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Optimization problems for curved mechanical structures

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Abstract

We study the optimization of three dimensional curved rods and of shells under minimal regularity assumptions for the geometry. The results that we establish concern the existence of optimal shapes and the sensitivity analysis. We also compute several numerical examples for the curved rods. The models that we use have been investigated in our previous work [11], [16] and a complete study of the Kirchhoff-Love arches and their optimization has been performed in [10].

1 Introduction

The scientific literature concerning the modeling of curved mechanical structures offers presently a variety of mathematical models for the study of the displacement of such an elastic body under the action of various internal or external forces and tractions. We refer just to the monographs of Ciarlet [9], Trabucho and Viaño [17], Antman [2], where a very rich material can be found for investigations in this direction.

It is a natural question now to develop a research program concerning the optimization of such objects, including numerical experiments. It should be mentioned that there are already several works of interest discussing such problems, Chenais and Rousselet [8], Rousselet [14], Myslinski, Piekarski and Rousselet [12], Sprekels and Tiba [15], Ignat, Sprekels and Tiba [10], etc.

In this article, we attempt a general analysis of optimization problems associated to curved rods and shells. The generality of our setting is related to the consideration of a general performance index, of general constraints on the geometry, the relaxation of the regularity assumptions, and the implementation of numerical experiments. In particular, we are assuming just C^2-regularity, instead of the usual C^3-hypotheses from the literature. For shells, we obtain this by using the generalized Naghdi-type model introduced in Sprekels and Tiba [16]. For rods, this is achieved by replacing the classical Frenet frame with a new general algebraic construction that will be introduced in Section 2. Other variants of local coordinates systems associated to three-dimensional curves under low regularity conditions may be found in Cartan [6] (the Darboux frame) or in Ignat, Sprekels and Tiba [11] from where we take the linear model that we are using. It consists of a system of nine ordinary differential equations with clamped boundary conditions, written in the weak form. Comparing with the regularity assumptions from the modeling process, we see that the optimization hypotheses are minimal.

Our approach allows to minimize within the class of curved rods of a prescribed length, which is a natural condition in applications. This is preserved as well by the variations that we are using, according to Section 4. We also show how to avoid certain degenerate cases: rods of zero length or with multiple points.

It is also to be noticed that, besides the fact that we have general constraints on the geometry, in certain important examples the parametrization used here allows to re-express them in a convex way. The optimization problems considered in this paper are nonconvex, but the convexity of the constraints set is very helpful in the numerical experiments.

The plan of the paper is as follows. We start with the theoretical discussion of optimization problems for curved rods. In Section 2, we indicate the necessary preliminaries and the formulation of the problem. In Section 3, we prove the existence of the solution (while uniqueness is not valid, in general), and in Section 4 we perform a sensitivity study.

A similar program is carried out in Sections 5, 6 and 7, in connection with the study of optimal shell configurations. Our basic assumption is that the geometry of the shell can be described by some mapping in $C^2(\bar{\omega})$. While this setting still allows for many applications, it is also helpful as it reduces the complexity of the problem and of the notations.

We underline that, in order to prove the existence of optimal shapes, coercivity inequalities of Korn-type, uniform with respect to the geometry, are to be established (in Sections 3 and 6). In particular, for shells the extension property in Lipschitz domains, Adams [1], plays an essential role.

In the last section, we present some numerical experiments for optimization problems for three-dimensional curved rods. Their difficulty is already quite big and shows that the numerical treatment of the shells optimization (which is not considered here) would require very special numerical approximation methods.

2 Description of the curved rods problem

Let $\bar{\theta} = (\theta_1, \theta_2, \theta_3) \in C^k[0, L]^3; k \in I\!N$, be a three-dimensional Jordan curve of length $L > 0$, and let $\bar{t} = (t_1, t_2, t_3) \in C^{k-1}[0, L]^3$ be its tangent vector. We shall always assume that $\bar{\theta}$ originates in the origin of the coordinates system and that it is parametrized with respect to its arc length, i.e. $|\bar{t}(x_3)|_{I\!R^3} = 1 \quad \forall\, x_3 \in [0, L]$.

Then, alternatively, we may consider $\varphi \in C^{k-1}[0, L]$ and $\psi \in C^{k-1}[0, L]$ to be some spherical coordinates of a unit vector given by $(\sin\varphi\cos\psi, \sin\varphi\sin\psi, \cos\varphi) \in C^{k-1}[0, L]$ which we denote again by $\bar{t}$. The corresponding three-dimensional curve is obtained by

$$\bar{\theta}(x_3) = \int_0^{x_3} \bar{t}(\tau)\, d\tau\,, \quad x_3 \in [0, L]\,. \tag{2.1}$$

Notice that, although the polar coordinates may not be uniquely determined in certain cases, relation (2.1) with arbitrary φ, ψ generates a rich class of three-dimensional regular curves having C^k-regularity, which is enough for optimization applications.

One advantage of the form (2.1) is that the curve is automatically parametrized with respect to its arc length, and that a local frame may be defined by purely algebraic means

$$\bar{n} = (\cos\varphi\cos\psi\,,\ \cos\varphi\sin\psi\,,\ -\sin\varphi) \tag{2.2}$$

$$\bar{b} = (-\sin\psi\,,\ \cos\psi\,,\ 0) \tag{2.3}$$

in all the points of the curve.

We denote by A the orthogonal matrix having the columns $\bar{t}, \bar{n}, \bar{b}$. The geometric meaning of this construction is that we perform a rotation of the global axes system, corresponding to the angles φ and ψ and indicated by A, i.e., $\bar{t} = A(1,0,0)^T$, $\bar{n} = A(0,1,0)^T$, $\bar{b} = A(0,0,1)^T$.

Remark 2.1 It is possible to apply (2.1)– (2.3) to absolutely continuous regular (i.e. with non-zero tangent) curves, after reparametrization with respect to the arc length. Although we employ the same notations, the vectors $\bar{n}$, $\bar{b}$ are different, in general, from the normal and binormal vectors of the classical Frenet frame obtained under stronger regularity assumptions, Bloch [4]. Other useful variants of local frames with low smoothness hypotheses may be found in Cartan [6], Ignat, Sprekels and Tiba [11].

We introduce the open set

$$\Omega = \bigcup_{x_3 \in]0,L[} \Big(\omega(x_3) \times \{x_3\}\Big) \subset I\!R^3 \tag{2.4}$$

with $\omega(x_3) \subset I\!R^2$, $x_3 \in [0, L]$, being a bounded domain, not necessarily simply connected, such that $\omega(x_3) \supset \omega$, an open subset of $I\!R^2$ satisfying the symmetry relations

$$0 = \int_\omega x_1\, dx_1\, dx_2 = \int_\omega x_2\, dx_1\, dx_2 = \int_\omega x_1\, x_2\, dx_1\, dx_2\,. \tag{2.5}$$

The curved rod $\tilde{\Omega}$ associated with $\bar{\theta}$ is obtained by the one-to-one transformation $F : \Omega \to \tilde{\Omega}$,

$$\begin{aligned} & (x_1, x_2, x_3) = \bar{x} \in \Omega \mapsto F\bar{x} = \tilde{x} = (\tilde{x_1}, \tilde{x_2}, \tilde{x_3}) \\ = \ & \bar{\theta}(x_3) + x_1\, \bar{n}(x_3) + x_2\, \bar{b}(x_3) \in \tilde{\Omega}, \quad \forall\, \bar{x} \in \Omega; \\ & \text{where } \tilde{\Omega} = \{\tilde{x} = F\,\bar{x}\,;\ \bar{x} \in \Omega\}. \end{aligned} \tag{2.6}$$

In the sequel, we will always assume that $\varphi, \psi \in C^1[0, L]$, i.e. $k = 2$. Then we get

$$\Big\langle \bar{t}(x_3)\,,\ \bar{t}'(x_3) \Big\rangle_{I\!R^3} = \Big\langle \bar{n}(x_3)\,,\ \bar{n}'(x_3) \Big\rangle_{I\!R^3} = \Big\langle \bar{b}(x_3)\,,\ \bar{b}'(x_3) \Big\rangle_{I\!R^3} = 0\,, \tag{2.7}$$

which yields the "equations of motion" of the considered local frame:

$$\begin{aligned} \bar{t}'(x_3) &= a(x_3)\,\bar{b}(x_3) + \beta(x_3)\,\bar{n}(x_3)\,, \\ \bar{b}'(x_3) &= -\,a(x_3)\,\bar{t}(x_3) + c(x_3)\,\bar{n}(x_3)\,, \\ \bar{n}'(x_3) &= -\,\beta(x_3)\,\bar{t}(x_3) + c(x_3)\,\bar{b}(x_3)\,, \end{aligned} \tag{2.8}$$

with $a\,,\ \beta\,,\ c \in C[0,L]$ expressing the curvature and torsion properties of the curved rod.

The Jacobian of F, denoted by $J(\bar{x}) = D\,F(\bar{x})$, is given by

$$J(\bar{x}) = \begin{bmatrix} n_1(x_3) & b_1(x_3) & t_1(x_3) + x_1\,n_1'(x_3) + x_2\,b_1'(x_3) \\ n_2(x_3) & b_2(x_3) & t_2(x_3) + x_1\,n_2'(x_3) + x_2\,b_2'(x_3) \\ n_3(x_3) & b_3(x_3) & t_3(x_3) + x_1\,n_3'(x_3) + x_2\,b_3'(x_3) \end{bmatrix}. \tag{2.9}$$

By (2.8), (2.9), we have

$$\det J(\bar{x}) = 1 - \beta(x_3)\,x_1 - a(x_3)\,x_2\,, \quad \forall\, \bar{x} \in \bar{\Omega}\,. \tag{2.10}$$

If $\omega(x_3)\,,\ x_3 \in [0,L]$, is contained in a sufficiently small disk in $I\!R^2$, then we may assume that

$$\det J(\bar{x}) \geq c > 0\,, \quad \forall\, \bar{x} \in \bar{\Omega}\,, \tag{2.11}$$

which justifies the introduction of the curved rod $\tilde{\Omega}$ via the geometric transformation F, Ciarlet [9], Thm. 3.1–1.

We assume that it is clamped at both ends, and that it is subjected to body forces $\tilde{f} \in L^2(\tilde{\Omega})^3$ (weight, electromagnetic field, etc.), as well as to surface tractions $\tilde{g}$ on the lateral surface of the rod, denoted by $\tilde{\Sigma}$, $\tilde{g} \in L^2(\tilde{\Sigma})^3$. On the "inside" lateral face of $\tilde{\Omega}$ (i.e. corresponding to the possible holes), we take $\tilde{g} \equiv 0$.

Denote by $\bar{y} : \tilde{\Omega} \to I\!R^3$ the corresponding displacement of each point $\tilde{x} \in \tilde{\Omega}$. In Ignat, Sprekels and Tiba [11], the general geometrical assumption that

$$\bar{y}(\tilde{x}) = \bar{\tau}(x_3) + x_1\,\bar{N}(x_3) + x_2\,\bar{B}(x_3)\,, \quad \forall\, \tilde{x} \in \tilde{\Omega}\,, \tag{2.12}$$

with $\bar{x} = (x_1\,, x_2\,, x_3) = F^{-1}(\tilde{x})$ and $\bar{\tau}\,,\ \bar{N}\,,\ \bar{B} \in H_0^1(0,L)$ unknown functions, is imposed. This is a special case of the so-called *polynomial approximation* of the displacement, see Trabucho and Viaño [17]. Then, the following boundary value problem is obtained from the elasticity problem:

$$\begin{aligned} B(\bar{y},\bar{v}) &= \tilde{\lambda} \int_\Omega \sum_{i,j=1}^{3} \Big[N_i(x_3)\,h_{1i}(\bar{x}) + B_i(x_3)\,h_{2i}(\bar{x}) \\ &+ \Big(\tau_i'(x_3) + x_1\,N_i'(x_3) + x_2\,B_i'(x_3)\Big) h_{3i}(\bar{x}) \Big] \Big[M_j(x_3)\,h_{1j}(\bar{x}) \\ &+ D_j(x_3)\,h_{2j}(\bar{x}) + \Big(\mu_j'(x_3) + x_1\,M_j'(x_3) + x_2\,D_j'(x_3)\Big)\,h_{3j}(\bar{x}) \Big] \Big|\det J(\bar{x})\Big|\,d\bar{x} \\ &+ \tilde{\mu} \int_\Omega \sum_{i<j} \Big[N_i(x_3)\,h_{1j}(\bar{x}) + B_i(x_3)\,h_{2j}(\bar{x}) + \Big(\tau_i'(x_3) + x_1\,N_i'(x_3) \end{aligned}$$

$$
\begin{aligned}
& + x_2\, B_i'(x_3)\Big)\, h_{3j}(\bar{x}) + N_j(x_3)\, h_{1i}(\bar{x}) + B_j(x_3)\, h_{2i}(\bar{x}) \\
+ \quad & \Big(\tau_j'(x_3) + x_1\, N_j'(x_3) + x_2\, B_j'(x_3)\Big)\, h_{3i}(\bar{x})\Bigg]\Bigg[M_i(x_3)\, h_{1j}(\bar{x}) + D_i(x_3)\, h_{2j}(\bar{x}) \\
+ \quad & \Big(\mu_i'(x_3) + x_1\, M_i'(x_3) + x_2\, D_i'(x_3)\Big)\, h_{3j}(\bar{x}) + M_j(x_3)\, h_{1i}(\bar{x}) + D_j(x_3)\, h_{2i}(\bar{x}) \\
+ \quad & \Big(\mu_j'(x_3) + x_1\, M_j'(x_3) + x_2\, D_j'(x_3)\Big)\, h_{3i}(\bar{x})\Bigg]\Big|\det J(\bar{x})\Big|\, d\bar{x} \\
+ \quad & 2\,\tilde{\mu}\int_\Omega \sum_{i=1}^3 \Bigg[N_i(x_3)\, h_{1i}(\bar{x}) + B_i(x_3)\, h_{2i}(\bar{x}) + \Big(\tau_i'(x_3) + x_1\, N_i'(x_3) \\
& + x_2\, B_i'(x_3)\Big)\, h_{3i}(\bar{x})\Bigg]\Bigg[M_i(x_3)\, h_{1i}(\bar{x}) + D_i(x_3)\, h_{2i}(\bar{x}) \\
+ \quad & \Big(\mu_i'(x_3) + x_1\, M_i'(x_3) + x_2\, D_i'(x_3)\Big) h_{3i}(\bar{x})\Bigg]\Big|\det J(\bar{x})\Big|\, d\bar{x} \\
= \quad & \sum_{l=1}^3 \int_\Omega f_l(\bar{x})\Big(\mu_l(x_3) + x_1\, M_l(x_3) + x_2\, D_l(x_3)\Big)\,|\det J(\bar{x})|\, d\bar{x} \\
+ \quad & \sum_{i,j=1}^3 \sum_{l=1}^3 \int_{\partial\Omega} g_l(\bar{x})\Big(\mu_l(x_3) + x_1\, M_l(x_3) + x_2\, D_l(x_3)\Big)|\det J(\bar{x})| \\
& \sqrt{\nu_i(\bar{x})\, g^{ij}(\bar{x})\, \nu_j(\bar{x})}\, d\tau\, . \qquad (2.13)
\end{aligned}
$$

Above, $\tilde{\lambda} \geq 0\,, \tilde{\mu} > 0$ are the Lamé coefficients of the material, the matrices $(h_{ij}(\bar{x})) = J(\bar{x})^{-1}$ and $(g^{ij}(\bar{x})) = (g_{ij}(\bar{x}))^{-1}\,,\ (g_{ij}(\bar{x})) = J(\bar{x})^T\, J(\bar{x})$ and $\bar{\mu}\,, \bar{M}\,, \bar{D} \in H_0^1(0,L)^3$ are arbitrary test functions with $\bar{v}(\bar{x}) = \bar{\mu}(x_3) + x_1\, \bar{M}(x_3) + x_2\, \bar{D}(x_3)\,$. More details and the proof of the coercivity of the bilinear functional $B(\,\cdot\,,\,\cdot\,)$: $H_0^1(0,L)^9 \times H_0^1(0,L)^9 \to I\!R$, given by (2.13), may be found in Ignat, Sprekels and Tiba [11], where different local bases are used. This yields the existence and the uniqueness of the solution $\bar{y}$ of (2.13), in $H_0^1(0,L)^9$.

In the sequel, we shall suppose that $\omega(x_3) = \omega$ for $x_3 \in [0,L]$, with ω satisfying (2.5).

For given $\bar{f}\,,\, \bar{g}$, a general shape optimization problem associated to (2.13) is

$$
\operatorname*{Min}_{\varphi,\psi}\ \Big\{\Pi(\varphi,\psi)\ =\ j(\bar{\theta},\bar{y})\Big\}\,, \qquad (\mathbf{P})
$$

subject to $\bar{\theta} \in \mathcal{K} \subset C^2[0,L]^3$, a closed bounded subset, and $\bar{y} = (\bar{\tau},\bar{N},\bar{B}) \in H_0^1(0,L)^9$ obtained as the solution of (2.13). We assume that the mapping j : $C^2[0,L]^3 \times H_0^1(0,L)^9 \to I\!R$ and satisfies some regularity properties. An important case for a cost functional j is the quadratic case. For instance, if

$$j(\bar{\theta}, \bar{y}) = |\tau_1|^2_{H_0^1(0,L)} + |\tau_2|^2_{H_0^1(0,L)} + |\tau_3|^2_{H_0^1(0,L)}, \tag{2.14}$$

then **(P)** aims at finding the shape of the curved rod that minimizes the displacement of the line of centroids under the prescribed forces and tractions. This is a natural safety requirement in many applications.

Concerning the constraints to which the curved rod may be submitted, we underline that our formalism automatically ensures a prescribed length $L > 0$. This eliminates possible trivial cases, such as $\bar{\theta}$ constant in $[0, L]$, and is also important from the optimization point of view, since otherwise the cost may depend on L. A simple sufficient condition under which $\bar{\theta}$ has no multiple points is

$$0 \leq \varphi(x_3) \leq \frac{\pi}{2} - \varepsilon, \quad x_3 \in [0, L], \tag{2.15}$$

with $\varepsilon > 0$ small. It may be used in problems concerning the optimization of strings, where the periodicity condition (for θ_1, θ_2)

$$\int_0^L t_1 \, dx_3 = \int_0^L t_2 \, dx_3 = 0 \tag{2.16}$$

is also important.

Notice that relations (2.14), (2.15) correspond to convex optimization problems, while relation (2.16) is nonlinear in φ, ψ and, consequently, the corresponding set $\mathcal{K}$ is nonconvex. Relation (2.11) should also be included in the definition of $\mathcal{K}$.

Remark 2.2 A very simple variant of representation of the unit tangent vector is $\bar{t} = (u_1, u_2, \sqrt{1 - u_1^2 - u_2^2})$, but this already assumes a prescribed sign for t_3 and requires the most restrictive hypothesis

$$u_1^2 + u_2^2 \leq 1 - \varepsilon$$

for the differentiability of the local frame. However, under this representation relation (2.16) becomes linear, which may be useful in some applications.

3 Existence of optimal curved rods

We prove the following continuous dependence result:

Theorem 3.1 *Assume that* $\varphi_n \to \varphi$, $\psi_n \to \psi$ *strongly in* $C^1[0, L]$. *If* $\bar{y}_n$, $\check{y}$ *denote the solutions to* (2.13) *associated with* (φ_n, ψ_n), *respectively with* (φ, ψ), *then*

$$\bar{y}_n \to \check{y} \quad \textit{strongly in} \;\; H_0^1(0, L)^9. \tag{3.1}$$

Proof. Clearly, we have

$$\bar{t}_n = (\cos\psi_n \,\sin\varphi_n \,,\, \sin\psi_n \,\sin\varphi_n \,,\, \cos\varphi_n) \to \bar{t} = (\cos\psi \,\sin\varphi \,,\, \sin\psi \,\sin\varphi \,,\, \cos\varphi) \tag{3.2}$$

in $C^1[0,L]^3$. Then, (2.1) shows that $\bar{\theta}_n \to \bar{\theta}$ in $C^2[0,L]^3$. By (2.2), (2.3), and with obvious notations, we get that $\bar{n}_n \to \bar{n}$ and $\bar{b}_n \to \bar{b}$ in $C^1[0,L]^3$.

From (2.8) it is easy to infer that

$$a_n \;=\; \langle \bar{t}_n' \,,\, \bar{b}_n \rangle_{I\!R^3} \;\to\; a \;=\; \langle \bar{t}' \,,\, \bar{b} \rangle_{I\!R^3} \quad \text{strongly in } C[0,L]\,. \tag{3.3}$$

Here, $\langle \cdot \,,\, \cdot \rangle$ is the inner product in $I\!R^3$, and we also have $\beta_n \to \beta$, $c_n \to c$ in $C[0,L]$.

Relation (2.10) shows that

$$\det J_n(\bar{x}) \;\to\; \det J(\bar{x})\,, \quad \text{in } C(\bar{\Omega}) \tag{3.4}$$

and, by (2.11), we see that $\{\det J_n(\bar{x})\}$ is bounded from below by some positive constant.

Moreover, (2.9) gives clearly that $J_n(\bar{x}) \;\to\; J(\bar{x})$ in $C(\bar{\Omega})^9$, and, likewise, that $J_n^{-1}(\bar{x}) \to J^{-1}(\bar{x})$, by (3.4) and the above observations. In particular, we have that

$$h_{ij}^n(\bar{x}) \;\to\; h_{ij}(\bar{x}) \quad \text{in } C(\bar{\Omega})\,. \tag{3.5}$$

Let B_n denote the bilinear functional (2.13) with coefficients h_{ij}^n, $\det J_n$.

Lemma 3.2 *There are $c_1 > 0$, $c_2 > 0$ such that*

$$B_n(\bar{y},\bar{y}) \;\geq\; c_1|\bar{y}|^2_{H_0^1(0,L)^9} \;-\; c_2|\bar{y}|^2_{L^2(0,L)^9} \tag{3.6}$$

for any $\bar{y} \in H_0^1(0,L)^9$ and any $n \in I\!N$.

Proof. By (3.4) and (2.11), we have

$$\begin{aligned} B_n(\bar{y},\bar{y}) \;\;\geq\;\; & \tilde{\mu}\, c \int_\Omega \sum_{i<j} \Big[N_i\, h_{1j}^n + B_i\, h_{2j}^n + \Big(\tau_i' + x_1\, N_i' + x_2\, B_i'\Big)\, h_{3j}^n \\ & \qquad + N_j\, h_{1i}^n + B_j\, h_{2i}^n + \Big(\tau_j' + x_1\, N_j' + x_2\, B_j'\Big)\, h_{3i}^n \Big]^2 d\bar{x} \\ & + 2\,\tilde{\mu}\, c \int_\Omega \sum_{i=1}^{3} \Big[N_i\, h_{1i}^n + B_i\, h_{2i}^n + \Big(\tau_i' + x_1\, N_i' + x_2 B_i'\Big)\, h_{3i}^n \Big]^2 d\bar{x}\,. \end{aligned}$$

By the uniform boundedness of the coefficients due to (3.5) and by usual binomial inequalities, we get

$$\begin{aligned} \frac{1}{\tilde{\mu}\, c}\, B_n(\bar{y},\bar{y}) \;\;\geq\;\; & \frac{1}{2} \int_\Omega \sum_{i<j} \Big[\Big(\tau_i' + x_1\, N_i' + x_2\, B_i'\Big)\, h_{3j}^n + \Big(\tau_j' + x_1\, N_j' + x_2\, B_j'\Big)\, h_{3i}^n \Big]^2 d\bar{x} \\ & + \int_\Omega \sum_{i=1}^{3} \Big[\Big(\tau_i' + x_1\, N_i' + x_2\, B_i'\Big)\, h_{3i}^n \Big]^2 d\bar{x} \;-\; \tilde{c}|\bar{y}|^2_{L^2(0,L)^9}\,. \end{aligned}$$

We use the following algebraic identity:

$$\frac{1}{2}\left[\left(z_1\,h^n_{32}+z_2\,h^n_{31}\right)^2+\left(z_2\,h^n_{33}+z_3\,h^n_{32}\right)^2+\left(z_1\,h^n_{33}+z_3\,h^n_{31}\right)^2\right]$$

$$+\frac{3}{2}\left[z_1^2(h^n_{31})^2+z_2^2(h^n_{32})^2+z_3^2(h^n_{33})^2\right]$$

$$=\ \frac{1}{2}\left(z_1^2+z_2^2+z_3^2\right)\left[(h^n_{31})^2+(h^n_{32})^2+(h^n_{33})^2\right]+\frac{1}{2}\left(z_1\,h^n_{31}+z_2\,h^n_{32}\right)^2$$

$$+\frac{1}{2}\left(z_1\,h^n_{31}+z_3\,h^n_{33}\right)^2+\frac{1}{2}\left(z_2\,h^n_{32}+z_3\,h^n_{33}\right)^2,$$

with $z_i := \tau_i' + x_1\,N_i' + x_2\,B_i'$, $i=\overline{1,3}$. It follows

$$\frac{1}{\tilde\mu\,c}B_n(\bar y,\bar y)\ \geq\ \frac{1}{4}\int_\Omega\sum_{i=1}^3\left(\tau_i'+x_1\,N_i'+x_2\,B_i'\right)^2\sum_{i=1}^3(h^n_{3i})^2\,d\bar x-\tilde c\,|\bar y|^2_{L^2(0,L)^9}\,. \tag{3.7}$$

A direct calculus allows to find h^n_{ij} and to check that, for some $k>0$,

$$\sum_{i=1}^3(h^n_{3i})^2\ =\ \left[\det J_n\right]^{-2}\sum_{i=1}^3(t^n_i)^2\ =\ \left[\det J_n\right]^{-2}\ \geq\ k\ >\ 0 \tag{3.8}$$

since $|\bar t_n|_{I\!R^3}=1$.

Relations (3.7), (3.8) give

$$\frac{1}{\tilde\mu\,c}B_n(\bar y,\bar y)\ \geq\ \frac{k}{4}\int_\Omega\sum_{i=1}^3\left(\tau_i'+x_1\,N_i'+x_2\,B_i'\right)^2d\bar x\ -\ \tilde c\,|\bar y|^2_{L^2(0,L)^9}\,.$$

Performing the computations in the right-hand side, and integrating with respect to x_1, x_2, we obtain the inequality (3.6) by means of (2.5). □

Proof of Theorem 3.1 (continued). We use a contradiction argument to show that the functionals B_n are uniformly coercive. We assume that there is a sequence $\varepsilon_n\to 0$ and a sequence $\tilde y_n\in H^1_0(0,L)^9$, $|\tilde y_n|_{H^1_0(0,L)^9}=1$, such that

$$0\ \leq\ B_n(\tilde y_n,\tilde y_n)\ \leq\ \varepsilon_n|\tilde y_n|^2_{H^1_0(0,L)^9}\,. \tag{3.9}$$

Let $\hat y$ be the weak limit of $\tilde y_n$ in $H^1_0(0,L)^9$, which may be supposed to exist.

We give a detailed computation for the last integral in the definition of $B_n(\tilde y_n,\tilde y_n)$:

$$I_n\ =\ 2\,\tilde\mu\int_\Omega\sum_{i=1}^3\left[\tilde N_i^n\,h^n_{1i}+\tilde B_i^n\,h^n_{2i}+\left(\tilde\tau_i^{n'}+x_1\,\tilde N_i^{n'}+x_2\,\tilde B_i^{n'}\right)h^n_{3i}\right]^2\det J^n\,d\bar x$$

$$
\begin{aligned}
= \;& 2\,\tilde{\mu} \int\limits_{\Omega} \sum_{i=1}^{3} \left[\tilde{N}_i^n\, h_{1i} + \tilde{B}_i^n\, h_{2i} + \left(\tilde{\tau}_i^{n'} + x_1\, \tilde{N}_i^{n'} + x_2\, \tilde{B}_i^{n'} \right) h_{3i} \right]^2 \det J\, d\bar{x} \\
& + 2\,\tilde{\mu} \int\limits_{\Omega} \sum_{i=1}^{3} \Bigg[\tilde{N}_i^n \left(h_{1i}^n \sqrt{\det J^n} - h_{1i} \sqrt{\det J} \right) + \tilde{B}_i^n \left(h_{2i}^n \sqrt{\det J^n} - h_{2i} \sqrt{\det J} \right) \\
& \quad + \left(\tilde{\tau}_i^{n'} + x_1\, \tilde{N}_i^{n'} + x_2\, \tilde{B}_i^{n'} \right) \left(h_{3i}^n \sqrt{\det J^n} - h_{3i} \sqrt{\det J} \right) \Bigg] \\
& \Bigg[\tilde{N}_i^n \left(h_{1i}^n \sqrt{\det J^n} + h_{1i} \sqrt{\det J} \right) + \tilde{B}_i^n \left(h_{2i}^n \sqrt{\det J^n} + h_{2i} \sqrt{\det J} \right) \\
& \quad + \left(\tilde{\tau}_i^{n'} + x_1\, \tilde{N}_i^{n'} + x_2\, \tilde{B}_i^{n'} \right) \left(h_{3i}^n \sqrt{\det J^n} + h_{3i} \sqrt{\det J} \right) \Bigg] d\bar{x}\,.
\end{aligned}
$$

Here, $\tilde{y}_n = (\tilde{\tau}_n, \tilde{N}_n, \tilde{B}_n)$ belong to the unit ball in $H_0^1(0,L)^9$. The uniform convergence of the coefficients (see (3.4), (3.5)) shows that the last integral converges to zero. The weak lower semicontinuity of quadratic forms gives

$$
\begin{aligned}
\liminf_{n\to\infty} I_n \;=\;& \liminf_{n\to\infty} 2\tilde{\mu} \int\limits_{\Omega} \sum_{i=1}^{3} \left[\tilde{N}_i^n\, h_{1i} + \tilde{B}_i^n\, h_{2i} + \left(\tilde{\tau}_i^{n'} + x_1\, \tilde{N}_i^{n'} + x_2\, \tilde{B}_i^{n'} \right) h_{3i} \right]^2 \\
& \times \det J d\bar{x} \\
\geq\;& 2\,\tilde{\mu} \int\limits_{\Omega} \sum_{i=1}^{3} \left[\hat{N}_i\, h_{1i} + \hat{B}_i\, h_{2i} + \left(\hat{\tau}_i' + x_1\, \hat{N}_i' + x_2\, \hat{B}_i' \right) h_{3i} \right]^2 \det J\, d\bar{x}\,,
\end{aligned}
$$

where $(\hat{\tau}\,,\, \hat{N}\,,\, \hat{B}) \in H_0^1(0,L)^9$ is the detailed notation of $\hat{y}$.

Computing the other terms in a similar way, we get

$$
\begin{aligned}
\liminf_{n\to\infty} B_n(\tilde{y}_n, \tilde{y}_n) \;\geq\;& 2\,\tilde{\mu} \int\limits_{\Omega} \sum_{i=1}^{3} \Bigg[\hat{N}_i\, h_{1i} + \hat{B}_i\, h_{2i} \\
& \quad + \left(\hat{\tau}_i' + x_1\, \hat{N}_i' + x_2\, \hat{B}_i' \right) h_{3i} \Bigg]^2 \det J\, d\bar{x} \\
& + \tilde{\lambda} \int\limits_{\Omega} \sum_{i,j=1}^{3} \left[\hat{N}_i\, h_{1i} + \hat{B}_i\, h_{2i} + \left(\hat{\tau}_i' + x_1\, \hat{N}_i' + x_2\, \hat{B}_i' \right) h_{3i} \right] \\
& \quad \left[\hat{N}_j\, h_{1j} + \hat{B}_j\, h_{2j} + \left(\hat{\tau}_j' + x_1\, \hat{N}_j' + x_2\, \hat{B}_j' \right) h_{3j} \right] \det J\, d\bar{x} \\
& + \tilde{\mu} \int\limits_{\Omega} \sum_{i<j} \Bigg[\hat{N}_i\, h_{1j} + \hat{B}_i\, h_{2j} + \left(\hat{\tau}_i' + x_1\, \hat{N}_i' + x_2\, \hat{B}_i' \right) h_{3j}
\end{aligned}
$$

$$+\hat{N}_j\, h_{1i} + \hat{B}_j\, h_{2i} + \left(\hat{\tau}_j' + x_1\, \hat{N}_j' + x_2\, \hat{B}_j'\right) h_{3i}\Bigg]^2$$

$$\det J\, d\bar{x} \;=\; B(\hat{y}, \hat{y})\,. \tag{3.10}$$

By assumption (3.9) and by (3.10), we have $B(\hat{y}, \hat{y}) = 0$.

It is known that such a relation yields $\hat{y} = 0$ (see, for instance, Lemma 2.3 in [11]).

We use again the inequality (3.9) with Lemma 3.2,

$$\varepsilon_n \;\geq\; B_n(\tilde{y}_n, \tilde{y}_n) \;\geq\; c_1 \;-\; c_2 |\tilde{y}_n|^2_{L^2(0,L)^9}\,, \tag{3.11}$$

since $|\tilde{y}_n|_{H_0^1(0,L)^9} = 1$.

Notice that $\tilde{y}_n \to \hat{y} = 0$ strongly in $L^2(0,L)^9$, by the above argument. Then, combining (3.10) and (3.11), we obtain the contradiction $0 \geq c_1$. We conclude that there is some $\delta > 0$ such that, $\forall\, n \geq 1$:

$$B_n(\bar{y}, \bar{y}) \;\geq\; \delta |\bar{y}|^2_{H_0^1(0,L)^9}\,, \quad \forall\; \bar{y} \;\in\; H_0^1(0,L)^9\,. \tag{3.12}$$

Let us fix in the corresponding to $B_n(\cdot,\cdot)$ state equations (2.13), $\bar{v} = \bar{y}_n$. Taking into account (3.12), we immediately obtain that $\{\bar{y}_n\}$ is bounded in $H_0^1(0,L)^9$. We may take a subsequence such that $\bar{y}_n \to \check{y}$ weakly in $H_0^1(0,L)^9$. Due to the uniform convergence of the coefficients h_{ij}^n, $\det J_n$, g_n^{ij}, one may pass to the limit in (2.13) and see that $\check{y}$ is indeed the solution of (2.13) associated to $(\varphi\,,\,\psi)$.

The last step of the proof is to show that the convergence is valid in the strong topology of $H_0^1(0,L)^9$. We substract the equations corresponding to (τ^n, N^n, B^n), respectively to $(\check{\tau}, \check{N}, \check{B})$, we intercalate advantageous terms and, finally, we take test functions of the form $(\tau^n, N^n, B^n) - (\check{\tau}, \check{N}, \check{B}) \in H_0^1(0,L)^9$. We write in detail just the simplest term:

$$2\,\tilde{\mu} \int_\Omega \sum_{i=1}^3 \left[N_i^n\, h_{1i}^n + B_1^n\, h_{2i}^n + \left(\tau_i^{n'} + x_1\, N_i^{n'} + x_2\, B_i^{n'}\right) h_{3i}^n\right]$$

$$\Bigg[(N_i^n - \check{N}_i)\, h_{1i}^n + (B_i^n - \check{B}_i)\, h_{2i}^n + \left(\tau_i^{n'} - \check{\tau}_i' + x_1(N_i^{n'} - \check{N}_i') + x_2(B_i^{n'} - \check{B}_i')\right)$$

$$h_{3i}^n\Bigg] \det J^n\, d\bar{x} - 2\,\tilde{\mu} \int_\Omega \sum_{i=1}^3 \left[\check{N}_i\, h_{1i} + \check{B}_i\, h_{2i} + \left(\check{\tau}_i' + x_1\, \check{N}_i' + x_2\, \check{B}_i'\right) h_{3i}\right]$$

$$\Bigg[(N_i^n - \check{N}_i) h_{1i} + (B_i^n - \check{B}_i)\, h_{2i} + \left(\tau_i^{n'} - \check{\tau}_i' + x_1(N_i^{n'} - \check{N}_i') + x_2(B_i^{n'} - \check{B}_i)'\right)$$

$$h_{3i}\Bigg] \det J\, d\bar{x} = 2\,\tilde{\mu} \int_\Omega \sum_{i=1}^3 \Bigg[(N_i^n - \check{N}_i)\, h_{1i} + (B_i^n - \check{B}_i)\, h_{2i}$$

$$+\left(\tau_i^{n'} - \check{\tau}_i' + x_1(N_i^{n'} - \check{N}_i') + x_2(B_i^{n'} - \check{B}_i'\right) h_{3i}\Bigg]^2 \det J\, d\bar{x}$$

$$+ 2\,\tilde{\mu} \int_\Omega \sum_{i=1}^3 \left[N_i^n\, h_{1i} + B_i^n\, h_{2i} + \left(\tau_i^{n'} + x_1\, N_i^{n'} + x_2\, B_i^{n'}\right)\, h_{3i}\right]$$

$$\Bigg[(N_i^n - \check{N}_i)(h_{1i}^n - h_{1i}) + (B_i^n - \check{B}_i)(h_{2i}^n - h_{2i}) + \Big(\tau_i^{n'} - \check{\tau}_i' + x_1(N_i^{n'} - \check{N}_i')$$

$$+ x_2(B_i^{n'} - \check{B}_i')\Big)(h_{3i}^n - h_{3i})\Bigg] \det J\, d\bar{x}$$

$$+ 2\,\tilde{\mu} \int_\Omega \sum_{i=1}^3 \left[N_i^n\, h_{1i} + B_i^n\, h_{2i} + \left(\tau_i^{n'} + x_1\, N_i^{n'} + x_2\, B_i^{n'}\right)\, h_{3i}\right]$$

$$\left[(N_i^n - \check{N}_i)\, h_{1i}^n + (B_i^n - \check{B}_i)\, h_{2i}^n + \left(\tau_i^{n'} - \check{\tau}_i^{n'} + x_1(N_i^{n'} - \check{N}_i') + x_2(B_i^{n'} - \check{B}_i')\right) h_{3i}^n\right]$$

$$\left[\det J^n - \det J\right] d\bar{x}$$

$$+ 2\,\tilde{\mu} \int_\Omega \sum_{i=1}^3 \left[N_i^n(h_{1i}^n - h_{1i}) + B_i^n(h_{2i}^n - h_{2i}) + \left(\tau_i^{n'} + x_1\, N_i^{n'} + x_2\, B_i^{n'}\right)(h_{3i}^n - h_{3i})\right]$$

$$\Bigg[(N_i^n - \check{N}_i)h_{1i}^n + (B_i^n - \check{B}_i)h_{2i}^n$$

$$+\left(\tau_i^{n'} - \check{\tau}_i' + x_1(N_i^{n'} - \check{N}_i') + x_2(B_i^{n'} - \check{B}_i')\right)h_{31}^n\Bigg] \det J^n\, d\bar{\tau}.$$

All the terms above, except the first one after the equality sign (the quadratic one), converge to zero due to the weak convergence of (τ^n, N^n, B^n) and to the uniform convergence of the coefficients. Similar computations may be performed for all the integrals in the variational equations, and we conclude that

$$\lim_{n\to\infty} B(\bar{y}_n - \check{y}\,,\, \bar{y}_n - \check{y}) \;=\; 0\,. \tag{3.13}$$

By (3.12), (3.13) the proof is finished. □

Corollary 3.1 *If $\mathcal{K} \subset C^2[0,L]^3$ is generated by a compact in $C^1[0,L]^3$ subset of $\{\varphi,\psi\}$ and $j : C^2(0,L)^3 \times H_0^1(0,L)^9 \to I\!R$ is lower semicontinuous, then the shape optimization problem* **(P)** *admits at least one optimal curved rod solution in $\mathcal{K}$.*

4 Sensitivity analysis of curved rods

We study first some differentiability properties of the mapping $(\varphi,\psi) \in C^1[0,L]^2 \mapsto \bar{y} \in H_0^1(0,L)^9$, with $\bar{y}$ being the solution of (2.13) corresponding to (φ,ψ). We consider $(\varphi_\lambda, \psi_\lambda) = (\varphi + \lambda\,\gamma\,,\, \psi + \lambda\,\xi) \in C^1[0,1]^2\,,\ \lambda \in I\!R_+\,$, to be some variation around (φ, ψ), and we denote by $\bar{y}_\lambda = (\bar{\tau}_\lambda, \bar{N}_\lambda, \bar{B}_\lambda) \in H^1_0(0,L)^9$ the corresponding solution of (2.13). Similarly, we denote by $\bar{t}_\lambda, \bar{\theta}_\lambda, \bar{n}_\lambda, \bar{b}_\lambda, a_\lambda, \beta_\lambda, c_\lambda, J_\lambda, h^\lambda_{ij}, g^{ij}_\lambda$ all the quantities defined in Section 2, starting from $(\varphi_\lambda, \psi_\lambda)$. Notice that, by our construction, the perturbed curved rod $\bar{\theta}_\lambda$ has length L and is parametrized with respect to its arc length, i.e. $|\bar{t}_\lambda|_{I\!R^3} = 1$.

It is elementary, though tedious, to check that all the below listed limits and operators exist and satisfy the indicated properties

$$\lim_{\lambda\to 0} \frac{\bar{t}_\lambda - \bar{t}}{\lambda} = \tilde{t}(\gamma,\xi)\,;\ \tilde{t}\,:\, C^1[0,L]^2 \to C^1[0,L]^3\,, \tag{4.1}$$
$$\lim_{\lambda\to 0} \frac{\bar{\theta}_\lambda - \bar{\theta}}{\lambda} = \tilde{\theta}(\gamma,\xi)\,;\ \tilde{\theta}\,:\, C^1[0,L]^2 \to C^2[0,L]^3\,, \tag{4.2}$$
$$\lim_{\lambda\to 0} \frac{\bar{n}_\lambda - \bar{n}}{\lambda} = \tilde{n}(\gamma,\xi)\,;\ \tilde{n}\,:\, C^1[0,L]^2 \to C^1[0,L]^3\,, \tag{4.3}$$
$$\lim_{\lambda\to 0} \frac{\bar{b}_\lambda - \bar{b}}{\lambda} = \tilde{b}(\gamma,\xi)\,;\ \tilde{b}\,:\, C^1[0,L]^2 \to C^1[0,L]^3\,, \tag{4.4}$$

$$\lim_{\lambda\to 0} \frac{a_\lambda - a}{\lambda} = \tilde{a}(\gamma,\xi)\,;\ \tilde{a}\,:\, C^1[0,L]^2 \to C[0,L]\,, \tag{4.5}$$
$$\lim_{\lambda\to 0} \frac{\beta_\lambda - \beta}{\lambda} = \tilde{\beta}(\gamma,\xi)\,;\ \tilde{\beta}\,:\, C^1[0,L]^2 \to C[0,L]\,, \tag{4.6}$$
$$\lim_{\lambda\to 0} \frac{c_\lambda - c}{\lambda} = \tilde{c}(\gamma,\xi)\,;\ \tilde{c}\,:\, C^1[0,L]^2 \to C[0,L]\,, \tag{4.7}$$
$$\lim_{\lambda\to 0} \frac{\det J_\lambda - \det J}{\lambda} = \tilde{\mathcal{D}}(\gamma,\xi)\,;\ \tilde{\mathcal{D}}\,:\, C^1[0,L]^2 \to C(\bar{\Omega})\,, \tag{4.8}$$
$$\lim_{\lambda\to 0} \frac{J_\lambda - J}{\lambda} = \tilde{J}(\gamma,\xi)\,;\ \tilde{J}\,:\, C^1[0,L]^2 \to C(\bar{\Omega})^9\,, \tag{4.9}$$
$$\lim_{\lambda\to 0} \frac{J^{-1}_\lambda - J^{-1}}{\lambda} = \tilde{I}(\gamma,\xi)\,;\ \tilde{I}\,:\, C^1[0,L]^2 \to C(\bar{\Omega})^9\,, \tag{4.10}$$
$$\lim_{\lambda\to 0} \frac{h^\lambda_{ij} - h_{ij}}{\lambda} = \tilde{h}_{ij}(\gamma,\xi)\,;\ \tilde{h}_{ij}\,:\, C^1[0,L]^2 \to C(\bar{\Omega})\,, \tag{4.11}$$
$$\lim_{\lambda\to 0} \frac{g^{ij}_\lambda - g^{ij}}{\lambda} = \tilde{g}^{ij}(\gamma,\xi)\,;\ \tilde{g}^{ij}\,:\, C^1[0,L]^2 \to C(\bar{\Omega})\,. \tag{4.12}$$

All the operators $\tilde{t}, \tilde{\theta}, \tilde{n}, \tilde{b}, \tilde{a}, \tilde{\beta}, \tilde{c}, \tilde{\mathcal{D}}, \tilde{J}, \tilde{I}, \tilde{h}_{ij}, \tilde{g}^{ij}$ are linear and bounded in the indicated spaces.

By Theorem 3.1, we also have that

$$\bar{y}_\lambda \to \bar{y} \text{ strongly in } H^1_0(0,L)^9\,. \tag{4.13}$$

In order to prove the differentiability properties of $\bar{y}_\lambda$, we substract the equations of $\bar{y}_\lambda$, $\bar{y}$, we divide by λ, and we intercalate advantageous terms. Later, we shall also fix the test functions of the form $\lambda^{-1}(\bar{y}_\lambda - \bar{y}) \in H^1_0(0,L)^9$.

In the right-hand side of (2.13), it is possible to pass to the limit

$$\lim_{\lambda\to 0}\Bigg\{\sum_{l=1}^{3}\int\limits_{\Omega} f_l(\bar{x})\Big(\mu_l(x_3)+x_1\,M_l(x_3)+x_2\,D_l(x_3)\Big)\frac{\det J_\lambda-\det J}{\lambda}\,d\bar{x}$$

$$+\sum_{i,j=1}^{3}\sum_{l=1}^{3}\int\limits_{\partial\Omega} g_l(\bar{x})\Big(\mu_l(x_3)+x_1\,M_l(x_3)$$

$$+x_2\,D_l(x_3)\Big)\frac{\det J_\lambda\sqrt{\nu_i(\bar{x})\,g_\lambda^{ij}\,\nu_j(\bar{x})}-\det J\sqrt{\nu_i(\bar{x})g^{ij}\nu_j(\bar{x})}}{\lambda}\,d\tau\Bigg\}$$

$$=\sum_{l=1}^{3}\int\limits_{\Omega} f_l(\bar{x})\Big(\mu_l(x_3)+x_1\,M_l(x_3)+x_2\,D_l(x_3)\Big)\,\tilde{\mathcal{D}}(\gamma,\xi)\,d\bar{x}$$

$$+\sum_{i,j=1}^{3}\sum_{l=1}^{3}\int\limits_{\partial\Omega} g_l(\bar{x})\Big(\mu_l(x_3)+x_1\,M_l(x_3)+x_2\,D_l(x_3)\Big)\Bigg[\tilde{\mathcal{D}}(\gamma,\xi)\sqrt{\nu_i\,g^{ij}\,\nu_j}$$

$$+\det J\frac{\nu_i\,\tilde{g}^{ij}(\gamma,\xi)\nu_j}{2\sqrt{\nu_i\,g^{ij}\,\nu_j}}\Bigg]\,d\tau\,. \tag{4.14}$$

We also write the corresponding transformation of the simplest term from $\mathcal{B}_\lambda(\,\cdot\,,\,\cdot\,)$, the bilinear functional (2.13) obtained from $(\varphi_\lambda,\psi_\lambda)$:

$$\frac{1}{\lambda}\Bigg\{2\,\tilde{\mu}\int\limits_{\Omega}\sum_{i=1}^{3}\Bigg[N_i^\lambda\,h_{1i}^\lambda+B_i^\lambda\,h_{2i}^\lambda+\Big(\tau_i^{\lambda'}+x_1\,N_i^{\lambda'}+x_2\,B_i^{\lambda'}\Big)\,h_{3i}^\lambda\Bigg]$$

$$\Bigg[M_i\,h_{1i}^\lambda+D_i\,h_{2i}^\lambda+\Big(\mu_i'+x_1\,M_i'+x_2\,d_i'\Big)\,h_{3i}^\lambda\Bigg]\;\det J_\lambda\,d\bar{x}$$

$$-\,2\,\tilde{\mu}\int\limits_{\Omega}\sum_{i=1}^{3}\Bigg[N_i\,h_{1i}+B_i\,h_{2i}+\Big(\tau_i'+x_1\,N_i'+x_2\,B_i'\Big)\,h_{3i}\Bigg]$$

$$\Bigg[M_i\,h_{1i}+D_i\,h_{2i}+\Big(\mu_i'+x_1\,M_i'+x_2\,D_i'\Big)\,h_{3i}\Bigg]\;\det J\,d\bar{x}\Bigg\}$$

$$=\;2\,\tilde{\mu}\int\limits_{\Omega}\sum_{i=1}^{3}\Bigg[\frac{N_i^\lambda-N_i}{\lambda}\,h_{1i}+\frac{B_i^\lambda-B_i}{\lambda}h_{2i}$$

$$+\Big(\frac{\tau_i^{\lambda'}-\tau_i'}{\lambda}+x_1\frac{N_i^{\lambda'}-N_i'}{\lambda}+x_2\frac{B_i^{\lambda'}-B_i'}{\lambda}\Big)\,h_{3i}\Bigg]$$

$$\Bigg[M_i\,h_{1i}+D_i\,h_{2i}+\Big(\mu_i'+x_1\,M_i'+x_2\,D_i'\Big)\,h_{3i}\Bigg]\det J\,d\bar{x}$$

$$+\,2\,\tilde{\mu}\int\limits_{\Omega}\sum_{i=1}^{3}\Bigg[N_i^\lambda\,h_{1i}+B_i^\lambda\,h_{2i}+\Big(\tau_i^{\lambda'}+x_1\,N_i^{\lambda'}+x_2\,B_i^{\lambda'}\Big)\,h_{3i}\Bigg]$$

$$\left[M_i \frac{h_{1i}^\lambda - h_{1i}}{\lambda} + D_i \frac{h_{2i}^\lambda - h_{2i}}{\lambda} + \left(\mu_i' + x_1 M_i' + x_2 D_i' \right) \frac{h_{3i}^\lambda - h_{3i}}{\lambda} \right] \det J \, d\bar{x}$$

$$+ 2 \tilde{\mu} \int_\Omega \sum_{i=1}^3 \left[N_i^\lambda \frac{h_{1i}^\lambda \det J_\lambda - h_{1i} \det J}{\lambda} + B_i^\lambda \frac{h_{2i}^\lambda \det J_\lambda - h_{2i} \det J}{\lambda} \right.$$

$$\left. + \left(\tau_i^{\lambda'} + x_1 N_i^{\lambda'} + x_2 B_i^{\lambda'} \right) \frac{h_{31}^\lambda \det J_\lambda - h_{3i} \det J}{\lambda} \right]$$

$$\left[M_i h_{1i}^\lambda + D_i h_{2i}^\lambda + \left(\mu_i' + x_1 M_i' + x_2 D_i' \right) h_{31}^\lambda \right] d\bar{x} \,. \tag{4.15}$$

The important term in (4.15) is the first term after the equality sign. Taking into account (4.14), and similar transformations in the other integrals defining $\mathcal{B}_\lambda(\,\cdot\,,\,\cdot\,)$, we obtain the relation

$$\mathcal{B}\left(\frac{\bar{y}_\lambda - \bar{y}}{\lambda} \,,\, \bar{v} \right) \;=\; Z_\lambda(\bar{v}) \tag{4.16}$$

for any test function $\bar{v} = (\bar{\mu}, \bar{M}, \bar{D}) \in H_0^1(0,L)^9$, and with some linear bounded operator $Z_\lambda : H_0^1(0,L)^9 \to I\!R$ for any $\lambda \in I\!R_+$. The relations (4.14), (4.15) show that the following estimate is valid

$$|Z_\lambda(\bar{v})| \;\le\; C |\bar{v}|_{H_0^1(0,L)^9} \,, \tag{4.17}$$

with some constant independent of $\lambda > 0$. Here, we use the differentiability properties of the coefficients, given in (4.1)–(4.12), and the convergence of $\bar{y}_\lambda$, according to (4.13). By fixing $\bar{v} = \lambda^{-1}(\bar{y}_\lambda - \bar{y})$, relations (4.16) and (4.17) show that $\{\frac{\bar{y}_\lambda - \bar{y}}{\lambda}\}$ is bounded in $H_0^1(0,L)^9$ for $\lambda > 0$, by the coercivity of $\mathcal{B}$. We may take a weakly convergent subsequence

$$\frac{\bar{y}_\lambda - \bar{y}}{\lambda} \to \hat{y} \,, \quad \text{weakly in } H_0^1(0,L)^9 \,. \tag{4.18}$$

As in the previous section, one may see that the convergence is valid in the strong topology of $H_0^1(0,L)^9$. The equation in variations has the form

$$\mathcal{B}(\hat{y}, \bar{v}) \;=\; Z(\bar{v}) \,, \quad \forall\, \bar{v} \,\in\, H_0^1(0,L)^9 \,, \tag{4.19}$$

with $Z(\bar{v}) = \lim_{\lambda \to 0} Z_\lambda(\bar{v})$, which exists by the above discussion. Z depends linearly and boundedly on $(\gamma, \xi) \in C^1[0,L]^2$.

Remark that (4.19) has a unique solution $\hat{y} \in H_0^1(0,L)^9$. We have proved the following result:

Proposition 4.1 *The mapping* $(\varphi, \psi) \in C^1[0,L]^2 \mapsto \bar{y} \in H_0^1(0,L)^9$ *is Gâteaux differentiable, and the derivative* $\hat{y}$ *satisfies* (4.19).

We introduce now the so-called adjoint sytem, with unknowns $\bar{T} = (\bar{R}, \bar{P}, \bar{Q}) \in H_0^1(0,L)^9$, and defined by

$$\mathcal{B}(\bar{T}, \bar{v}) \;=\; \nabla_2\, j(\bar{\theta}, \bar{y})(\bar{v})\,, \quad \forall\; \bar{v} \,\in\, H_0^1(0,L)^9\,. \tag{4.20}$$

In (4.20), we assume that $j : C^2[0,L]^3 \times H_0^1(0,L)^9 \to I\!R$ is Fréchet-differentiable, and that $\nabla_2 j$ denotes the second component of ∇j or, equivalently, the partial Fréchet differential with respect to $\bar{y}$. The existence and uniqueness of a solution $\bar{T} \in H_0^1(0,L)^9$ to (4.20) is obvious, due to the coercivity and boundedness of $\mathcal{B}(\,\cdot\,,\,\cdot\,)$.

Proposition 4.2 *If j is Fréchet differentiable, then the directional derivative of the cost functional Π in the problem* **(P)** *at the point $(\varphi, \psi) \in C^1[0,L]^2$ and in the direction $(\gamma, \xi) \in C^1[0,L]^2$ is given by*

$$\begin{aligned}
&\nabla\Pi(\varphi,\psi)\,(\gamma,\xi) \\
=\;& \nabla_1 j(\bar{\theta},\bar{y})\,\tilde{\theta}(\gamma,\xi) + \sum_{l=1}^{3}\int_{\Omega} f_l(\bar{x})\Big(R_l(x_3) + x_1\,P_l(x_3) + x_2\,Q_l(x_3)\Big)\,\tilde{\mathcal{D}}(\gamma,\xi)\,d\bar{x} + \\
& \sum_{i,j=1}^{3}\sum_{l=1}^{3}\int_{\partial\Omega} g_l(\bar{x})\Big(R_l(x_3) + x_1\,P_l(x_3) + x_2\,Q_l(x_3)\Big)\,\tilde{\mathcal{D}}(\gamma,\xi)\,\sqrt{\nu_i\,g^{ij}\,\nu_j}\,d\tau \\
& + \sum_{i,j=1}^{3}\sum_{l=1}^{3}\int_{\partial\Omega} g_l(\bar{x})\Big(R_l(x_3) + x_1\,P_l(x_3) + x_2\,Q_l(x_3)\Big) \\
& \qquad \det J\,\frac{1}{\sqrt{\nu_i\,g^{ij}\,\nu_j}}\,\nu_i\tilde{g}^{ij}(\gamma,\xi)\nu_j\,d\tau \\
& - 2\,\tilde{\mu}\int_{\Omega}\sum_{i=1}^{3}\Big[N_i\,h_{1i} + B_i\,h_{2i} + \Big(\tau_i' + x_1\,N_i' + x_2\,B_i'\Big)\,h_{3i}\Big] \\
& \qquad \Big[P_i\,\tilde{h}_{1i}(\gamma,\xi) + Q_i\,\tilde{h}_{2i}(\gamma,\xi) + \Big(R_i' + x_1\,P_i' + x_2\,Q_i'\Big)\,\tilde{h}_{3i}(\gamma,\xi)\Big]\,\det J\,d\bar{x} \\
& - 2\,\tilde{\mu}\int_{\Omega}\sum_{i=1}^{3}\Big[N_i\,\tilde{h}_{1i}(\gamma,\xi) + B_i\,\tilde{h}_{2i}(\gamma,\xi) + \Big(\tau_i' + x_1\,N_i' + x_2\,B_i'\Big)\,\tilde{h}_{3i}(\gamma,\xi)\Big] \\
& \qquad \Big[P_i\,h_{1i} + Q_i\,h_{2i} + \Big(R_i' + x_1\,P_i' + x_2\,Q_i'\Big)\,h_{3i}\Big]\,\det J\,d\bar{x} \\
& - 2\,\tilde{\mu}\int_{\Omega}\sum_{i=1}^{3}\Big[N_i\,h_{1i} + B_i\,h_{2i} + \Big(\tau_i' + x_1\,N_i' + x_2\,B_i'\Big)\,h_{3i}\Big] \\
& \qquad \Big[P_i\,h_{1i} + Q_i\,h_{2i} + \Big(R_i' + x_1\,P_i' + x_2\,Q_i'\Big)\,h_{3i}\Big]\,\tilde{\mathcal{D}}(\gamma,\xi)\,d\bar{x}
\end{aligned}$$

$$
\begin{aligned}
&- \tilde{\lambda} \int_{\Omega} \sum_{i,j=1}^{3} \left[N_i\, h_{1i} + B_i\, h_{2i} + \left(\tau_i' + x_1\, N_i' + x_2\, B_i' \right) h_{3i} \right] \\
&\quad \left[P_j\, \tilde{h}_{1j}(\gamma,\xi) + Q_j\, \tilde{h}_{2j}(\gamma,\xi) + \left(R_j' + x_1\, P_j' + x_2\, Q_j' \right) \tilde{h}_{3j}(\gamma,\xi) \right] \det J\, d\bar{x} \\
&- \tilde{\lambda} \int_{\Omega} \sum_{i,j=1}^{3} \left[N_i\, \tilde{h}_{1i}(\gamma,\xi) + B_i\, \tilde{h}_{2i}(\gamma,\xi) + \left(\tau_i' + x_1\, N_i' + x_2\, B_i' \right) \tilde{h}_{3i}(\gamma,\xi) \right] \\
&\quad \left[P_j\, h_{1j} + Q_j\, h_{2j} + \left(R_j' + x_1\, P_j' + x_2\, Q_j' \right) h_{3j} \right] \det J\, d\bar{x} \\
&- \tilde{\lambda} \int_{\Omega} \sum_{i,j=1}^{3} \left[N_i\, h_{1i} + B_i\, h_{2i} + \left(\tau_i' + x_1\, N_i' + x_2\, B_i' \right) h_{3i} \right] \\
&\quad \left[P_j\, h_{1j} + Q_j\, h_{2j} + \left(R_j' + x_1\, P_j' + x_2\, Q_j' \right) h_{3j} \right] \tilde{\mathcal{D}}(\gamma,\xi)\, d\bar{x} \\
&- \tilde{\mu} \int_{\Omega} \sum_{i<j} \Big[N_i\, h_{1j} + B_i\, h_{2j} + \left(\tau_i' + x_1\, N_i' + x_2\, B_i' \right) h_{3j} + N_j\, h_{1i} + B_j\, h_{2i} \\
&\qquad + \left(\tau_j' + x_1\, N_j' + x_2\, B_j' \right) h_{3i} \Big] \\
&\quad \Big[P_i\, \tilde{h}_{1j}(\gamma,\xi) + Q_i\, \tilde{h}_{2j}(\gamma,\xi) + \left(R_i' + x_1\, P_i' + x_2\, Q_i' \right) \tilde{h}_{3j}(\gamma,\xi) + P_j\, \tilde{h}_{1i}(\gamma,\xi) \\
&\qquad + Q_j\, \tilde{h}_{2i}(\gamma,\xi) + \left(R_j' + x_1\, P_j' + x_2\, Q_j' \right) \tilde{h}_{3i}(\gamma,\xi) \Big] \det J\, d\bar{x} \\
&- \tilde{\mu} \int_{\Omega} \sum_{i<j} \Big[N_i\, \tilde{h}_{1j}(\gamma,\xi) + B_i\, \tilde{h}_{2j}(\gamma,\xi) + \left(\tau_i' + x_1\, N_i' + x_2\, B_i' \right) \tilde{h}_{3j}(\gamma,\xi) \\
&\qquad + N_j\, \tilde{h}_{1i}(\gamma,\xi) + B_j\, \tilde{h}_{2i}(\gamma,\xi) + \left(\tau_j' + x_1\, N_j' + x_2\, B_j' \right) \tilde{h}_{3i}(\gamma,\xi) \Big] \\
&\quad \Big[P_i\, h_{1j} + Q_i\, h_{2j} + \left(R_i' + x_1\, P_i' + x_2\, Q_i' \right) h_{3j} + P_j\, h_{1i} + Q_j\, h_{2i} \\
&\qquad + \left(R_j' + x_1\, P_j' + x_2\, Q_j' \right) h_{3i} \Big] \det J\, d\bar{x} \\
&- \tilde{\mu} \int_{\Omega} \sum_{i<j} \Big[N_i\, h_{1j} + B_i\, h_{2j} + \left(\tau_i' + x_1\, N_i' + x_2\, B_i' \right) h_{3j} \\
&\qquad + N_j\, h_{1i} + B_j\, h_{2i} + \left(\tau_j' + x_1\, N_j' + x_2\, B_j' \right) h_{3i} \Big]
\end{aligned}
$$

$$\left[P_i\, h_{1j} + Q_i\, h_{2j} + \left(R_i' + x_1\, P_i' + x_2\, Q_i' \right) h_{3j} + P_j\, h_{1i} + Q_j\, h_{2i} \right.$$

$$\left. + \left(R_j' + x_1\, P_j' + x_2\, Q_j' \right) h_{3i} \right] \tilde{\mathcal{D}}(\gamma, \xi)\, d\bar{x}\,. \tag{4.21}$$

Remark 4.1 In order to compute (4.21), from (φ, ψ) and $(\gamma, \xi) \in C^1[0, L]^2$, one has to compute $\bar{\theta} \in C^2[0, L]^3$ by (2.1), $\bar{y} = (\bar{\tau}, \bar{N}, \bar{B}) \in H_0^1(0, L)^9$ by (2.13), $\bar{T} = (\bar{R}, \bar{P}, \bar{Q}) \in H_0^1(0, L)^9$ by (4.20), and to use (4.1)–(4.12). Since spaces of continuous functions are taken into account, it is not advantageous to rewrite (4.21) by using adjoint operators.

It is also to be noticed that the above argument holds for $\varphi, \psi, \gamma, \xi$ piecewise in $C^1[0, L]$. This is important for the numerical experiments in Section 8.

Remark 4.2 Assuming that the cross-section of the rod is not constant, one may study optimization problems with respect to the cross-section as well, under appropriate regularity conditions.

Let $\mathcal{C} = \{(\varphi, \psi) \in C^1[0, L]^2\,;\ \bar{\theta}(\varphi, \psi) \in \mathcal{K}\}$ and $u_0 = (\varphi_0, \psi_0) \in \mathcal{C}$ be arbitrarily fixed. We denote by

$$T(\mathcal{C}; u_0) = \{u \in C^1[0, L]^2\,;\ u = \lim_{n \to \infty} \lambda_n (u_n - u_0)\,,\ \lambda_n \geq 0\,,\ u_n \in \mathcal{C}\,,\ \text{ and } u_n \to u_0\}$$

the cone of tangents to $\mathcal{C}$ at u_0, Barbu and Precupanu [3]. It is known that if $\mathcal{C}$ is convex (see examples (2.15),(2.16) and Remark 2.2), then $T(\mathcal{C}; u_0) = \overline{\bigcup_{\lambda > 0} \lambda(\mathcal{C} - u_0)}$.

Corollary 4.3 *Assume that* $u^* = (\varphi^*, \psi^*)$ *is a (local) optimum point for* **(P)**. *Then, the following statements are valid:*

i) If Π *is Fréchet differentiable on* $C^1[0, L]^2$, *then*

$$\nabla \Pi(\varphi^*, \psi^*)(\gamma, \xi) \geq 0\,, \quad \forall\ (\gamma, \xi)\ \in\ T(\mathcal{C}; u^*)\,.$$

ii) If $\mathcal{C}$ *is convex, then the directional derivative of* Π *satisfies*

$$\nabla \Pi(\varphi^*, \psi^*)(\gamma, \xi) \geq 0\,, \quad \forall\ (\gamma, \xi)\ \in\ \mathcal{C} - u^*\,.$$

Remark 4.3 Corollary 4.3 gives the standard first order optimality conditions for the problem **(P)**, Tröltzsch [18]. Relations (4.21), (4.20), etc., indicate the explicit calculation of the directional derivative of the cost functional and will be used in the last section in the numerical experiments.

5 Formulation of the shell optimization problem

Let $\omega \subset \mathbb{R}^2$ denote a bounded domain, not necessarily simply connected, with Lipschitz boundary $\partial\omega$. Define

$$\Omega = \omega \times]-\varepsilon, \varepsilon[\subset \mathbb{R}^3$$

for some "small" $\varepsilon > 0$. We denote by $(x_1, x_2) \in \omega$ and $x_3 \in]-\varepsilon, \varepsilon[$, $\bar{x} = (x_1, x_2, x_3) \in \Omega$, the independent variables.

Let $p : \omega \to \mathbb{R}$ be a $C^2(\bar{\omega})$ mapping, whose graph represents the middle surface $\mathcal{S}$ of a shell. We introduce the geometrical transformation

$$F : \Omega \to \mathbb{R}^3,$$

$$F(\bar{x}) = \bar{\pi}(x_1, x_2) + x_3\, \bar{n}(x_1, x_2), \tag{5.1}$$

with $\bar{\pi} = (\pi_1, \pi_2, \pi_3) = (x_1, x_2, p(x_1, x_2))$, and with $\bar{n} = (n_1, n_2, n_3)$ denoting the normal vector to $\mathcal{S}$ in the point $\bar{\pi}(x_1, x_2)$. Since the tangent vectors $\dfrac{\partial \bar{\pi}}{\partial x_1} = (1, 0, p_1)$ and $\dfrac{\partial \bar{\pi}}{\partial x_2} = (0, 1, p_2)$, with $p_1 = \dfrac{\partial p}{\partial x_1}$ and $p_2 = \dfrac{\partial p}{\partial x_2}$, are always linearly independent, we may take $\bar{n}$ as the normalization of $\dfrac{\partial \bar{\pi}}{\partial x_1} \wedge \dfrac{\partial \bar{\pi}}{\partial x_2}$, that is

$$\bar{n} = \frac{1}{\sqrt{1 + p_1^2 + p_2^2}}(-p_1, -p_2, 1). \tag{5.2}$$

Assume that $\partial\omega = \bar{\gamma}_0 \cup \bar{\gamma}_1$, with γ_0, γ_1 nonoverlapping open parts of $\partial\omega$ such that meas $(\gamma_0) > 0$, and let $\Gamma_0 := \gamma_0 \times]-\varepsilon, \varepsilon[$, $\Gamma_1 := \partial\Omega \setminus \Gamma_0$. We introduce the notations

$$\hat{\Omega} := F(\Omega), \hat{\Gamma}_0 := F(\Gamma_0), \hat{\Gamma}_1 := F(\Gamma_1).$$

We argue later (see (5.9)) that F is a homeomorphism for small ε, and the open set $\hat{\Omega}$ will represent a shell. We assume that body forces $\hat{f} \in L^2(\hat{\Omega})^3$ and surface tractions $\hat{g} \in L^2(\hat{\Gamma}_1)^3$ act on the shell. Our main mechanical assumption is that the corresponding displacement $\hat{u} \in V(\hat{\Omega}) = \{\hat{v} \in H^1(\hat{\Omega})^3 \,;\, \hat{v}|_{\hat{\Gamma}_0} = 0\}$ has the form

$$\hat{u}(\hat{x}) = \bar{u}(x_1, x_2) + x_3\, \bar{r}(x_1, x_2), \quad x \in \hat{\Omega}. \tag{5.3}$$

Here, $\bar{x} = (x_1, x_2, x_3) = F^{-1}(\hat{x}) \in \Omega$ and $\bar{u} = (u_1, u_2, u_3)$, $\bar{r} = (r_1, r_2, r_3)$ belong to the Hilbert space

$$V(\omega) = \{\bar{v} = (v_1, v_2, v_3) \in H^1(\omega)^3 \,;\, \bar{v}|_{\gamma_0} = 0\}, \tag{5.4}$$

equipped with the norm

$$|\bar{v}|_{V(\omega)} := \int_\omega \left(|\nabla v_1|^2 + |\nabla v_2|^2 + |\nabla v_3|^2\right) dx_1 dx_2 .$$

If we denote by $\tilde{V}(\hat{\Omega})$ the subspace of $V(\hat{\Omega})$ defined by (5.3), (5.4), we can see that $\tilde{V}(\hat{\Omega})$ can be simply identified with $V(\omega) \times V(\omega)$, and we shall do this repeatedly later in this paper.

Clearly, $\bar{u}$ represents the displacement of the middle surface $\mathcal{S}$ of the shell, while $\bar{r}$ is the modification of the points along the normal $\bar{n}(x_1, x_2)$, assumed to remain on a line. The form (5.3) allows for both dilation and contraction of the elastic material; it is a generalization of the classical Naghdi model, Ciarlet [9], Blouza [5].

The Jacobian $J = DF$ of F is given by

$$J(\bar{x}) = \begin{bmatrix} 1 + x_3 \frac{\partial n_1}{\partial x_1} & x_3 \frac{\partial n_1}{\partial x_2} & n_1 \\ x_3 \frac{\partial n_2}{\partial x_1} & 1 + x_3 \frac{\partial n_2}{\partial x_2} & n_2 \\ p_1 + x_3 \frac{\partial n_3}{\partial x_1} & p_2 + x_3 \frac{\partial n_3}{\partial x_2} & n_3 \end{bmatrix}. \tag{5.5}$$

As $|\bar{n}|^2_{I\!R^3} = 1$, we get $\langle \bar{n}, \frac{\partial \bar{n}}{\partial x_i} \rangle_{I\!R^3} = 0$, $i = 1, 2$, which shows that $\dfrac{\partial \bar{n}}{\partial x_i}$ can be generated by $\dfrac{\partial \bar{\pi}}{\partial x_1}$ and $\dfrac{\partial \bar{\pi}}{\partial x_2}$. We get the relations

$$\frac{\partial \bar{n}}{\partial x_1}(x_1, x_2) = \frac{\partial n_1}{\partial x_1} \frac{\partial \bar{\pi}}{\partial x_1} + \frac{\partial n_2}{\partial x_1} \frac{\partial \bar{\pi}}{\partial x_2}, \tag{5.6}$$

$$\frac{\partial \bar{n}}{\partial x_2}(x_1, x_2) = \frac{\partial n_1}{\partial x_2} \frac{\partial \bar{\pi}}{\partial x_1} + \frac{\partial n_2}{\partial x_1} \frac{\partial \bar{\pi}}{\partial x_2}, \tag{5.7}$$

which are special cases of the equations of motion of the local frame on the surface $\mathcal{S}$, Cartan [6]. The coefficients $\dfrac{\partial n_i}{\partial x_\alpha}$, $i = \overline{1,3}$, $\alpha = 1, 2$ are related to the curvatures of $\mathcal{S}$.

Equalities (5.5)–(5.7) yield

$$\det J(\bar{x}) = \left[1 + x_3 \left(\frac{\partial n_1}{\partial x_1} + \frac{\partial n_2}{\partial x_2} \right) + x_3^2 \left(\frac{\partial n_1}{\partial x_1} \frac{\partial n_2}{\partial x_2} - \frac{\partial n_1}{\partial x_2} \frac{\partial n_2}{\partial x_1} \right) \right] \cdot \sqrt{1 + p_1^2 + p_2^2}. \tag{5.8}$$

Since $p \in C^2(\bar{\omega})$, for "small" $\varepsilon > 0$ we get that

$$\det J(\bar{x}) \geq c > 0, \quad \forall\, \bar{x} \in \Omega. \tag{5.9}$$

Let us notice that (5.9) justifies the definition of the shell $\hat{\Omega}$ via the transformation F, see Ciarlet [9], Thm. 3.1–1.

We denote the elements of $J(\bar{x})^{-1}$ by

$$J(\bar{x}) = (h_{ij}(\bar{x}))_{i,j=\overline{1,3}}. \tag{5.10}$$

In Sprekels and Tiba [16], the following generalized Naghdi model is obtained:

$$
\begin{aligned}
&\mathcal{B}\left((\bar u,\bar r)\,,\,(\bar\mu,\bar\rho)\right)\\
&= \tilde\lambda \int_\Omega \left\{ \sum_{i=1}^3 \left[\left(\frac{\partial u_i}{\partial x_1} + x_3 \frac{\partial r_i}{\partial x_1} \right) h_{1i} + \left(\frac{\partial u_i}{\partial x_2} + x_3 \frac{\partial r_i}{\partial x_2} \right) h_{2i} + r_i\, h_{3i} \right] \right\}\\
&\times \left\{ \sum_{j=1}^3 \left[\left(\frac{\partial \mu_j}{\partial x_1} + x_3 \frac{\partial \rho_j}{\partial x_1} \right) h_{1j} + \left(\frac{\partial \mu_j}{\partial x_2} + x_3 \frac{\partial \rho_j}{\partial x_2} \right) h_{2j} + \rho_j\, h_{3j} \right] \right\} |\det J(\bar x)|\, d\bar x\\
&+ 2\,\tilde\mu \int_\Omega \sum_{i=1}^3 \left[\left(\frac{\partial u_i}{\partial x_1} + x_3 \frac{\partial r_i}{\partial x_1} \right) h_{1i} + \left(\frac{\partial u_i}{\partial x_2} + x_3 \frac{\partial r_i}{\partial x_2} \right) h_{2i} + r_i\, h_{3i} \right]\\
&\left[\left(\frac{\partial \mu_i}{\partial x_1} + x_3 \frac{\partial \rho_i}{\partial x_1} \right) h_{1i} + \left(\frac{\partial \mu_i}{\partial x_2} + x_3 \frac{\partial \rho_i}{\partial x_2} \right) h_{2i} + \rho_i\, h_{3i} \right] |\det J(\bar x)|\, d\bar x\\
&+ \tilde\mu \int_\Omega \sum_{1\le i<j\le 3} \left[\left(\frac{\partial u_i}{\partial x_1} + x_3 \frac{\partial r_i}{\partial x_1} \right) h_{1j} + \left(\frac{\partial u_i}{\partial x_2} + x_3 \frac{\partial r_i}{\partial x_2} \right) h_{2j} + r_i\, h_{3j} \right.\\
&+ \left.\left(\frac{\partial u_j}{\partial x_1} + x_3 \frac{\partial r_j}{\partial x_1} \right) h_{1i} + \left(\frac{\partial u_j}{\partial x_2} + x_3 \frac{\partial r_j}{\partial x_2} \right) h_{2i} + r_j\, h_{3i} \right]\\
&\times \left[\left(\frac{\partial \mu_i}{\partial x_1} + x_3 \frac{\partial \rho_i}{\partial x_1} \right) h_{1j} + \left(\frac{\partial \mu_i}{\partial x_2} + x_3 \frac{\partial \rho_i}{\partial x_2} \right) h_{2j} + \rho_i\, h_{3j} \right.\\
&+ \left.\left(\frac{\partial \mu_j}{\partial x_1} + x_3 \frac{\partial \rho_j}{\partial x_1} \right) h_{1i} + \left(\frac{\partial \mu_j}{\partial x_2} + x_3 \frac{\partial \rho_j}{\partial x_2} \right) h_{2i} + r_j\, h_{3i} \right] |\det J(\bar x|\, d\bar x\\
&= \int_\Omega \sum_{l=1}^3 f_l(\mu_l + x_3\,\rho_l)\,|\det J(\bar x)|\, d\bar x + \int_{\Gamma_1} \sum_{l=1}^3 \sum_{i,j=1}^3 g_l(\mu_l + x_3\,\rho_l)\,|\det J(\bar x)|\\
&\times \sqrt{\nu_i(\bar x)\, g^{ij}(\bar x)\, \nu_j(\bar x)}\, d\tau\,, \quad \forall\ (\bar\mu,\bar\rho)\ \in\ V(\omega)^2\,.
\end{aligned}
\tag{5.11}
$$

Here, $\bar f(\bar x) = \hat f(F\bar x)\,,\ \bar g(\bar x) = \hat g(F\bar x)\,,\ \bar x \in \Omega\,,$ we use the assumed form (5.3) of the displacement, and $\bar\mu \in V(\omega)\,,\ \bar\rho \in V(\omega)$ are arbitrary test functions. The coefficients g^{ij} are obtained by

$$
\left(g^{ij}(\bar x)\right)_{i,j=\overline{1,3}} = J(\bar x)^{-1}\left[J(\bar x)^T\right]^{-1}, \tag{5.12}
$$

and $(\nu_i(\bar x))_{i=\overline{1,3}}$ is the unit outside normal to Γ_1 at $\bar x \in \Gamma_1$.

The coercivity of $\mathcal{B}$ on $V(\omega)\times V(\omega)$ was proved by Sprekels and Tiba [16], for ε small enough. This gives the existence and the uniqueness of the solution $(\bar u,\bar r) \in V(\omega)\times V(\omega)$ to (5.11).

For given $\bar f$ and $\bar g$ (defined in a sufficiently large ball in $I\!R^3$), we consider the following general shape optimization problem associated with (5.11):

$$\operatorname*{Min}_{p} \left\{ \Pi(p) \; = \; j\Big(\bar{y}(x_1,x_2)\,,\; p(x_1,x_2)\Big) \right\} \tag{$\mathbf{P}$}$$

with $\bar{y}(x_1,x_2) = (\bar{u}(x_1,x_2), \bar{r}(x_1,x_2)) \in V(\omega)^2$ given by (2.11), and subject to the "control" constraint $p \in \mathcal{K} \subset C^2(\bar{\omega})$, closed and bounded. Notice that (5.9) should be included in the definition of $\mathcal{K}$. The mapping $j : V(\omega)^2 \times C^2(\omega) \to \mathbb{R}$ satisfies certain regularity properties to be described later. One classical example is the quadratic case

$$2j(\bar{y},p) \;=\; |u_1|^2_{V(\omega)} \;+\; |u_2|^2_{V(\omega)} \;+\; |u_3|^2_{V(\omega)}\,. \tag{5.13}$$

Then $(\mathbf{P})$ aims at finding the shape of the shell (the surface $\mathcal{S}$) that minimizes the displacement of the middle surface under the prescribed body forces and tractions.

Concerning the constraints to which the shell itself may be submitted and which are abstractly written as $p \in \mathcal{K}$, there is a large variety of examples. We just list several:

$$0 \;\le\; p(x_1,x_2)\,, \quad \forall\; (x_1,x_2) \;\in\; \omega \tag{5.14}$$

(pointwise constraints),

$$\int_{\omega} p(x_1,x_2)\, dx_1\, dx_2 \;\ge\; c \tag{5.15}$$

(integral constraints). A special integral constraint is to prescribe limits for the area of $\mathcal{S}$:

$$\int_{\omega} \sqrt{1+p_1^2+p_2^2} \;\ge\; \beta\,. \tag{5.16}$$

Although all the examples (5.13)–(5.16) have a convex nature, the shape optimization problem $(\mathbf{P})$ is strongly nonconvex, since the dependence $p \mapsto \bar{y}$ is nonlinear. $(\mathbf{P})$ is a control-into-coefficients problem.

6 Existence of optimal shells

We prove first the following continuous dependence result:

Theorem 6.1 *Assume that $p_n : \bar{\omega} \to \mathbb{R}$ and $p_n \to p$ in $C^2(\bar{\omega})$. If $\bar{y}_n = (\bar{u}_n, \bar{r}_n)$ and $\bar{y} = (\bar{u}, \bar{r})$ are the solutions of* (5.11) *corresponding to p_n, p, then $\bar{y}_n \to \bar{y}$ strongly in $V(\omega)^2$, for sufficiently small $\varepsilon > 0$.*

Proof. Relations (5.1), (5.2), (5.5), (5.8) give (with obvious notations)

$$\bar{n}_n \;\to\; \bar{n} \quad \text{in } C^1(\bar{\omega})^3\,, \tag{6.1}$$

$$F_n \;=\; \bar{\pi}_n \;+\; x_3\,\bar{n}_n \;\to\; F \;=\; \bar{\pi} \;+\; x_3\,\bar{n} \quad \text{in } C^1(\bar{\Omega})^3\,, \tag{6.2}$$

$$J_n \;\to\; J \quad \text{in } C(\bar{\Omega})^9\,, \tag{6.3}$$

$$\det J_n \;\to\; \det J \quad \text{in } C(\bar{\Omega})\,. \tag{6.4}$$

Notice that

$$J(\bar{x}) = \begin{bmatrix} 1 & 0 & n_1 \\ 0 & 1 & n_2 \\ p_1 & p_2 & n_3 \end{bmatrix} \begin{bmatrix} 1 + x_3 \frac{\partial n_1}{\partial x_1} & x_3 \frac{\partial n_1}{\partial x_2} & 0 \\ x_3 \frac{\partial n_2}{\partial x_1} & 1 + x_3 \frac{\partial n_2}{\partial x_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = S\,R =: S(I + x_3 M) \tag{6.5}$$

(new matrix notations).

Similarly, we have

$$J_n = S_n R_n = S_n(I + x_3 M_n)\,. \tag{6.6}$$

A simple calculus gives

$$\begin{aligned} S_n^{-1} &= \frac{1}{\sqrt{1 + (p_1^n)^2 + (p_2^n)^2}} \begin{bmatrix} n_3^n - n_2^n p_2^n & n_1^n p_2^n & -n_1^n \\ n_2^n p_1^n & n_3^n - n_1^n p_1^n & -n_2^n \\ -p_1^n & -p_2^n & 1 \end{bmatrix} \\ &\longrightarrow \frac{1}{\sqrt{1 + p_1^2 + p_2^2}} \begin{bmatrix} n_3 - n_2 p_2 & n_1 p_2 & -n_1 \\ n_2 p_1 & n_3 - n_1 p_1 & -n_2 \\ -p_1 & -p_2 & 1 \end{bmatrix} = S^{-1}\,, \end{aligned}$$

strongly in $C^1(\bar{\omega})$. Moreover,

$$R_n^{-1} = (I + x_3 M_n)^{-1} = I - x_3 M_n + x_3^2 M_n^2 - x_3^3 M_n^3 + \dots \tag{6.7}$$

for ε small. Clearly, we have

$$M_n = \begin{bmatrix} \dfrac{\partial n_1^n}{\partial x_1} & \dfrac{\partial n_1^n}{\partial x_2} & 0 \\ \dfrac{\partial n_2^n}{\partial x_1} & \dfrac{\partial n_2^n}{\partial x_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow M = \begin{bmatrix} \dfrac{\partial n_1}{\partial x_1} & \dfrac{\partial n_1}{\partial x_2} & 0 \\ \dfrac{\partial n_2}{\partial x_1} & \dfrac{\partial n_2}{\partial x_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.8}$$

in $C(\bar{\omega})$. Relations (6.7) and (6.8) show (by a passage to the limit in the infinite sum, $n \to \infty$) that $R_n^{-1} \to R^{-1}$ in $C(\bar{\Omega})$, for ε small.

Then, (6.6) and the above argument give

$$J_n^{-1} \longrightarrow J^{-1} \quad \text{in } C(\bar{\Omega})^9\,. \tag{6.9}$$

In particular, we have that:

$$h_{ij}^n(\bar{x}) \longrightarrow h_{ij}(\bar{x}) \quad \text{in } C(\bar{\Omega})\,, \quad \forall\, i,j = \overline{1,3}\,, \tag{6.10}$$

$$g_n^{ij}(\bar{x}) \longrightarrow g^{ij}(\bar{x}) \quad \text{in } C(\bar{\Omega})\,, \quad \forall\, i,j = \overline{1,3}\,, \tag{6.11}$$

according to (5.10), (5.12), (6.3) and (6.4).

Let $\mathcal{B}_n$ denote the bilinear form $\mathcal{B}$ from (5.11), with coefficients h_{ij}^n, $\det J_n$. We show that it has a coercivity constant independent of $n \in I\!N$, for $\varepsilon > 0$ small enough (again independently of n).

Proposition 6.2 *Assume that $\mathcal{K}$ is bounded in $C^2(\bar{\omega})$ and that $\varepsilon < \varepsilon(\mathcal{K})$ and $\delta << \varepsilon$ are given positive numbers. There are $c = c(\mathcal{K}) > 0$ and $m = m(\mathcal{K}) > 0$ such that*

$$\mathcal{B}_p(\hat{u}, \hat{u}) \ \geq \ c\Big[\varepsilon|\bar{u}|^2_{V(\omega)} + \varepsilon^3|\bar{r}|^2_{V(\omega)}\Big] \ - \ \frac{m}{\delta}\left[|\bar{r}|^2_{L^2(\omega)^3} + |\bar{u}|^2_{L^2(\omega)^3}\right], \tag{6.12}$$

for any $p \in \mathcal{K}$ and any $\hat{u} \in H^1(\hat{\Omega})^3$ given by (5.3).

The constant $\varepsilon(\mathcal{K}) > 0$ depends on $c_i > 0$, $i = 1, 2$, defined in (6.23) and in Lemma 6.3. It should be small enough such that (5.8) is fulfilled, which is possible due to the boundedness of $\mathcal{K}$ in $C^2(\bar{\omega})$. The precise significance of $\varepsilon(\mathcal{K})$, $c(\mathcal{K})$, $m(\mathcal{K})$ is indicated in the proof.

The notation $\mathcal{B}_p(\,\cdot\,,\,\cdot\,)$ signifies the bilinear functional (5.11) associated to some $p \in \mathcal{K}$. We prove Proposition 6.2 only for the case $\bar{u}, \bar{r} \in H^1_0(\omega)^3 = V(\omega)$, in order to avoid more technical arguments related to the extension of $\hat{u}$ to $H^1_0(I\!R)^3$.

Proof. We consider the mapping $\bar{w} \in H^1(\Omega)^3$, given by

$$\bar{w}(x_1, x_2, x_3) \ = \ \bar{u}(x_1, x_2) \ + \ x_3\, \bar{r}(x_1, x_2)\,, \tag{6.13}$$

such that $\hat{u}(\hat{x}) = \bar{w}(F^{-1}\hat{x})$, $\hat{x} \in \hat{\Omega}$, $\bar{x} = F^{-1}\hat{x} \in \Omega$. Denote by

$$S^+ \ = \ [\varepsilon\,,\, \varepsilon + \delta] \ \times \ \bar{\omega}\,, \quad S^- \ = \ [-\varepsilon - \delta\,,\, -\varepsilon] \ \times \ \bar{\omega}\,. \tag{6.14}$$

We extend $\bar{w}$ to $\Omega \cup S^+ \cup S^-$ by $\tilde{w}|_\Omega = \bar{w}$ and:

$$\tilde{w}(\bar{x}) \ = \ \delta^{-1}\left\{ \Big[(\varepsilon + \delta) - x_3\Big]\, \bar{u}(x_1, x_2) + \varepsilon\,(\varepsilon + \delta - x_3)\, \bar{r}(x_1, x_2) \right\} \tag{6.15}$$

for $\bar{x} \in S^+$,

$$\tilde{w}(\bar{x}) \ = \ \delta^{-1}\left\{ (\varepsilon + \delta + x_3)\, \bar{u}(x_1, x_2) - \varepsilon\,(\varepsilon + \delta + x_3)\, \bar{r}(x_1, x_2) \right\} \tag{6.16}$$

for $\bar{x} \in S^-$.

Then, we may extend $\tilde{w}$ by 0 to $I\!R^3$ as $\bar{u}, \bar{r} \in H^1_0(\omega)^3$. In the general case of a partially clamped shell, one has to use an extension procedure around $\omega \subset I\!R^2$, too (for instance the Calderon extension, Adams [1], since $\partial\omega$ is assumed Lipschitzian).

We may assume that F_p, i.e. the transformation (5.1) associated to any $p \in \mathcal{K}$, is still one-to-one on $\Omega \cup S^+ \cup S^-$, since $\varepsilon + \delta$ is "small" and $\mathcal{K}$ is bounded (see (5.8)). We denote

$$\Sigma_p^+ \ := \ F_p(S^+)\,,\, \Sigma_p^- \ := \ F_p(S^-)\,. \tag{6.17}$$

Above, the index $p \in \mathcal{K}$ puts into evidence the dependence on p of the geometrical transformation and of the sets. We introduce the extension of $\hat{u} \in H^1(\hat{\Omega}_p)^3$ by

$$\tilde{u}(\hat{x}) \ = \ \tilde{w}\Big(F_p^{-1}(\hat{x})\Big)\,. \tag{6.18}$$

Clearly, it holds $\tilde{u} \in H_0^1(\hat{\Omega}_p \cup \Sigma_p^+ \cup \Sigma_p^-)$.

As $\mathcal{K}$ is bounded in $C^2(\bar{\omega})$, there is a ball O in $I\!R^3$ such that $O \supset \hat{\Omega}_p \cup \Sigma_p^+ \cup \Sigma_p^-$, for any $p \in \mathcal{K}$. We may extend $\tilde{u}$ by 0 to O so that $\tilde{u} \in H_0^1(O)$. We have

$$\mathcal{B}_p(\hat{u}, \hat{u}) + \tilde{\mu} \int\limits_{\Sigma_p^+ \cup \Sigma_p^-} \sum_{i,j=1}^3 \left| \hat{e}_{ij}(\tilde{u}) \right|^2 d\hat{x} \ \geq \ \tilde{\mu} \int\limits_O \sum_{i,j=1}^3 \left| \hat{e}_{ij}(\tilde{u}) \right|^2 d\hat{x}$$

since $\tilde{\lambda} \geq 0$, $\tilde{\mu} \geq 0$. The Korn's inequality, applied to the last integral, gives that

$$\begin{aligned} \mathcal{B}_p(\hat{u}, \hat{u}) \ &\geq \ c |\tilde{u}|^2_{H_0^1(O)} - \tilde{\mu} \int\limits_{\Sigma_p^+ \cup \Sigma_p^-} \sum_{i,j=1}^3 \left| \hat{e}_{ij}(\tilde{u}) \right|^2 d\hat{x} \\ &\geq \ c |\tilde{u}|^2_{H^1(\hat{\Omega}_p)} - \tilde{\mu} \int\limits_{\Sigma_p^+ \cup \Sigma_p^-} \sum_{i,j=1}^3 \left| \hat{e}_{ij}(\tilde{u}) \right|^2 d\hat{x} \,, \end{aligned} \tag{6.19}$$

with $c > 0$ being independent of $p \in \mathcal{K}$.

We have to estimate the last term in (6.19). To this end, we compute

$$\begin{aligned} \int\limits_{\Sigma_p^+ \cup \Sigma_p^-} \left| \frac{\partial \tilde{u}_i}{\partial \hat{x}_j} \right|^2 d\hat{x} \ &= \int\limits_{\Sigma_p^+ \cup \Sigma_p^-} \Bigg\langle \Bigg(\frac{\partial \tilde{w}_i}{\partial x_1} (\bar{x}(\hat{x})) \,, \\ &\qquad \frac{\partial \tilde{w}_i}{\partial x_2} (\bar{x}(\hat{x})) \,, \frac{\partial \tilde{w}_i}{\partial x_3} (\bar{x}(\hat{x})) \Bigg) \,, \left(d_{1j}^p(\hat{x}) \,, d_{2j}^p(\hat{x}) \,, d_{3j}^p(\hat{x}) \right) \Bigg\rangle^2_{I\!R^3} d\hat{x} \\ &= \int\limits_{S^+ \cup S^-} \left\langle \left(\frac{\partial \tilde{w}_i}{\partial x_1}, \frac{\partial \tilde{w}_i}{\partial x_2}, \frac{\partial \tilde{w}_i}{\partial x_3} \right) , \left(h_{1j}^p \,, h_{2j}^p \,, h_{3j}^p \right) \right\rangle^2_{I\!R^3} |\det J_p| d\bar{x} \,, \end{aligned}$$

where $(d_{ij}^p)_{i,j=\overline{1,3}} := D\, F_p^{-1}(\hat{x})$, $(h_{ij}^p)_{i,j=\overline{1,3}} := J_p^{-1}(\bar{x})$, and where we have performed a standard change of variables in the integral (see Sprekels and Tiba [16] for a detailed calculation). Notice that the extension of h_{ij}^p to $S^+ \cup S^-$ is obvious by (5.5).

As $\{\det J_p\}$, $\{h_{ij}^p\}$ are bounded for $p \in \mathcal{K}$, we have to estimate the gradient of $\tilde{w}$ in $L^2(S^+ \cup S^-)$. We compute it in S^+, for example:

$$\frac{\partial \tilde{w}}{\partial x_\alpha} \ = \ \delta^{-1} \left[(\varepsilon + \delta - x_3) \frac{\partial \bar{u}}{\partial x_\alpha} + \varepsilon \, (\varepsilon + \delta - x_3) \frac{\partial \bar{r}}{\partial x_\alpha} \right], \quad \alpha \ = \ 1, 2 \,, \tag{6.20}$$

$$\frac{\partial \tilde{w}}{\partial x_3} \ = \ - \, \delta^{-1} (\bar{u} + \varepsilon \, \bar{r}) \,. \tag{6.21}$$

Thus, we get

$$\left| \frac{\partial \tilde{w}}{\partial x_3} \right|_{L^2(S^+ \cup S^-)^3} \leq \sqrt{2} \delta^{-\frac{1}{2}} |\bar{u}|_{L^2(\omega)^3} + \sqrt{2} \varepsilon \, \delta^{-\frac{1}{2}} |\bar{r}|_{L^2(\omega)^3} \,.$$

For $\alpha = 1, 2$, we have:

$$\left| \frac{\partial \tilde{w}}{\partial x_\alpha} \right|_{L^2(S^+ \cup S^-)^3} \leq \frac{\sqrt{2}}{\sqrt{3}} \delta^{\frac{1}{2}} \left| \frac{\partial \bar{u}}{\partial x_\alpha} \right|_{L^2(\omega)^3} + \frac{\sqrt{2}}{\sqrt{3}} \varepsilon \, \delta^{\frac{1}{2}} \left| \frac{\partial \bar{r}}{\partial x_\alpha} \right|_{L^2(\omega)^3} . \tag{6.22}$$

Consequently, we can find some $c_1 > 0$, independent of $p \in \mathcal{K}$, such that

$$\mathcal{B}_p(\hat{u}, \hat{u}) \geq c |\hat{u}|^2_{H^1(\hat{\Omega}_p)} - c_1 \left[\delta |\bar{u}|^2_{V(\omega)} + \varepsilon^2 \, \delta |\bar{r}|^2_{V(\omega)} + \delta^{-1} |\bar{u}|^2_{L^2(\omega)^3} + \varepsilon^2 \, \delta^{-1} |\bar{r}|^2_{L^2(\omega)^3} \right] . \tag{6.23}$$

Lemma 6.3 *If $\hat{\Omega}_p = F_p(\Omega)$, there are $c_2 > 0$, $c_3 \in I\!R$, independent of $p \in \mathcal{K}$, that*

$$|\hat{u}|^2_{H^1(\hat{\Omega}_p)} \geq c_2 \left[\varepsilon |\bar{u}|^2_{V(\omega)} + \varepsilon^3 |\bar{r}|^2_{V(\omega)} \right] - c_3 \, \varepsilon |\bar{r}|^2_{L^2(\omega)^3} , \quad \forall \, \hat{u}(\hat{x}) = \bar{w}(F_p^{-1} \hat{x}) \in H^1(\Omega_p) ,$$

for $\varepsilon \leq \varepsilon_0$ and with $\varepsilon_0 > 0$ independent of $p \in \mathcal{K}$.

Proof. The proof of this Lemma is quite technical, and we quote Sprekels and Tiba [16, Sect. 3] in this respect. It is possible to check that all the constants appearing there may be chosen independently of $p \in \mathcal{K}$. We indicate here just a precise quantitative argument which replaces the qualitative proof of Lemma 3.3 in Sprekels and Tiba [16], in order to preserve the control of the constants. We have:

$$|\hat{u}|^2_{H^1(\hat{\Omega}_p)} = \int_\Omega \sum_{i,j=1}^3 \left[\left(\frac{\partial u_i}{\partial x_1} + x_3 \frac{\partial r_i}{\partial x_1} \right) h^p_{1j}(\bar{x}) + \left(\frac{\partial u_i}{\partial x_2} + x_3 \frac{\partial r_i}{\partial x_2} \right) h^p_{2j}(\bar{x}) + r_i(\bar{x}) \, h^p_{3j}(\bar{x}) \right]^2 |\det J_p(\bar{x})| \, d\bar{x} , \tag{6.24}$$

after the change of variables via $F_p : \Omega \to \hat{\Omega}_p$.

We define the quadratic form

$$Q_p(\bar{u}, \bar{r}) = 2 \varepsilon \int_\omega \sum_{i,j=1}^3 \left(\frac{\partial u_i}{\partial x_1} h^{p,0}_{1j} + \frac{\partial u_i}{\partial x_2} h^{p,0}_{2j} + r_1 \, h^{p,0}_{3j} \right)^2 \sqrt{1 + p_1^2 + p_2^2} \, dx_1 \, dx_2 + \frac{2 \varepsilon^3}{3} \int_\omega \sum_{i,j=1}^3 \left(\frac{\partial r_i}{\partial x_1} h^{p,0}_{1j} + \frac{\partial r_i}{\partial x_2} h^{p,0}_{2j} \right)^2 \sqrt{1 + p_1^2 + p_2^2} \, dx_1 \, dx_2 , \tag{6.25}$$

and we estimate it first. Here, $(h^{p,0}_{ij})$ are the elements of the matrix S_p^{-1} (see (6.5), (6.6)), that is, they constitute an approximation of (h^p_{ij}). Taking into account the structure of S_p^{-1}, we get

$$\begin{aligned} \frac{\partial r_i}{\partial x_1} &= \frac{p_1}{\sqrt{1+p_1^2+p_2^2}}\left(\frac{\partial r_i}{\partial x_1}\, h_{13}^{p,0} + \frac{\partial r_i}{\partial x_2}\, h_{23}^{p,0}\right) \\ &\quad + \frac{1}{\sqrt{1+p_1^2+p_2^2}}\left(\frac{\partial r_i}{\partial x_1}\, h_{11}^{p,0} + \frac{\partial r_i}{\partial x_2}\, h_{21}^{p,0}\right), \end{aligned} \tag{6.26}$$

and similarly for $\frac{\partial r_i}{\partial x_2}$, $\frac{\partial u_i}{\partial x_\alpha}$, $i = \overline{1,3}$, $\alpha = 1, 2$.

Then, simple algebraic manipulations in (6.25), (6.26), involving the triangle inequality (and the fact that the coefficients of the parentheses in the right-hand side of (6.26) are less than one), put into evidence a constant, independent of $p \in \mathcal{K}$, such that:

$$Q_p(\bar{u}, \bar{r}) \geq c\Big(\varepsilon |\bar{u}|^2_{V(\omega)} + \varepsilon^3 |\bar{r}|^2_{V(\omega)} - \varepsilon |\bar{r}|^2_{L^2(\omega)^3}\Big), \quad c > 0. \tag{6.27}$$

Taking the difference between (6.24), (6.25), estimates similar to Sprekels and Tiba [16, Sect. 3] show that it will be dominated by the right-hand side in (6.27), for ε small. This ends the proof of **Lemma 6.3**. □

Combining it with (6.23), we get (6.12), for $\delta << \varepsilon$, and the proof of Proposition 6.2 is finished. □

Proposition 6.4 *Let $\tilde{\mathcal{K}} \subset \mathcal{K}$ be a compact subset. There are $\hat{\varepsilon} > 0$, such that for $\varepsilon < \hat{\varepsilon}$, there is $c_\varepsilon > 0$, independent of $p \in \tilde{\mathcal{K}}$,*

$$\mathcal{B}_p(\hat{u}, \hat{u}) \geq c_\varepsilon\Big[|\bar{u}|^2_{V(\omega)} + |\bar{r}|^2_{V(\omega)}\Big], \quad \hat{u}(\hat{x}) = \bar{w}(F_p^{-1}(\hat{x})) \in H^1(\hat{\Omega}_p), \tag{6.28}$$

for any $p \in \tilde{\mathcal{K}}$.

Proof. We fix $\hat{\varepsilon}$ and $\varepsilon < \hat{\varepsilon}$, $\delta << \varepsilon$, such that (6.12) is valid.

Assume that (6.28) is false, i.e. there is no $c_\varepsilon > 0$ with the indicated property. Therefore, for any $a > 0$, there is $p_a \in \mathcal{K}$ and $\bar{u}_a$, $\bar{r}_a$, $\hat{u}_a(\hat{x}) = \bar{w}_a(F_{p_a}^{-1}(\hat{x}))$, such that

$$0 \leq \mathcal{B}_{p_a}(\hat{u}_a, \hat{u}_a) \leq a\Big[|\bar{u}_a|^2_{V(\omega)} + |\bar{r}_a|^2_{V(\omega)}\Big]. \tag{6.29}$$

In (6.29), we can assume that $|(\bar{u}_a, \bar{r}_a)|_{V(\omega)^2} = 1$, and, consequently, that $\mathcal{B}_{p_a}(\hat{u}_a, \hat{u}_a) \to 0$ for $a \to 0$. Moreover, we can suppose that $\bar{u}_a \to \hat{u}$, $\bar{r}_a \to \hat{r}$, both weakly in $V(\omega)$, and $p_a \to \hat{p} \in \tilde{\mathcal{K}}$ strongly in $C^2(\bar{\omega})$, due to the compactness of $\tilde{\mathcal{K}}$. In particular, we get $h_{ij}^a \to \hat{h}_{ij}$ strongly in $C(\bar{\Omega})$, where $(h_{ij}^a)_{i,j=1,3} = J_{p_a}^{-1}$, $(\hat{h}_{ij})_{i,j=\overline{1,3}} = J_{\hat{p}}^{-1}$.

It is simple to see, due to the uniform convergence of the coefficients h_{ij}^a, that

$$\mathcal{B}_{p_a}(\hat{u}_a, \hat{u}_a) - \mathcal{B}_{\hat{p}}(\hat{u}_a, \hat{u}_a) \to 0 \tag{6.30}$$

(see (5.11)). The weak lower semicontinuity in $H^1(\omega)^3 \times H^1(\omega)^3$ of $\mathcal{B}_{\hat{p}}(\cdot, \cdot)$ and (6.29), (6.30) show that

$$0 \geq \liminf_{a\to 0} \mathcal{B}_{p_a}(\hat{u}_a, \hat{u}_a) = \liminf_{a\to 0} \mathcal{B}_{\hat{p}}(\hat{u}_a, \hat{u}_a) \geq \mathcal{B}_{\hat{p}}\Big((\hat{u}, \hat{r});\, (\hat{u}, \hat{r})\Big) \geq 0. \tag{6.31}$$

Clearly, (6.31) shows that $\mathcal{B}_{\hat{p}}((\hat{u}\,,\,\hat{r})\,;\,(\hat{u}\,,\,\hat{r}))=0$, and the coercivity of $\mathcal{B}_{\hat{p}}$ gives $\hat{u}=0$, $\hat{r}=0$, according to Sprekels and Tiba [16]. We conclude that $\bar{u}_a\to 0$, $\bar{r}_a\to 0$, both weakly in $V(\omega)$ and strongly in $L^2(\omega)^3$.

We combine (6.29) and (6.12) to obtain that

$$\begin{aligned} a &\geq c\Big[\varepsilon|\bar{u}_a|^2_{V(\omega)}+\varepsilon^2|\bar{r}_a|^2_{V(\omega)}\Big]-\frac{m}{\delta}\Big[|\bar{r}_a|^2_{L^2(\omega)^3}+|\bar{u}_a|^2_{L^2(\omega)^3}\Big]\\ &\geq c\,\varepsilon^3-\frac{m}{\delta}\Big[|\bar{r}_a|^2_{L^2(\omega)^3}+|\bar{u}_a|^2_{L^2(\omega)^3}\Big]\,. \end{aligned}$$

Taking $a\to 0$, we get the contradiction

$$0\;\geq\;c\,\varepsilon^3$$

which ends the proof. □

Proof of Theorem 6.1 We notice that the assumptions of Proposition 6.4 are fulfilled and that (6.28) is valid for $\{p_n\}$, for any $n\in\mathbb{N}$. Then, if we fix $(\bar{\mu},\bar{\rho})=\bar{y}_n=(\bar{u}_n,\bar{r}_n)$ in (5.11) with $p=p_n$, we get immediately that $\{\bar{y}_n\}$ is bounded in $V(\omega)^2$. We may assume that $\bar{u}_n\to\bar{u}$, $\bar{r}_n\to\bar{r}$, both weakly in $V(\omega)$, on a subsequence. Due to the uniform convergence of the coefficients, one may pass to the limit in (5.11) and see that $\bar{y}=(\bar{u},\bar{r})$ is indeed the solution of (5.11) associated to p. As the solution of (5.11) is unique, $\bar{y}$ is the weak limit of the whole sequence.

Now, we have to show that the convergence is valid in the strong topology of $V(\omega)^2$. We substract the equations corresponding to $\bar{y}_n$, $\bar{y}$, we intercalate advantageous terms (see the last step in the proof of Theorem 3.1) and, finally, we take test functions of the form $\bar{y}_n-\bar{y}\in V(\omega)^2$. As the difference of the corresponding right-hand sides converges to 0 (by the above weak convergence property), a detailed calculus gives that

$$\lim_{n\to\infty}\mathcal{B}_p(\bar{y}_n-\bar{y},,\,\bar{y}_n-\bar{y})\;=\;0\,. \tag{6.32}$$

By (6.28), (6.32), the proof is finished. □

Corollary 6.5 *If $\mathcal{K}\subset C^2(\bar{\omega})$ is compact and $j:V(\omega)^2\times C^2(\bar{\omega})\to R$ is lower semicontinuous, then the shape optimization problem* **(P)** *admits at least one optimal solution $p\in\mathcal{K}$.*

7 Sensitivity analysis for shells

We investigate some differentiability properties of the mapping $p\in C^2(\bar{\omega})\mapsto\bar{y}\in V(\omega)^2$ defined by (5.11). We consider $p+\lambda\,q$, $\lambda\in\mathbb{R}_+$, and $q\in C^2(\bar{\omega})$, a small perturbation of $p\in C^2(\bar{\omega})$, and we denote by $\bar{y}_\lambda=(\bar{u}^\lambda,\bar{r}^\lambda)\in V(\omega)^2$ the corresponding solution of (5.11). Similarly, we denote by $\bar{n}_\lambda\in C^1(\bar{w})^3$, $F_\lambda\in C^1(\bar{\Omega})^3$, $J_\lambda\in C(\bar{\Omega})^9$, $h^\lambda_{ij}\in C(\bar{\Omega})$, $g^{ij}_\lambda\in C(\bar{\Omega})$, $\mathcal{B}_\lambda$, etc., all the quantities defined in **Section 5**, starting from $p_\lambda=p+\lambda\,q$. We shall simply write $\mathcal{B}$ for $\mathcal{B}_p$.

It is elementary, though tedious, to check that the below listed limits, and linear and bounded operators, exist in the indicated spaces:

$$\lim_{\lambda\to 0}\frac{\bar n_\lambda-\bar n}{\lambda} = \tilde n(q)\,;\ \tilde n\ :\ C^2(\bar\omega)\ \to C^1(\bar\omega)^3\,, \tag{7.1}$$

$$\lim_{\lambda\to 0}\frac{J_\lambda-J}{\lambda} = \tilde J(q)\,;\ \tilde J\ :\ C^2(\bar\omega)\ \to C(\bar\Omega)^9\,, \tag{7.2}$$

$$\lim_{\lambda\to 0}\frac{J_\lambda^{-1}-J^{-1}}{\lambda} = \tilde I(q)\,;\ \tilde I\ :\ C^2(\bar\omega)\ \to C(\bar\Omega)^9\,, \tag{7.3}$$

$$\lim_{\lambda\to 0}\frac{h_{ij}^\lambda-h_{ij}}{\lambda} = \tilde h_{ij}(q)\,;\ \tilde h_{ij}\ :\ C^2(\bar\omega)\ \to C(\bar\Omega)\,, \tag{7.4}$$

$$\lim_{\lambda\to 0}\frac{\det J_\lambda-\det J}{\lambda} = \mathcal D(q)\,;\ \mathcal D\ :\ C^2(\bar\omega)\ \to C(\bar\Omega)\,, \tag{7.5}$$

$$\lim_{\lambda\to 0}\frac{g_\lambda^{ij}-g^{ij}}{\lambda} = \tilde g^{ij}(q)\,;\ \tilde g^{ij}\ :\ C^2(\bar\omega)\ \to C(\bar\Omega)\,. \tag{7.6}$$

By Theorem 6.1, we also know that

$$\bar y_\lambda \longrightarrow \bar y \quad \text{strongly in } V(\omega)^2\,. \tag{7.7}$$

Now, we subtract the equations for $\bar y_\lambda$ and for $\bar y$, we divide by λ, and we shall prove that it is possible to take $\lambda\to 0$. In the right-hand side, we have

$$\begin{aligned}
&\lim_{\lambda\to 0}\Bigg\{\int_\Omega\sum_{l=1}^3 f_l(\mu_l+x_3\,\rho_l)\frac{\det J_\lambda-\det J}{\lambda}\,d\bar x\\
&+\int_{\Gamma_1}\sum_{l=1}^3\sum_{i,j=1}^3 g_l(\mu_l+x_3\,\rho_l)\frac{\det J_\lambda\sqrt{\nu_i\,g_\lambda^{ij}\,\nu_j}-\det J\sqrt{\nu_i\,g^{ij}\,\nu_j}}{\lambda}\,d\tau\Bigg\}\\
=\ &\sum_{l=1}^3\int_\Omega f_l(\mu_l+x_3\,\rho_l)\mathcal D(q)\,d\bar x\\
&+\sum_{l=1}^3\sum_{i,j=1}^3\int_{\Gamma_1} g_l(\mu_l+x_3\,\rho_l)\left[\mathcal D(q)\sqrt{\nu_i\,g^{ij}\,\nu_j}+\det J\frac{\nu_i\,\tilde g^{ij}(q)\,\nu_j}{2\sqrt{\nu_i\,g^{ij}\,\nu_j}}\right]d\tau\,.
\end{aligned} \tag{7.8}$$

Here $\bar v=(\bar\mu,\bar\rho)\in V(\omega)^2$ is an arbitrary test function.

As the computation of $\frac{1}{\lambda}[\mathcal B_\lambda-\mathcal B]$ is quite lengthy, we write in detail just the terms from the bilinear functionals associated with the coefficient $2\,\tilde\mu$, namely:

$$\begin{aligned}
&\frac{1}{\lambda}\Bigg\{\int_\Omega\sum_{i=1}^3\left[\left(\frac{\partial u_i^\lambda}{\partial x_1}+x_3\,\frac{\partial r_i^\lambda}{\partial x_1}\right)h_{1i}^\lambda+\left(\frac{\partial u_i^\lambda}{\partial x_2}+x_3\,\frac{\partial r_i^\lambda}{\partial x_2}\right)h_{2i}^\lambda+r_i^\lambda\,h_{3i}^\lambda\right]\\
&\left[\left(\frac{\partial \mu_i}{\partial x_1}+x_3\,\frac{\partial \rho_i}{\partial x_1}\right)h_{1i}^\lambda+\left(\frac{\partial \mu_i}{\partial x_2}+x_3\,\frac{\partial \rho_i}{\partial x_2}\right)h_{2i}^\lambda+\rho_i\,h_{3i}^\lambda\right]|\det J_\lambda|d\bar x
\end{aligned}$$

$$
\begin{aligned}
&-\int_\Omega \left[\Big(\frac{\partial u_i}{\partial x_1}+x_3\frac{\partial r_i}{\partial x_1}\Big)h_{1i}+\Big(\frac{\partial u_i}{\partial x_2}+x_3\frac{\partial r_i}{\partial x_2}\Big)h_{2i}+r_ih_{3i}\right]\\
&\left[\Big(\frac{\partial \mu_i}{\partial x_1}+x_3\frac{\partial \rho_i}{\partial x_1}\Big)h_{1i}+\Big(\frac{\partial \mu_i}{\partial x_2}+x_3\frac{\partial \rho_i}{\partial x_2}\Big)h_{2i}+\rho_ih_{3i}\right]|\det J|d\bar{x}\Bigg\}\\
&=\int_\Omega\sum_{i=1}^3\left[\left(\frac{\partial\frac{u_i^\lambda-u_i}{\lambda}}{\partial x_1}+x_3\frac{\partial\frac{r_i^\lambda-r_i}{\lambda}}{\partial x_1}\right)h_{1i}+\left(\frac{\partial\frac{u_i^\lambda-u_i}{\lambda}}{\partial x_2}+x_3\frac{\partial\frac{r_i^\lambda-r_i}{\lambda}}{\partial x_2}\right)h_{2i}+\frac{r_i^\lambda-r_i}{\lambda}h_{3i}\right]\\
&\left[\Big(\frac{\partial \mu_i}{\partial x_1}+x_3\frac{\partial \rho_i}{\partial x_1}\Big)h_{1i}+\Big(\frac{\partial \mu_i}{\partial x_2}+x_3\frac{\partial \rho_i}{\partial x_2}\Big)h_{2i}+\rho_ih_{3i}\right]|\det J|d\bar{x}\\
&+\int_\Omega\sum_{i=1}^3\left[\Big(\frac{\partial u_i^\lambda}{\partial x_1}+x_3\frac{\partial r_i^\lambda}{\partial x_1}\Big)\frac{h_{1i}^\lambda\det J_\lambda-h_{1i}\det J}{\lambda}\right.\\
&\left.+\Big(\frac{\partial u_i^\lambda}{\partial x_2}+x_3\frac{\partial r_i^\lambda}{\partial x_2}\Big)\frac{h_{2i}^\lambda\det J_\lambda-h_{2i}\det J}{\lambda}+r_i^\lambda\frac{h_{3i}^\lambda\det J_\lambda-h_{3i}\det J}{\lambda}\right]\\
&\left[\Big(\frac{\partial \mu_i}{\partial x_1}+x_3\frac{\partial \rho_i}{\partial x_1}\Big)h_{1i}^\lambda+\Big(\frac{\partial \mu_i}{\partial x_2}+x_3\frac{\partial \rho_i}{\partial x_2}\Big)h_{2i}^\lambda+\rho_ih_{3i}^\lambda\right]d\bar{x}\\
&+\int_\Omega\sum_{i=1}^3\left[\Big(\frac{\partial u_i^\lambda}{\partial x_1}+x_3\frac{\partial r_i^\lambda}{\partial x_1}\Big)h_{1i}+\Big(\frac{\partial u_i^\lambda}{\partial x_2}+x_3\frac{\partial r_i^\lambda}{\partial x_2}\Big)h_{2i}+r_i^\lambda h_{3i}\right] \qquad (7.9)\\
&\left[\Big(\frac{\partial \mu_i}{\partial x_1}+x_3\frac{\partial \rho_i}{\partial x_1}\Big)\frac{h_{1i}^\lambda-h_{1i}}{\lambda}+\Big(\frac{\partial \mu_i}{\partial x_2}+x_3\frac{\partial \rho_i}{\partial x_2}\Big)\frac{h_{2i}^\lambda-h_{2i}}{\lambda}+\rho_i\frac{h_{3i}^\lambda-h_{3i}}{\lambda}\right]|\det J|d\bar{x}\,.
\end{aligned}
$$

According to (7.4), (7.5) and (6.10) the last two integrals are of the form $Z_\lambda(\bar{y}_\lambda\,,\,\bar{v})$, and there is a constant independent of $\lambda>0$ such that the bilinear forms Z_λ satisfy:

$$|Z_\lambda(\bar{y}_\lambda\,,\,\bar{v}|\leq C|\bar{y}_\lambda|_{V(\omega)^2}|\bar{v}|_{V(\omega)^2}\,. \tag{7.10}$$

Applying the same technique to all of the terms of $\mathcal{B}_\lambda-\mathcal{B}$, (7.8)–(7.10) give

$$\mathcal{B}\Big(\frac{\bar{y}_\lambda-\bar{y}}{\lambda}\,,\,\bar{v}\Big)\;=\;\tilde{Z}_\lambda(\bar{y}_\lambda,\bar{v})\,,\quad\forall\;\bar{v}\in V(\omega)^2\,, \tag{7.11}$$

where $\tilde{Z}_\lambda$ is obtained by adding together all the terms from (7.8)–(7.10).

By fixing $\bar{v}=\frac{\bar{y}_\lambda-\bar{y}}{\lambda}$ in (7.11), and taking into account (7.10) and (7.7), we see that $\left\{\frac{\bar{y}_\lambda-\bar{y}}{\lambda}\right\}$ is bounded in $V(\omega)^2$, due to Proposition 6.4. We may take a weakly convergent subsequence,

$$\frac{\bar{y}_\lambda-\bar{y}}{\lambda}\to\hat{y}\quad\text{weakly in}\quad V(\omega)^2\,, \tag{7.12}$$

and we can pass to the limit in (7.11). The obtained equation in variations has the form:

$$\mathcal{B}(\hat{y},\bar{v})=Z(\bar{v})\,,\quad\forall\;\bar{v}\in V(\omega)^2\,, \tag{7.13}$$

where $Z(\bar{v}) = \lim_{\lambda\to 0} \tilde{Z}_\lambda(\bar{y}_\lambda, \bar{v})$ and $Z : V(\omega)^2 \to I\!R$ is a linear bounded functional. Notice that (7.13) has a unique solution $\hat{y} \in V(\omega)^2$, due to (6.28). We thus have proved:

Proposition 7.1 *The mapping $p \in C^2(\bar{\omega}) \mapsto \bar{y} \in V(\omega)^2$ given by (5.11) is Gâteaux differentiable, and the directional derivative $\hat{y}$ satisfies (7.13).*

We introduce now the so-called adjoint system with unknowns $\bar{s} = (\bar{a}, \bar{b}) \in V(\omega)^2$,

$$\mathcal{B}(\bar{s}, \bar{v}) = \nabla_1 j(\bar{y}, p)(\bar{v}), \ \forall\ \bar{v} \in V(\omega)^2. \tag{7.14}$$

The existence and the uniqueness of the solution to (7.14) is clear due to the properties of $\mathcal{B}$. We have assumed that j is Fréchet differentiable on $V(\omega)^2 \times C^2(\bar{\omega})$, and $\nabla_1 j$, $\nabla_2 j$ denote the partial differentials with respect to $\bar{y}$, p.

Proposition 7.2 *If j is Fréchet differentiable, then the directional derivative of the cost functional Π in the problem* **(P)***, at the point $p \in C^2(\bar{\omega})$ and in the direction $q \in C^2(\bar{\omega})$, is given by:*

$$\nabla\Pi(p)q = \nabla_2 j(\bar{y}, p)q + Z(\bar{s}). \tag{7.15}$$

Proof.

$$\lim_{\lambda\to 0} \frac{\Pi(p+\lambda q) - \Pi(p)}{\lambda} = \nabla_2 j(\bar{y}, p)q + \nabla_1 j(\bar{y}, p)\hat{y},$$

by the chain rule and Proposition 7.1. Moreover, by (7.14), (7.13), we have

$$\nabla_1 j(\bar{y}, p)\hat{y} = \mathcal{B}(\bar{s}, \hat{y}) = \mathcal{B}(\hat{y}, \bar{s}) = Z(\bar{s}).$$

□

Remark. In order to compute (7.15) from $p, q \in C^2(\bar{\omega})$, one has to compute $\bar{y}$ by (5.11), $\bar{s}$ by (7.14) and Z by (7.13). The computation of Z is standard (see (7.9), (7.8)), but tedious and we do not detail it here.

Corollary 7.3 *Assume that p^* is a (local) optimal shape for* **(P)***, and $\bar{y}^*$ is the associated deformation, and that all the above assumptions are fulfilled. Then*

i) If $\mathcal{K} \subset C^2(\bar{\omega})$ is convex, we have

$$\nabla_2 j(\bar{y}^*, p^*)q + Z(\bar{s}) \geq 0, \ \forall\ q \in \mathcal{K} - p^*.$$

ii) If $\mathcal{K}$ is not convex, we have:

$$\nabla_2 j(\bar{y}^*, p^*)q + Z(\bar{s}) \geq 0, \quad \forall\ q \in T(\mathcal{K}, p^*).$$

Remark. Corollary 7.3 gives the standard optimality conditions for the problem **(P)**. The directional derivative obtained in Proposition 7.2 may be used, in principle, in the numerical computations, as in the case of the curved rods. However, the coercivity properties of the bilinear functional $\mathcal{B}_p$ are valid just for small thickness ε, and the coercivity constant depends in a very bad manner on ε (see Proposition 6.2 or Sprekels and Tiba [16]). This shows that instabilities (the locking problem) may appear in the numerical experiments and special numerical schemes are to be used. The interested reader may consult Paumier and Chenais [7], Pitkäranta and Leino [13], for a discussion on the approximation of the state equation (5.11).

8 Numerical experiments

In the papers of Ignat, Sprekels and Tiba [10], [11], many numerical examples concerning the deformation of three-dimensional curved rods and the optimization of planar arches are reported. Here, we concentrate on the problem discussed in §2-§4. The "locking phenomenon" [7], [13] is avoided in our experiments by allowing the thickness of the curved rod to be "larger" then the division that we consider for the interval $[0, L]$ $L = 4\pi\sqrt{2}$. Namely, we have divided the interval $[0, L]$ in 100 equal parts and we have taken the cross section of the curved rod to be always given by a disk with radius $R = 0.3$. For the integrals over the cross section, the usual change of variables to polar coordinates leads to the integration over the rectangle $[0, R] \times [0, 2\pi]$ which allows the use of simple numerical integration formulae corresponding to the discrete grids. We have divided it into 8, respectively 80 parts, and we have used Simpson's iterative formula.

In general, as initial iteration to the optimization algorithm, we have considered the spiral, lying on the cylinder $x_1^2 + x_2^2 = 1$, given by:

$$\psi^0(x_3) = \frac{\pi}{4}\,,\ \psi^0(x_3) = \frac{\pi}{2} + \frac{x_3}{\sqrt{2}}\,, \quad x_3 \in [0, L]\,. \tag{8.1}$$

A simple calculus shows that the rod parametrization corresponding to (8.1) is

$$\bar{\theta}(x_3) = \left(\cos\frac{x_3}{\sqrt{2}}\,,\ \sin\frac{x_3}{\sqrt{2}}\,,\ \frac{x_3}{\sqrt{2}}\right), \quad x_3 \in [0, L]\,. \tag{8.2}$$

Deformations for this example of a curved rod, under the action of various body forces, have been computed in Ignat, Sprekels and Tiba [11]. The Lamé constants taken into account are $\lambda = 50$, $\mu = 100$. The solution of the state system (2.13) in the Sobolev space $H_0^1(0, L)^9$ is approximated by linear splines from V_h^9, where $h > 0$ is the division norm of $[0, L]$, and where

$$V_h = \{v_h \in C[0, L]\,;\ v_h(0) = v_h(L) = 0\,,\ v_h \text{ is piecewise linear in } [0, L]\}\,. \tag{8.3}$$

The same matrix governs both (2.13) and the adjoint system (4.20). We underline that finding the matrix (which has to be recomputed in each optimization iteration) is the most time consuming step of the algorithm. This is due to the three-dimensional character of the objects that we are studying. The model (2.13) provides a dimension reduction up to O.D.E.'s, and this is reflected in that the coefficients involve the computation of many integrals over the cross section. One can compute the gradient of the cost functional, and use descent algorithms for the optimization of the geometry of the 3D rods, as explained in Section 4. We have used the Uzawa algorithm combined with the Armijo line search rule.

A first class of examples is obtained when the force $\bar{f} = (0, 0, f_3)$ with the variants:

$$f_3(x_3) = \begin{cases} 10 \ , & x_3 \in \left[0, \dfrac{L}{2}\right], \\ -10 \ , & x_3 \in \left]\dfrac{L}{2}, L\right], \end{cases} \tag{8.4}$$

$$f_3(x_3) \equiv 10 \quad \text{in } [0, L], \tag{8.5}$$

$$f_3(x_3) = \begin{cases} 10 \ , & x_3 \in \left[0, \dfrac{L}{2}\right], \\ 0 \ , & x_3 \in \left]\dfrac{L}{2}, L\right], \end{cases} \tag{8.6}$$

$$f_3(x_3) = \begin{cases} 0 \ , & x_3 \in \left[0, \dfrac{L}{2}\right], \\ 10 \ , & x_3 \in \left]\dfrac{L}{2}, L\right]. \end{cases} \tag{8.7}$$

The cost functional was $\Pi = \frac{1}{2}|\tau_i|^2_{L^2(0,L)}$ with $i = 2, 3$ (compare with (2.14)). We have also imposed the constraint (2.15), with $\varepsilon = \frac{\pi}{8}$, to avoid the appearance of self intersecting curves. We have neglected (2.11), but it may be checked a posteriori that $\det J \neq 0$.

In all the cases (8.4)–(8.7), the vertical column, which corresponds to $\varphi \equiv 0$, was the geometric solution of the given problem. Indeed, the vertical column is the most resistant structure with respect to vertical forces as in (8.4)–(8.7). In this case also, the lateral displacements τ_1, τ_2 are with several orders of magnitude smaller than the vertical displacement.

Figure 1 shows the initial and the final geometries, obtained in one or two iterations. In Figures 2–5, the values of τ_3 (in the final iteration) are shown and one can see its dependence on the forces (8.4)–(8.7), respectively.

In another set of numerical tests, we have considered $\bar{f} = 10\bar{b}$ (recall (2.3)). Again the initial iteration was given by (8.1) (or (8.2)) or by some perturbation of it

$$\varphi^0(x_3) = \begin{cases} \dfrac{\pi}{4} + 0,1 \ , & x_3 \in \left[0, \dfrac{L}{2}\right], \\ \dfrac{\pi}{4} - 0,1 \ , & x_3 \in \left]\dfrac{L}{2}, L\right], \end{cases} \tag{8.8}$$

and the objective functional was the same as above.

Notice that, under our parametrization, it is very simple to change the initial iteration, which is an important advantage in nonconvex optimization problems. The main property of this choice of $\bar{f}$ is that it acts always in the horizontal plane, although in various directions. It is also very easy to be constructed, under our approach. For the constraints, we have taken $\varepsilon = 0$ in (2.15). This allows horizontal curves as well, but self intersections may appear (which indeed was the case). That is, in this set of experiments (2.11) is violated. In the examples that we have computed, a clear decrease in the cost was observed and the tendency to produce an horizontal curve as the solution. Although self intersections are present, horizontal curves will deform just in the horizontal plane under the action of $\bar{f} = 10\bar{b}$. That is, a mechanical interpretation is still possible (and due to this, it was necessary to allow $\varepsilon = 0$ in (2.15)).

An interesting feature of this type of experiments was that the optimal φ was bang-bang. The figures 6, 7 show this when the initial iteration was given by (8.8), respectively (8.1). Figures 8 and 9 show the last two iterations of another example with first iteration (8.1) and cost functional $\frac{1}{2}(\tau_2)^2_{L^2(0,L)}$. As the optimal φ is very close to $\frac{\pi}{2}$, then the obtained (self intersecting) structure is almost horizontal. In this example, the optimal values of τ_3 are very small too, which corresponds well to the mechanical interpretation.

In general, one experiment took between two and three hours, on a powerful Compaq GS80 workstation.

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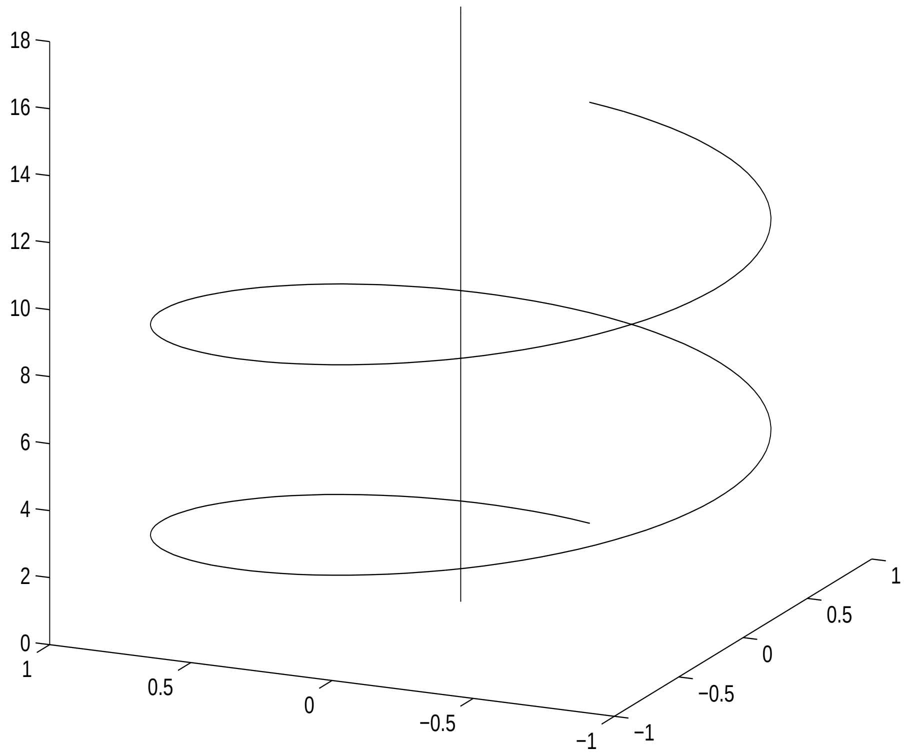


Figure 1

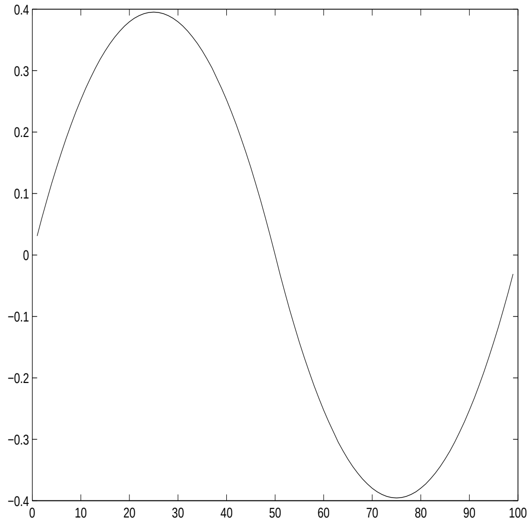


Figure 2

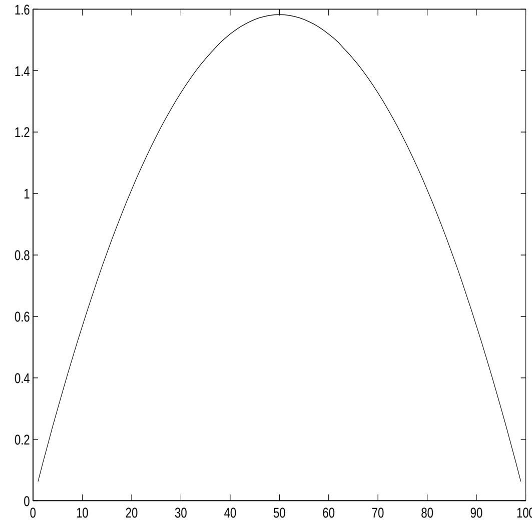


Figure 3

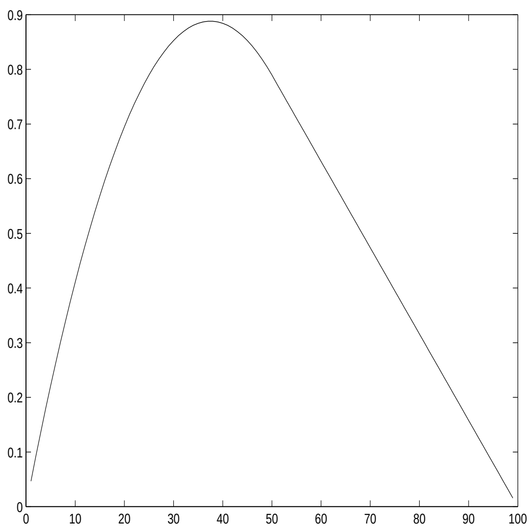


Figure 4

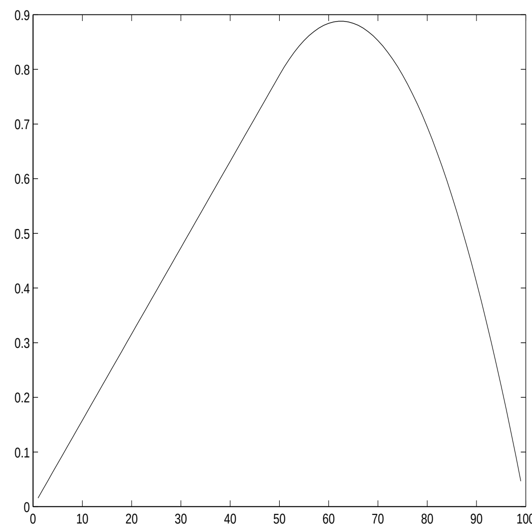
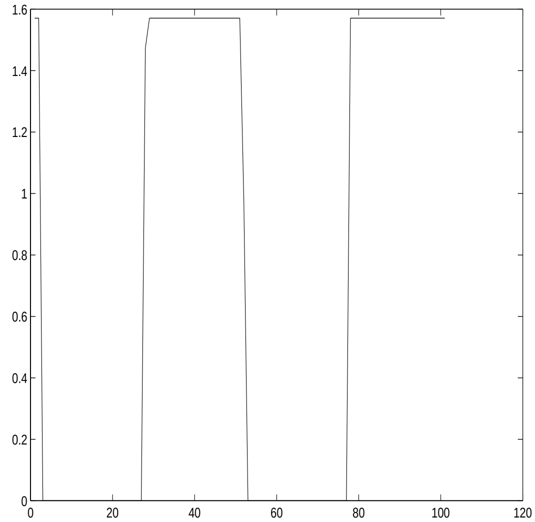


Figure 5



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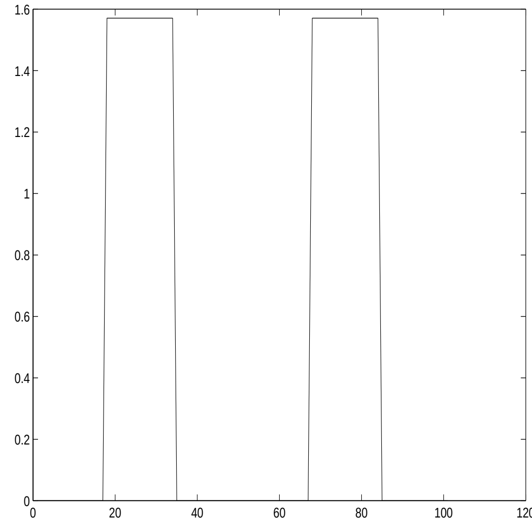

Figure 7

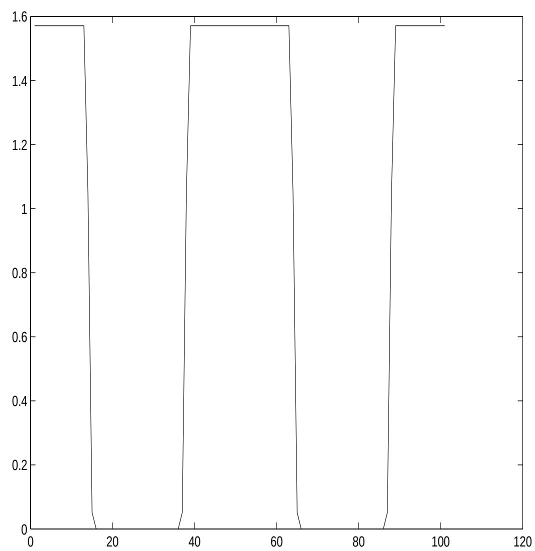

Figure 8

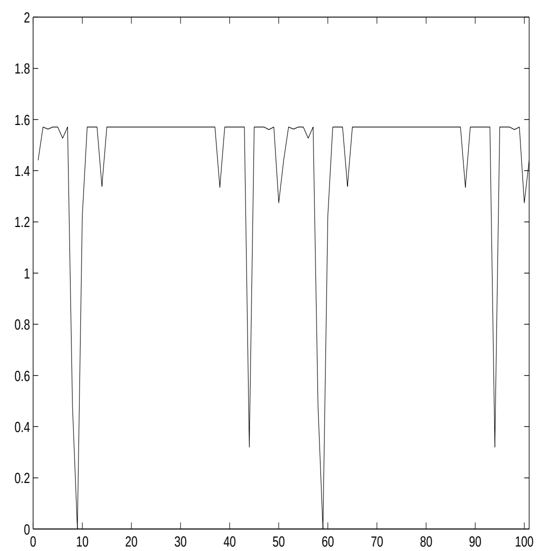

Figure 9