 0$ for $|x| \uparrow \infty$. To see this note that for the initial conditions $\varphi$, considered here, $u_t^{(\beta_i)}(x) \to 1$ for $|x|\uparrow\infty$ $(i = 1,2)$ since, using the mild form of the solutions and the monotonicity in the initial condition,

$$\begin{aligned} T_t^\alpha(\varphi\wedge 1)(x) \;\le\; u_t^{(\beta_i)}(\varphi\wedge 1)(x) \;&\le\; u_t^{(\beta_i)}(\varphi)(x) \\ \le\; u_t^{(\beta_i)}(\varphi\vee 1)(x) \;&\le\; T_t^\alpha(\varphi\vee 1)(x), \end{aligned} \qquad (59)$$

where we have used Lemma 5. The lower and upper bounds converge appropriately to 1.

At the minimum $(t_{\min}, x_{\min})$ we would have that $\frac{\partial}{\partial t}v_{t_{\min}}(x_{\min}) \le 0$ as well as $\Delta_\alpha v_{t_{\min}}(x_{\min}) > 0$ by the positive maximum principle (see Theorem 2.2 Chapter 4 of Ethier and Kurtz (1986) [7]). Recalling the choice of $R$ and $f$ we obtain a contradiction to the equality in (58), and therefore may conclude that $v$ is indeed non-negative on $[0,T]\times\mathbb{R}^d$ and so also on $\mathbb{R}_+\times\mathbb{R}^d$. Finally, to obtain the same statement for $\varphi \in C_{\mathrm{b}}^{+}(\mathbb{R}^d)$ we use the fact that there exists a sequences $\varphi_n \in C_{\mathrm{b}}^{2,+}(\mathbb{R}^d)$ such that $||\varphi - \varphi_n||_\infty \to 0$ as $n\uparrow\infty$. Arguments analogous to those in (51) to (54) then show immediately that $||u^{(\beta_i)}(\varphi) - u^{(\beta_i)}(\varphi_n)||_\infty \to 0$, and so we are done. The convergence in (49) stated in Lemma 10 now finishes the proof. □

In order to show that mass does not escape to infinity, we need to consider the behaviour of $u$ started from *runaway test functions* $r_k$, $k\ge 1$. We first define an auxiliary function $r_k^{(\epsilon)}$ for some $0 < \epsilon < \frac{1}{2}$ by

$$r_k^{(\epsilon)}(x) \tag{60}$$

$$:= \begin{cases} \frac{1}{k} & \text{for} \quad |x| \leq k+\epsilon, \\ (\frac{1-k^{-1}}{1-2\epsilon})|x| + (\frac{-k+1-\epsilon+(1-\epsilon)k^{-1}}{1-2\epsilon}) & \text{for} \quad k+\epsilon < |x| \leq k+1-\epsilon, \\ 1 & \text{for} \quad |x| > k+1-\epsilon. \end{cases}$$

In short, $r_k^{(\epsilon)}$ is radially symmetric and linearly non-decreasing in $|x|$ between its two constant values $\frac{1}{k}$ and 1. Note also that $r_k^{(\epsilon)}$ is monotonously non-increasing in $k$. Now let $\Phi \in C^{\infty,+}(\mathbb{R}^d)$ with support in $B(0,\epsilon)$, the open ball around 0 with $\epsilon$ radius, and so that $\int_{\mathbb{R}^d} \Phi(x)dx = 1$. We then define

$$r_k(x) := \int_{\mathbb{R}^d} \Phi(x-y)\, r_k^{(\epsilon)}(y)\, dy, \tag{61}$$

as the mollification of $r_k^{(\epsilon)}$. As an immediate consequence of the properties of $r_k^{(\epsilon)}$, we obtain that $r_k$ belongs to $C^{\infty,++}(\mathbb{R}^d)$, is also radially symmetric, monotonously non-increasing in $k$, and that it is constantly $\frac{1}{k}$ (respectively 1) for $|x| \leq k$ (respectively $|x| \geq k+1$).

**Lemma 12 (Runaway solutions).** *We have $u_t(r_k)(x) \downarrow 0$ as $k \uparrow \infty$, for any $0 \leq t < \rho^{-1}$ and $x \in \mathbb{R}^d$. The same statement holds for $r_k$ replaced by $|\Delta_\alpha r_k| \vee r_k$ and $|g(r_k)| \vee r_k$.*

*Proof.* Let $t \geq 0$. We note that $u_t(r_k)(x)$ is monotonously non-increasing in $k$ for every $x$, and bounded below by zero, so that a pointwise limit exists, which we call $u_t(r_\infty)(x)$. From the radial symmetry in the definition of $r_k$ as well as in the equation (9) we can immediately observe that, for all $k$, $u_t(r_k)(0) = \min_{x\in\mathbb{R}^d} u_t(r_k)(x)$.
Now consider a positive test function $\psi \in C_b^{2,++}(\mathbb{R}^d)$ with $\psi(x) = \exp(-|x|)$ for $|x| \geq 1$. We will first show that there exists a constant $\kappa = \kappa(\alpha) > 0$, such that

$$\Delta_\alpha \psi(x) \leq \kappa \psi(x), \tag{62}$$

for all $x \in \mathbb{R}^d$. Indeed, for $\alpha = 2$ this follows from the fact that $\Delta\psi(x) = (1 - \frac{d-1}{|x|})\psi(x) \leq \psi(x)$ for all $|x| \geq 1$. For $0 < \alpha < 2$, we use the well-known representation (see, for example, (5) of Section IX.11 in Yosida (1980) [25]),

$$\Delta_\alpha \psi(x) = \frac{1}{\Gamma(-\alpha/2)} \int_0^\infty s^{-1-\alpha/2} \big[\psi(x) - T_s^{(2)}\psi(x)\big]\, ds \tag{63}$$

where $\Gamma$ is Euler's Gamma function. Thus, we obtain

$$\Delta_\alpha \psi(x) \leq c \int_0^1 s^{-1-\alpha/2} \big[\psi(x) - T_s^{(2)}\psi(x)\big]\, ds + c\psi(x) \int_1^\infty s^{-1-\alpha/2} ds. \tag{64}$$

Here, the integral of the second term is finite. The first term can be estimated by Taylor's Formula,

$$\begin{aligned}
&\int_0^1 s^{-1-\alpha/2}\big[\psi(x) - T_s^{(2)}\psi(x)\big]\,ds \qquad (65)\\
&\le \sup_{0\le s<1} T_s^{(2)}\Delta\psi(x)\int_0^1 s^{-1-\alpha/2}s\;ds \;\le\; c\psi(x),
\end{aligned}$$

where, in the second inequality, we have used (62) for $\alpha = 2$ together with the well known fact that

$$\sup_{0\le s\le 1} T_s^{(2)}\psi \le c\psi. \qquad (66)$$

It is well known that the mild solution $u$ is also a solution in the weak form for an appropriate class of test functions including our $\psi$. Thus, we obtain for any $t \ge 0$,

$$\begin{aligned}
\langle u_t(r_k),\psi\rangle &= \langle r_k,\psi\rangle + \int_0^t \langle u_s(r_k), \Delta_\alpha\psi - (\rho\log u_s(r_k))\psi\rangle ds \qquad (67)\\
&\le \langle r_k,\psi\rangle + (\kappa+\rho\log k)\int_0^t \langle u_s(r_k),\psi\rangle ds.
\end{aligned}$$

Note that $\frac{1}{k} = u_0(r_k)(0) \le u_t(r_k)(x) \le 1$ implies that $-\log u_t(r_k)(x) \le \log k$. Using (62) we can therefore apply Gronwall's Inequality in order to obtain for all $t \ge 0$,

$$\begin{aligned}
\langle u_t(r_k),\psi\rangle &\le \Big(\int_{\mathbb{R}^d} r_k(x)\psi(x)dx\Big)\,\mathrm{e}^{(\kappa+\rho\log k)t} \qquad (68)\\
&\le \Big(\int_{|x|<k}\tfrac{1}{k}\psi(x)dx + \int_{|x|\ge k}\psi(x)dx\Big)\mathrm{e}^{\kappa t}k^{\rho t}\\
&\le \big(ck^{\rho t-1} + c(d)k^{\rho t+d-1}\mathrm{e}^{-k}\big)\,\mathrm{e}^{\kappa t},
\end{aligned}$$

provided that $k \ge 1$. The expression converges to zero as $k\uparrow\infty$ for all $t < \rho^{-1}$. This implies that for $t < \rho^{-1}$,

$$\int_{\mathbb{R}^d} u_t(r_\infty)(x)\psi(x)dx = 0. \qquad (69)$$

Taken together with the monotonicity of the limit in $|x|$ we obtain $u_t(r_\infty)(x) = 0$ for all $t < \rho^{-1}$ and $x \in \mathbb{R}^d$.

The statement of the lemma for $|\Delta_\alpha r_k| \vee r_k$ and $|g(r_k)| \vee r_k$ in $C^{++}(\mathbb{R}^d)$ follows by repeating the same line of arguments. The estimates of (67) hold true unchanged since both initial conditions are still bounded below by $\frac{1}{k}$ which is hence also true for the solutions $u$. The only changes in the calculations given in (68) occur thus in the estimates of the initial condition. Since $\sup_k |\Delta_\alpha r_k| \vee r_k \le c < \infty$, we now estimate

$$\int_{\mathbb{R}^d}(|\Delta_\alpha r_k| \vee r_k)(x)\psi(x)dx) \le \int_{|x|<k}\tfrac{1}{k}\psi(x)dx + c\int_{|x|\ge k}\psi(x)dx, \qquad (70)$$

with the additional constant $c$ being inconsequential in the following. Because $\sup_x |g(r_k(x))| \vee r_k = \sup_{0\le a\le 1} g(a) \le c < \infty$, we estimate in this case,

$$\int_{\mathbb{R}^d} (|g(r_k)| \vee r_k)(x)\psi(x)dx) \le \int_{|x|<k} \tfrac{1}{k}(\log k)\psi(x)dx + c\int_{|x|\ge k} \psi(x)dx. \quad (71)$$

The constant in the second integral on the right hand side is once again unimportant. The first term now leads to $k^{\rho t-1}\log k$ (instead of $k^{\rho t-1}$), which still converges to zero for $t < \rho^{-1}$. □

## 2.6 Tightness of the one-dimensional processes

In order to show part (a) of Proposition 4, we use Kurtz' criterion for tightness, see Ethier and Kurtz (1986) [7] Theorem 3.8.6. The compact containment condition for a given time $t$ is already implied by Lemma 7. It thus suffices to verify that for $1 > \delta \downarrow 0$,

$$\sup_{0<\beta\le 1} \sup_{t\le T} \mathbb{E}\left[\left(\left|\langle X_{t+\delta}^{(\beta)}, \varphi\rangle - \langle X_t^{(\beta)}, \varphi\rangle\right| \wedge 1\right)^2\right] \to 0. \quad (72)$$

For each $m, \beta, t$ we define the event $A^{m,\beta,t} := \left\{\langle X_t^{(\beta)}, \varphi\rangle \ge m\right\}$. We then bound the quantity in (72) by

$$\begin{aligned} &2 \sup_{0<\beta\le 1} \sup_{t\le T+1} \mathbb{P}\left[A^{m,\beta,t}\right] \qquad (73) \\ &+ c(m) \sup_{0<\beta\le 1} \sup_{t\le T} \mathbb{E}\left[\left|\exp\langle X_{t+\delta}^{(\beta)}, -\varphi\rangle - \exp\langle X_t^{(\beta)}, -\varphi\rangle\right|^2\right]. \end{aligned}$$

Note that the first term converges to zero as $m \uparrow \infty$ because of Lemma 7. Now let $0 < \theta < \theta_0 e^{-\rho T}$. Then there exists a constant $c(\theta)$ so that, for all $x, y \ge 0$, we have

$$\left|\exp(-x) - \exp(-y)\right| \le c(\theta)\,|x-y|^\theta. \quad (74)$$

With this in mind we bound the expectation in the second line of (73) by

$$\begin{aligned} &\left|\mathbb{E}\left[\exp\langle X_{t+\delta}^{(\beta)}, -2\varphi\rangle - \exp(\langle X_{t+\delta}^{(\beta)}, -\varphi\rangle + \langle X_t^{(\beta)}, -\varphi\rangle)\right]\right| \\ &\quad + \left|\mathbb{E}\left[\exp\langle X_t^{(\beta)}, -2\varphi\rangle - \exp(\langle X_{t+\delta}^{(\beta)}, -\varphi\rangle + \langle X_t^{(\beta)}, -\varphi\rangle)\right]\right| \\ &\le \left|\mathbb{E}\left[\exp\langle X_t^{(\beta)}, -u_\delta^{(\beta)}(2\varphi)\rangle - \exp\langle X_t^{(\beta)}, -u_\delta^{(\beta)}(\varphi) - \varphi\rangle\right]\right| \\ &\quad + \left|\mathbb{E}\left[\exp\langle X_t^{(\beta)}, -2\varphi\rangle - \exp\langle X_t^{(\beta)}, -u_\delta^{(\beta)}(\varphi) - \varphi\rangle\right]\right| \\ &\le c(\theta)\mathbb{E}\left[\langle X_t^{(\beta)}, |u_\delta^{(\beta)}(2\varphi) - u_\delta^{(\beta)}(\varphi) - \varphi|\rangle^\theta\right] + \mathbb{E}\left[\langle X_t^{(\beta)}, |u_\delta^{(\beta)}(\varphi) - \varphi|\rangle^\theta\right] \\ &\le c(\theta)\left(||u_\delta^{(\beta)}(2\varphi) - 2\varphi||_\infty^\theta + 2||u_\delta^{(\beta)}(\varphi) - \varphi||_\infty^\theta\right) \mathbb{E}\left[\langle X_t^{(\beta)}, 1\rangle^\theta\right]. \qquad (75) \end{aligned}$$

Here, we have used conditioning and the Markov property of the processes as well as the log-Laplace relation (15). Since (75) converges to zero uniformly over $0 < \beta \leq 1$ and $t \leq T$ by Lemma 7 and Lemma 9 we obtain (72).

## 2.7 Compact containment and convergence

In this section, we show part (b) of Proposition 4, thus establishing tightness in law. Convergence then follows by identifying the unique limit of any convergent subsequence.

According to the characterisation of compact sets in $M_{\mathrm{f}}$ (see Kallenberg (1976) [16] A 7.5), claim (b) is implied by the following two statements:

(i) For all $\epsilon > 0$ there exists an $N_\epsilon \geq 1$ so that

$$\sup_{0<\beta\leq 1} \mathbb{P}\left[\sup_{0\leq t\leq T} \bar{X}_t^{(\beta)} > N_\epsilon\right] < \epsilon. \tag{76}$$

(ii) For all $\epsilon > 0$ there exists a $k_\epsilon$ such that for the Borel set $A_{k_\epsilon} := \{x \in \mathbb{R}^d : |x| > k_\epsilon\}$,

$$\sup_{0<\beta\leq 1} \mathbb{P}\left[\sup_{0\leq t\leq T} X_t^{(\beta)}(A_{k_\epsilon}) > \epsilon\right] < \epsilon. \tag{77}$$

We remark that (i) is satisfied according to Lemma 7. For condition (ii) consider the test function $r_k \in C^{\infty,++}(\mathbb{R}^d)$ defined in (61), which has been chosen so that $r_k \geq 1_{A_k}$. Thus, it suffices to show (77) with $A_{k_\epsilon}$ replaced by $r_{k_\epsilon}$.

We will first show the statement in (77) for $T = \tilde{t} < \rho^{-1}$ since we want to use Lemma 12. For each $K \geq 1$, we define a stopping time $\tau_K = \tau_K(k,\beta) := \inf\left\{t \geq 0 : \left\langle X_t^{(\beta)}, |\Delta_\alpha r_k| + |g(r_k)|\right\rangle \geq K\right\}$. For each sample $\omega$, either $\tau_K \leq T$ or $\tau_K > T$, hence we can make the following estimate involving the process stopped at $\tau_K$ :

$$\mathbb{P}\left[\sup_{0\leq t\leq T} \langle X_t^{(\beta)}, r_k\rangle > \epsilon\right] \leq \mathbb{P}\left[\tau_K \leq T\right] + \mathbb{P}\left[\sup_{0\leq t\leq T} \left\langle X_{t\wedge\tau_K}^{(\beta)}, r_k\right\rangle > \epsilon\right]. \tag{78}$$

Since there is a constant $c$ independent of $k$ so that $|\Delta_\alpha r_k| + |g(r_k)| < c$, Lemma 7 implies that as $K \uparrow \infty$,

$$\sup_{0<\beta\leq 1}\sup_{k\geq 1} \mathbb{P}\left[\tau_K(k,\beta) \leq T\right] \leq \sup_{0<\beta\leq 1} \mathbb{P}\left[\sup_{0\leq t\leq T} \bar{X}_t^{(\beta)} \geq \tfrac{K}{c}\right] \to 0. \tag{79}$$

In order to deal with the second probability in (78), we define the martingale

$$\begin{aligned} M_t^{(\beta)}(r_k) := {} & \exp\langle X_t^{(\beta)}, -r_k\rangle \\ & - \int_0^t \exp\langle X_s^{(\beta)}, -r_k\rangle \left\langle X_s^{(\beta)}, -\Delta_\alpha r_k + g^{(\beta)}(r_k)\right\rangle ds. \end{aligned} \tag{80}$$

Thus, the stopped process, defined by

$$M_t^{(\beta,\tau_K)}(r_k) := \exp\langle X_{t\wedge\tau_K}^{(\beta)}, -r_k\rangle - \int_0^{t\wedge\tau_K} \exp\langle X_s^{(\beta)}, -r_k\rangle \Big\langle X_s^{(\beta)}, -\Delta_\alpha r_k + g^{(\beta)}(r_k)\Big\rangle ds, \tag{81}$$

is also a martingale. For some $\epsilon' > 0$, the second term in (78) is equal to

$$\begin{aligned}
&\mathbb{P}\left[\sup_{0\le t\le T}\Big(1-\exp\langle X_{t\wedge\tau_K}^{(\beta)}, -r_k\rangle\Big) > \epsilon'\right] \\
&= \mathbb{P}\bigg[\sup_{0\le t\le T}\Big(1 - M_t^{(\beta,\tau_K)}(r_k) \\
&\qquad - \int_0^{t\wedge\tau_K} \exp\langle X_s^{(\beta)}, -r_k\rangle \langle X_s^{(\beta)}, -\Delta_\alpha r_k + g^{(\beta)}(r_k)\rangle ds\Big) > \epsilon'\bigg] \\
&\le \frac{1}{\epsilon'}\mathbb{E}\bigg[\sup_{0\le t\le T}\big(1 - M_t^{(\beta,\tau_K)}(r_k)\big) \\
&\qquad + \sup_{0\le t\le T}\int_0^{t\wedge\tau_K} \exp\langle X_s^{(\beta)}, -r_k\rangle \Big\langle X_s^{(\beta)}, \big| -\Delta_\alpha r_k + g^{(\beta)}(r_k)\big|\Big\rangle ds\bigg] \\
&\le \frac{1}{\epsilon'}\bigg(\mathbb{E}\Big[\big|1 - M_T^{(\beta,\tau_K)}(r_k)\big|\Big] \\
&\qquad + \mathbb{E}\Big[\int_0^{T\wedge\tau_K} \exp\langle X_s^{(\beta)}, -r_k\rangle \Big\langle X_s^{(\beta)}, \big| -\Delta_\alpha r_k + g^{(\beta)}(r_k)\big|\Big\rangle ds\Big]\bigg) \\
&\le \frac{1}{\epsilon'}\bigg(\mathbb{E}\Big[1 - \exp\langle X_T^{(\beta,\tau_K)}, -r_k\rangle\Big] \\
&\qquad + 2\mathbb{E}\Big[\int_0^{T\wedge\tau_K} \exp\langle X_s^{(\beta)}, -r_k\rangle \langle X_s^{(\beta)}, |\Delta_\alpha r_k| + |g(r_k)|\rangle ds\Big]\bigg).
\end{aligned} \tag{82}$$

Consider the first expectation of the last expression. It is bounded by

$$\mathbb{P}\left[\tau_K \le T\right] + \mathbb{E}\Big[1 - \exp\langle X_0^{(\beta)}, -u_T^{(\beta)}(r_k)\rangle\Big]. \tag{83}$$

By (79), the probability term becomes small uniformly in $\beta$ and $k$ as $K \uparrow \infty$. The rest of the expression can be bounded by $\mathbb{E}[1 - \exp\langle X_0^{(\beta)}, -u_T(r_k)\rangle]$ by Lemma 11. As $u_T(r_k) \le 1$, the expectation converges to zero as $k \uparrow \infty$ for each $\beta$ by Lemma 12 and Lebesgue's Dominated Convergence Theorem. Furthermore $X_0^{(\beta)} \Rightarrow X_0$ and the convergence of $u_T(r_k) \downarrow 0$ is monotone in $k$ yielding convergence of the expectation uniformly over all $0 < \beta \le 1$.

Using the fact that $\sup_{0\le a\le K} a(1-\exp(-a))^{-1} =: c(K) < \infty$ the expectation in the last line of the array (82) is bounded by

$$c(K)\int_0^T \mathbb{E}\left[1-\exp\langle X_s^{(\beta)}, -|\Delta_\alpha r_k| - |g(r_k)|\rangle\right] ds \tag{84}$$
$$\le c(K)\int_0^T \mathbb{E}\left[1-\exp\left\langle X_0^{(\beta)}, -u_s^{(\beta)}\Big(\big(|\Delta_\alpha r_k|\vee r_k\big)+\big(|g(r_k)|\vee r_k\big)\Big)\right\rangle\right] ds$$
$$\le c(K)\int_0^T \mathbb{E}\left[1-\exp\left\langle X_0^{(\beta)}, -u_s\Big(\big(|\Delta_\alpha r_k|\vee r_k\big)+\big(|g(r_k)|\vee r_k\big)\Big)\right\rangle\right] ds.$$

Here, we have exploited the log-Laplace representation (15) and the monotonicity of $u_s^{(\beta)}$ in the initial condition in the first inequality, as well as Lemma 11 in the second inequality. Again, by Lemma 12 together with the convergence of $X_0^{(\beta)}$ to $X_0$ and the uniform boundedness of the solutions in $k$, we obtain $\beta$-uniform convergence of the integrand to zero as $k\uparrow\infty$ for each $s\le T$. Since the integrand is bounded by 1 a further application of Lebesgue's Dominated Convergence Theorem leads to the appropriate convergence of the entire expression.

Thus, we can finally conclude that there exists a $k_\epsilon$ such that the left hand side of (78) is smaller than $\epsilon$ for all $\beta$. First, choose $K$ large enough keeping in mind (79) and then $k_\epsilon$ large enough. This concludes the proof of (77) and hence of claim (ii) for $T=\tilde t<\rho^{-1}$. Taken together with (a) of Proposition 4 we obtain tightness in law in $D([0,\tilde t], M_{\rm f})$. We show subsequently that any subsequence convergent in law on the space $D([0,T], M_{\rm f})$, denoted by $X^{(\beta_n)}$, where $\beta_n\downarrow 0$ as $n\uparrow\infty$, converges to a unique limit $X$ that satisfies (17). It suffices to identify the finite dimensional distributions of $X$. As $\{\langle\,\cdot\,,\varphi\rangle : \varphi\in C_{\rm b}^{++}(\mathbb{R}^d)\}$ is separating in $M_{\rm f}$, any $X_t\in M_{\rm f}$ can be characterised by $\langle X_t,\varphi\rangle$ for $\varphi\in C_{\rm b}^{++}(\mathbb{R}^d)$.

For $m\in\mathbb{N}$, let $0\le t_1\le\cdots\le t_m\le T$, as well as $\varphi_i\in C_{\rm b}^{++}(\mathbb{R}^d)$ $(1\le i\le m)$ and define recursively

$$u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m) := u_{t_1,\dots,t_{m-1}}(\varphi_1,\dots,\varphi_{m-1}+u_{t_m-t_{m-1}}(\varphi_m)). \tag{85}$$

Analogously, we define $u^{(\beta)}_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)$ and note that by the Markov property and (15),

$$\mathbb{E}\left[\prod_{i=1}^m \exp\langle X_{t_i}^{(\beta)}, -\varphi_i\rangle\right] = \mathbb{E}\left[\exp\langle X_0^{(\beta)}, -u^{(\beta)}_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)\rangle\right]. \tag{86}$$

We can further show that as $n\to\infty$,

$$\left\|u^{(\beta_n)}_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m) - u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)\right\|_\infty \to 0. \tag{87}$$

This follows by induction using (44) and (50) upon noting that for any sequence $(\varphi_n)_{n\ge1}$ with $0<c_1\le\varphi_n\le c_2<\infty$ continuous and with $||\varphi_n-\varphi||_\infty\to 0$ we have $||u_t^{(\beta_n)}(\varphi_n) - u_t(\varphi)||_\infty \to 0$ for any $t \geq 0$. To see this, consider that the expression is bounded by

$$||u_t^{(\beta_n)}(\varphi_n) - u_t(\varphi_n)||_\infty + ||u_t(\varphi_n) - u_t(\varphi)||_\infty,$$

where the first term converges to zero as in (51) to (54) in the proof of Lemma 10. The convergence to zero of the second term uses $||T_t^\alpha \varphi_n - T_t^\alpha \varphi||_\infty \to 0$ along with similar arguments. We may now conclude that

$$\begin{aligned}
&\left|\mathbb{E}\Big[\prod_{i=1}^{m} \exp\langle X_{t_i}, -\varphi_i\rangle - \exp\big\langle X_0, -u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)\big\rangle\Big]\right| \qquad (88)\\
&= \lim_{n\uparrow\infty}\left|\mathbb{E}\Big[\prod_{i=1}^{m} \exp\big\langle X_{t_i}^{(\beta_n)}, -\varphi_i\big\rangle - \exp\big\langle X_0, -u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)\big\rangle\Big]\right|\\
&= \lim_{n\uparrow\infty}\left|\mathbb{E}\Big[\exp\big\langle X_0^{(\beta_n)}, -u_{t_1,\dots,t_m}^{(\beta_n)}(\varphi_1,\dots,\varphi_m)\big\rangle - \exp\big\langle X_0, -u_t(\varphi)\big\rangle\Big]\right|\\
&\leq \lim_{n\uparrow\infty}\Bigg(c(\theta_0)\mathbb{E}\Big[\Big\langle X_0^{(\beta_n)}, \big|u_{t_1,\dots,t_m}^{(\beta_n)}(\varphi_1,\dots,\varphi_m) - u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)\big|\Big\rangle^{\theta_0}\Big]\\
&\qquad + \bigg|\mathbb{E}\Big[\exp\big\langle X_0^{(\beta_n)}, -u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)\big\rangle\\
&\qquad\qquad - \exp\big\langle X_0, -u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)\big\rangle\Big]\bigg|\Bigg),
\end{aligned}$$

where we have used (86) in the second equality and (74) in the third inequality. Both terms in the last expression converges to 0 as $n \to \infty$, the first by (87) since $\sup_n \mathbb{E}[\langle X_0^{\beta_n}, 1\rangle^{\theta_0}] < \infty$ by assumption. But $u_{t_1,\dots,t_m}(\varphi_1,\dots,\varphi_m)$ is unique and so any limit point of the $(X^{(\beta)})_{0<\beta\leq 1}$ equals the unique process $X$ satisfying (17) on $D([0,T], M_{\mathrm{f}})$ for $\varphi \in C_{\mathrm{b}}^{++}(\mathbb{R}^d)$.

We can now reiterate these arguments in order to lift the restriction of the assumption $T = \tilde{t} < \rho^{-1}$. From the above, we know that $X_{\tilde{t}}^{(\beta)} \Rightarrow X_{\tilde{t}}$, and from Lemma 7 we obtain $\sup_{0<\beta\leq 1}\mathbb{E}[\langle X_{\tilde{t}}^{(\beta)}, 1\rangle^{\theta}] < \infty$ for any $0 < \theta < \theta_0 \mathrm{e}^{\rho\tilde{t}}$. Thus, we can apply the same arguments to the process started at $\tilde{t}$ which converges again on $D([0,\tilde{t}\,], M_{\mathrm{f}})$. This implies convergence of the processes started at time 0 on $D([0,2\tilde{t}\,], M_{\mathrm{f}})$. Further reiteration yields convergence on any arbitrary time interval $[0,T]$, and therefore on $\mathbb{R}_+$.

It finally remains to note that $X$ also satisfies (17) with $\varphi \in C_{\mathrm{b}}^{+}(\mathbb{R}^d)$ and $u(\varphi)$ the unique solution as given in Theorem 1(b). This can be seen by considering $\varphi_n \downarrow \varphi$ with $\varphi_n \in C_{\mathrm{b}}^{++}(\mathbb{R}^d)$. In this case, both sides of (17) converge appropriately due to Lebesgue's Dominated Convergence Theorem, and so we are done.

## 2.8 Log-Laplace equations (continued)

Here we complete part (b) of Theorem 1. The uniqueness of the extension to non-negative initial conditions relies on the existence of the process $X$ constructed before. We restate our objective:

**Lemma 13 (Solutions for non-negative $\varphi$).** *If $\varphi_n \in C_b^{++}(\mathbb{R}^d)$, $n \geq 1$, such that boundedly pointwise $\varphi_n \downarrow \varphi \in C_b^+(\mathbb{R}^d)$ as $n \uparrow \infty$, then pointwise $u(\varphi_n) \downarrow u(\varphi) \in C([0,\infty), C_b^+(\mathbb{R}^d))$ as $n \uparrow \infty$, and the limit $u = u(\varphi)$ solves equation (9), satisfies (11), and is independent of the choice of the sequence $(\varphi_n)_{n\geq 1}$ converging to $\varphi$.*

*Proof.* By Lemma 10, $u(\varphi_n)$, exists for all $n \geq 1$ and is bounded below by $\inf_{x\in\mathbb{R}^d} \varphi(x) \wedge 1$. >From the log-Laplace representation (17) we see that the sequence is monotonously non-increasing as $n \uparrow \infty$ and that, for each $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, there exists a limit $u_t(\varphi)(x) := \lim_{n\uparrow\infty} u_t(\varphi_n)(x)$. Clearly, the limit is independent of the choice of the sequence $(\varphi_n)_{n\geq 1}$ since the left hand side of (17) converges to a unique limit by the Dominated Convergence Theorem. This implies that $g(u_t(\varphi_n)(x))$ converges boundedly pointwise to $g(u_t(\varphi)(x))$. Thus by Lebesgue's Dominated Convergence Theorem,

$$\int_0^t \int_{\mathbb{R}^d} p_{t-s}^\alpha(x-y) g(u_s(\varphi_n)(y)) dy ds \to \int_0^t \int_{\mathbb{R}^d} p_{t-s}^\alpha(x-y) g(u_s(\varphi)(x)) dy ds.$$

Hence, $u(\varphi)$ fulfills the mild form of (9) pointwise.

Like the approximating sequence, $(t,x) \mapsto u_t(\varphi)(x)$ is a uniformly bounded positive function on $\mathbb{R}_+ \times \mathbb{R}^d$. It only remains to show joint continuity in $t$ and $x$. The right continuity at $t=0$ follows immediately from the strong continuity of $T_t^\alpha$ as well as the boundedness of the solutions. Otherwise, we consider for some $0 < \epsilon < T$, $\epsilon < t \leq t' \leq T$ and $x, x' \in \mathbb{R}^d$,

$$\begin{aligned}
\left|u_{t'}(x') - u_t(x)\right| \;&\leq\; \int_{\mathbb{R}^d} \left|p_{t'}^\alpha(x'-y) - p_t^\alpha(x-y)\right| \varphi(y) dy \\
&\quad + \int_t^{t'} \int_{\mathbb{R}^d} p_{t'-s}^\alpha(x'-y) \left|g(u_s(y)\right| dy ds \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \left|p_{t'-s}^\alpha(x'-y) - p_{t-s}^\alpha(x-y)\right| \left|g(u_s(y)\right| dy ds \\
&\leq\; c\Big( \int_{\mathbb{R}^d} \left|p_{t'}^\alpha(x'-y) - p_t^\alpha(x-y)\right| dy + |t'-t| \qquad (89) \\
&\quad + \int_0^t \int_{\mathbb{R}^d} \left|p_{t'-s}^\alpha(x'-y) - p_{t-s}^\alpha(x-y)\right| dy ds \Big). \qquad (90)
\end{aligned}$$

Now, let $|t'-t| \downarrow 0$ as well as $|x'-x| \downarrow 0$. We note that

$$\sup_{\epsilon < t \leq T} \sup_{x\in\mathbb{R}^d} p_t^\alpha(x) < \infty, \qquad (91)$$

and that $p_t^\alpha(x)$ is jointly continuous on $(0,\infty) \times \mathbb{R}^d$ (see Appendix in Fleischmann and Gärtner (1986) [11]). Thus, by Lebesgue's Dominated Convergence Theorem, the spatial integrals in (89) and (90) converge to zero, the latter for all $s < t$. Since the spatial integral in (90) is further bounded by 2, another application of Lebesgue's Theorem concludes the proof. □

## 3 Immortality and infinite biodiversity

As already mentioned in Section 1.3, our process $X$ is immortal and propagates instantaneously:

**Proposition 14 (Immortality and instantaneous propagation).** *Take $\mu \neq 0$, $t > 0$, and $\varphi \in C_{\mathrm{com}}^{+}$ with $\varphi \neq 0$. Then $\langle X_t, \varphi \rangle > 0$, $\mathbb{P}_\mu$–a.s.*

In other words, almost surely the Lebesgue measure is absolutely continuous with respect to $X_t$. Recall that this is quite different from the behaviour of the approximating supercritical $X^{(\beta)}$ processes.

*Proof.* By the Markov property of $X$, we may fix $0 < t < \rho^{-1}$. Clearly,

$$\mathbb{P}_\mu\left[\langle X_t, \varphi \rangle = 0\right] = \lim_{\theta \uparrow \infty} \mathbb{E}_\mu\left[\mathrm{e}^{\langle X_t, -\theta\varphi \rangle}\right] = \exp\Big[-\lim_{\theta \uparrow \infty} \big\langle \mu, u_t(\theta\varphi) \big\rangle\Big]. \tag{92}$$

Hence, by Monotone Convergence it suffices to show that for each $x \in \mathbb{R}^d$,

$$u_t(\theta\varphi)(x) \uparrow \infty \qquad \text{as} \qquad \theta \uparrow \infty. \tag{93}$$

Let us now consider a sequence $(\varphi_n)_{n \geq 1}$ with $\varphi_n \in C_b^{++}(\mathbb{R}^d)$ and $\varphi_n \downarrow \varphi$ pointwise as well as $\|\varphi_n\|_\infty \to \|\varphi\|_\infty$. By the Feynman-Kac representation of solutions to (9) in the Lipschitz region,

$$u_t(\theta\varphi_n)(x) = \theta\, E_x\left[\varphi_n(\xi_t) \exp\Big(-\int_0^t \rho \log\left[u_{t-s}(\theta\varphi_n)(\xi_s)\right]\Big)\right], \tag{94}$$

where $(\xi, P_x)$ is a motion with generator $\Delta_\alpha$ started at $x$. Consequently, by Lemma 3.2,

$$u_s(\theta\varphi_n)(\xi_s) \leq \theta\, \|\varphi_n\|_\infty, \qquad s \geq 0. \tag{95}$$

Therefore,

$$\begin{aligned} u_t(\theta\varphi_n)(x) &\geq \theta E_x\left[\varphi_n(\xi_t) \exp\Big(-\rho t \log\left[\theta\|\varphi_n\|_\infty\right]\Big)\right] \\ &= \theta\left(\theta\|\varphi_n\|_\infty\right)^{-\rho t} E_x\left[\varphi_n(\xi_t)\right] \\ &= \theta^{1-\rho t}\|\varphi_n\|_\infty^{-\rho t} T_t^\alpha \varphi_n(x). \end{aligned} \tag{96}$$

By Theorem 1(b) the left hand side converges to $u_t(\theta\varphi)(x)$ as $n \to \infty$. The right hand side converges by assumption implying

$$u_t(\theta\varphi_n)(x) \geq \theta^{1-\rho t}\|\varphi\|_\infty^{-\rho t} T_t^\alpha \varphi(x), \tag{97}$$

which becomes infinite as $\theta \uparrow \infty$ giving (93). This completes the proof. □

Proposition 14 implies that $X$ has countably infinite biodiversity. This we want to make precise now. Recall that an infinitely divisible random measure $Y \in M_{\mathrm{f}}$ has a *clustering representation*

$$Y \;=\; \gamma + \sum_i \chi_i \tag{98}$$

(see, for instance, Lemma 6.5 in Kallenberg (1976) [16]). Here $\gamma \in M_\mathrm{f}$ is the deterministic component of $Y$ (or the essential infimum of $Y$), and the clusters (families) $\chi_i \in M_\mathrm{f}$ are the "points" of a Poissonian point measure on $\mathcal{M}_\mathrm{f}(\mathbb{R}^d) \setminus \{0\}$ with some intensity measure $\mathbf{Q}$, , which is called the *canonical measure* of $Y$. We can reformulate (98) as the *classical Lévy-Hincin formula* for the log-Laplace transforms,

$$-\log \mathbb{E}_\mu \left[ \mathrm{e}^{\langle Y, -\varphi \rangle} \right] = \langle \gamma, \varphi \rangle + \int_{\mathcal{M}_\mathrm{f}(\mathbb{R}^d)} \mathbf{Q}(d\chi)(1 - \mathrm{e}^{\langle \chi, -\varphi \rangle}) \tag{99}$$

(see Theorem 6.1 of Kallenberg [16]). Let $B$ be a bounded Borel subset of $\mathbb{R}^d$. If $\gamma = 0$, then the number $\#\{i : \chi_i(B) > 0\}$ of families in $B$ has a Poisson distribution with expectation $\mathbf{Q}(\chi : \chi(B) > 0)$. If $\gamma(B) > 0$ then one could say a "continuum of families" contributes to $Y(B)$. Therefore in [12] the following terminology was introduced:

**Definition 15 (Biodiversity).** We say that the (local) *biodiversity* of the infinitely divisible random measure $Y$ is

- *finite*, if $\gamma = 0$ and $\mathbf{Q}(\chi : \chi(B) > 0) < \infty$ for every compact set $B$,
- *countably infinite*, if $\gamma = 0$ and $\mathbf{Q}(\chi : \chi(B) > 0) = \infty$ for every open set $B \neq \emptyset$,
- *uncountably infinite*, if $\gamma(B) > 0$ for every open set $B \neq \emptyset$. ◇

Armed with this terminology, we can now prove the following result:

**Corollary 16 (Countably infinite biodiversity).** *For every fixed $\mu \neq 0$ and $t > 0$, the random measure $X_t$ has (locally) countably infinite biodiversity.*

Recall that this is in contrast to the finite biodiversity of the random states of the approximating processes $X^{(\beta)}$.

*Proof.* For $Y$ to have finite local biodiversity, it is necessary and sufficient that

$$\mathbb{P}_\mu \left[ Y(B) = 0 \right] > 0 \quad \text{for any compact set } B. \tag{100}$$

This follows from the simple observation that

$$\mathbf{Q}\left(\chi : \ \chi(B) > 0\right) = -\log \mathbb{P}_\mu \left[ Y(B) = 0 \right], \tag{101}$$

provided that $\gamma = 0$. Then from Proposition 14 it follows that the $X_t$ have infinite biodiversity. Finally, $X_t$ does not have a deterministic component, since $X_t(\mathbb{R}^d)$ has a stable distribution with index $\mathrm{e}^{-\rho t}$ [recall (23) and (26)]. This finishes the proof. ☐

# 4 Appendix

*Proof of Proposition* 8. We first note that $t \mapsto M_t(\lambda) := \exp\big(-\bar{u}_{-t}(\lambda)\bar{X}_t\big) = \exp(-\lambda^{(e^{\rho t})}\bar{X}_t)$ is a martingale, for each $\lambda > 0$, since for $s \le t$,

$$\begin{aligned} \mathbb{E}_m\big[\exp\big(-\bar{u}_{-t}(\lambda)\bar{X}_t\big) \mid \mathcal{F}_s\big] &= \exp\big(-\bar{u}_{t-s}(\bar{u}_{-t}(\lambda))\bar{X}_s\big) \\ &= \exp\big(-\bar{u}_{-s}(\lambda)\bar{X}_s\big) \end{aligned} \tag{102}$$

by the Markov and branching property of the process $\bar{X}$ and the semigroup property of the solution $\bar{u}$. Since $M_t(\lambda)$ takes values in $[0,1]$ the limit as $t \uparrow \infty$ exists a.s., and we denote it by $W(\lambda)$. By Lebesgue's Dominated Convergence Theorem, for all $\theta > 0$,

$$\begin{aligned} \mathbb{E}_m[W^\theta(\lambda)] &= \lim_{t\uparrow\infty} \mathbb{E}_m[\exp(-\theta\bar{u}_{-t}(\lambda)\bar{X}_t)] \\ &= \lim_{t\uparrow\infty} \exp(-\bar{u}_t(\theta\bar{u}_{-t}(\lambda))m)] \\ &= \lim_{t\uparrow\infty} \exp(-(\theta\lambda^{(e^{\rho t})})^{e^{-\rho t}} m)] = \exp(-\lambda m). \end{aligned} \tag{103}$$

This implies that $W(\lambda)$ takes the value 1 with probability $\exp(-\lambda m)$ and is 0 otherwise. Since $M_t(\lambda)$ is monotonously non-increasing in $\lambda$ for each $t \ge 0$, the limit $W(\lambda)$ is non-increasing in $\lambda$. Also note that $W(\lambda)$ is defined a.s. for all rational $\lambda$. With the exception of a null set, we can therefore define the threshold variable $V := \inf\big\{\text{rational } \lambda : W(\lambda) = 0\big\}$. From $\mathbb{P}_m[V < \lambda] = \lim_{\lambda'\uparrow\lambda} \mathbb{P}_m[W(\lambda') = 0] = 1 - \exp(-\lambda m)$, we obtain that $V$ is exponentially distributed with mean $1/m$. It follows that a.s.

$$\lambda^{(e^{\rho t})}\bar{X}_t \to \begin{cases} 0 & \text{for } \lambda < V, \\ \infty & \text{for } \lambda > V \end{cases} \tag{104}$$

as $t \uparrow \infty$. This implies that for any random variables $V_0$ and $V_1$ with rational values so that $V_0 < V < V_1$,

$$V_1^{-e^{\rho t}} \le \bar{X}_t \le V_0^{-e^{\rho t}}, \tag{105}$$

a.s. for $t = t(\omega)$ large enough. Hence, we have a.s.,

$$\log(\tfrac{1}{V_1}) \le \liminf_{t\uparrow\infty} e^{-\rho t}\log(\bar{X}_t) \le \limsup_{t\uparrow\infty} e^{-\rho t}\log(\bar{X}_t) \le \log(\tfrac{1}{V_0}). \tag{106}$$

The statement now follows by letting almost surely $V_0$ and $V_1$ tend to $V$. □

*Acknowledgement.* We are grateful to Ingo Mattheis for aiding us in generating the figure and to Wolfgang Höppner for references to PDE literature.

# References

[1] J. Bertoin and J.-F. L. Gall. The Bolthausen-Sznitman coalescent and the genealogy of continuous-state branching processes. *Probab. Theory Relat. Fields*, 117:249–266, 2000.

[2] E. Bolthausen and A.-S. Sznitman. On Ruelle's probability cascades and an abstract cavity method. *Comm. Math. Phys.*, 197(2):247–276, 1998.

[3] D. A. Dawson. Measure-valued Markov processes. In *École d'Été de Probabilités de Saint-Flour XXI—1991*, volume 1541 of *Lecture Notes in Math.*, pages 1–260. Springer, Berlin, 1993.

[4] D. A. Dawson and V. Vinogradov. Almost-sure path properties of $(2, d, \beta)$-superprocesses. *Stochastic Process. Appl.*, 51(2):221–258, 1994.

[5] E. B. Dynkin. *An Introduction to Branching Measure-Valued Processes*, volume 6 of *CRM Monograph series*. AMS, 1994.

[6] A. M. Etheridge. *An Introduction to Superprocesses*, volume 20 of *University Lecture series*. American Mathematical Society, 2000.

[7] S. N. Ethier and T. G. Kurtz. *Markov Processes: Characterization and Convergence*. Wiley Series in Probability and Mathematical Statistics. Wiley, 1986.

[8] P. I. Fitzsimmons. Construction and regularity of measure-valued Markov branching processes. *Israel J. Math.*, 64(3):337–361, 1988.

[9] P. I. Fitzsimmons. Corrections to "Construction and regularity of measure-valued Markov branching processes". *Israel J. Math.*, 73(1):127, 1991.

[10] K. Fleischmann. Critical behaviour of some measure-valued processes. *Math. Nachr.*, 135:131–147, 1988.

[11] K. Fleischmann and J. Gärtner. Occupation time processes at a critical point. *Math. Nachr.*, 125:275–290, 1986.

[12] K. Fleischmann and A. Klenke. The biodiversity of catalytic super-Brownian motion. *Ann. Appl. Probab.*, 10(1):1121–1136, 2000.

[13] D. R. Grey. Almost sure convergence in Markov branching processes with infinite mean. *J. Appl. Prob.*, 14:702–716, 1977.

[14] I. Iscoe. A weighted occupation time for a class of measure-valued branching processes. *Probab. Theory Relat. Fields*, 71(1):85–116, 1986.

[15] A. Jakubowski. On the Skorokhod topology. *Ann. Inst. Henri Poincaré*, 22(3):263–285, 1986.

[16] O. Kallenberg. *Random Measures*. Academic Press, 1976.

[17] J.-F. LeGall. Random trees and spatial branching processes. *Lecture Notes for the course on "Lévy processes and branching processes" in Aarhus*, 2000.

[18] L. Mytnik and E. Perkins. Regularity and irregularity of beta-stable super-Brownian motion. *Preprint*, 2003.

[19] J. Neveu. A continuous state branching process in relation with the GREM model of spin glasses theory. *Rapport Interne no 267, CMP École Polytechnique*, 1992.

[20] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*, volume 44 of *Applied Mathematical Sciences*. Springer, 1983.

[21] J. Pitman. Coalescents with multiple collisions. *Ann. Probability*, 27(4):1870–1902, 1999.

[22] D. Ruelle. A mathematical reformulation of Derrida's REM and GREM. *Comm. Math. Physics*, 108:225–239, 1987.

[23] J. Smoller. *Shock Waves and Reaction-Diffusion Equations*, volume 258 of *A series of comprehensive studies in Mathematics*. Springer, 1983.

[24] S. Watanabe. A limit theorem of branching processes and continuous state branching. *J. Math. Kyoto University*, 8:141–167, 1968.

[25] K. Yosida. *Functional Analysis*, volume 123 of *A series of Comprehensive Studies in Mathematics*. Springer, 1971.

[26] V. M. Zolotarev. *One-Dimensional Stable Distributions*, volume 65 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1986.