Weierstraß-Institut
für Angewandte Analysis und Stochastik

im Forschungsverbund Berlin e.V.

Preprint ISSN 0946 – 8633

An L^1-stability and Uniqueness Result for Balance Laws with Multifunctions: A Model from the Theory of Granular Media

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submitted: 10th February 2003

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No. 815
Berlin 2003

2000 *Mathematics Subject Classification.* 35L65, 35L45, 35B35.

Key words and phrases. system of hyperbolic conservation law, multifunction, weak entropy solutions, L^1-stability, uniqueness, well-posedness.

Edited by
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Abstract

In this paper it is studied uniqueness and L^1-stability for 2×2 system coming out from the theory of granular media. The investigations are done in a class of weak entropy solutions. The appearance of multifunction in a source term, given by Coulomb-Mohr friction law, requires a modification of definition of the solution.

1 Introduction

We consider the system describing the motion of an avalanche down a slope, which will be denoted by the following values:

- the height of an avalanche $h : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$,
- the density of an avalanche $\rho : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$,
- the velocity of an avalanche $v : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$.

The system consists of differential inclusion

$$\begin{aligned} \frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x}(\rho h v) &= 0, \\ \frac{\partial}{\partial t}(\rho h v) + \frac{\partial}{\partial x}\left(\rho h v^2 + \frac{1}{2}\beta \rho h^2\right) &\in \rho h g, \end{aligned} \tag{1.1}$$

where $\beta := \beta(x)$ is a given function and $g := g(x, v)$ is a given multifunction. The first equation in (1.1) describes the conservation of mass whereas the second differential inclusion describes the balance of linear momentum. For simplicity the dependence of g and β on x will be ignored, what does not influence the problem. The constant β and the multifunction $g(v)$ are defined by

$$\beta = k \cos(\gamma),$$

$$g(v) = \begin{cases} \sin(\gamma) + [-\cos(\gamma), +\cos(\gamma)] & \text{for} \quad v = 0, \\ \sin(\gamma) - \frac{v}{|v|}\cos(\gamma) & \text{for} \quad v \neq 0, \end{cases}$$

where $\frac{\pi}{2} < \gamma < \frac{\pi}{2}$ is an angle between gravitational force and a constant slope ground and k is a positive constant. The evolution of three variables (ρ, h, v) cannot be determined uniquely by these two balance laws, so an additional constitutive relation has to be added. One possible approach is to assume that ρ is a function of h and v. Then we obtain a system of two differential inclusions for the two independent variables (h, v). We consider the constitutive relation $\rho = h^{-\frac{1}{2}}$. Following the nonlinear transformations of the above system in [7] we obtain the new system

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}F(u) \in \tilde{G}(u), \tag{1.2}$$

where

$$F(u) = \left(\begin{array}{c} u_1 u_2 \\ \dfrac{u_2^2}{2} + \dfrac{u_1^2}{2} \end{array} \right) \quad \text{and} \quad \tilde{G}(u) = \left(\begin{array}{c} 0 \\ \tilde{g}(u_2) \end{array} \right).$$

Let us introduce the class of weak entropy solutions, which is the proper one for the above system.

Definition 1.1. Suppose that $\eta = \eta(u_1, u_2)$, $q = q(u_1, u_2)$ are scalar C^1-functions satisfying

$$\nabla_{(u_1,u_2)}\eta(u_1, u_2) \cdot \nabla_{(u_1,u_2)}F(u_1, u_2) = \nabla_{(u_1,u_2)}q(u_1, u_2).$$

Such functions (η, q) are called Entropy-Flux Pairs. If η is convex, then (η, q) is called a Convex Entropy-Flux Pair.

Definition 1.2. We call $u \in \mathbb{L}^\infty([0, T) \times \mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ a Weak Entropy Solution to the system

$$u_t + F(u)_x = G(x, t)$$

with the initial data $u^0 \in \mathbb{L}^\infty(\mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ and source term $G \in \mathbb{L}^\infty([0, T) \times \mathbb{R}; \mathbb{R}^2)$ iff

1. u is a weak solution, i.e.

$$\begin{aligned} \int\limits_{[0,T)\times\mathbb{R}} & \left[u(t, x) \cdot \frac{\partial}{\partial t}\psi(t, x) + F(u(t, x)) \cdot \frac{\partial}{\partial x}\psi(t, x)\right. \\ & \left. + G(t, x) \cdot \psi(t, x)\right] dtdx + \int\limits_{\mathbb{R}} u^0(x) \cdot \psi(0, x)dx = 0 \end{aligned}$$

for all test functions $\psi \in C_c^1([0, T) \times \mathbb{R}; \mathbb{R}^2)$.

2. The entropy inequality

$$\begin{aligned} \int\limits_{[0,T)\times\mathbb{R}} & \left[\eta(u(t, x))\frac{\partial}{\partial t}\varphi(t, x) + q(u(t, x))\frac{\partial}{\partial x}\varphi(t, x)\right. \\ & \left. + \nabla_u\eta(u(t, x)) \cdot G(t, x)\varphi(t, x)\right] dtdx + \int\limits_{\mathbb{R}} \eta(u^0(x))\varphi(0, x)dx \geq 0 \end{aligned}$$

holds for all non-negative test functions $\varphi \in C_c^1([0, T) \times \mathbb{R}; \mathbb{R})$ and all convex entropy-flux pairs (η, q).

Remark
The above definition is standard in the theory of conservation laws. However it cannot be used for system (1.2) because of the multifunction in a source term. Therefore we need the following extension.

Definition 1.3. We call $u \in \mathbb{L}^\infty([0,T) \times \mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ a Weak Entropy Solution to the system (1.2) with the initial data $u^0 \in \mathbb{L}^\infty(\mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ iff

1. $\exists\, G(t,x) \in \tilde{G}(u(t,x))$ for a.a. $(t,x) \in [0,T) \times \mathbb{R}$.
2. u is a Weak Entropy Solution according to Definition 1.2 to a system

$$u_t + F(u)_x = G(x,t)$$

In paper [7] there has been shown existence of solutions to the problem (1.2). We will recall this theorem

Notation
By $C^0_\omega(\Omega; X)$ we denote the Banach space of the continuous functions $u : \Omega \to X$ with the norm weighted by ω $(\omega(x) = 1 + |x|)$, $\|u\|_{C^0_\omega} = sup_{x\in\Omega}\|\omega(x)u(x)\|$. Note that X does not have to be a linear space. By $C^r_b(\Omega; X)$ we denote the Banach space of r-times differentiable functions with the usual norm.

Theorem 1.1. *Assume that the initial data* $u^0 = (u^0_1, u^0_2) \in C^3_b(\mathbb{R}; \mathbb{R}^2)$, $\inf_{x\in\mathbb{R}} u^0_1(x) \geq 0$, $(u^0_1 - \overline{u}, u^0_2) \in C^0_\omega(\mathbb{R}; \mathbb{R}^2)$ *for some positive real constant* $\overline{u}$.
Then the problem (1.2) possesses the Weak Entropy Solution in the sense of Definition 1.3 for all positive T, *with* $\inf_{x\in\mathbb{R}} u^0_1(t,x) \geq 0$ *for a.a.* $t \in [0,T)$.

For further consideration it is useful to observe that the system (1.2) is equivalent to the following system of two independent inclusions coupled only by a right-hand side

$$\begin{aligned} \frac{\partial}{\partial t}(u_1 - u_2) - \frac{1}{2}\frac{\partial}{\partial x}(u_1 - u_2)^2 &\in -\tilde{g}(u_2) \\ \frac{\partial}{\partial t}(u_1 + u_2) + \frac{1}{2}\frac{\partial}{\partial x}(u_1 + u_2)^2 &\in \tilde{g}(u_2) \end{aligned} \tag{1.3}$$

Using new variables $u_1 - u_2 = w_1, u_1 + u_2 = w_2$ the system can be rewritten

$$\begin{aligned} \frac{\partial}{\partial t}w_1 - \frac{1}{2}\frac{\partial}{\partial x}w_1^2 &\in -\tilde{g}\left(\frac{w_2 - w_1}{2}\right) \\ \frac{\partial}{\partial t}w_2 + \frac{1}{2}\frac{\partial}{\partial x}w_2^2 &\in \tilde{g}\left(\frac{w_2 - w_1}{2}\right) \end{aligned} \tag{1.4}$$

One could expect that the system consisting of differential inclusion instead of equation should bring a big amount of solutions. It can be easily shown that there exist

a lot of stationary solutions (see [6] for details), contrary to the system of two differential equations, where only one stationary solution has been obtained. For the solutions $u \in \mathbb{L}^\infty([0,T) \times \mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ as in the Definition 1.2 the result of uniqueness cannot be obtained because of the possible occurrence of initial layer to such a solution. But together the method of vanishing viscosity and Lemma 3.3 with some additional assumption on the initial data (c.f. assumption on ω in Lemma 3.3) bring the solution being in $C^0([0,T); \mathbb{L}^1_{loc}(\mathbb{R}))$. Note that this is the typical time regularity for the uniqueness results for the scalar conservation law (c.f. [9], [11]).
Therefore we obtain the uniqueness of solutions, what is the main result of this paper

Theorem 1.2. *Let* (w_1, w_2), $(\overline{w}_1, \overline{w}_2) \in C^0([0,T); \mathbb{L}^1_{loc}(\mathbb{R}; \mathbb{R}^2)) \cap \mathbb{L}^\infty([0,T) \times \mathbb{R}; \mathbb{R}^2)$ *be two weak entropy solutions to the system (1.4) with* $(w_1 - \overline{w}_1, w_2 - \overline{w}_2) \in \mathbb{L}^\infty([0,T); \mathbb{L}^1(\mathbb{R}; \mathbb{R}^2))$ *and initial data* (w_1^0, w_2^0), $(\overline{w}_1^0, \overline{w}_2^0) \in \mathbb{L}^1_{loc}(\mathbb{R}; \mathbb{R}^2)$. *Then for any* $0 < t < T$

$$\|w_1(t) - \overline{w}_1(t)\|_{L^1(\mathbb{R})} + \|w_2(t) - \overline{w}_2(t)\|_{L^1(\mathbb{R})} \leq \|w_1^0 - \overline{w}_1^0\|_{L^1(\mathbb{R})} + \|w_2^0 - \overline{w}_2^0\|_{L^1(\mathbb{R})}.$$

Remark
Similar results for strongly coupled 2×2 - system are not straightforward. Even for the homogeneous system the uniqueness of solutions is conditional under the assumption on BV norm $\|u^0\|_{BV(\mathbb{R};\mathbb{R}^2)} << 1$ c.f. [1], [2], [3]. This yields a global in time estimate on BV norm. However in the case of nonhomogenous equations some "dissipative properties" of the right-hand side in sense of a proper BV norm are meaningful, they do not appear in this case.

2 Technical Lemma

For technical reasons it will be convenient to formulate the following lemma

Lemma 2.1. *Let* $g : \mathbb{R} \to 2^{\mathbb{R}}$ *be a monotone multifunction where* $g(b) \in [-1,1]$ *is multivalued only if* $b = 0$ *and*

$$I(a, b, \overline{a}, \overline{b}) = [g(b) - \overline{g}(\overline{b})][\eta'_\delta(a + b - \overline{(a+b)}) - \eta'_\delta(a - b - \overline{(a-b)})]$$

where $\eta_\delta : \mathbb{R} \to \mathbb{R}$ *is a function, for* $\delta > 0$ *defined by*

$$\eta_\delta(y) = \begin{cases} 0 & y \in (-\infty, 0] \\ \frac{y^2}{4\delta} & y \in (0, 2\delta] \\ y - \delta & y \in (2\delta, \infty) \end{cases} \tag{2.1}$$

Then $I \leq 0$ *for every* $a, \overline{a}, b, \overline{b} \in \mathbb{R}$.

Remark
By monotone multifunction we understand that g has the following property

$$\forall a, b \in \mathbb{R} \qquad a < b \Rightarrow g_a \geq g_b \qquad \forall g_a \in g(a),\, g_b \in g(b).$$

Proof
Let observe that η'_δ is a nondecreasing function and

$$[\eta'_\delta(a+b-\overline{(a+b)})-\eta'_\delta(a-b-\overline{(a-b)})] = [\eta'_\delta((a-\overline{a})+(b-\overline{b}))-\eta'_\delta((a-\overline{a})-(b-\overline{b}))]$$

then consider three cases:

1. $b < \overline{b}$ then $[\eta'_\delta(a+b-\overline{(a+b)})-\eta'_\delta(a-b-\overline{(a-b)})] \leq 0$
2. $b = \overline{b}$ then $[\eta'_\delta(a+b-\overline{(a+b)})-\eta'_\delta(a-b-\overline{(a-b)})] = 0$
3. $b > \overline{b}$ then $[\eta'_\delta(a+b-\overline{(a+b)})-\eta'_\delta(a-b-\overline{(a-b)})] \geq 0$.

□

3 Additional Estimates for Vanishing Viscosity Solutions

We will observe the stability of solutions to the parabolic system

$$\begin{aligned} \frac{\partial}{\partial t} w_1 - \frac{1}{2}\frac{\partial}{\partial x} w_1^2 &= \varepsilon \frac{\partial^2}{\partial x^2} w_1 - g^\varepsilon\left(\frac{w_2 - w_1}{2}\right) \\ \frac{\partial}{\partial t} w_2 + \frac{1}{2}\frac{\partial}{\partial x} w_2^2 &= \varepsilon \frac{\partial^2}{\partial x^2} w_2 + g^\varepsilon\left(\frac{w_2 - w_1}{2}\right) \end{aligned} \tag{3.1}$$

with $\varepsilon \searrow 0$. The function g has been replaced by a smooth bounded function g^ε, which was constructed by mollifying g with some smooth function with compact support. The above problem possesses a classical solution. We only recall the proper theorem from [7]

Theorem 3.1. *Assume that the initial data $w^0 = (w_1^0, w_2^0) \in C_b^3(\mathbb{R};\mathbb{R}^2)$, with $(w_1^0 + w_2^0 - 2\overline{u}, w_1^0 - w_2^0) \in C_\omega^0(\mathbb{R};\mathbb{R}^2)$ for some positive real constant $\overline{u}$.*
Then the problem (3.1) possesses a classical global in time solution, i.e. $w \in C^0([0,T); C_b^2(\mathbb{R};\mathbb{R}^2))$, $\frac{\partial}{\partial t} w \in C_b^0([0,T) \times \mathbb{R};\mathbb{R}^2)$ where T is an arbitrary time.

For further consideration we need a new, independent of ε estimate for the solution to the system (3.1).

Lemma 3.2. *Let (w_1, w_2) and $(\overline{w}_1, \overline{w}_2)$ be two different solutions to the system (3.1) as in Theorem 3.1 with initial data (w_1^0, w_2^0) and $(\overline{w}_1^0, \overline{w}_2^0)$ from $\mathbb{L}^1_{loc}(\mathbb{R};\mathbb{R}^2)$. Then for any $0 < t < T$*

$$\|w_1(t) - \overline{w}_1(t)\|_{L^1(\mathbb{R})} + \|w_2(t) - \overline{w}_2(t)\|_{L^1(\mathbb{R})} \leq \|w_1^0 - \overline{w}_1^0\|_{L^1(\mathbb{R})} + \|w_2^0 - \overline{w}_2^0\|_{L^1(\mathbb{R})}.$$

Proof
Let η_δ be an entropy function defined as the function in Lemma 2.1. With help of simple algebraic tricks we arrive

$$
\begin{aligned}
&\partial_t[\eta_\delta(w_1-\overline{w}_1)+\eta_\delta(w_2-\overline{w}_2)]-\frac{1}{2}\partial_x\{\eta_\delta'(w_1-\overline{w}_1)(w_1^2-\overline{w}_1^2)\\
&+\eta_\delta'(w_2-\overline{w}_2)(w_2^2-\overline{w}_2^2)\}+\frac{1}{2}\eta_\delta''(w_1-\overline{w}_1)(w_1^2-\overline{w}_1^2)\partial_x(w_1-\overline{w}_1)\\
&-\frac{1}{2}\eta_\delta''(w_2-\overline{w}_2)(w_2^2-\overline{w}_2^2)\partial_x(w_2-\overline{w}_2)=\varepsilon\Delta[\eta_\delta(w_1-\overline{w}_1)+\eta_\delta(w_2-\overline{w}_2)]\\
&-\varepsilon\eta_\delta''(w_1-\overline{w}_1)[\partial_x(w_1-\overline{w}_1)]^2-\varepsilon\eta_\delta''(w_2-\overline{w}_2)[\partial_x(w_2-\overline{w}_2)]^2\\
&+[g^\varepsilon\left(\frac{w_2-w_1}{2}\right)-g^\varepsilon\left(\frac{\overline{w}_2-\overline{w}_1}{2}\right)][\eta_\delta'(w_2-\overline{w}_2)-\eta_\delta'(w_1-\overline{w}_1)]
\end{aligned}
\tag{3.2}
$$

Since the terms $\varepsilon\eta_\delta''(w_i-\overline{w}_i)$, $\quad i=1,2.$ are nonnegative and the last term on the right-hand side is nonpositive by Lemma 2.1, integrating the above inequality over $\mathbb{R}\times(0,t)$ with $0<t<T$ fixed leads to the following inequality

$$
\begin{aligned}
&\int_{\mathbb{R}}[\eta_\delta(w_1(t,x)-\overline{w}_1(t,x))+\eta_\delta(w_2(t,x)-\overline{w}_2(t,x))]dx\\
&-\int_{\mathbb{R}}[\eta_\delta(w_1^0(x)-\overline{w}_1^0(x))+\eta_\delta(w_2^0(x)-\overline{w}_2^0(x))]dx\\
&\le\int_{\mathbb{R}\times(0,t)}\frac{1}{2}\{\eta_\delta''(w_2-\overline{w}_2)(w_2^2-\overline{w}_2^2)\partial_x(w_2-\overline{w}_2)\\
&-\eta_\delta''(w_1-\overline{w}_1)(w_1^2-\overline{w}_1^2)\partial_x(w_1-\overline{w}_1)\}dtdx
\end{aligned}
\tag{3.3}
$$

$\eta_\delta(w_i(t,x)-\overline{w}_i(t,x))$ converges pointwise to $[w_i(t,x)-\overline{w}_i(t,x)]^+$ as $\delta\to 0$ and $\eta_\delta''(w_i(t,x)-\overline{w}_i(t,x))$ converges pointwise to 0. Then

$$
\begin{aligned}
&\int_{\mathbb{R}}[w_1(t,x)-\overline{w}_1(t,x)]^+dx+\int_{\mathbb{R}}[w_2(t,x)-\overline{w}_2(t,x)]^+dx\\
&\le\int_{\mathbb{R}}[w_1^0(x)-\overline{w}_1^0(x)]^+dx+\int_{\mathbb{R}}[w_2^0(x)-\overline{w}_2^0(x)]^+dx
\end{aligned}
\tag{3.4}
$$

Interchanging w_i with $\overline{w}_i$ leads to the analogous inequality. Adding both inequalities yields the assertion of the theorem. □

Lemma 3.3. *Let (w_1,w_2) be the strong solution to system (3.1) with initial data (w_1^0,w_2^0). Moreover there exists $\omega\in C^0(\mathbb{R};\mathbb{R})$ with $\omega(0)=0$ such that*

$$\int_{\mathbb{R}}|w_1^0(x)-w_1^0(x+h)|dx+\int_{\mathbb{R}}|w_2^0(x)-w_2^0(x+h)|dx\le\omega(h).$$

Then for any $0 < t < T - h$ *and* $r > 0$

$$\int_{-r}^{r} \{|w_1(t+h,x) - w_1(t,x)| + |w_2(t+h,x) - w_2(t,x)|\} dx$$
$$\le c_1(h + h^{2/3} + h^{1/3})(r+1)\left(\|(w_1)^2\|_{L^\infty([t,t+h)\times\mathbb{R})} + \|(w_2)^2\|_{L^\infty([t,t+h)\times\mathbb{R})} + 1\right)$$
$$+ c_2\omega(h^{1/3}),$$

Proof

Let observe that by Lemma 3.2

$$\int_{\mathbb{R}} \{|w_1(t,x) - w_1(t,x+y)| + |w_2(t,x) - w_2(t,x+y)|\} dx$$
$$\le \int_{\mathbb{R}} \{|w_1^0(x) - w_1^0(x+y)| + |w_2^0(x) - w_2^0(x+y)|\} dx \le \omega(|y|).$$

First each of the equations of the system (3.1) will be treated separately. Multiplying each by a bounded test function $\varphi(x)K(x)$, where

$$K(x) = \begin{cases} 1 & \text{for } |x| \le r \\ 0 & \text{for } |x| > r+1 \\ (x+r+1)^2(x+r-1) & \text{for } -(r+1) \le x < -r \\ (x-r-1)^2(x-r+1) & \text{for } r < x \le r+1 \end{cases}$$

and φ are the functions in $W^{2,\infty}(\mathbb{R})$. Integrating over $\mathbb{R} \times (t, t+h)$ leads to

$$\int_{\mathbb{R}} \varphi(x)K(x)[w_i(t+h,x) - w_i(t,x)] dx$$
$$= \int_{(t,t+h)\times\mathbb{R}} \frac{(-1)^i}{2}[\varphi_x(x)K(x) + \varphi(x)K_x(x)] w_i^2(t,x) dt dx$$
$$+ \int_{(t,t+h)\times\mathbb{R}} \{\varepsilon[\varphi_{xx}(x)K(x) + \varphi_x(x)K_x(x) + \varphi(x)K_{xx}] w_i(t,x) + \varphi(x)K(x)g(t,x)\} dt dx,$$

for $i = 1, 2$. A good way leading to the assertion of the theorem would be using as a test function $\varphi_i(x) = \operatorname{sgn} v_i(x)$, with $v_i(x) = [w_i(t+h,x) - w_i(t,x)]$. Because of its discontinuity it has to be mollified first. So we choose as a test function $\varphi_i(x) = \sigma * \operatorname{sgn} v_i$ where $\sigma = h^{-1/3}\xi(\frac{x}{h^{1/3}})$, with some smooth and nonnegative function ξ of compact support and total mass one. Note $|K|, |K_x|, |K_{xx}|, |\varphi_i|, |h^{\frac{1}{3}}(\varphi_i)_x|, |h^{\frac{2}{3}}(\varphi_i)_{xx}|, |g|$ are uniformly bounded in $\mathbb{L}^\infty([0,T) \times \mathbb{R}; \mathbb{R})$. Then we have

$$\int_{-r}^{r} \varphi_i(x)v_i(x) dx \le c_1(h + h^{2/3} + h^{1/3})(r+1)(\|(w_i)^2\|_{L^\infty([t,t+h)\times\mathbb{R})} + 1)$$

Note $|v_i(x)| - v_i(x)\text{sgn}v_i(z) \leq 2|v_i(x) - v_i(z)|$ then

$$|v_i(x)|-v_i(x)\text{sgn}v_i(z) \leq 2\int_{\mathbb{R}} \sigma(\xi)\{|w_1(t,x)-w_1(t,x-h^{\frac{1}{3}}\xi)|+|w_2(t,x)-w_2(t,x-h^{\frac{1}{3}}\xi)|\}d\xi.$$

This yields the assertion of the lemma. □

4 Stability of solutions

Observing the condition which has to be satisfied by an entropy-flux pair (Definition 1.1)

$$\nabla_{(w_1,w_2)}\eta(w_1,w_2)\cdot\nabla_{(w_1,w_2)}F(w_1,w_2) = \nabla_{(w_1,w_2)}q(w_1,w_2)$$

one can find out that the matrix $\nabla F_{(w_1,w_2)}(w_1,w_2)$ in this case is diagonal so the above vector equation has the form

$$(\nabla_{w_1}\eta(w_1,w_2), \nabla_{w_2}\eta(w_1,w_2))\cdot\begin{pmatrix} -w_1 & 0 \\ 0 & w_2 \end{pmatrix} = (\nabla_{w_1}q(w_1,w_2), \nabla_{w_2}q(w_1,w_2)).$$

It follows the possibility of finding entropy-flux pairs (η^1, q^1) dependent only on w_1 and (η^2, q^2) on w_2. Hence the vector equation could be decoupled into scalar equations

$$\begin{aligned} \nabla_{w_1}\eta^1(w_1)\cdot(-w_1) &= \nabla_{w_1}q^1(w_1), \\ \nabla_{w_2}\eta^2(w_2)\cdot w_2 &= \nabla_{w_2}q^2(w_2). \end{aligned}$$

Following the notation for capital letters g will denote some selection from $\tilde{g}$ i. e. $g(t,x) \in \tilde{g}(w_i(t,x))$ for a. a. $(t,x) \in [0,T)\times\mathbb{R}$. Then the entropy inequality takes the following form

$$\begin{aligned} &\int_{[0,T)\times\mathbb{R}} [\eta^1(w_1(t,x)) + \eta^2(w_2(t,x))]\varphi_t(t,x) + [q^1(w_1(t,x)) + q^2(w_2(t,x))]\varphi_x(t,x) \\ &+g(t,x)[\nabla_{w_2}\eta^2(w_2(t,x)) - \nabla_{w_1}\eta^1(w_1(t,x))]\varphi(t,x)dtdx \\ &+\int_{\mathbb{R}}[\eta^1(w_1^0(x)) + \eta^2(w_2^0(x))]\varphi(0,x)dx \geq 0 \end{aligned} \tag{4.1}$$

From now on $\eta^i_\delta(w_i, \overline{w}_i)$, denoted below, becomes the entropy function, where $\overline{w}_i$ is a parameter taking values in $\mathbb{R}, i = 1,2$.

$$\eta^i_\delta(w_i,\overline{w}_i) = \begin{cases} 0 & \text{for} \quad w_i \leq \overline{w}_i \\ \frac{(w_i-\overline{w}_i)^2}{4\delta} & \text{for} \quad \overline{w}_i < w_i \leq \overline{w}_i + 2\delta \\ w_i - \overline{w}_i - \delta & \text{for} \quad w_i > \overline{w}_i + 2\delta \end{cases}$$

$$\nabla_{w_i}\eta^i_\delta(w_i,\overline{w}_i) = \begin{cases} 0 & \text{for} \quad w_i \leq \overline{w}_i \\ \frac{w_i-\overline{w}_i}{2\delta} & \text{for} \quad \overline{w}_i < w_i \leq \overline{w}_i + 2\delta \\ 1 & \text{for} \quad w_i > \overline{w}_i + 2\delta \end{cases}$$

$$\nabla_{\overline{w}_i}\eta^i_\delta(w_i,\overline{w}_i) = \begin{cases} 0 & \text{for} \quad w_i \le \overline{w}_i \\ -\frac{w_i-\overline{w}_i}{2\delta} & \text{for} \quad \overline{w}_i < w_i \le \overline{w}_i + 2\delta \\ -1 & \text{for} \quad w_i > \overline{w}_i + 2\delta \end{cases}$$

Proof of Theorem 1.2

To the above entropy inequality as a test function we can insert $\varphi(t,x,\overline{t},\overline{x}) \in C^1_c(((0,T)\times\mathbb{R})^2,\mathbb{R})$. Then for some fixed $(\overline{t},\overline{x})$ the inequality takes the form

$$\int\limits_{[0,T)\times\mathbb{R}} \{\partial_t\varphi(t,x,\overline{t},\overline{x})[\eta^1_\delta(w_1,\overline{w}_1)+\eta^2_\delta(w_2,\overline{w}_2)] + \partial_x\varphi(t,x,\overline{t},\overline{x})[q^1_\delta(w_1,\overline{w}_1)+q^2_\delta(w_2,\overline{w}_2)]$$
$$+\varphi(t,x,\overline{t},\overline{x})g(t,x)[\nabla_{w_2}\eta^2_\delta(w_2,\overline{w}_2) - \nabla_{w_1}\eta^1_\delta(w_1,\overline{w}_1)]\}dtdx \ge 0$$

In the same manner with (t,x) fixed, the following inequality can be obtained

$$\int\limits_{[0,T)\times\mathbb{R}} \{\partial_{\overline{t}}\varphi(t,x,\overline{t},\overline{x})[\eta^1_\delta(w_1,\overline{w}_1)+\eta^2_\delta(w_2,\overline{w}_2)] + \partial_{\overline{x}}\varphi(t,x,\overline{t},\overline{x})[q^1_\delta(w_1,\overline{w}_1)+q^2_\delta(w_2,\overline{w}_2)]$$
$$-\varphi(t,x,\overline{t},\overline{x})\overline{g}(\overline{t},\overline{x})[\nabla_{w_2}\eta^2_\delta(w_2,\overline{w}_2) - \nabla_{w_1}\eta^1_\delta(w_1,\overline{w}_1)]\}d\overline{t}d\overline{x} \ge 0$$

Then further on integrating both inequalities with respect to $(\overline{t},\overline{x})$ the first one and with respect to (t,x) the second one, then adding them leads to the following result

$$\int\limits_{([0,T)\times\mathbb{R})^2} \{(\partial_t+\partial_{\overline{t}})\varphi(t,x,\overline{t},\overline{x})[\eta^1_\delta(w_1,\overline{w}_1)+\eta^2_\delta(w_2,\overline{w}_2)]$$
$$+(\partial_x+\partial_{\overline{x}})\varphi(t,x,\overline{t},\overline{x})[q^1_\delta(w_1,\overline{w}_1)+q^2_\delta(w_2,\overline{w}_2)]$$
$$+\varphi(t,x,\overline{t},\overline{x})[g(t,x)-\overline{g}(\overline{t},\overline{x})][\nabla_{w_2}\eta^2_\delta(w_2,\overline{w}_2) - \nabla_{w_1}\eta^1_\delta(w_1,\overline{w}_1)]\}dtdxd\overline{t}d\overline{x} \ge 0 \tag{4.2}$$

To do the next step we choose the test function in such a way that for a fixed $\xi:\mathbb{R}\longrightarrow\mathbb{R}_+$ smooth function satisfying

$$\int\limits_{\mathbb{R}} \xi(x)dx = 1$$

$$\varphi(t,x,\overline{t},\overline{x}) = \frac{1}{\epsilon^2}\psi\left(\frac{t+\overline{t}}{2},\frac{x+\overline{x}}{2}\right)\xi\left(\frac{t-\overline{t}}{2\epsilon}\right)\xi\left(\frac{x-\overline{x}}{2\epsilon}\right)$$

where $\psi:\mathbb{R}\times\mathbb{R}_+\longrightarrow\mathbb{R}_+$ is a Lipschitz continuous function. Letting $\epsilon\searrow 0$ we find

$$(\partial_t+\partial_{\overline{t}})\varphi(t,x,\overline{t},\overline{x}) = \frac{1}{\epsilon^2}\partial_x\psi\left(\frac{t+\overline{t}}{2},\frac{x+\overline{x}}{2}\right)\xi\left(\frac{t-\overline{t}}{2\epsilon}\right)\xi\left(\frac{x-\overline{x}}{2\epsilon}\right)$$

$$(\partial_x+\partial_{\overline{x}})\varphi(t,x,\overline{t},\overline{x}) = \frac{1}{\epsilon^2}\partial_x\psi\left(\frac{t+\overline{t}}{2},\frac{x+\overline{x}}{2}\right)\xi\left(\frac{t-\overline{t}}{2\epsilon}\right)\xi\left(\frac{x-\overline{x}}{2\epsilon}\right)$$

the inequality (4.2) yields to

$$\int\limits_{[0,T)\times\mathbb{R}} \{\partial_t\psi(t,x)[\eta^1_\delta(w_1,\overline{w}_1)+\eta^2_\delta(w_2,\overline{w}_2)]+\partial_x\psi(t,x)[q^1_\delta(w_1,\overline{w}_1)+q^2_\delta(w_2,\overline{w}_2)]$$
$$+\psi(t,x)[g(t,x)-\overline{g}(\overline{t},\overline{x})][\nabla_{w_2}\eta^2_\delta(w_2,\overline{w}_2)-\nabla_{w_1}\eta^1_\delta(w_1,\overline{w}_1)]\}dtdx\geq 0 \tag{4.3}$$

In above inequality let us use as a test function $\psi(t,x)=\zeta_r(x)\theta_\epsilon(t)$ where

$$\zeta_r(x)=\begin{cases} 0 & |x|>r+1 \\ r+1-|x| & r<|x|<r+1 \\ 1 & |x|<r \end{cases} \tag{4.4}$$

and

$$\theta_\epsilon(t)=\begin{cases} 0 & 0<t\leq s \text{ or } t>\tau+\epsilon \\ 1 & s+\epsilon<t\leq\tau \\ \frac{1}{\epsilon}t-\frac{s}{\epsilon} & s<t\leq s+\epsilon \\ -\frac{1}{\epsilon}t+1+\frac{\tau}{\epsilon} & \tau<t\leq\tau+\epsilon \end{cases} \tag{4.5}$$

for $r>0,\ 0<s<\tau,\ 0<\epsilon<\tau-\epsilon$.
According to Lemma 2.1 the nonpositive term containing function g can be omitted.

$$\frac{1}{\epsilon}\int\limits_{-r}^{r}\int\limits_{\tau}^{\tau+\epsilon}[\eta_\delta(w_1,\overline{w}_1)+\eta_\delta(w_2,\overline{w}_2)]dtdx-\frac{1}{\epsilon}\int\limits_{\{r<|x|<r+1\}}\int\limits_{s}^{\tau+\epsilon}\theta_\epsilon(t)[q_\delta(w_1,\overline{w}_1)+q_\delta(w_2,\overline{w}_2)]dtdx$$
$$\leq\frac{1}{\epsilon}\int\limits_{-r}^{r}\int\limits_{s}^{s+\epsilon}[\eta_\delta(w_1,\overline{w}_1)+\eta_\delta(w_2,\overline{w}_2)]dtdx$$

Let $s\searrow 0,\ \epsilon\searrow 0$. Then using continuity with respect to t (i.e. $w,\ \overline{w}\in C^0\left([0,T);L^1_{loc}(\mathbb{R};\mathbb{R}^2)\right)$) we have

$$\int\limits_{-r}^{r}[\eta_\delta(w_1,\overline{w}_1)+\eta_\delta(w_2,\overline{w}_2)]dx-\int\limits_{0}^{\tau}\int\limits_{\{r<|x|<r+1\}}\theta_\epsilon(t)[q_\delta(w_1,\overline{w}_1)+q_\delta(w_2,\overline{w}_2)]dxdt$$
$$\leq\int\limits_{-(r+1)}^{(r+1)}[\eta_\delta(w^0_1,\overline{w}^0_1)+\eta_\delta(w^0_2,\overline{w}^0_2)]dx$$

for all $t\in[0,T)$. Letting $\delta\searrow 0,\ r\to\infty$ we conclude with standard dominated Lebesgue convergence theorem argument

$$\int\limits_{\mathbb{R}}\{[w_1(\tau,x)-\overline{w}_1(\tau,x)]^+ + [w_2(\tau,x)-\overline{w}_2(\tau,x)]^+\}dx$$
$$\leq\int\limits_{\mathbb{R}}\{[w^0_1(x)-\overline{w}^0_1(x)]^+ + [w^0_2(x)-\overline{w}^0_2(x)]^+\}dx$$

Interchanging w_i with $\overline{w}_i$ leads to the analogous inequality. Adding both inequalities yields the assertion of the theorem. □

Acknowledgment
One of the authors (P.G.) appreciates the grant of SFB 298 at Damstadt University of Technology and the grant of Weierstrass Institute for Applied Analysis and Stochastics, Berlin. He would like to thank Professor Dr Krzysztof Wilmański for hospitality. The other one (A.Ś.) appeciates DFG Gradiuertenkolleg - Modellierung und numerische Beschreibung technischer Stroemungen at Darmstadt University of Technology. Both authors would like to thank Professor Dr Reinhard Farwig for advising and support.

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