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On unique solvability of nonlocal drift–diffusion type problems

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Abstract

We prove a priori estimates in $L^2(0,T;W^{1,2}(\Omega))$ and $L^\infty(Q_T)$, existence and uniqueness of solutions to Cauchy–Neumann problems for parabolic equations

$$\frac{\partial\sigma(u)}{\partial t}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Big\{\rho(u)\;b_i\Big(t,x,\frac{\partial(u-v)}{\partial x}\Big)\Big\}+a\big(t,x,v,u\big)=0, \tag{0.1}$$

$(t,x)\in Q_T=(0,T)\times\Omega\subset R^{n+1}$, where $\rho(u)=\frac{\partial\sigma(u)}{\partial u}>0$ and the function v is defined by the nonlocal expression

$$v(t,x)=-\int_\Omega K(x,y)\big[\sigma(u(t,y))-f(t,y)\big]\;dy, \tag{0.2}$$

instead of solving an elliptic boundary problem as in the corresponding local case. Such problems arise as mathematical models of various diffusion-drift processes driven by gradients of local particle concentrations and nonlocal interaction potentials. An example is the transport of electrons in semiconductors, where u has to be interpreted as chemical and v as electro–statical potential.

1 Introduction

We prove a priori estimates, existence and uniqueness of weak solutions to initial–boundary value problems of the form

$$\frac{\partial\sigma(u)}{\partial t}-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\Big\{\rho(u)\;b_i\Big(t,x,\frac{\partial(u-v)}{\partial x}\Big)\Big\}+a\big(t,x,v,u\big)=0,\quad (t,x)\in Q_T, \tag{1.1}$$

$$v(t,x)=-\int_\Omega K(x,y)\big[\sigma(u(t,y))-f(t,y)\big]\;dy,\quad (t,x)\in Q_T, \tag{1.2}$$

$$\rho(u)\;\sum_{i=1}^{n}b_i\Big(t,x,\frac{\partial(u-v)}{\partial x}\Big)\cos(\nu,x_i)=0,\quad (t,x)\in\Gamma=(0,T)\times\partial\Omega, \tag{1.3}$$

$$u(0,x)=h(x),\quad x\in\Omega, \tag{1.4}$$

where $\sigma(u)=\int_0^u\rho(s)\;ds,\;\rho>0,\;\Omega$ is a bounded open set in $\mathbb{R}^n$ and $Q_T=(0,T)\times\Omega$, $T>0$. In the case of smooth boundary $\partial\Omega$ of the set Ω, ν is the outer unit normal on $\partial\Omega$ and (ν,x_i) is the angle between ν and the x_i-axis.

In the physical motivation and derivation (cf. [1, 6]) of systems like (1.1) – (1.3) the 'free energy'

$$F(c) = \int_\Omega \{\Lambda(\sigma^{-1}(c)) + c\int_\Omega K(x,y)[\frac{c(t,y)}{2} - f(t,y)]dy\}dx,\ \Lambda(u) = \int_0^u s\rho(s)ds$$

plays an important role. Here $c = \sigma(u)$ can be seen as particle concentration and the respective terms model local and nonlocal particle interaction. Then, provided the reaction term a vanishes, the system (1.1) – (1.4) describes the mass conservating evolution of c from the initial value $c_0 = \sigma(h)$ towards critical points or even minimizers of F under diffusion and drift forces, caused by the local and the global term in F, respectively. Moreover, the functional F will be also the key for our mathematical analysis of the system (cf. Theorem 1). In particular, in the case that: the kernel K is symmetrical, the vector field $\{b_i(t,x,\cdot)\} \in (\mathbb{R}^n \to \mathbb{R}^n)$ is monotone and $b_i(t,x,0) = a = 0$, we find for solutions u, v of (1.1) – (1.3)

$$\frac{dF(c)}{dt} = \int_\Omega \frac{\partial\sigma(u)}{\partial t}(u-v)\ dx = -\int_\Omega \rho(u)\sum_{i=1}^n b_i(t,x,\frac{\partial(u-v)}{\partial x})\ \frac{\partial(u-v)}{\partial x_i}\ dx \le 0,$$

that means, F is Lyapunov functional in that case.

Problems of the form (1.1) – (1.4) arise as nonlocal mathematical models of various applied problems, for instance reaction–drift–diffusion processes of electrically charged species, phase transition processes and transport processes in porous media. The investigation of nonlinear nonlocal problems has received much attention in last years. In the papers [6, 7, 11, 12] nonlocal models of phase separation were formulated and studied.

Corresponding local problems were studied by many authors (cf. [4, 5]). See also the papers [2], [3, 16], where degenerate parabolic equations were studied. Most strong results for local drift–diffusion type problems have been recently proved in [10]. Such local problems result from (1.1) – (1.4) by replacing the integral equation (1.2) by an elliptic differential equation like

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i}\left[\kappa(x)\frac{\partial v}{\partial x_i}\right] + \sigma(u) = f(t,x), \quad (t,x) \in Q_T\ , \tag{1.5}$$

completed by some boundary condition for the function v.

We consider problem (1.1) – (1.4) under standard conditions for the functions b_i and some conditions for the function a to be formulated in Section 2. Our main specific assumption concerning the equation (1.1) reads:

ρ_1) $\rho \in (\mathbb{R}^1 \to \mathbb{R}^1)$ with $\rho(u) > 0,\ u \in \mathbb{R}^1$, is continuous and has a piecewise continuous derivative ρ' such that $\frac{\rho'(u)}{\rho(u)}$ is nonincreasing on $\mathbb{R}^1$.

This condition seems natural in view of properties of probability particle distribution functions arising in mathematical physics. So in the semiconductor theory [1, 4] relevant examples for functions ρ satisfying condition $\rho_1)$ are given by $\sigma = \mathcal{F}_{\gamma+1}$, $\rho = \sigma' = \mathcal{F}_\gamma$, where $\mathcal{F}_\gamma$ denotes the Fermi integral

$$\mathcal{F}_\gamma(u) = \frac{1}{\Gamma(\gamma+1)} \int_0^\infty \frac{s^\gamma \, ds}{1+\exp(s-u)} \quad \gamma > -1 . \tag{1.6}$$

Another example comes from phase separation problems [6, 7], where the Fermi function

$$\sigma(u) = \frac{1}{1+exp(-u)}, \quad \rho(u) = \sigma'(u) = \frac{1}{(1+e^u)(1+e^{-u})}, \tag{1.7}$$

plays a role corresponding to that of $\mathcal{F}_{\gamma+1}$.

Our main assumption on the kernel $K(x,y)$ is:

$K_1)$ the function $K(x,y)$ is defined for $x, y \in \Omega$, $K(x,y) = K(y,x)$ and $K(\cdot,y) \in W^{1,1}(\Omega)$ for almost every $y \in \Omega$ such that

$$\operatorname*{ess\,sup}_{x\in\Omega} \int_\Omega \left\{ |K(x,y)| + \left| \frac{\partial K(x,y)}{\partial x} \right| \right\} dy + \operatorname*{ess\,sup}_{y\in\Omega} \int_\Omega \left| \frac{\partial K(x,y)}{\partial x} \right| dx \le \varkappa . \tag{1.8}$$

Remark that condition $K_1)$ implies (cf. Lemma 1 below) properties as assumed in [6, 7] for integral operators generated by kernels $K(x,y) = K(|x-y|)$.
Remark also that kernels $|x-y|^{2-n}$, $\log \frac{1}{|x-y|}$, corresponding to Newton potentials and fundamental solutions of equation (1.5) with bounded measurable function κ satisfy condition $K_1)$ [14]. The Green function for equation (1.5) satisfies condition $K_1)$ in the cases of Dirichlet or Neumann boundary conditions for sufficient smooth $\partial\Omega$ and κ. Conditions on κ guarantying condition $K_1)$ for the Green function can be formulated also in terms of smallness of the number

$$\operatorname*{ess\,sup}_{x\in\Omega} \kappa(x) \, [\operatorname*{ess\,inf}_{x\in\Omega} \kappa(x)]^{-1} - 1 .$$

We formulate our assumptions and main results in Section 2. First a priori estimates for the solution (u,v) are given in Section 3. In that Section we prove also regularity properties of the function v, important for further considerations. An estimate of u in $L^\infty(Q_T)$ is given in Section 4. Sections 5 and 6 are devoted to proofs of existence and uniqueness of solutions to problem (1.1) – (1.4) respectively.
We are planning in forthcoming papers to apply our approach to systems of equations describing reaction–drift–diffusion processes in isothermal and non–isothermal cases.

2 Formulation of assumptions and main results

Let Ω be a bounded open set in $\mathbb{R}^n$ and $Q_T = (0,T) \times \Omega$, $T > 0$. We assume that $n > 2$. For $n \le 2$ it is necessary to make simple changes in our conditions that are connected with Sobolev's embedding theorem.

We assume following condition on the set Ω:

∂) Ω is such that the embeddings $W^{1,1}(\Omega) \subset L^{\frac{n}{n-1}}(\Omega)$, $W^{1,p}(\Omega) \subset L^{\infty}(\Omega)$ hold for $p > n$.

In view of the proof of a priori estimates for solutions of problem (1.1) – (1.4) we need restrictions on growth and on degeneration of the function ρ as $u \to \pm\infty$. From condition ρ_1) the existence of

$$\alpha_{\pm} = \lim_{u \to \pm\infty} \rho(u) \tag{2.1}$$

follows. For nonconstant functions $\rho(u)$ at least one of the limits α_-, α_+ is zero [8]. Studying the problem (1.1) – (1.4) we have to distinguish the cases of zero or non–zero values of $\alpha_{\pm}$. Therefore we shall consider two cases:

$$\alpha_1) \quad \alpha_- = 0,\ \alpha_+ \neq 0\ , \qquad \alpha_2) \quad \alpha_- = \alpha_+ = 0\ .$$

Note that examples for α_1) and α_2) are given by (1.6) and (1.7), respectively. Our additional restrictions on the function ρ are following:

ρ_2) if condition α_1) holds, then a positive constant ρ_1 exists such that

$$\rho_1^{-1}(u^{\gamma}+1) \leq \rho(u) \leq \rho_1(u^{\gamma}+1), \quad u > 0, \quad 0 \leq \gamma \leq \frac{2}{n-1}\ , \tag{2.2}$$

ρ_3) there exists a positive constant ρ_2 such that

$$|\rho'(u)| \leq \rho_2\, \rho(u) \tag{2.3}$$

for $u < 0$ in the case of condition α_1) and $\forall u \in \mathbb{R}^1$ if condition α_2) holds.

Let the coefficients a , b_i from (1.1) satisfy the assumptions:

i) $a(t,x,v,u)$, $b_i(t,x,\xi)$, $i = 1,\dots,n$, are measurable with respect to t, x for every $u, v \in \mathbb{R}^1$, $\xi \in \mathbb{R}^n$ and continuous with respect to $u, v \in \mathbb{R}^1$, $\xi \in \mathbb{R}^n$, for almost every $(t,x) \in Q_T$; $b_i(t,x,0) = 0$, $i = 1,\dots,n$;

ii) there exist positive constants ν_1, ν_2 such that $\forall\ \xi', \xi'' \in \mathbb{R}^n$ and $(t,x) \in Q_T$

$$\sum_{i=1}^{n} \big[b_i(t,x,\xi') - b_i(t,x,\xi'')\big](\xi_i' - \xi_i'') \geq \nu_1 |\xi' - \xi''|^2,$$

$$|b_i(t,x,\xi)| \leq \nu_2(|\xi|+1), \quad i = 1,\dots,n;$$

iii) there exist nonnegative functions $\alpha_1 \in L^1(Q_T)$, $\alpha \in L^{p_1}(Q_T)$, $p_1 > \frac{n+2}{2}$, such that for arbitrary $(t,x) \in Q_T$, $v, u \in \mathbb{R}^1$

$$a(t,x,v,u)u \geq \nu_1\ \varepsilon(u)|u|^m - \nu_2|v|^m - \alpha_1(t,x),$$

$$|a(t,x,v,u)| \leq \nu_2(\varepsilon(u)|u| + |v|)^{m-1} + \alpha(t,x)\ ,$$

where $m = \frac{2+\gamma}{1+\gamma}$ under condition α_1) and $m = 2$ under condition α_2), here $\varepsilon(u)$ is a nonnegative function bounded on $\mathbb{R}^1$.

We assume also an additional condition on the kernel $K(x,y)$:

K_2) if condition α_1) is satisfied, then

$$\int_\Omega \int_\Omega K(x,y)\, g(x)\, g(y)\, dx\, dy \geq 0 \,, \quad \forall g \in L^2(\Omega).$$

Remark 1 *In relevant applications the kernel K models nonlocal particle interaction. Positive sign of K as assumed in condition K_2), corresponds to repulsive interaction between particles and implies, roughly speaking, global existence of solutions, whereas negative sign models attraction forces and may be cause blow of solutions (cf. [5]). However, under condition α_2) assumed in the papers [6, 7] ρ turns out to be bounded, so global existence can be proved without condition K_2).*

We consider problem (1.2) – (1.4) with f, h such that

$$f \in C\big([0,T], L^{p_2}(\Omega)\big), \quad \frac{\partial f}{\partial t} \in L^m\big(0,T,[W^{1,m}(\Omega)]^*\big)\,, \tag{2.4}$$

$$h(x) \in L^\infty(\Omega) \tag{2.5}$$

and $p_2 > n + \frac{2\gamma}{\gamma+1}$ in the case of condition α_1) and $p_2 > n$ under condition α_2).

Definition 1 *A pair of functions (u,v), $u,v \in L^2(0,T;W^{1,2}(\Omega))$, is called solution of problem (1.1) – (1.4), if following conditions are satisfied:*

i) the derivative $\frac{\partial \sigma(u)}{\partial t}$ exists in the sense of distributions,

$$\int_{Q_T}\int \rho(u)\Big[\Big|\frac{\partial u}{\partial x}\Big|^2 + \Big|\frac{\partial v}{\partial x}\Big|^2\Big]\, dx\, dt < \infty\,, \tag{2.6}$$

$$\sigma(u) \in C\big([0,T], L^2(\Omega)\big), \quad \frac{\partial \sigma(u)}{\partial t} \in L^2\big(0,T;[W^{1,2}(\Omega)]^*\big)\,; \tag{2.7}$$

ii) $\forall \varphi \in C^\infty(\overline{Q}_T)$ and almost every $\tau \in (0,T)$, $Q_\tau = (0,\tau)\times\Omega$,

$$\int_0^\tau < \frac{\partial \sigma(u)}{\partial t}, \varphi > dt +$$

$$+ \int_{Q_\tau}\int \Big\{ \sum_{i=1}^n \rho(u)\, b_i\Big(t,x,u,\frac{\partial(u-v)}{\partial x}\Big)\frac{\partial \varphi}{\partial x_i} + a(t,x,v,u)\varphi\Big\}\, dx\, dt = 0\,, \tag{2.8}$$

equality (1.2) is satisfied for almost all $(t,x) \in Q_T$;

iii) $\forall \varphi \in C^\infty(\overline{Q}_T)$, satisfying $\varphi(T,x) = 0$ for $x \in \Omega$,

$$\int_0^T < \frac{\partial \sigma(u)}{\partial t}, \varphi > dt \; + \int_{Q_T}\int \big[\sigma(u) - \sigma(h)\big]\frac{\partial \varphi}{\partial t}\, dx\, dt = 0\,. \tag{2.9}$$

Remark 2 *Let (u,v) be a solution of problem (1.1) – (1.4). Since the space $C^\infty(\overline{Q}_T)$ is dense in the weighted space $L^2(0,T;W^{1,2}(\Omega,\rho(u))$, the integral identity (2.8) holds for all $\varphi \in L^2(0,T;W^{1,2}(\Omega))$ such that*

$$\int_{Q_T}\int \rho(u)\Big|\frac{\partial \varphi}{\partial x}\Big|^2\,dx\,dt < \infty.$$

Remark 3 *Lemma 1 below guarantees that the right hand side of equality (1.2) is well–defined under our conditions on the functions σ and f.*

In what follows we shall understand as known parameters all numbers from the conditions *ii), iii),* K_1), norms of functions f, h, α_1, α in respective spaces, numbers that depend only on Ω, T, n, the numbers $\rho_1, \rho_2,\ \rho_3 = \max\{\rho(u) : |u| \le m_0\}$ and $\rho_4 = \min\{\rho(u) : |u| \le m_0\}$, where

$$m_0 = \|h(x)\|_{L^\infty(\Omega)} + 1. \tag{2.10}$$

Further we shall denote by c_j constants depending only on known parameters.

Theorem 1 *Let the conditions i) – iii), $K_1), K_2), \rho_1), \partial)$, (2.4), (2.5) be satisfied. Then there exists a constant M_1 depending only on known parameters, such that each solution (u,v) of problem (1.1) – (1.4) satisfies*

$$ess\sup_{t\in(0,T)}\int_\Omega \Lambda(u(t,x))\,dx + \int_{Q_T}\int \Big\{\rho(u)\Big|\frac{\partial(u-v)}{\partial x}\Big|^2\Big\}\,dt\,dx \le M_1, \tag{2.11}$$

where

$$\Lambda(u) = \int_0^u s\,\rho(s)\,ds\ . \tag{2.12}$$

Theorem 2 *Let the assumptions of Theorem 1 and condition ρ_2) be satisfied. Then there exists a constant M_2, depending only on known parameters, such that each solution (u,v) of problem (1.1) – (1.4) satisfies*

$$\int_{Q_T}\int \rho(u)\Big[\Big|\frac{\partial u}{\partial x}\Big|^2 + \Big|\frac{\partial v}{\partial x}\Big|^2\Big]\,dx\,dt \le M_2. \tag{2.13}$$

Theorem 3 *Let the assumptions of Theorem 2 be satisfied. Then there exist constants M_3, p_3 depending only on known parameters such that $p_3 > n$ and each solution (u,v) of problem (1.1) – (1.4) satisfies*

$$\|v\|_{L^\infty(Q_T)} + \Big\|\frac{\partial v}{\partial x}\Big\|_{L^{p_3+2}(Q_T)} + \Big\|\frac{\partial v}{\partial x}\Big\|_{L^\infty(Q_T,L^{p_3}(\Omega))} \le M_3. \tag{2.14}$$

In order to prove a priori estimates for $u(t,x)$ we need an additional condition with respect to the function a. In view of our uniqueness result we assume a stronger condition than needed here:

a) the function $\frac{1}{\rho(u)}\, a(t,x,v,u)$ is nondecreasing with respect to $u \in \mathbb{R}^1$, for arbitrary $(t,x) \in Q_T$, $v \in \mathbb{R}^1$.

Theorem 4 *Let the conditions i) – iii), $\rho_1) - \rho_3), K_1), K_2)$, a), ∂), (2.4), (2.5) be satisfied. Then there exists a constant M_4, depending only on known parameters, such that each solution (u,v) of problem (1.1) – (1.4) satisfies*

$$ess\sup\left\{|u(t,x)| : (t,x) \in Q_T\right\} \leq M_4. \tag{2.15}$$

Theorem 5 *Let the conditions of Theorem 4 be satisfied. Then the initial–boundary value problem (1.1) – (1.4) has at least one solution in the sense of Definition 1.*

Theorem 6 *Let the conditions of Theorem 4 be satisfied and assume additionally that the functions $b_i(t,x,\xi)$, $\rho'(u)$, $a(t,x,v,u)$ are locally Lipschitzian with respect to ξ, u, v, respectively. Then the solution of problem (1.1) – (1.4) in the sense of Definition 1 is unique.*

Corollary 1 *Let the conditions of Theorem 6 be satisfied and assume additionally that the functions $f(t,x)$, $b_i(t,x,\xi)$, $a(t,x,v,u)$ are Lipschitzian with respect to t. Then the solution u of problem (1.1) – (1.4) is regular in the sense that*

$$t \to t\,\frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)).$$

Remark 4 *Corollary 1 and Theorem 4 imply that $t \to t\,\frac{\partial \sigma(u)}{\partial t} \in L^\infty(0,T;L^2(\Omega))$. Consequently, (1.1) can be understand not only in the sense of distributions, but even as an equation in $L^2(0,T;L^2(\Omega))$.*

Proofs of the theorems 1, 2, 3 are given in Section 3, proofs of the theorems 4, 5, 6 are given in Sections 4, 5, 6, respectively.

3 Integral estimates of the solution

We start from auxiliary lemmas needed in the proofs of the Theorems 1– 6.
Let us define operators K_0, K_1 for $g \in L^\infty(\Omega)$ by

$$K_0\, g(x) = \int_\Omega |K(x,y)|\, g(y)\, dy, \quad K_1\, g(x) = \int_\Omega \left|\frac{\partial K(x,y)}{\partial x}\right| g(y)\, dy. \tag{3.1}$$

Lemma 1 *The operators K_0, K_1 are well defined by (3.1) for $g \in L^p(\Omega)$, $p \in [1,\infty]$, and they are bounded operators in following spaces*

$$K_0 : L^p(\Omega) \to L^{\frac{np}{n-p}}(\Omega) \quad for \quad 1 \leq p < n\,, \tag{3.2}$$

$$K_1 : L^p(\Omega) \to L^p(\Omega) \quad for \quad 1 \leq p \leq \infty. \tag{3.3}$$

Proof. Firstly, we prove (3.3). For $p = 1$ and $p = \infty$ (3.3) is a simple consequence of (1.8). For $1 < p < \infty$ we find for $g \in L^p(\Omega)$ by Hölder's inequality

$$\int_\Omega |K_1\, g(x)|^p\, dx \le \int_\Omega \Big[\int_\Omega \Big|\frac{\partial K(x,y)}{\partial x}\Big| |g(y)|\, dy\Big]^p dx \le$$
$$\le \int_\Omega \Big[\int_\Omega \Big|\frac{\partial K(x,y)}{\partial x}\Big| |g(y)|^p\, dy\Big]\Big[\int_\Omega \Big|\frac{\partial K(x,z)}{\partial x}\Big|\, dz\Big]^{p-1} dx \le \varkappa^p \int_\Omega |g(y)|^p\, dy\ ,$$

that is (3.3).

For proving (3.2) we use the embedding theorem for $W^{1,1}(\Omega)$ to infer from (1.8)

$$\operatorname*{ess\,sup}_{x\in\Omega} \int_\Omega |K(x,y)|^{\frac{n}{n-1}}\, dy \le c_1. \tag{3.4}$$

Now by Hölder's inequality we have for $1 \le p < n,\ g \in L^\infty(\Omega)$,

$$\int_\Omega |K_0\, g(x)|^{\frac{np}{n-p}}\, dx \le$$
$$\le \int_\Omega \Big\{\int_\Omega |K(x,y)|^{\frac{n(p-1)}{(n-1)p}} \cdot \Big[|K(x,y)|^{\frac{n-p}{(n-1)p}} \cdot |g(y)|^{\frac{n-p}{n}}\Big] \cdot |g(y)|^{\frac{p}{n}}\, dy\Big\}^{\frac{np}{n-p}} dx \le$$
$$\le \int_\Omega \Big\{\int_\Omega |K(x,y)|^{\frac{n}{n-1}}\, dy\Big\}^{\frac{n(p-1)}{n-p}} \cdot \Big\{\int_\Omega |K(x,\overline{y})|^{\frac{n}{n-1}} \cdot |g(\overline{y})|^p\, d\overline{y}\Big\} \times$$
$$\times \Big\{\int_\Omega |g(\widetilde{y})|^p\, d\widetilde{y}\Big\}^{\frac{p}{n-p}} dx \le c_2 \Big\{\int_\Omega |g(x)|^p\, dx\Big\}^{\frac{n}{n-p}}.$$

This inequality implies (3.2) and the proof of Lemma 1 is complete. □

Lemma 2 *Let the assumptions of Theorem 1 be satisfied. Then the estimate*

$$\int_0^\tau < \frac{\partial\sigma(u)}{\partial t}, v > dt \le M_5\,\big(1 + \|\sigma(u(\tau,x))\|_{L^1(\Omega)} + \|\sigma(u)\|_{L^2(Q_\tau)}\big) \tag{3.5}$$

holds for each $\tau \in (0,T)$ with a constant M_5 depending only on known parameters.

Proof. Let $\tau \in (0,T)$ and define for $0 < \delta < T - \tau$

$$I(\delta) = \int_0^\tau \int_\Omega \big[\sigma(u(t+\delta,x)) \cdot v(t+\delta,x) - \sigma(u(t,x))v(t,x)\big]\, dx\, dt. \tag{3.6}$$

By writing $I(\delta)$ as difference of two integrals and changing the integration variable in the first integral we get

$$I(\delta) = \int_\tau^{\tau+\delta} \int_\Omega \sigma(u(t,x))v(t,x)\, dx\, dt - \int_0^\delta \int_\Omega \sigma(u(t,x))v(t,x)\, dx\, dt. \tag{3.7}$$

On the other hand we can rewrite $I(\delta)$ as

$$I(\delta) = I_1(\delta) + I_2(\delta), \tag{3.8}$$

where

$$
\begin{aligned}
I_1(\delta) &= \int_0^\tau \int_\Omega \big[\sigma(u(t+\delta,x)) - \sigma(u(t,x))\big] v(t+\delta,x)\; dx\; dt\;,\\
I_2(\delta) &= \int_0^\tau \int_\Omega \sigma(u(t,x)) \cdot \big[v(t+\delta,x) - v(t,x)\big]\; dx\; dt.
\end{aligned}
$$

Using (1.2) and setting $v_1(t,x) = \int_\Omega K(x,y)\sigma(u(t,y))\; dy$, we can rewrite I_2 as

$$
\begin{aligned}
I_2(\delta) = &\int_0^\tau \int_\Omega \big[\sigma(u(t+\delta,x)) - \sigma(u(t,x))\big] v(t,x)\; dx\; dt + \\
&+ \int_0^\tau \int_\Omega \big[f(t+\delta,x) - f(t,x)\big] v_1(t,x)\; dx\; dt - \\
&- \int_\delta^\tau \int_\Omega \big[f(t-\delta,x) - f(t,x)\big] v_1(t,x)\; dx\; dt - \\
&- \int_\tau^{\tau+\delta} \int_\Omega f(t-\delta,x) v_1(t,x)\; dx\; dt\; + \int_0^\delta \int_\Omega f(t,x) v_1(t,x)\; dx\; dt\;.
\end{aligned}
\tag{3.9}
$$

From (3.8) – (3.9), (2.4), (2.7) and Lemma 1 we see that dividing $I(\delta)$ by δ and passing to the limit $\delta \to +0$ gives

$$
\begin{aligned}
&\int_\Omega \sigma(u(\tau,x)) v(\tau,x)\; dx - \int_\Omega \sigma(h(x)) v(0,x)\; dx \;= 2\int_0^\tau < \frac{\partial \sigma(u)}{\partial t}, v > \; dt + \\
&+ 2\int_0^\tau < \frac{\partial f}{\partial t}, v_1 > \; dt \; - \int_\Omega f(\tau,x) v_1(\tau,x)\; dx \; + \int_\Omega f(0,x) v_1(0,x)\; dx.
\end{aligned}
\tag{3.10}
$$

We shall estimate the summands in (3.10). In the case of condition α_1) we have by (2.3), (3.2) and condition K_2)

$$
\begin{aligned}
&\int_\Omega \sigma(u(\tau,x)) v(\tau,x) dx = \int_\Omega \Big\{ f(\tau,x) v_1(\tau,x) - \int_\Omega K(x,y) \sigma(u(\tau,x)) \times \\
&\times \sigma(u(\tau,y))\; dy \Big\}\; dx \le \int_\Omega f(\tau,x) v_1(\tau,x)\; dx \le c_3\, \|\sigma(u(\tau,x)\|_{L^1(\Omega)}.
\end{aligned}
\tag{3.11}
$$

An analogous estimate is true under condition α_2), because of the boundedness of the function σ in that case. Further, using Lemma 1 we get

$$
\Big| \int_0^\tau < \frac{\partial f}{\partial t}, v_1 > \; dt \Big| \le c_4 \Big\{ \int_0^\tau \int_\Omega \Big[\Big| \frac{\partial v_i}{\partial x} \Big|^2 + v_1^2 \Big]\; dx\; dt \Big\}^{\frac{1}{2}} \le c_5\, \|\sigma(u)\|_{L^2(Q_\tau)}. \tag{3.12}
$$

Estimating the remaining summands in (3.10) and using (3.11), (3.12), we obtain from (3.10) the desired estimate (3.5) and the proof of Lemma 2 is complete. □

Proof of Theorem 1. Condition (2.6) and Remark 2 allow us to use the test function $\varphi = u - v$ in the integral identity (2.8). Then, evaluating the resulting terms by conditions $ii), iii)$ and Lemma 2, we obtain

$$\begin{aligned}
&\int_0^\tau <\frac{\partial\sigma(u)}{\partial t}, u> \ dt \ + \int_{Q_\tau}\int \rho(u)\Big|\frac{\partial(u-v)}{\partial x}\Big|^2 \ dx \ + \\
&+\int_{Q_\tau}\int \varepsilon(u)|u(t,x)|^m \ dx \ dt \le c_6\Big\{1+\|\sigma(u(\tau,x))\|_{L^1(\Omega)}+ \\
&+\|\sigma(u)\|_{L^m(Q_\tau)}+\int_{Q_\tau}\int \big[|v(t,x)|^m+\alpha_1(t,x)+\alpha^{m'}(t,x)\big] \ dx \ dt\Big\}.
\end{aligned} \tag{3.13}$$

We transform the first integral in (3.13) in following way

$$\int_0^\tau <\frac{\partial\sigma(u)}{\partial t}, u> \ dt \ = \int_\Omega \Lambda(u,(\tau,x)) \ dx - \int_\Omega \Lambda(h(x)) \ dx \tag{3.14}$$

with $\Lambda(u)$ defined by (2.9). The proof of equality (3.14) is analogous to the proof of Lemma 1 in [9].

Remarking that condition $\rho_2)$ implies

$$c_7^{-1}\big[\sigma(u)\big]^m - c_8 \le \Lambda(u) \le c_7\big[\sigma(u)\big]^m + c_8 \ , \tag{3.15}$$

using (3.14) and Lemma 1, we obtain from (3.13)

$$\begin{aligned}
&\int_\Omega \sigma^m(u(\tau,x)) \ dx + \int_{Q_\tau}\int \rho(u)\Big|\frac{\partial(u-v)}{\partial x}\Big|^2 \ dx \ dt \ + \\
&+\varepsilon\int_{Q_\tau}\int (u(t,x))^m \ dx \ dt \le c_9\Big\{1+\int_{Q_\tau}\int \sigma^m(u(t,x)) \ dx \ dt\Big\}.
\end{aligned} \tag{3.16}$$

Now the estimate (2.11) follows from (3.15), (3.16) and Gronwall's Lemma and the proof of Theorem 1 is complete. □

Proof of Theorem 2. The assertion of Theorem 2 follows simply under $\alpha_2)$. Indeed, in this case the functions ρ, σ are bounded such that (2.4) and Lemma 1 imply $\frac{\partial v}{\partial x} \in C([0,T], L^{p_3}(\Omega))$. Hence (2.13) follows immediately from (2.11).
Let us now assume that condition $\alpha_1)$ is satisfied. In this case we define

$$u_+(t,x) = \max\{u(t,x), 0\}, \quad Q_T^\pm = \{(t,x) \in Q_T : \pm\, u(t,x) > 0\}. \tag{3.17}$$

Theorem 1 implies

$$\operatorname*{ess\ sup}_{t\in(0,T)}\int_\Omega \big[1+u_+(t,x)\big]^{\gamma+2} \ dx \ dt + \int_{Q_T^+}\int \big[1+u(t,x)\big]^\gamma\Big|\frac{\partial(u-v)}{\partial x}\Big|^2 \ dx \ dt \le c_{10} \ . \tag{3.18}$$

Thus for proving the desired inequality (2.13), it suffices to show that

$$\int_{Q_T}\int \big[1+u_+(t,x)\big]^\gamma\Big|\frac{\partial v}{\partial x}\Big|^2 \ dx \ dt \le c_{11} \ . \tag{3.19}$$

Define for $q \ge \frac{2}{\gamma+1}, \ \lambda \ge \gamma+2$,

$$I(q) = \int_{Q_T}\int \big[1+u_+(t,x)\big]^\gamma\Big|\frac{\partial v}{\partial x}\Big|^q \ dx \ dt, \quad J(\lambda) = \int_{Q_T}\int u_+^\lambda(t,x) \ dx \ dt \ . \tag{3.20}$$

We shall need the following assertions:

1) the estimate $I(\frac{2}{\gamma+1}) \leq c_{12}$ holds;

2) if for $\overline{\lambda} \geq \gamma+2$, $J(\overline{\lambda}) \leq c_{13}$, then $I(\overline{q}) \leq c_{14}$ with $\overline{q} = \min\{\frac{\overline{\lambda}-\gamma}{\gamma+1}, \frac{p_2-\gamma}{\gamma+1}\}$;

3) if for $\widetilde{q} \in [\frac{2}{\gamma+1}, 2]$, $I(\widetilde{q}) \leq c_{15}$, then $J(\widetilde{\lambda}) \leq c_{16}$, with $\widetilde{\lambda} = \widetilde{q} + \gamma + \frac{q}{n}(2+\gamma)$.

To prove assertion 1) we apply (1.8), Theorem 1, Hölder's and Young's inequalities:

$$I\Big(\frac{2}{\gamma+1}\Big) \leq \varkappa^{\frac{2}{\gamma+1}-1} \int_{Q_T}\!\!\int \big[1+u_+(t,x)\big]^\gamma \int_\Omega \Big|\frac{\partial K(x,y)}{\partial y}\Big| \times$$
$$\times\ |\sigma(u(t,y)) - f(t,y)|^{\frac{2}{\gamma+1}}\, dy\, dx\, dt \leq \tag{3.21}$$
$$\leq c_{17}\, \varkappa^{\frac{2}{\gamma+1}} \int_{Q_T}\!\!\int \big\{1+u_+(t,x)+|f(t,x)|\big\}^{2+\gamma}\, dx\, dt\ ,$$

where we used also (2.2) and the simple inequality

$$|\sigma(u)| \leq c_{18}\,(1+u_+)^{\gamma+1}, \quad u_+ = \max(u,0). \tag{3.22}$$

Now (3.21), (3.18) and (2.4) imply assertion 1).

Assertion 2) follows from the next inequality that is obtained analogously to (3.21).

$$I(\overline{q}) \leq \varkappa^{\overline{q}-1} \int_{Q_T}\!\!\int \big[1+u_+(t,x)\big]^\gamma \int_\Omega \Big|\frac{\partial K(x,y)}{\partial y}\Big| |\sigma(u(t,y)) - f(t,y)|^{\overline{q}}\, dy\, dx\, dt \leq$$
$$\leq c_{19}\, \varkappa^{q} \int_{Q_T}\!\!\int \big\{1+u_+(t,x)+|f(t,x)|\big\}^{\gamma+\overline{q}(\gamma+1)}\, dx\, dt \leq c_{20}. \tag{3.23}$$

Assertion 3) follows by Hölder's inequality and Sobolev's embedding theorem. Indeed, we get from (3.18)

$$J\Big(\widetilde{q}+\gamma+\frac{\widetilde{q}}{h}(2+\gamma)\Big) \leq \int_0^T \Big\{\int_\Omega \big[1+u_+(t,x)\big]^{2+\gamma}\, dx\Big\}^{\frac{\widetilde{q}}{n}} \times$$
$$\times\ \Big\{\int_\Omega \big[1+u_+(t,x)\big]^{(\widetilde{q}+\gamma)\frac{n}{n-\widetilde{q}}}\, dx\Big\}^{\frac{n-\widetilde{q}}{n}}\, dt \leq$$
$$\leq c_{21} \int_{Q_T^+}\!\!\int \big[1+u(t,x)\big]^\gamma \Big|\frac{\partial u}{\partial x}\Big|^{\widetilde{q}}\, dx\, dt + c_{21}\Big\{\int_{Q_T}\!\!\int \big[1+u_+(t,x)\big]^{2+\gamma}\, dx\, dt\Big\}^{\frac{\gamma+\widetilde{q}}{\gamma+2}}. \tag{3.24}$$

Since $I(\widetilde{q}) \leq c_{15}$ and (3.18) imply

$$\int_{Q_T^+}\!\!\int \big[1+u(t,x)\big]^\gamma \Big|\frac{\partial u}{\partial x}\Big|^{\widetilde{q}}\, dx\, dt \leq c_{22}\ , \tag{3.25}$$

we obtain assertion 3) from (3.24), (3.25) and (3.18).
Let us define sequences $\{q_i\},\ \{\lambda_i\},\ i = 1, \dots, N$, such that

$$q_1 = \frac{2}{\gamma + 1},\ \lambda_i = q_i + \gamma + \frac{q_i}{n}(2 + \gamma),\ q_{i+1} = \frac{\lambda_i - \gamma}{\gamma + 1},\ q_{N-1} < 2,\ q_N \geq 2. \tag{3.26}$$

This definition is justified by (2.2) and

$$q_{i+1} - q_i = \frac{q_i}{n(\gamma + 1)}\big[2 - \gamma(n-1)\big] \geq \frac{2}{n(\gamma+1)^2}\big[2 - \gamma(n-1)\big] > 0.$$

Now, using the assertions 1) – 3), we get by iteration that $I(q_N) \leq c_{23}$ and hence (3.19). This ends the proof of Theorem 2. $\square$

Proof of Theorem 3. Analogously as in the proof of Theorem 2, we can restrict us to the case of condition α_1). We test the integral identity (2.8) with

$$\varphi(t,x) = \big[\sigma(u_k(t,x)) - \sigma_0\big]_+ \big\{1 + [\sigma(u_k(t,x)) - \sigma_0]^2\big\}^r\ , \tag{3.27}$$

where $u_k(t,x) = \min\{u(x,t), k\},\ k > m_0,\ m_0$ is given by 2.10), $\sigma_0 = \sigma(m_0)$ and $r \in \big(-\frac{1}{2}, \infty\big)$ is an arbitrary number.

Analogously to Lemma 1 in [9] we have

$$\int_0^\tau < \frac{\partial \sigma(u)}{\partial t}, \varphi > dt = \int_\Omega \Lambda^{(r)}(u(\tau,x))\, dx\ , \tag{3.28}$$

where

$$\Lambda^{(r)}(u) = \int_0^u \rho(s)\big[\sigma(s_k) - \sigma_0\big]_+ \big\{1 + [\sigma(s_k) - \sigma_0]^2\big\}^r\, ds,\ s_k = \min\{s,k\}, \tag{3.29}$$

and

$$\Lambda^{(r)}(u) \geq \frac{1}{2(r+1)}\left\{1 + [\sigma(u_k) - \sigma_0]^2\right\}^{r+1} - 1 \quad \text{for} \quad u > m_0\ . \tag{3.30}$$

We write the derivative of φ in the form

$$\frac{\partial \varphi}{\partial x_i} = \rho(u)\Phi^{(r)}(u_k)\ \frac{\partial u}{\partial x_i} \cdot \chi(m_0 < u < k)\ , \tag{3.31}$$

where $\chi(m_0 < u < k)$ is the characteristic function of the set $\{(t,x) \in Q_T : m_0 < u(t,x) < k\}$ and the function $\Phi^{(r)}(u)$ satisfies

$$c_{23}\ \underline{r}\big\{1 + [\sigma(u) - \sigma_0]^2\big\}^r \leq \Phi^{(r)}(u) \leq c_{24}\,(r+1)\big\{1 + [\sigma(u) - \sigma_0]^2\big\}^r \tag{3.32}$$

with $\underline{r} = \min(1+2r, 1)$. Using (3.28) – (3.32) and the conditions *ii), iii)*, we obtain

$$
\begin{aligned}
&\int_\Omega \{1 + [\sigma(u_k(\tau, x)) - \sigma_0]^2\}^{r+1} \chi(m_0 < u)\, dx + \\
&+ \int_0^\tau \int_\Omega \rho^2(u)\{1 + [\sigma(u_k) - \sigma_0]^2\}^r \cdot \Big|\frac{\partial u}{\partial x}\Big|^2 \chi(m_0 < u < k)\, dx\, dt \le \\
&\le c_{24} \Big\{ \Big[\frac{r+1}{\underline{r}}\Big]^2 \int_0^\tau \int_\Omega \rho^2(u)\{1 + [\sigma(u_k) - \sigma_0]^2\}^r \Big|\frac{\partial v}{\partial x}\Big|^2 \times \\
&\times \chi(m_0 < u < k)\, dx\, dt + \frac{r+1}{\underline{r}}\Big[1 + \int_{Q_\tau}\int \big[|u|^{m-1} + |v|^{m-1} + \alpha(t,x)\big] \times \\
&\times \big\{1 + [\sigma(u_k) - \sigma_0]_+^2\big\}^{r+\frac{1}{2}} \chi(m_0 < u)\, dx\, dt\Big]\Big\}.
\end{aligned}
\tag{3.33}
$$

We introduce the notations $\{u > 1\} = \{(t,x) \in Q_T : u(t,x) > 1\}$ and

$$
\begin{aligned}
&I^*(q) = \operatorname*{ess\,sup}_{t \in (0,T)} \int_\Omega \sigma^q(u_+(t,x))\, dx \; + \iint\limits_{\{u>1\}} \rho^2(u)\sigma^{q-2}(u)\Big|\frac{\partial u}{\partial x}\Big|^2 dx\, dt \\
&J^*(\lambda) = \int_{Q_T}\int [1 + u_+(t,x)]^\lambda\, dx\, dt\,, \quad q \ge \frac{2+\gamma}{1+\gamma},\ \lambda \ge \gamma + 2\,.
\end{aligned}
\tag{3.34}
$$

We shall need the following assertions:

1) $I^*(\frac{2+\gamma}{1+\gamma}) \le c_{25}$;

2) if $J^*(\overline{\lambda}) \le c_{26}$, $\overline{\lambda} \ge \gamma + 2$, then $I^*(\overline{q}) \le c_{27}$, $\overline{q} = \min\{\frac{\overline{\lambda}-2\gamma}{\gamma+1}, p_2 - \frac{2\gamma}{\gamma+1}, \frac{\overline{\lambda}}{\rho_1'(\gamma+1)} + 1\}$;

3) if $I^*(\widetilde{q}) \le c_{28}$ for $\widetilde{q} \ge \frac{2+\gamma}{1+\gamma}$, then $J^*(\widetilde{\lambda}) \le c_{29}$ for $\widetilde{\lambda} = \frac{1}{n}\widetilde{q}(n+2)(1+\gamma)$.

Remarking that $\rho^2(u)\sigma^{q-2}(u) \le c_{30}\, \rho(u)$ for $u > 1$, $q = \frac{2+\gamma}{1+\gamma}$, we obtain assertion 1) immediately from the Theorems 1, 2.

To prove assertion 2) we start estimating the first integral on the right hand side of (3.33) with $r \le \min\{\frac{\overline{\lambda}-2\gamma}{2(\gamma+1)} - 1,\ \frac{p_2}{2} - \frac{\gamma}{\gamma+1} - 1\}$. Analogously to the inequality (3.21) we have

$$
\begin{aligned}
&\int_0^\tau \int_\Omega \rho^2(u)\{1 + [\sigma(u_k) - \sigma_0]^2\}^r \Big|\frac{\partial v}{\partial x}\Big|^2 \chi(m_0 < u < k)\, dx\, dt \le \\
&\le c_{31} \int_{Q_\tau^+}\int [1 + u_k(t,x)]^{2\gamma + 2r(1+\gamma)} \Big|\frac{\partial v}{\partial x}\Big|^2 dx\, dt \le \\
&\le c_{31}\, \varkappa \int_{Q_\tau^+}\int [1 + u_k(t,x)]^{2\gamma+2r(1+\gamma)} \int_\Omega \Big|\frac{\partial K(x,y)}{\partial y}\Big| \times \\
&\qquad \times |\sigma(u(t,y)) - f(t,y)|^2\, dy\, dx\, dt \le \\
&\le c_{32} \int_{Q_T}\int \big\{1 + |\sigma(u(t,x)| + |f(t,x)|\big\}^{\frac{2\gamma}{\gamma+1} + 2r + 2} dx\, dt \le \\
&\le c_{33} \int_{Q_T}\int \big\{1 + [u_+(t,x)]^{\gamma+1} + |f(t,x)|\big\}^{\frac{2\gamma}{\gamma+1} + 2r + 2} dx\, dt \le c_{34}.
\end{aligned}
\tag{3.35}
$$

Let us now estimate the last integral in (3.33). Using Theorem 1, Lemma 1, Hölder's inequality and supposing r such that

$$2(r+\frac{m}{2}) \le p_2, \quad 2(\gamma+1)\Big(r+\frac{m}{2}\Big) \le \overline{\lambda}, \quad (2r+1)(\gamma+1)p_1' \le \overline{\lambda}\,,$$

we obtain

$$\begin{aligned}&\int_{Q_\tau^+}\int \big[|u|^{m-1}+|v|^{m-1}+\alpha(t,x)\big]\big\{1+[\sigma(u_k)-\sigma_0]_+^2\big\}^{r+\frac{1}{2}}\,dx\,dt \le \\ &\le c_{34}\Big\{\int_{Q_\tau^+}\int [1+u_k]^{2(\gamma+1)(r+\frac{m}{2})}\,dt\,dx + \int_{Q_\tau}\int |\sigma(u)-f(t,x)|^{2(r+\frac{m}{2})}\,dx\,dt+ \\ &+\Big[\int_{Q_\tau^+}\int [1+u_k]^{(2r+1)(\gamma+1)p_1'}\,dx\,dt\Big]^{\frac{1}{p_1'}}\Big\} \le c_{35}\,.\end{aligned} \tag{3.36}$$

From (3.35), (3.36) we see that the left hand side in (3.33) is bounded by some constant depending only on known parameters and independent of k, r, provided $J^*(\overline{\lambda}) \le c_{26}$ and r is defined by

$$r = \frac{1}{2}\min\Big\{\frac{\overline{\lambda}-2\gamma}{\gamma+1}, p_2-\frac{2\gamma}{\gamma+1}, \frac{\overline{\lambda}}{p_1'(\gamma+1)}+1\Big\}-1. \tag{3.37}$$

So we are able to pass to the limit $k \to +\infty$ in (3.33) to obtain $I^*(\overline{q}) \le c_{27}$. That is assertion 2).

Assertion 3) follows from Hölder's inequality and Sobolev's embedding theorem analogously to inequality (3.24).

Now we define numbers $\{q_i\}, \{\lambda_i\},\ i=1,\dots,N$, such that

$$\begin{aligned}&q_1 = \frac{2+\gamma}{1+\gamma}, \quad \lambda_i = \frac{1}{n}\,q_i(n+2)(1+\gamma)\\ &q_{i+1} = \min\Big\{\frac{\lambda_i-2\gamma}{1+\gamma},\ p_2-\frac{2\gamma}{1+\gamma},\ \frac{\lambda_i}{p_1'(\gamma+1)}+1\Big\},\\ &q_{N-1} < p_2-\frac{2\gamma}{\gamma+1},\ q_N = p_2-\frac{2\gamma}{\gamma+1}.\end{aligned} \tag{3.38}$$

This definition is justified, since $\{q_i\}$ is increasing by

$$\begin{aligned}&\frac{\lambda_i-2\gamma}{\gamma+1}-q_i = \frac{2q_i}{n}-\frac{2\gamma}{\gamma+1} \ge \frac{2}{n(\gamma+1)}[\gamma+2-n\gamma] > 0\,,\\ &\frac{\lambda_i}{p_1'(\gamma+1)}+1-q_i = q_i\Big[\frac{p_1-1}{p_1}\cdot\frac{n+2}{n}-1\Big]+1 > 1\,.\end{aligned} \tag{3.39}$$

Note also that $\lambda_N > (n+2)(1+\gamma)$.
So the assertions 1) – 3) imply $I^*(q_i) \le c_{36},\ J^*(\lambda_i) \le c_{37}$ for $i=1,\dots,N$. By $I^*(q_N) \le c_{36},\ J^*(\lambda_N) \le c_{37}$ we have

$$\operatorname*{ess\,sup}_{t\in(0,T)}\int_\Omega \big[\sigma(u(t,x))\big]^{p_2-\frac{2\gamma}{\gamma+1}}dx \le c_{36}, \quad \int_{Q_T}\int |\sigma(u(t,x))|^{\frac{\lambda_N}{1+\gamma}}dxdt \le c_{37}\,. \tag{3.40}$$

Hence the conditions (2.4), φ) and Lemma 1 imply (2.14) with $p_3 = \min\{p_2-\frac{2\gamma}{\gamma+1}, \frac{\lambda_N}{\gamma+1}-2\}$ and the proof of Theorem 3 is complete. □

4 Boundedness of the function u

Firstly we want to estimate $u(t,x)$ from above under condition α_1).

Lemma 3 *Let the conditions of Theorem 4 and α_1) be satisfied. Then there exists a constant M_6 depending only on known parameters such that*

$$ess\sup \big\{u(t,x) : (t,x) \in Q_T\big\} \le M_6. \tag{4.1}$$

Proof. We apply (3.33) and estimate the integrals on the right hand side by Hölder's inequality. Using the properties of the function α, (2.14), and (3.40), we get

$$\begin{aligned}
&\int_{Q_\tau}\int \rho^2(u)\big\{1+[\sigma(u_k)-\sigma_0]^2\big\}\Big|\frac{\partial v}{\partial x}\Big|^2\chi(m_0<u<k)\;dx\;dt\;+\\
&+\int_{Q_\tau}\int \big[|u|^{m-1}+|v|^{m-1}+\alpha(t,x)\big]\big\{1+[\sigma(u_k)-\sigma_0]^2\big\}^{r+\frac12}\chi(m_0<u)\;dx\;dt\le\\
&\le c_{39}\Big\{\int_{Q_\tau}\int \big\{1+[\sigma(u_k)-\sigma_0]^2\big\}^{(r+1)\overline{p}}\chi(m_0<u)\;dx\;dt\Big\}^{\frac{1}{\overline{p}}}
\end{aligned}\tag{4.2}$$

with $\overline{p} < \frac{n+2}{n}$ depending only on known parameters. (3.33), (4.2) imply for $r \ge 1$

$$\begin{aligned}
&\int_\Omega \big\{1+[\sigma(u_k(\tau,x))-\sigma_0]^2\big\}^{r+1}\chi(m_0<u)\;dx\;+\\
&+\int_{Q_\tau}\int \rho^2(u)\big\{1+[\sigma(u)-\sigma_0]^2\big\}^{r}\Big|\frac{\partial u}{\partial x}\Big|^2\chi(m_0<u<k)\;dx\;dt\;\le\\
&\le c_{40}\,r^2\Big\{1+\int_{Q_\tau}\int \big\{1+[\sigma(u_k)-\sigma_0]^2\big\}^{(r+1)\overline{p}}\chi(m_0<u)\;dx\;dt\Big\}^{\frac{1}{\overline{p}}}.
\end{aligned}\tag{4.3}$$

(4.3), Hölder's inequality and Sobolev's embedding inequalities yield for $r \ge 1$

$$\begin{aligned}
&\int_{Q_T}\int \big\{1+[\sigma(u_k)-\sigma_0]^2\big\}^{(r+1)\frac{n+2}{n}}\cdot\chi(m_0<u)\;dx\;dt\le\\
&\le c_{41}\cdot\int_0^T\Big\{\int_\Omega \big\{1+[\sigma(u_k)-\sigma_0]^2\big\}^{r+1}\cdot\chi(m_0<u)\;dx\Big\}^{\frac2n}\times\\
&\qquad\times\Big\{\int_\Omega \big\{1+[\sigma(u_k)-\sigma_0]^2\big\}^{(r+1)\frac{n}{n-2}}\;dx\Big\}^{\frac{n}{n-2}}\;dt\le\\
&\le c_{42}\,r^2\mathrm{ess}\sup_{0<t<T}\Big\{\int_\Omega \big\{1+[\sigma(u_k(t,x))-\sigma_0]^2\big\}^{r+1}\chi(m_0<u)\;dx\Big\}^{\frac2n}\times\\
&\qquad\times\int_{Q_T}\int\Big[\rho^2(u)\big\{1+[\sigma(u)-\sigma_0]^2\big\}^{r}\Big|\frac{\partial u}{\partial x}\Big|^2\chi(m_0<u<k)\;+\\
&+\big\{1+[\sigma(u_k)-\sigma_0]_+^2\big\}^{r+1}\Big]\;dx\;dt\;\le\\
&\le c_{43}\,r^{4+\frac4n}\Big\{\int_{Q_T}\int \big\{1+[\sigma(u_k)-\sigma_0]_+^2\big\}^{(r+1)\overline{p}}\;dx\;dt\Big\}^{(1+\frac2n)\frac{1}{\overline{p}}}.
\end{aligned}\tag{4.4}$$

The inequalities (4.4), (3.40) justify the application of Moser's iteration process to verify (4.1) and the proof of Lemma 3 is complete. $\square$

For arbitrary $k \in \mathbb{R}$ and functions w on Q_T we define:

$$w^{(k)} = w^{(k)}(t,x) = \max\{w(t,x), k\}, \quad w_- = w_-(t,x) = \min\{w(t,x), 0\}. \tag{4.5}$$

Lemma 4 *Let the conditions of the Theorem 4 be satisfied. Then there exists a constant M_7 depending only on known parameters such that for an arbitrary $k \in \mathbb{R}$*

$$ess \sup_{t\in(0,T)} \int_\Omega |u^{(k)}(t,x)|\, dx \; + \int_{Q_T}\!\!\int \Big|\frac{\partial u^{(k)}(t,x)}{\partial x}\Big|^2 dx\, dt \;\le M_7. \tag{4.6}$$

The proof of this lemma is analogous to the proof of Lemma 5 in [10].

Lemma 5 *Let the conditions of Theorem 4 be satisfied. Then the estimate*

$$ess \inf \{u(t,x) : (t,x) \in Q_T\} \ge -M_8 \tag{4.7}$$

holds with a positive constant M_8 depending only on known parameters.

Proof. We test the integral identity (2.8) with

$$\varphi = \frac{1}{\rho(u^{(k)})}\big[\sigma(u^{(k)}) - \sigma(-m_0)\big]_- \cdot |u^{(k)} + m_0|^r, \quad k < -m_0,\ r > 0.$$

Then, analogously to the proof of the inequality (4.32) in [10], we obtain

$$\begin{aligned}
&\int_\Omega |[u^{(k)}(\tau,x) + m_0]_-|^{r+1}\, dx \,+ \\
&+ \int_{Q_\tau}\!\!\int |u+m_0|^r \Big|\frac{\partial u}{\partial x}\Big|^2 \chi(k<u<-m_0)\, dx\, dt \;\le \\
&\le c_{44}\,(r+1)^2 \int_{Q_\tau}\!\!\int \Big\{\Big[|u+m_0|^r\Big|\frac{\partial v}{\partial x}\Big|^2 + \Big|\frac{\partial u}{\partial x}\Big|^2 + \Big|\frac{\partial v}{\partial x}\Big|^2\Big]\chi(k<u<-m_0) \,+ \\
&+ [1+\alpha(t,x)]\cdot |[u^{(k)}+m_0]_-|^r\Big\}\, dx\, dt.
\end{aligned} \tag{4.8}$$

Using Lemma 4 we have from (4.8)

$$\begin{aligned}
&\int_\Omega |[u^{(k)}(\tau,x) + m_0]_-|^{r+1}\, dx + \int_{Q_\tau}\!\!\int |u+m_0|^r \Big|\frac{\partial u}{\partial x}\Big|^2 \chi(k<u<-m_0)\, dx\, dt \\
&\le c_{45}\,(r+1)^2 \Big\{ \int_{Q_\tau}\!\!\int |[u^{(k)}+m_0]_-|^r \widetilde{\alpha}(t,x)\, dx\, dt \;+ 1\Big\},
\end{aligned} \tag{4.9}$$

where $\widetilde{\alpha}(t,x) = \alpha(t,x) + |\frac{\partial v}{\partial x}|^2 + 1$. The condition on α and Theorem 3 imply $\widetilde{\alpha} \in L^{\widetilde{p}}(Q_T)$ with $\widetilde{p} > \frac{n+2}{2}$.

The inequality (4.9) allows to apply Moser's iteration process for proving

$$|[u^{(k)}(t,x)+m_0]_-| \le c_{46}.$$

This implies (4.7) with $M_8 = m_0 + c_{46}$ and Lemma 5 is proved. □

Proof of Theorem 4. The assertion of Theorem 4 follows immediately from the lemmas 3, 5 if condition α_1) is satisfied. In the case of condition α_2) Lemma 5 yields a lower bound of $u(t,x)$. The existence of an upper bound in that case can be analogously shown. The proof of Theorem 4 is complete. □

5 Proof of the existence Theorem

Firstly we shall assume that condition α_1) is satisfied. In this case we regularize the problem (1.1)-(1.4) by replacing ρ, a, σ by ρ^*, a^*, σ^* in the following way: Let M_4 be the constant from Theorem 4 and $(t,x) \in Q_T,\ v \in R^1$, then

$$\rho^*(u) = \rho(u),\ a^*(t,x,v,u) = a(t,x,v,u),\ \sigma^*(u) = \sigma(u),\ if\ u \le M_4; \tag{5.1}$$

$$\begin{aligned} &\rho^*(u) = \rho(M_4)e^{M_4-u},\ a^*(t,x,v,u) = a(t,x,v,M_4)e^{M_4-u}, \\ &\sigma^*(u) = \sigma(M_4) + \rho(M_4)[1-e^{M_4-u}],\ if\ u > M_4\ . \end{aligned} \tag{5.2}$$

We consider the regularized problem in Q_T, i. e.,

$$\frac{\partial \sigma^*(u)}{\partial t} - \sum_{c=i}^{n} \frac{\partial}{\partial x_i}\Big\{\rho^*(u)\ b_i\Big(t,x,\frac{\partial(u-v)}{\partial x}\Big)\Big\} + a^*(t,x,v,u), \tag{5.3}$$

$$v(t,x) = -\int_\Omega K(x,y)[\sigma^*(u(t,y)) - f(t,y)]\,dy, \tag{5.4}$$

$$\sum_{i=i}^{n} b_i\Big(t,x,\frac{\partial(u-v)}{\partial x}\Big)\cos(\nu,x_i) = 0 \quad (t,x) \in (0,T)\times\partial\Omega, \tag{5.5}$$

$$u(0,x) = h(x),\ x \in \Omega. \tag{5.6}$$

This problem satisfies all conditions of Section 2 with the same known parameters as problem (1.1) – (1.4). Therefore each solution (u,v) of problem (5.3) – (5.6) satisfies the priori estimate (2.15). So from (5.1) we see that a solution (u,v) of problem (5.3) – (5.6) is automatically solution of problem (1.1) – (1.4). Therefore it is sufficient to establish the existence of a solution of problem (5.3) – (5.6) in order to prove Theorem 5.

Let $X(k),\ k \in [0,1]$, be the Banach space of functions such that

$$\|u\|^2_{X(k)} = \|u\|^2_{L^2(0,T;W^{1,2}(\Omega))} + \sup_{0<\delta<\frac{T}{2}} \int_{Q_{T-\delta}} \int \frac{|u(t+\delta,x)-u(t,x)|^2}{\delta^k}\,dx\,dt < \infty\ .$$

To study the solvability of problem (5.3) – (5.6) we introduce the operator $A : X(\frac{1}{2}) \to X(\frac{1}{2})$ transforming a function $g \in X(\frac{1}{2})$ into the solution $U = Ag$ of the following problem in Q_T

$$\frac{\partial \sigma^*(U)}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big\{\rho^*(U)\, b_i\Big(t,x,\frac{\partial (U-G)}{\partial x}\Big)\Big\} + a^*(t,x,G,U) = 0, \tag{5.7}$$

$$G(t,x) = -\int_\Omega K(x,y)[\sigma^*(g(t,y)) - f(t,y)]\, dy, \tag{5.8}$$

$$\sum_{i=1}^{n} b_i\Big(t,x,\frac{\partial (U-G)}{\partial x}\Big)\cos(\nu,x_i) = 0, \quad (t,x) \in (0,T)\times\partial\Omega, \tag{5.9}$$

$$U(0,x) = h(x), \quad x \in \Omega. \tag{5.10}$$

Taking into account the boundedness of the function σ^*, the assumptions (2.4), ∂) and Lemma 1 we have

$$\operatorname*{ess\,sup}_{t\in(0,T)} \int_\Omega \Big|\frac{\partial G(t,x)}{\partial x}\Big|^{p_2} dx \;+ \operatorname*{ess\,sup}_{(t,x)\in Q_T} |G(t,x)| \le c_{47}\ , \tag{5.11}$$

with a constant c_{47} depending only on known parameters and independent of g. In order to guaranty the unique solvability of problem (5.7), (5.9)– (5.10) for given function G satisfying (5.11), the Theorems 3, 4 in [9] can be adapted. Indeed, the functions

$$b_i^*(t,x,\xi) = b_i\Big(t,x,\xi - \frac{\partial G(t,x)}{\partial x}\Big), \quad i = 1,\dots,n$$

satisfy the inequalities

$$\sum_{i=1}^{n}\big[b_i^*(t,x,\xi') - b_i^*(t,x,\xi'')\big](\xi_i' - \xi_i'') \ge \nu|\xi' - \xi''|^2, \tag{5.12}$$

$$|b_i^*(t,x,\xi)| \le \nu_2|\xi| + \beta(t,x) \tag{5.13}$$

with $\beta(t,x) = \nu_2(1 + |\frac{\partial G}{\partial x}|) \in L^\infty(0,T;L^{p_2}(\Omega))$, which essentially coincide with the conditions $ii)_2$ and ii^*) ensuring in [9] existence and uniqueness in the case of Dirichlet boundary conditions. But it is simple to check that the Theorems 3, 4 in [9] are also true for Neumann boundary conditions.

The estimate (5.11) and adaptations of the Theorems 1, 2 from [9] imply

$$\operatorname{ess\,sup}\big\{|U(t,x)| : (t,x)\in Q_T\big\} \le M_9, \quad \int_{Q_T}\!\!\int \Big|\frac{\partial G(t,x)}{\partial x}\Big|^2 dx\, dt \le M_9\ , \tag{5.14}$$

where $U(t,x)$ is the solution of problem (5.7) – (5.10) and M_9 is a constant depending only on known parameters and independent of g.

Using the estimates (5.14), (5.11) we can show analogously to [13] that

$$\sup_{0<\delta<\frac{T}{2}} \int_{Q_{T-\delta}}\!\!\int \frac{|U(t+\delta,x) - U(t,x)|^2}{\delta}\, dx\, dt \le M_{10}\ , \tag{5.15}$$

with a constant M_{10} depending only on known parameters and independent of g.

So the solution U of problem (5.7) – (5.10) belongs to the space $X(1)$ and therefore the operator $A : X(\frac{1}{2}) \to X(\frac{1}{2})$ is well defined. From the definition of this operator we see immediately that the solvability of problem (5.3) – (5.6) is equivalent to the existence of a fixed point

$$Ag = g, \quad g \in X\Big(\frac{1}{2}\Big). \tag{5.16}$$

We shall prove the existence of a solution of (5.16) by using the Leray–Schauder principle. The Leray–Schauder degree theory implies (cf. [13, 16]) that for the solvability of the equation (5.16) it is sufficient to establish following statements:

1) there exists a family $\{A_\theta\}$, $\theta \in [0,1]$, of operators $A_\theta : X(\frac{1}{2}) \to X(\frac{1}{2})$ such that $A_1 = A$, $A_0 = 0$, A_θ is completely continuous $\forall \theta \in [0,1]$ and $\{A_\theta\}$ satisfies following continuity condition: for arbitrary sequences $\{\theta_j\}$, $\{u_j\}$ such that $\theta_j \to \theta_0$, $u_j \to u_0$ we have $A_{\theta_j} u_j \to A_{\theta_0} u_0$, where $\to$ denotes strong convergence in $X(\frac{1}{2})$.

2) there exists a positive number R such that

$$A_\theta g \neq g \quad \text{for} \quad \theta \in [0,1], \quad \|g\|_{X(\frac{1}{2})} = R\,. \tag{5.17}$$

We define $A_\theta g = U_\theta$, where U_θ is the solution of the problem

$$\begin{aligned} &\frac{\partial \sigma^*(U_\theta)}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\Big\{\rho^*(U_\theta)\, b_i\Big(t, x, \frac{\partial (U_\theta - \theta G)}{\partial x}\Big)\Big\} + \\ &\quad + a^*(t, x, G, U_\theta) - (1-\theta) a^*(t, x, G, 0) = 0,\ (t,x) \in Q_T\,, \end{aligned} \tag{5.18}$$

$$\sum_{i=1}^{n} b_i\Big(t, x, \frac{\partial (U_\theta - \theta G)}{\partial x}\Big) \cos(\nu, x_i) = 0\,, \quad (t,x) \in (0,T) \times \partial\Omega\,, \tag{5.19}$$

$$U_\theta(0,x) = \theta h(x), \quad x \in \Omega\,. \tag{5.20}$$

The unique solvability of this problem can be seen as that of (5.7) – (5.10). Hence the operator $A_\theta : X(\frac{1}{2}) \to X(\frac{1}{2})$ is well defined.

We shall check firstly statement 2) formulated above. Let us assume that θ, g are such that $\theta \in [0,1]$, $g \in X(\frac{1}{2})$ and $A_\theta g = g$. Then from (5.18) – (5.20), (5.8) we see that the pair (U, G) is solution of a nonlocal nonlinear problem being analogous to problem (1.1) – (1.4). Consequently, by Theorem 2 there exists a constant M_{11} depending only on known parameters and independent of $\theta \in [0,1]$ such that $\|U_\theta\|_{L^2(0,T;W^{1,2}(\Omega))} \leq M_{11}$. From the corresponding inequality (5.15) we have $\|U_\theta\|_{X(\frac{1}{2})} \leq M_{12}$ with a constant M_{12} depending only on known parameters and independent of $g(t,x), \theta$. Since the equality $A_\theta g = g$ implies $\|g\|_{X(\frac{1}{2})} \leq M_{12}$, the desired relation (5.17) is fulfilled for $R = M_{12} + 1$.

Now we shall check statement 1) formulated above. The equalities $A_1 = A$ and $A_0 = 0$ hold because of the unique solvability of problem (5.7), (5.10). Thus it remains to prove compactness and continuity of the operator A. To this aim we prove the following lemma.

Lemma 6 *Assume that the conditions of Theorem 4 are satisfied. Then the operator* $A : X(\frac{1}{2}) \to X(\frac{1}{2})$ *defined by the map* $g \to U = Ag$*, where* U *is the solution of problem (5.7)– (5.10), is completely continuous.*

Proof. Firstly we remark that the operator A is bounded. This follows immediately from (5.14), (5.15).

Next we prove auxiliary inequalities. Let for this purpose $g_i \in X(\frac{1}{2})$, $i = 1, 2$, and set

$$U_i = Ag_i, \quad G_i(t,x) = -\int_\Omega K(x,y)\big[\sigma^*(g_i(t,y)) - f(t,y)\big]\, dy\,. \tag{5.21}$$

We put the test functions φ_i given by

$$\varphi_1 = \frac{1}{\rho^*(U_1)}\big[\sigma^*(U_1) - \sigma^*(U_2)\big], \quad \varphi_2 = U_1 - U_2,$$

into the integral identities corresponding to U_i. Taking the difference of the two resulting equalities, we get

$$\begin{aligned}
&\int_0^\tau \Big\{ < \frac{\partial \sigma^*(U_1)}{\partial t}, \frac{1}{\rho^*(U_1)}\big[\sigma^*(U_1) - \sigma^*(U_2)\big] > - < \frac{\partial \sigma^*(U_2)}{\partial t}, U_1 - U_2 > \Big\}\, dt + \\
&+ \sum_{i=1}^n \int_{Q_\tau}\!\!\int \Big\{ \rho^*(U_1)\, b_i\Big(t, x, \frac{\partial (U_1 - G_1)}{\partial x}\Big) \frac{\partial}{\partial x_i}\Big[\frac{1}{\rho^*(U_1)}\big(\sigma^*(U_1) - \sigma^*(U_2)\big)\Big] - \\
&- \rho^*(U_2)\, b_i\Big(t, x, \frac{\partial (U_2 - G_2)}{\partial x}\Big) \frac{\partial}{\partial x_i}[U_1 - U_2]\Big\}\, dx\, dt + \\
&+ \int_{Q_\tau}\!\!\int \Big\{ a^*(t,x,G_1,U_1) \frac{\sigma^*(U_1) - \sigma^*(U_2)}{\rho^*(U_1)} - a^*(t,x,G_2,U_2)(U_1 - U_2) \Big\}\, dx\, dt = 0\,.
\end{aligned} \tag{5.22}$$

We transform the first integral in (5.22) analogously to Lemma 2 in [9] to obtain

$$\begin{aligned}
&\int_0^\tau \Big\{ < \frac{\partial \sigma^*(U_1)}{\partial t}, \frac{1}{\rho^*(U_1)}\big[\sigma^*(U_1) - \sigma^*(U_2)\big] > - < \frac{\partial \sigma^*(U_2)}{\partial t}, U_1 - U_2 > \Big\}\, dt \;= \\
&= \int_\Omega \int_{U_2(\tau,x)}^{U_1(\tau,x)} \big[U_1(\tau,x) - s\big] \rho^*(s)\, ds\, dx \ge c_{48} \int_\Omega |U_1(\tau,x) - U_2(\tau,x)|^2\, dx.
\end{aligned} \tag{5.23}$$

To estimate the second integral in (5.22), we note that by condition $\rho_1)$

$$-\frac{(\rho^*)'(U_1)}{\rho^*(U_1)}\big[\sigma^*(U_1) - \sigma^*(U_2)\big] \ge -\int_{U_2}^{U_1} \frac{(\rho^*)'(s)}{\rho^*(s)} \rho^*(s)\, ds \;= \rho^*(U_2) - \rho^*(U_1)\;, \tag{5.24}$$

such that

$$
\begin{aligned}
&\sum_{i=1}^{n} \rho^*(U_1)\, b_i\Big(t,x,\frac{\partial(U_1-G_1)}{\partial x}\Big)\frac{\partial}{\partial x_i}\Big[\frac{1}{\rho^*(U_1)}\big(\sigma^*(U_1)-\sigma^*(U_2)\big)\Big] \geq \\
&\geq \sum_{i=1}^{n}\Big[b_i\Big(t,x,\frac{\partial(U_1-G_1)}{\partial x}\Big)-b_i\Big(t,x,\frac{\partial G_1}{\partial x}\Big)\Big]\frac{\partial U_1}{\partial x_i}\big[\rho^*(U_2)-\rho^*(U_1)\big]+ \\
&+\sum_{i=1}^{n} b_i\Big(t,x,\frac{\partial(U_1-G_1)}{\partial x}\Big)\Big[\rho^*(U_1)\frac{\partial U_1}{\partial x_i}-\rho^*(U_2)\frac{\partial U_2}{\partial x_i}\Big]- \\
&-\sum_{i=1}^{n} b_i\Big(t,x,-\frac{\partial G_1}{\partial x}\Big)\frac{\partial U_1}{\partial x_i}\cdot\frac{(\rho^*)'(U_1)}{\rho^*(U_1)}\big[\sigma^*(U_1)-\sigma^*(U_2)\big].
\end{aligned}
\tag{5.25}
$$

Since the properties of the function ρ ensure that

$$
\Big|\rho^*(U_1)-\rho^*(U_2)-\frac{(\rho^*)'(U_1)}{\rho^*(U_1)}\cdot\big[\sigma^*(U_1)-\sigma^*(U_2)\big]\Big| \leq c_{49}\,|U_1-U_2|\,, \tag{5.26}
$$

we get from (5.11), (5.25), (5.26) and condition *ii)*

$$
\begin{aligned}
&\sum_{i=1}^{n}\Big\{\rho^*(U_1)\, b_i\Big(t,x,\frac{\partial(U_1-G_1)}{\partial x}\Big)\frac{\partial}{\partial x_i}\Big[\frac{1}{\rho^*(U_1)}\big(\sigma^*(U_1)-\sigma^*(U_2)\big)\Big]- \\
&-\rho^*(U_2)\, b_i\Big(t,x,\frac{\partial(U_2-G_2)}{\partial x}\Big)\frac{\partial}{\partial x_i}(U_1-U_2)\Big\} \geq c_{50}\,\Big|\frac{\partial(U_1-U_2)}{\partial x}\Big|^2- \\
&-c_{51}\Big\{\Big|\frac{\partial(G_1-G_2)}{\partial x}\Big|^2+\Big(1+\Big|\frac{\partial G_1}{\partial x}\Big|\Big)\Big|\frac{\partial U_1}{\partial x}\Big|\cdot|U_1-U_2|\Big\}.
\end{aligned}
\tag{5.27}
$$

Using condition *iii)* and (5.14), we can estimate the last integral in (5.22)

$$
\begin{aligned}
&\Big|a^*(t,x,G_1,U_1)\frac{\sigma^*(U_1)-\sigma^*(U_2)}{\rho^*(U_1)}-a^*(t,x,G_2,U_2)(U_1-U_2)\Big| \leq \\
&\leq c_{52}\,|U_1-U_2|[1+\alpha(t,x)]\,.
\end{aligned}
\tag{5.28}
$$

Finally, from (5.22), (5.23), (5.27) and (5.28) we see that

$$
\begin{aligned}
&\int_\Omega |U_1(\tau,x)-U_2(\tau,x)|^2\,dx\ +\int_{Q_\tau}\!\!\int\Big|\frac{\partial(U_1-U_2)}{\partial x}\Big|^2\,dx\,dt \leq \\
&\leq c_{53}\int_{Q_\tau}\!\!\int\Big\{\Big|\frac{\partial(G_1-G_2)}{\partial x}\Big|^2+\Big[1+\Big|\frac{\partial G_1}{\partial x}\Big|\Big]\Big|\frac{\partial U_1}{\partial x}\Big|\cdot|U_1-U_2|+ \\
&+\big[1+\alpha(t,x)\big]|U_1-U_2|\Big\}\,dx\,dt.
\end{aligned}
\tag{5.29}
$$

Now we are ready to return to the study of properties of the operator A. We begin with the compactness. Let $\{g_j\}$ be a bounded sequence in $X(\frac{1}{2})$. Then by the compactness of the embedding $X(\frac{1}{2}) \subset L^2(Q_T)$ we can assume that $\{g_j\}$ converges strongly in $L^2(Q_T)$ to some function g_0. This and Lemma 1 imply the

strong convergence of $\frac{\partial G_j}{\partial x}$ to $\frac{\partial}{\partial x}G_0$ in $[L^2(Q_T)]^n$, where G_j is defined analogously to (5.21). Using (5.14), (5.15) with $U_j = Ag_j$, we can assume that U_j converges to some $U_0 \in X(\frac{1}{2})$ weakly in $L^2(0,T;W^{1,2}(\Omega))$ and strongly in $L^q(Q_T)$ for an arbitrary $q < \infty$.
In order to prove strong convergence of $\{U_j\}$ in $L^2(0,T;W^{1,2}(\Omega))$, we use (5.29) with $U_1 = U_j$, $U_2 = U_i$, $G_1 = G_j$, $G_2 = G_i$ and we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{\tau\in(0,T)} \int_\Omega |U_j(\tau,x) - U_i(\tau,x)|^2\, dx + \int_{Q_T}\!\!\int \Big|\frac{\partial(U_j-U_i)}{\partial x}\Big|^2\, dx\, dt \le \\
&\le c_{54}\int_{Q_T}\!\!\int \Big\{\Big|\frac{\partial(G_j-G_i)}{\partial x}\Big|^2 + \Big[1+\Big|\frac{\partial G_j}{\partial x}\Big|\Big]\cdot\Big|\frac{\partial U_j}{\partial x}\Big|\cdot|U_j-U_i| + \\
&+ \big[1+\alpha(t,x)\big]|U_i-U_j|\Big\}\, dx\, dt.
\end{aligned} \tag{5.30}$$

Using already known convergence properties of the sequences $\{U_j\}$, $\{G_j\}$ and (5.11), we see that the right hand side of (5.30) tends to zero as $j,i\to\infty$. That means compactness of the sequence $\{U_j\}$ in $L^2(0,T;W^{1,2}(\Omega))$. The compactness of this sequence in $X(\frac{1}{2})$ follows now from (5.30) and (5.15) with U_j, U_i. So we have established the compactness of the operator A.
Now we shall check its continuity. Let $\{g_j\}$ be a sequence converging strongly in $X(\frac{1}{2})$ to g_0. Lemma 1 implies that $\frac{\partial}{\partial x}G_j \to \frac{\partial}{\partial x}G_0$ in $[L^2(Q_T)]^n$. Using the compactness of A we can assume that $\{U_j = Ag_j\}$ converges strongly in $X(\frac{1}{2})$ to some $\overline{U}_0 \in X(\frac{1}{2})$. We have to show $\overline{U}_0 = Ag_0$. From the integral identity for U_j

$$\begin{aligned}
&\int_0^\tau < \frac{\partial\sigma^*(U_j)}{\partial t},\varphi > dt\ + \int_{Q_\tau}\!\!\int \Big\{\sum_{i=1}^n \rho^*(U_j)\, b_i\Big(t,x,\frac{\partial(U_j-G_j)}{\partial x}\Big)\frac{\partial\varphi}{\partial x_i} + \\
&+ a^*\big(t,x,G_j,U_j\big)\varphi\Big\}\, dx\, dt = 0, \quad \varphi\in L^2\big(0,T;W^{1,2}(\Omega)\big)
\end{aligned} \tag{5.31}$$

we obtain the boundedness of the sequence $\{\sigma^*(U_j)\}$ in $L^2(0,T;[W^{1,2}(\Omega)]^*)$. Therefore we can assume that $\sigma^*(U_j)$ converges weakly in $H^1(0,T;[W^{1,2}(\Omega)]^*)$ to some functional h_0. Using the strong convergence of $\{U_j\}$ to $\overline{U}_0$ in $L^2(Q_T)$, it is simple to see that $h_0 = \sigma^*(\overline{U}_0)$.
Now we are able to pass to the limit $j\to\infty$ in (5.31) to get

$$\begin{aligned}
&\int_0^\tau < \frac{\partial\sigma^*(\overline{U}_0)}{\partial t},\varphi > dt\ + \int_{Q_\tau}\!\!\int \Big\{\sum_{i=1}^n \rho^*(\overline{U}_0)\, b_i\Big(t,x,\frac{\partial(\overline{U}_0-G_0)}{\partial x}\Big)\frac{\partial\varphi}{\partial x_i} + \\
&+ a^*\big(t,x,G_0,\overline{U}_0\big)\varphi\Big\}\, dx\, dt = 0,\ \varphi\in L^2\big(0,T;W^{1,2}(\Omega)\big),\ \overline{U}_0(0,x) = h(x),\ x\in\Omega\,.
\end{aligned}$$

Adapting the uniqueness result Theorem 4 from [9], we obtain from (5) $\overline{U}_0 = Ag_0$ and this ends the proof of Lemma 6. $\square$

End of the proof of Theorem 5. We have had reduced the solvability of problem (1.1) – (1.4) to that of equation (5.16). The solvability of the last equation follows via Leray–Schauders's principle from the above formulated statements 1), 2), which are consequences of Lemma 6. Therefore the proof of Theorem 5 is complete provided condition α_1) is satisfied. In the case of condition α_2) the same arguments can be used. But it is not necessary to pass to the regularized problem (5.3) – (5.5) in that case. $\square$

6 Proof of the Uniqueness Theorem

Assume by contradiction the existence of two solutions (u_j, v_j), $j = 1, 2$, of problem (1.1) – (1.4) in the sense of Definition 1. We shall show that $u_1(t,x) = u_2(t,x)$, $v_1(t,x) = v_2(t,x)$. By Theorem 2 – 4 we have

$$\|u_j\|_{L^\infty(Q_T)} + \|v_j\|_{L^\infty(Q_T)} + \left\|\frac{\partial u_j}{\partial x}\right\|_{L^2(Q_T)} + \left\|\frac{\partial v_j}{\partial x}\right\|_{L^2(Q_T)} \le M_{13} \tag{6.1}$$

with a constant M_{13} depending only on known parameters. Let us now prove two auxiliary estimates.

First auxiliary estimate: We have for almost all $\tau \in (0,T)$

$$\begin{aligned} &\int_\Omega |u_1(\tau,x) - u_2(\tau,x)|^2 + \int_{Q_\tau}\int \left|\frac{\partial(u_1-u_2)}{\partial x}\right|^2 dx\, dt \le \\ &\le c_{55}\int_{Q_\tau}\int \Big\{\left|\frac{\partial(v_1-v_2)}{\partial x}\right|^2 + |v_1-v_2|^2 + \\ &+ \Big[\left|\frac{\partial v_1}{\partial x}\right|\left|\frac{\partial u_1}{\partial x}\right| + 1 + \alpha(t,x)\Big]|u_1-u_2|^2\Big\}\, dx\, dt\,. \end{aligned} \tag{6.2}$$

We shall obtain this estimate from the equality (5.22) with $\rho(u_i), \sigma(u_i), a(t,x,v_i,u_i)$, u_i, v_i instead of $\rho^*(U_i), \sigma^*(U_i), a^*(t,x,G_i,U_i), U_i, G_i$ respectively. Indeed, using (6.1) the local Lipschitz conditions for ρ' resp. for $a(t,x,\cdot,u)$, we get

$$\left|\rho(u_1) - \rho(u_2) - \frac{\rho'(u_1)}{\rho(u_1)}\big[\sigma(u_1) - \sigma(u_2)\big]\right| \le c_{56}\,|u_1-u_2|^2\,, \tag{6.3}$$

and

$$\begin{aligned} &\left|a(t,x,v_1,u_2)\frac{\sigma(u_1)-\sigma(u_2)}{\rho(u_1)} - a(t,x,v_2,u_2)(u_1-u_2)\right| \le \\ &\le |a(t,x,v_1,u_2)|\cdot\left|\frac{\sigma(u_1)-\sigma(u_2)}{\rho(u_1)} - (u_1-u_2)\right| + \\ &+ |a(t,x,v_1,u_2) - a(t,x,v_2,u_2)|\cdot|u_1-u_2| \le \\ &\le c_{57}\Big\{\big[1+\alpha(t,x)\big]|u_1-u_2|^2 + |v_1-v_2|^2\Big\}\,. \end{aligned} \tag{6.4}$$

Now (6.2) follows by using (6.3) and (6.4) in the same way as (5.29) by using (5.23) and (5.27).

Second auxiliary inequality: We have for almost all $\tau \in (0,T)$

$$\begin{aligned} &\int_\Omega |u_1(\tau,x)-u_2(\tau,x)|^2\,dx + \int_{Q_\tau}\!\!\int |u_1-u_2|^2\Big[\Big|\frac{\partial u_1}{\partial x}\Big|^2+\Big|\frac{\partial u_2}{\partial x}\Big|^2\Big]\,dx\,dt \le \\ &\le c_{58}\int_{Q_\tau}\!\!\int \Big\{\Big|\frac{\partial(u_1-u_2)}{\partial x}\Big|^2+\Big|\frac{\partial(v_1-v_2)}{\partial x}\Big|^2+ \\ &+\Big[\Big|\frac{\partial v_1}{\partial x}\Big|^2+\Big|\frac{\partial v_2}{\partial x}\Big|^2\Big]|u_1-u_2|^2+\big[1+\alpha(t,x)\big]|u_1-u_2|^2\Big\}\,dx\,dt\,. \end{aligned} \tag{6.5}$$

The proof of inequality (6.5) coincides with that of inequality (6.8) in [10]. That proof is based on testing the integral identity (2.8) for $u=u_j, v=v_j$ with the test functions φ_j given by

$$\varphi_1=\frac{1}{\rho(u_1)}\big[\exp(N\sigma(u_1))-\exp(N\sigma(u_2))\big]_+\,,\quad \varphi_2=N[u_1-u_2]_+\exp(N\sigma(u_2))]\,,$$

where $N=\max\left\{\frac{|1-2\rho'(s)|}{\rho^2(s)}:|s|\le M_{13}\right\}$ and M_{13} is the constant from (6.1). Remark that this proof is independent of the equation for the function v.

We shall use also the estimate

$$\int_\Omega\Big|\frac{\partial(v_1-v_2)}{\partial x}\Big|^2\,dx+\int_\Omega|v_1-v_2|^2\,dx\le c_{59}\int_\Omega|u_1-u_2|^2\,dx \tag{6.6}$$

following from Lemma 1 and (6.1).

Now we can turn to the proof of Theorem 6. Applying Cauchy's inequality to the term with $|\frac{\partial v_1}{\partial x}|$ in (6.2), we obtain from (6.2), (6.5), (6.6)

$$\begin{aligned} &\int_\Omega|u_1(\tau,x)-u_2(\tau,x)|^2+\int_{Q_\tau}\!\!\int\Big|\frac{\partial(u_1-u_2)}{\partial x}\Big|^2\,dx\,dt\le \\ &\le c_{60}\int_{Q_\tau}\!\!\int\Big[1+\alpha(t,x)+\Big|\frac{\partial v_1}{\partial x}\Big|^2+\Big|\frac{\partial v_2}{\partial x}\Big|^2\Big]|u_1-u_2|^2\,dx\,dt. \end{aligned} \tag{6.7}$$

From condition *iii)* and Theorem 3 we have

$$1+\alpha(t,x)+\Big|\frac{\partial v_1}{\partial x}\Big|^2+\Big|\frac{\partial v_2}{\partial x}\Big|^2\in L^{\widetilde p}(Q_T),\quad \widetilde p=\min\Big(p_1,\frac{p_3+2}{2}\Big)>\frac{n+2}{2}.$$

Estimating the integral on the right hand side of (6.7) by Hölder's inequality, we get

$$\begin{aligned} &\int_{Q_\tau}\!\!\int\Big[1+\alpha(t,x)+\Big|\frac{\partial v_1}{\partial x}\Big|^2+\Big|\frac{\partial v_2}{\partial x}\Big|^2\Big]|u_1-u_2|^2\,dx\,dt\le \\ &\le c_{61}\Big\{\int_{Q_\tau}\!\!\int|u_1-u_2|^{2\widetilde p^1}\,dx\,dt\Big\}^{\frac{1}{\widetilde p}}. \end{aligned} \tag{6.8}$$

Applying Hölder's and Young's inequalities and the embedding $V^2(Q)\to L^{\frac{2(n+2)}{n}}(Q)$ (cf. [13]), we can estimate the last integral in (6.8) as follows

$$\begin{aligned} &\Big\{\int_{Q_\tau}\!\!\int|u_1-u_2|^{2\widetilde p'}\,dx\,dt\Big\}^{\frac{1}{\widetilde p'}}\le\varepsilon\Big\{\sup_{0<\theta<\tau}\int_\Omega|u_1(\theta,x)-u_2(\theta,x)|^2\,dx+ \\ &+\int_{Q_\tau}\!\!\int\Big|\frac{\partial(u_1-u_2)}{\partial x}\Big|^2\,dx\,dt\Big\}+c_{62}\,\varepsilon^{\frac{-(n+2)(\widetilde p^1-1)}{n+2-\widetilde p^1}}\int_{Q_\tau}\!\!\int|u_1-u_2|^2\,dx\,dt \end{aligned} \tag{6.9}$$

with an arbitrary positive number ε. With a suitable ε (6.7) – (6.9) imply

$$\int_\Omega |u_1(\tau,x)-u_2(\tau,x)|^2\,dx \le c_{63}\int_{Q_\tau}\int |u_1(t,x)-u_2(t,x)|^2\,dx\,dt \tag{6.10}$$

for all $\tau \in (0,T)$. Gronwall's lemma and the last estimate yield $u_1 = u_2$. By (6.6) this implies $v_1 = v_2$ and the proof of Theorem 6 is complete. $\square$

Proof of Corollary 1. With the solution u of (1.1)– (1.4) we define

$$u_1(t,x) = u(t,x)\ , \quad u_2(t,x) = u(t+\delta,x)\ , \quad \delta \in (0, T-t)$$

and test the integral identity (2.8) with the functions $\varphi_i,\ i = 1,2$, given by

$$\varphi_1(t,x) = \frac{t^2}{\rho^*(u_1(t,x))}\big[\sigma^*(u_1(t,x)) - \sigma^*(u_2(t,x))\big], \quad \varphi_2(t,x) = t^2(u_1(t,x) - u_2(t,x))\ .$$

Then, arguing essentially as in the proof of (6.7) and (6.10), we obtain

$$\tau^2\int_\Omega |u_1(\tau,x)-u_2(\tau,x)|^2\,dx + \int_{Q_\tau} t^2\int \left|\frac{\partial(u_1-u_2)}{\partial x}\right|^2 dx\,dt \le$$
$$\le c_{64}\int_{Q_\tau}\int |u_1-u_2|^2\,dx\,dt.$$

Now dividing by δ^2, applying Gronwall's lemma and taking the limit $\delta \to 0$, the corollary follows. $\square$

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