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Dynamical approach to complex regional economic growth based on Keynesian model for China

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Abstract

The paper addresses the problem of complex regional economic growth by using nonlinear Keynesian model with focusing on direct foreign investments effects. We investigate the dynamics of the model for the broad range of parameters which, in particular, contains the parameter values obtained recently by econometric analysis of the data for economic growth in China. For the single-region model we give conditions for which the dynamics of the model will be chaotic or regular. The parameters which prevent the economic stagnation are indicated. Further, we consider the model for two regions with a common trade as a coupling factor. The conditions are given for the two trading systems to exhibit chaotic synchronization, in-phase and out-of-phase behavior.

1 Introduction

Direct foreign investments and reform politics created rapid economic growth in China from 1988. In recent years the direct foreign investments (DFI), especially the joint ventures and external demand through export did play an important role in the Chinese regional development, cf. [1]. This article focuses on direct foreign investment as the driving force in a growth process. The growth process is here modelled in a multiplier-accelerator model.

A paradox of investment and growth was shown in a Keynesian model [1] with the origin in a declining marginal propensity to consume at increasing income. Alternatively this effect could be created by declining propensity to invest at decreasing capital labor ratio as widely discussed in literature.

Often alternative models can not be distinguished from each other in real life due to limited amount of data. The true model therefore only can be located within a range of parameter values and model specifications. This aspect is crucial in dynamic model building.

Parameter values for the model are estimated by the analysis of statistical data, cf. Sec. 2. In Sec. 3 we develop the single region model. Analysis of the model in Sec. 4 shows that different kinds of behavior of income level may be realised including instant growth, periodic or chaotic oscillations. The "critical" parameters which may alter the behavior of the model is shown to relate to the investment characteristics which essentially depend on the propensity of investors to response to fluctuations of the income level.

In Sec. 5 we introduce a two region model which accounts for the interaction between regions through a common trade. We indicate parameter regions where model exhibits synchronous chaotic or periodic behaviors, i.e. describes the phenomenon of equalizing of the regional economical growth even when initial regional investments are unequal [1].

2 Data and notations

The data used in this analysis are the regional data published by the China Statistical Yearbook (1989-1997). The data for gross domestic product (GDP) and GDP per capita measured in yuan are adjusted for inflation and are expressed in 1990 price deflated by consumer price indices for China. The data for "direct foreign investment and other foreign investment" (DFI) measured in US dollars are deflated by the US consumer price index in order to make DFI an index for the (real) inflow of world market investment goods. Both deflators are published by the IMF: "International Financial Statistics Yearbook, 1997".

The following notations are used for the development of data and model parameters estimation:

GDP – Gross Domestic Product for the region, measured in yuan;

POP – Population of the region;

DFI – Direct Foreign Investment and other foreign investment in the region measured in dollars;

DFLUS – Deflator – consumer price index for USA;

DFL – Deflator – consumer price index for China;

FGDP=GDP/DFL – fixed price GDP of the region;

FDFI=DFI/DFLUS – fixed price DFI in the region;

FPCY=FGDP/POP=Y_t – fixed price per capita income of the region at time t;

FDPIPC=FDFI/POP=DFI_t – fixed price per capita in the region at time t.

3 Individual region model

The model for income development in regions can be formed as a Keynesian model which includes direct foreign investments and a regional trade [1]. In this section we start with the model for an individual region which does not include the regional trade. We assume the consumption C_t at time t to be the following function of income

$$C_t = \alpha_0 + \tilde{\alpha} Y_t = \alpha_0 + (\alpha_1 - \alpha_2 Y_{t-1}) Y_t, \tag{1}$$

where α_0, α_1, and α_2 are constant parameters. The econometric data analysis [1] brings us to the following estimations for the parameters: $\alpha_0 \approx 11$, $\alpha_1 \approx 0.99$, and $\alpha_2 \approx 8 \cdot 10^{-7}$ with corresponding relative errors less than 10%. Figure 1 illustrates the equilibrium value of consumption versus income. It is nearlylinear for small Y and has a saturation point at $Y^* = \alpha_1 / 2\alpha_2$ which approximately admits the value $6 \cdot 10^6$ for the assumed parameters. Small nonzero value of consuming at $Y = 0$ can be maintained from gifts and stocks.

We assume that in the absence of the regional trade the income is formed mostly by

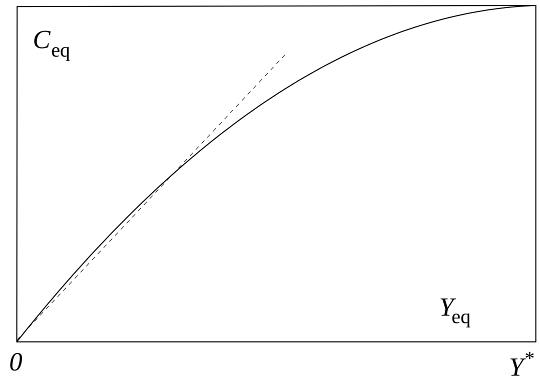

Figure 1: Dependence of consumption C_{eq} on income equilibrium value Y_{eq}. Y^* denotes an income level at the saturation point.

consumption and direct foreign investments, cf. [1]:

$$Y_t = C_t + DFI_t. \tag{2}$$

In order to take into account the adaption of DFI and its influence on the development of the region, we use the standard Keynesian adjustment algorithm

$$DFI_t = \gamma_1'(Y_{t-1} - Y_{t-2}) + \gamma_2'(Y_{t-2} - Y_{t-3}) + \gamma_0 DFI_{t-1}, \tag{3}$$

where the constants $\gamma_{1,2}'$ give the propensity of direct foreign investments to react on income changes for two preceding periods (*reaction coefficients* in the following). For simplicity, we confine our analysis to the case of $\gamma_2' = 0$. Using empirical data, the coefficients γ_0 and γ_1' are determined as 0.7 and 0.03, respectively, with relative errors 10%. In order to make our model more economically consistent we additionally equip it with the floorpreventing DFI to be negative:

$$DFI_t = \max\{\gamma_1'(Y_{t-1} - Y_{t-2}) + \gamma_0 DFI_{t-1}, 0\}. \tag{4}$$

Finally, the set of equations (1), (2), and (4) constitutes the closed dynamical model for the description of the fluctuations of consumption level, income, and direct foreign investments. An appropriate rescaling procedure $y = Y\left(\frac{\alpha_2}{\alpha_1}\right)$, $c = C\left(\frac{\alpha_2}{\alpha_1^2}\right)$, and $x = DFI\left(\frac{\alpha_2}{\alpha_1^2}\right)$ brings these equations to the form

$$c_t = \alpha + (1 - y_{t-1})y_t, \tag{5}$$

$$\alpha_1 y_t = c_t + x_t, \tag{6}$$

$$x_t = \max\{\gamma_0 x_{t-1} + \gamma_1(y_{t-1} - y_{t-2}), 0\}, \tag{7}$$

where $\alpha = \alpha_0\alpha_2/\alpha_1^2$ and $\gamma_1 = \gamma_1'/\alpha_1$ are new parameters. With respect to the new coordinates, the saturation point for consumption (cf. Fig. 1) corresponds to $y^* = 1/2$.

In order to derive the explicit form of the mapping that describes the dynamics of our system let us express x_t in terms of y_{t-1} and y_{t-2} by combining (5) and (6), then substitute the obtained expression for x_t into equation (7). Finally, we obtain two-dimensional piecewise-smooth map $F : R_+^2 \to R_+^2$:

$$F: \begin{pmatrix} y \\ z \end{pmatrix} \to \begin{pmatrix} \frac{1}{\beta+y}\left[\alpha + \max\{-\alpha\gamma_0 + \gamma_0(\beta+z)y + \gamma_1(y-z), 0\}\right] \\ y \end{pmatrix} \tag{8}$$

with four parameters α, β, γ_0, and γ_1. As we previously emphasized, we are interesting in the dynamics of the model (8) with the parameters that are given by econometric analysis in [1]. Numerical values for these parameters can be estimated as $\alpha = 0.9 \cdot 10^{-5}$, $\beta = (1-\alpha_1)/\alpha_1 = 0.01$, $\gamma_0 = 0.7$, and $\gamma_1 = 0.03$. The admissible interval for the parameters takes into account the relative error (about 10%) for their estimation. In fact, some results of the present paper are more general and involve a more broad range of the parameter values. A particular attention is paid to the role of the parameters γ_0 and γ_1, which we call in the following as DFI *adaption* and *reaction* coefficients, respectively.

Before proceeding to our main results, note that at a constant level of direct foreign investments, the dynamics of our model is described by only two equations (5) and (6). In this case we obtain one-dimensional map $f : R_+ \to R_+$ given by

$$f: \; y \to f(y) = \frac{\alpha + \bar{x}}{\beta + y}, \tag{9}$$

where we denoted a constant DFI level as $\bar{x}$. This map has a unique fixed point

$$\bar{y}_0 = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \alpha + \bar{x}}, \tag{10}$$

which is stable for all meaningful values of the parameters. In order to prove this let us write the derivative $f'(\bar{y}_0)$ in the form $|f'(\bar{y}_0)| = 1/(1 + \kappa(\alpha, \beta, \bar{x}))$, where $\kappa = (2(\beta/2)^2 + \beta\sqrt{(\beta/2)^2 + \alpha + \bar{x}})/(\alpha + \bar{x})$. It is easy to note that κ is positive for all positive α, β, and $\bar{x}$. Hence, we have $|f'(\bar{y}_0)| < 1$ that implies the stability of the fixed point. We conclude that, at a constant level of direct foreign investments and in the absence of a regional trade, our model asymptotically displays the expected equilibrium behavior.

4 Dynamics of the two-dimensional model

In this section we describe the dynamics of map (8) for the regional income development. Important features of this map are its noninvertibility and nondifferentiability on some curve in the phase space. Let us define the following regions in the phase space:

$$\Pi_0 = \{(y, z): \; \gamma_0(\beta + z)y + \gamma_1(y - z) < \alpha\gamma_0\},$$
$$\Pi_1 = \{(y, z): \; \gamma_0(\beta + z)y + \gamma_1(y - z) > \alpha\gamma_0\}.$$

Being confined to either of these regions, system (8) is differentiable while it losses differentiability on the boundary

$$\partial\Pi = \{(y, z): \; \gamma_0(\beta + z)y + \gamma_1(y - z) = \alpha\gamma_0\},$$

between the regions Π_0 and Π_1, cf. Fig. 2. Note that the map is essentially one-dimensional in the region Π_0 since the first equation in (8) does not involve the variable z in this case.

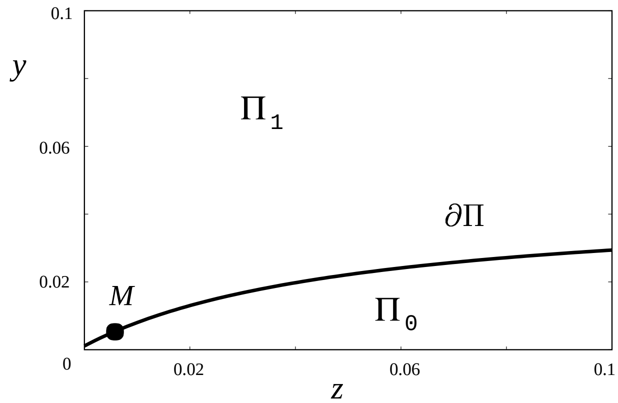

Figure 2: Splitting of the phase space of map (8) into regions Π_0 and Π_1. The fixed point M belongs to the boundary $\partial\Pi$.

Behavior near the equilibrium point

The map F has a unique equilibrium point

$$M = (y_0, z_0), \text{ where } y_0 = z_0 = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \alpha}.$$

This point corresponds to zero value of direct foreign investments and a relatively low value of income. It is of interest to study the stability of this state in order to provide conditions for the system either to converge to the unprofitable equilibrium or to exhibit some nontrivial behavior with nonzero DFI.

Note that M belongs to the curve $\partial\Pi$ for all values of parameters. Hence, any its neighborhood U_M can be splitted into two parts on which the map is smooth: $U_1 = U_M \cap \Pi_1$ and $U_0 = U_M \cap \Pi_0$. The fact that map (8) is not smooth at M does not allow us to make conclusions about the stability of M using the linear theory, i.e. just evaluating Jacobian. Instead, the nonlinear effects arising from the dynamical interplay between U_0 and U_1 should be taken into account.

Jacobi matrix restricted to the region U_0 has the following form:

$$J_0 = \begin{pmatrix} -\alpha/(\beta+y)^2 & 0 \\ 1 & 0 \end{pmatrix}, \tag{11}$$

The eigenvectors of this matrix evaluated at M have the form $v_1 = (\lambda_1^0, 1)^T$ and $v_2 = (0,1)^T$. $\lambda_1^0 = -\alpha/(\beta+y_0)^2$ and $\lambda_2^0 = 0$ are the corresponding eigenvalues. The eigenvectors v_1 and v_2 define local invariant manifolds near the equilibrium point M in U_0, cf. Fig. 3. It can be shown that for all positive α and β these manifolds are stable, i.e. $|\lambda_1^0| < 1$. The last inequality follows from the following representation of $|\lambda_1^0|$:

$$|\lambda_1^0| = \frac{\alpha}{(\beta+y_0)^2} = \frac{\alpha}{\alpha + \frac{\beta^2}{2} + \beta\sqrt{\left(\frac{\beta}{2}\right)^2 + \alpha}} = \frac{1}{1+\kappa_1(\alpha,\beta)},$$

where $\kappa_1(\alpha,\beta) = \left(\frac{\beta^2}{2} + \beta\sqrt{\left(\frac{\beta}{2}\right)^2 + \alpha}\right)/\alpha$ is positive for all positive α and β.

Note that all points from the neighborhood U_0 are mapped onto the local manifold of v_1-vector for one iteration because of the zero value of λ_2^0. Moreover, due to the negativeness of λ_1^0, points eventually escape the region Π_0, cf. Fig. 3.

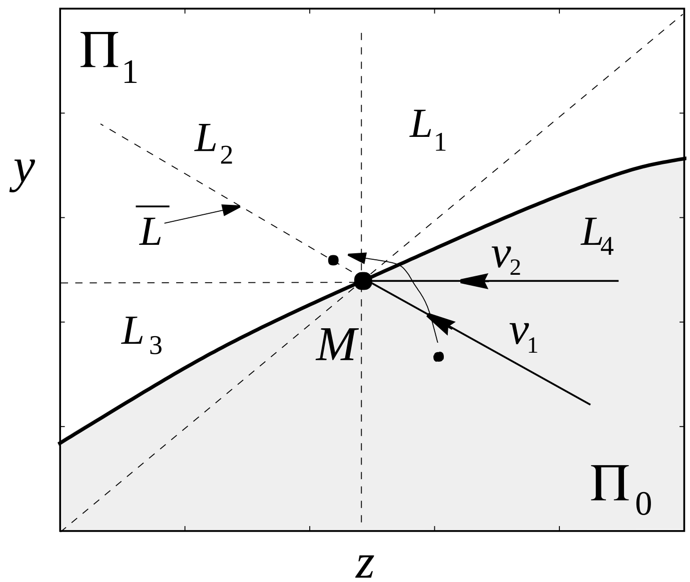

Figure 3: Dynamics near equilibrium point M. v_1 and v_2 denote vectors along the local invariant manifolds restricted to Π_0. L_1 – L_4 denote different regions that are used for the proof of Proposition 1.

Now consider the region $U_1 = U_M \cap \Pi_1$. The Jacobi matrix at the equilibrium point with respect to the region U_1 is

$$J_1 = \begin{pmatrix} -\alpha/(\beta+y_0)^2 + \gamma_1/(\beta+y_0) + \gamma_0 & \alpha\gamma_0/(\beta+y_0)^2 - \gamma_1/(\beta+y_0) \\ 1 & 0 \end{pmatrix} := \begin{pmatrix} s_1 & s_2 \\ 1 & 0 \end{pmatrix}. \tag{12}$$

Because of the involved complexity, we do not give the explicit expression for the eigenvalues of J_1 here. Instead, we show the corresponding diagram for eigenvalues of J_1 in Fig. 4. The dashed area in the figure corresponds to complex eigenvalues. The area where both eigenvalues have their modulus less than one is shown in gray (below the line H).

It is evident that we may expect the loss of stability for the equilibrium point M in the region, where Jacobian J_1 has complex eigenvalues, cf. Fig. 4. In Appendix A we prove Proposition 1 which gives analytical conditions for the stability of the equilibrium point.

In order to illustrate the obtained result, the numerical procedure was performed using Proposition 1. The two-dimensional grid (2000x2000) for parameters $\gamma_0 \in [0,1]$ and $\gamma_1 \in [0, 0.04]$ was introduced with fixed $\alpha = 8.98 \cdot 10^{-6}$ and $\beta = 0.0101$. The numerical procedure was applied with the use of Proposition 1 for each point of this grid to find the stability of the fixed point for each parameter value. The final two-dimensional stability diagram is shown in Fig. 5. We can observe that stability region for the point M (shown in black) is not only those parameters for which Jacobian J_1 has eigenvalues with modulus less than one (the area below white line), but also some set from another region where the linearized map restricted to region Π_1 is unstable. This phenomena appears due to the nonsmoothness of the neighborhood of M and the balanced effect from the two regions of smoothness: Π_0 and Π_1.

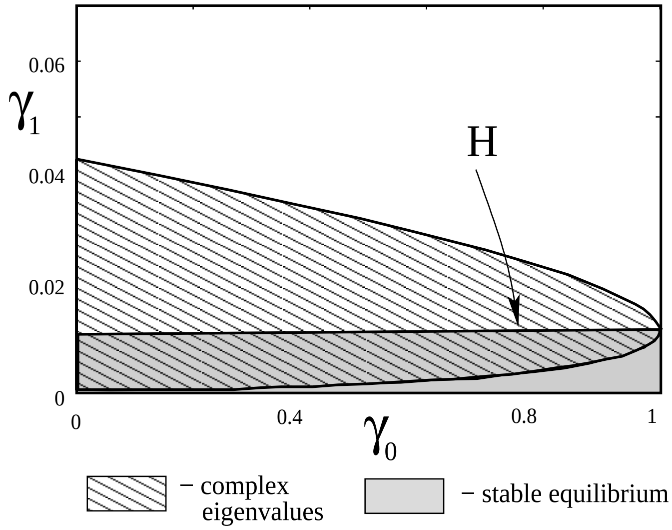

Figure 4: Bifurcation diagram for the equilibrium point with respect to region U_1.

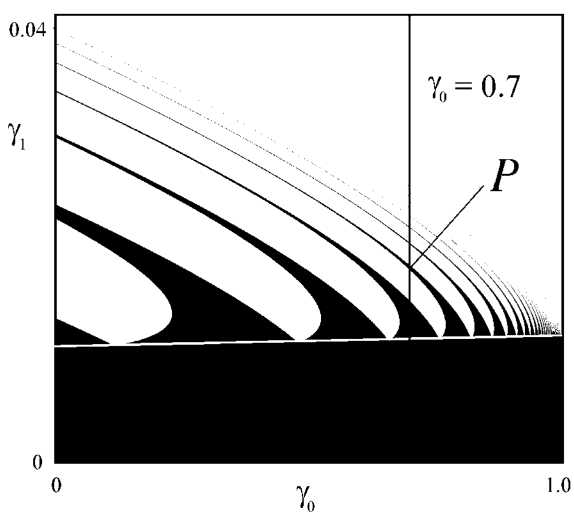

Figure 5: Stability region for the equilibrium point M (shown in black).

An interesting observation can be made from the above stability analysis: with increasing of the reaction coefficient γ_1 the dynamics of the system, in general, is ruled out from the equilibrium state, but we still may enter regions where this state becomes stable.

Reaction coefficient role in the nontrivial economic behavior

We shall observe from Fig. 5 that the threshold where the equilibrium point may loss its stability (white line) depends mainly on the value of the reaction coefficient γ_1. This fact brings us to the conclusion that independently on the adaption coefficient γ_0 the economic stagnation expressed here as asymptoting to the equilibrium value with relatively small level of income can be overcomed, in general, by increasing only the reaction coefficient.

The observed phenomena can be explained in the following way: for a low level of DFI (it may be assumed to be low for the initial time) the reaction component of DFI adjustment procedure becomes dominant, i.e. the expectations for DFI for the next period are mostly based on income changes: $DFI_t \approx \gamma_1(Y_{t-1} - Y_{t-2})$. We can therefore observe that in that approximation effect of the adaption coefficient γ_0 is not important. Instead, if reaction expectations are not involved in the adjustment process ($\gamma_1 = 0$) then the dynamics of the system displays convergence to $DFI = 0$ provided $\gamma_0 < 1$.

It is worth to note that the above mentioned threshold depends on inner properties of the system. In our case, these properties are determined by the parameters α and β.

Global behavior of the single region model

The analysis of the previous section was confined to the small neighborhood of the fixed point which, being stable, causes the stagnation process. Now we consider global properties of map (8). As we have mentioned before, we are particularly interesting in the influence of direct foreign investments on the dynamics of the region. Therefore we will pay attention mainly to the influence of γ_0 and γ_1 parameters. For this, we fix admissible values for consumption characteristics $\alpha = 9 \cdot 10^{-6}$ and $\beta = 0.01$ for the numerical simulations of the present section.

Fig. 6 shows the bifurcation diagram for $\gamma_0 = 0.7$ with γ_1 varying from 0.01 to 0.04. The path along which parameters are varied is shown also in Fig. 5. Figure 7 shows dependence of maximal Lyapunov exponent of the corresponding invariant sets versus γ_1. Intervals of chaotic behavior with positive Lyapunov exponent $\lambda_{\max} > 0$ can be clearly observed. Figure 8 illustrates the chaotic attractor that exists for $\gamma_1 = 0.035$.

As follows from the above analysis of the equilibrium, after the first stability loss ($\gamma_1 \approx 0.02$), the coexistence of the stable equilibrium and a periodic solution may take place on some countable number of parameter intervals. Figure 9 illustrates the basins of attraction of two coexistent sets for the parameters from the largest of these intervals, cf. point P in Fig. 5.

Let us now interpret obtained results from the economical point of view. For $\gamma_0 < 1$ and

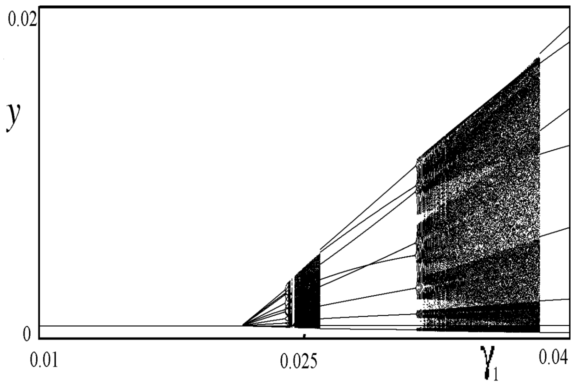

Figure 6: Bifurcation diagram for system (8). The parameters are $\gamma_0 = 0.7$, $\alpha = 9 \cdot 10^{-6}$, and $\beta = 0.01$.

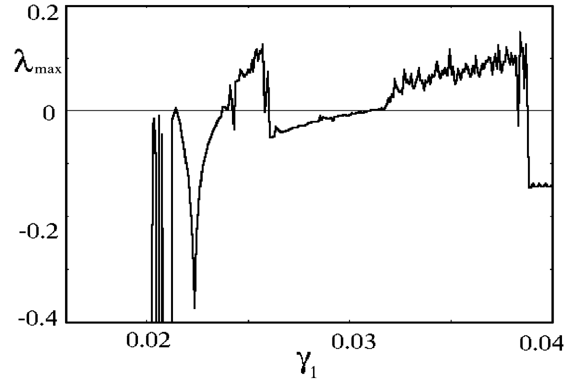

Figure 7: Maximal lyapunov exponent versus γ_1 for $\gamma_0 = 0.7$.

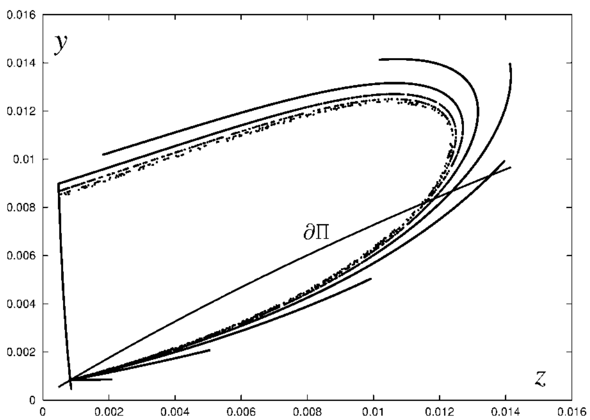

Figure 8: Chaotic attractor for system (8) which exists for $\gamma_1 = 0.035$.

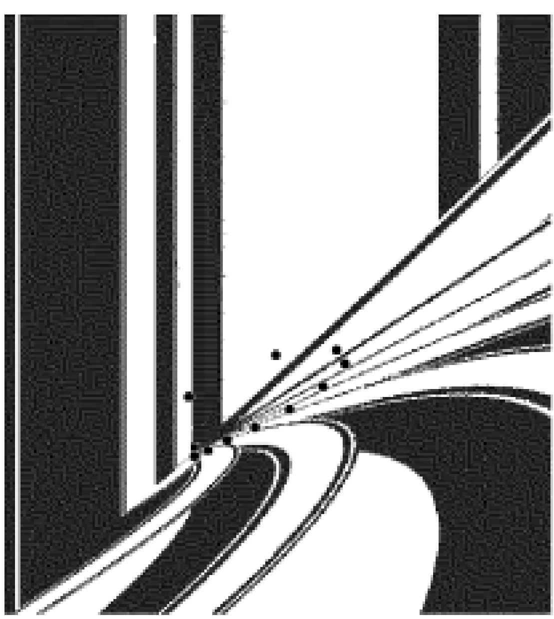

Figure 9: Basins of coexistent stable equilibrium point M (shown as square) and stable period-10 orbit (shown as circles) with multipliers $\mu_1 = 0.35$ and $\mu_2 = 0$. Black area corresponds to the basin of the equilibrium. Parameters are $\gamma_0 = 0.7$, $\gamma_1 = 0.022$.

low values of reaction coefficient γ_1, the equilibrium value of income is approached, where direct foreign investments vanish and income approaches some small positive value. By increasing the reaction coefficient, the dynamics of the model becomes more rich. First, the periodic changes of income with period 10 appear. At that time, it is still possible for the model to approach zero DFI level provided initial conditions (initial value of income Y_0 and initial level of DFI) correspond to the black area in Fig. 9, or asymptote the more "advanced" development situation that corresponds to periodic behavior. With further increasing of γ_1 the system can exhibit chaotic behavior, cf. Figs. 6,8. In general, it is more profitablefor the market, the dynamics of which is described by system (8), to be in the regime with higher reaction coefficient γ_1. This can be clearly seen from Fig. 10, where we plot the averaged value of income $< Y >$ versus γ_1. The averaging procedure involves calculation of the income over 1000 points after skipping 10^4 iterations.

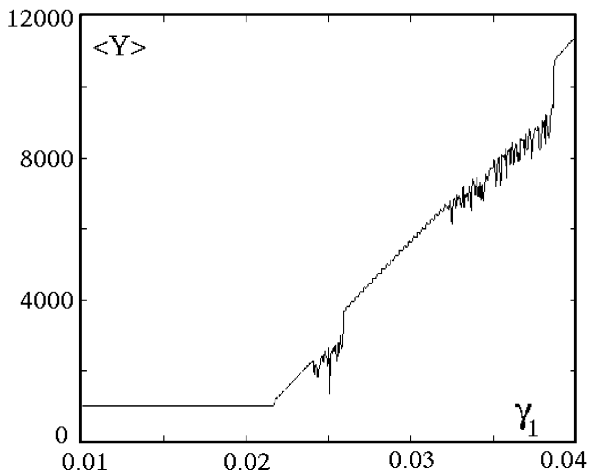

Figure 10: Averaged value of income $< Y >$ versus γ_1.

The case $\gamma_0 \approx 1$ as a model for finite-time economic growth

In this section we show that for $\gamma_0 \approx 1$ map (8) may be considered as a model for finite time economic growth. In particular, system (8) with $\gamma_0 = 1$ is shown to be structurally unstable and cannot be considered as a model describing long term behavior.

By substituting $\gamma_0 = 1$ into (8) it is possible to show that the system has the invariant set

$$\{(y,z): \; y = z \geq y_0\} \tag{13}$$

which consists of equilibrium points. This fact implies that our system is structurally unstable. Consider the stability properties of the equilibrium points (13). They all, except M, belong to region Π_1 and, therefore, their stability can be determined by inspecting the following eigenvalues of Jacobian J_1:

$$\lambda_1 = 1, \quad \lambda_2 = \frac{\gamma_1 - y}{\beta + y}. \tag{14}$$

The first eigenvalue corresponds to the direction along the invariant set $y = z$ and reflects the fact that there is no motion along this set. λ_2 corresponds to the transverse direction. Therefore the transverse stability of an equilibrium takes place provided

$$\left| \frac{\gamma_1 - y}{\beta + y} \right| < 1.$$

This inequality holds for all $y > y^* = (\gamma_1 - \beta)/2$. Thus, we conclude that for sufficiently large γ_1 (in fact, for $\gamma_1 > 2y_0 + \beta$) all equilibria satisfying $y > y^*$ are transversely stable and unstable otherwise.

For our analysis it is important to note that with increasing γ_1 the attracting set of all transversely stable equilibria movestoward the higher values of income level. The considered case provide us with the scenario when systems starting from different initial conditions (initial income level and direct foreign investments) will in general asymptote to different equilibrium levels. The larger initial income level or DFI system has the larger equilibrium level of income it is going to achieve. Figure 11 shows characteristic behavior of orbits for $\gamma_1 = 0.03$ and $\gamma_1 = 0.05$.

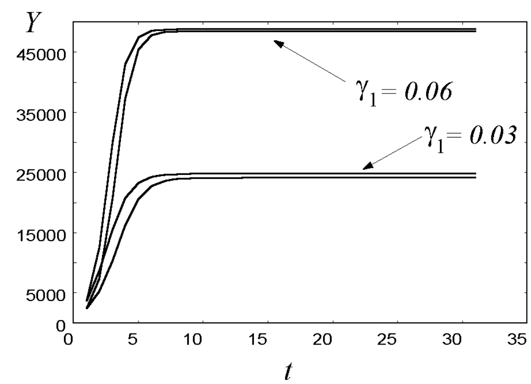

Figure 11: Orbits for $\gamma_0 = 1$ and γ_1 as indicated in the figure. The corresponding initial conditions are $y_0^1 = 0.002, z_0^1 = 0.001$ and $y_0^1 = 0.003, z_0^1 = 0.001$ for each value of parameter γ_1.

5 Two-region model

In practice, each economical region is interacting with other regions via common trade. In this section we consider the model for two regions with common trade. Let EXR_i, $i = 1, 2$ be region's i export to another region. The external trade is here assumed as pure DFI-driven. The regional trade is considered more important for the individual region than the external trade. Although this is a simplification, it is nevertheless to a great degree supported by the data for Chinese economic growth, cf. [1]. Therefore the model admits the form:

$$\begin{aligned} Y_{1,t} &= C_{1,t} + DFI_{1,t} + EXR_{1,t} - EXR_{2,t}, && (15)\\ C_{1,t} &= \alpha_{10} + (\alpha_{11} - \alpha_{12} Y_{1,t-1}) Y_{1,t}, && (16)\\ DFI_{1,t} &= \gamma_{10} DFI_{1,t} + \gamma_{11}(Y_{1,t-1} - Y_{1,t-2}), && (17)\\ EXR_{1,t} &= \delta_1 Y_{2,t}, && (18)\\ Y_{2,t} &= C_{2,t} + DFI_{2,t} + EXR_{2,t} - EXR_{1,t}, && (19)\\ C_{2,t} &= \alpha_{20} + (\alpha_{21} - \alpha_{22} Y_{2,t-1}) Y_{2,t}, && (20)\\ DFI_{2,t} &= \gamma_{20} DFI_{2,t} + \gamma_{21}(Y_{2,t-1} - Y_{2,t-2}), && (21)\\ EXR_{2,t} &= \delta_2 Y_{1,t}. && (22) \end{aligned}$$

Similarly to the previous section, we additionally assume, that $DFI_{i,t}$ turns to zero provided the right hand side of (17) or, respectively, (21) is negative. The additional equations (18) and (22) accounts for the export which is also included in equations (15) and (19).

Two identical regions with a common trade.

It is evident that being in the same law space (e.g. Chinese regions) the regions can be assumed to have the similar consumption function and export propensity. Moreover, we assume here that DFI adjustment process has the same form in both regions. Hence, we put here $\alpha_{i0} = \alpha_0$, $\alpha_{i1} = \alpha_1$, $\alpha_{i2} = \alpha_2$, $\gamma_{i0} = \gamma_0$, $\gamma_{i1} = \gamma_1'$, and $\delta_i = \delta'$.

Excluding C_1 and C_2 from equations (15)-(16) and (19)-(20), the system achieves the following form with respect to rescaled variables:

$$\begin{aligned} x_t &= -\alpha + y_t(\beta + y_{t-1}) + \delta(y_t - v_t), && (23)\\ x_t &= \max\{0, \gamma_0 x_{t-1} + \gamma_1(y_{t-1} - y_{t-2})\}, && (24)\\ u_t &= -\alpha + v_t(\beta + v_{t-1}) + \delta(v_t - y_t), && (25)\\ u_t &= \max\{0, \gamma_0 u_{t-1} + \gamma_1(v_{t-1} - v_{t-2})\}, && (26) \end{aligned}$$

where $\delta = \delta'/\alpha_1$, (x, y) are rescaled variables for DFI and income of the first region and (u, v) of the second.

System (23)-(26) implicitly determines the four-dimensional map which describes the economical development of the regions. In a way similar to Sec. 3 we shall choose y_t, v_t, $z_t = y_{t-1}$, and $w_t = v_{t-1}$ as independent variables. With respect to these variables x_t and u_t can be uniquely defined as

$$\begin{aligned} x_t &= G(y_{t-1}, z_{t-1}, v_{t-1}, w_{t-1}),\\ u_t &= G(v_{t-1}, w_{t-1}, y_{t-1}, z_{t-1}), \end{aligned} \qquad (27)$$

where function G is determined as

$$G(y, z, v, w) = \max\{0, \gamma_0(-\alpha + y(\beta + z) + \delta(y - v)) + \gamma_1(y - z)\}. \qquad (28)$$

By substituting (23) and (25) into (24) and (26) respectively, solving it with respect to y and v, the explicit mapping admits the following form

$$F_4 : \begin{pmatrix} y \\ z \\ v \\ w \end{pmatrix} \rightarrow \begin{pmatrix} \dfrac{\delta(G(v,w,y,z)+\alpha)+(\beta+\delta+v)(G(y,z,v,w)+\alpha)}{-\delta^2+(\beta+\delta+y)(\beta+\delta+v)} \\ y \\ \dfrac{\delta(G(y,z,v,w)+\alpha)+(\beta+\delta+y)(G(v,w,y,z)+\alpha)}{-\delta^2+(\beta+\delta+y)(\beta+\delta+v)} \\ v \end{pmatrix}, \qquad (29)$$

where function G is determined by (28).

General properties of the map F_4.

The mapping (29) for the dynamics of two regions with a common trade can be written in a general form as

$$F_4 : \begin{pmatrix} y \\ z \\ v \\ w \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{F}(y,z,v,w) \\ y \\ \mathcal{F}(v,w,y,z) \\ v \end{pmatrix} \qquad (30)$$

with some piecewise-smooth function $\mathcal{F}$. It is obvious that without common trade between regions, i.e. $\delta = 0$, system (29) is splitted into two independent 2-dimensional maps and each of them has the form (8) and describes the dynamics of the isolated regions. Thus, we obtain the following property of function $\mathcal{F}$ being evaluated at $\delta = 0$:

$$\mathcal{F}(y,z,v,w)|_{\delta=0} = \bar{\mathcal{F}}(y,z).$$

Hence, parameter δ stands for the coupling between the regions which measures the propensity of the regions to the competitive common trade.

System (29) has a symmetry with respect to changing of variables (y, z) onto (v, w). As a result, there exists a two-dimensional invariant manifold $SM : \{y = v, \ z = w\}$ which is usually called *synchronization* manifold [2-3]. This hyperplane corresponds to identically synchronized motions, i.e. when the development of both regions is identical. Synchronization between two trading regions can be usually expected for strong coupling δ. Since the dynamics in the hyperplane is described also by 2-dimensional map (8) all the results of previous section can be applied to the system (29) restricted to the synchronization manifold. From the economical point of view the identically synchronized motions correspond to the case when both regions starting from the same initial state (i.e. income level and DFI) are developing identically due to the assumed similarity of consumption function and DFI adjustment with the common trade being balanced: export equals to import.

Thus we may conclude that our system has a common feature of coupled symmetric systems and methods of the synchronization theory [4-10] can be applied to its investigation. In particular, in the following we give conditions for the synchronization of two interacting regions.

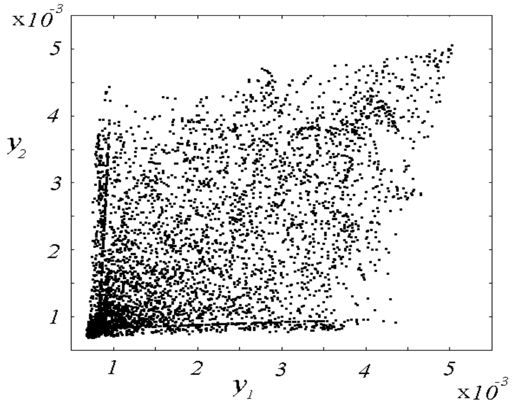

Figure 12: Asynchronous hyperchaotic attractor for parameter values $\gamma_1 = 0.0258$ and $\delta = 0.0005$.

The system (29) is piecewise-smooth. The following constrains determine boundary between regions of smoothness in the phase space:

$$\begin{array}{l} x^{\mathrm{ex}} = \gamma_0(-\alpha + y(\beta + z) + \delta(y - v)) + \gamma_1(y - z) = 0, \\ u^{\mathrm{ex}} = \gamma_0(-\alpha + v(\beta + w) + \delta(v - y)) + \gamma_1(v - w) = 0, \end{array} \tag{31}$$

that correspond to the case when expected DFI (x^{ex} and u^{ex} are its rescaled counterparts) for one of the regions given by (17) or (21) is zero.

The four regions where system (29) is smooth can be determined as follows:

$$\begin{array}{ll} \Pi_{00}: & x^{\mathrm{ex}} < 0,\ u^{\mathrm{ex}} < 0, \\ \Pi_{01}: & x^{\mathrm{ex}} < 0,\ u^{\mathrm{ex}} > 0, \\ \Pi_{10}: & x^{\mathrm{ex}} > 0,\ u^{\mathrm{ex}} < 0, \\ \Pi_{11}: & x^{\mathrm{ex}} > 0,\ u^{\mathrm{ex}} > 0. \end{array} \tag{32}$$

Conditions for synchronization

The particularly interesting phenomena occur when the coupled systems are chaotic [4-10]. In this case, because of the sensitive dependence on initial conditions systems will be desynchronized for low values of coupling, For example, taking the parameter values as the following $\gamma_1 = 0.0258$ and $\delta = 0.0005$, the asynchronous attractor exists as shown in Fig. 12. The attractor is hyperchaotic because it has two positive Lyapunov exponents [17-18].

We recall here that identical *chaotic synchronization* takes place when the systems exhibit asymptotically identical behavior, i.e. $\|(y,z)(t) - (v,w)(t)\| \to 0$ as $t \to \infty$. Synchronous motions approach asymptotically symmetrical synchronization manifold. The numerical criteria for this kind of synchronization can be the negativeness of the largest transverse Lyapunov exponent $\lambda_\perp < 0$, cf. [4-10]. We plot $\lambda_\perp$ versus δ in Fig. 13 for fixed $\gamma_1 = 0.025$, $\gamma_0 = 0.7$ which corresponds to an interval of chaotic behavior for the single system, cf. Fig. 7. $\lambda_{||}$ denotes the largest Lyapunov exponent for the motion along the synchronization manifold. We can observe that for $\delta > 0.0015$ the coupling make the two systems to be synchronized.

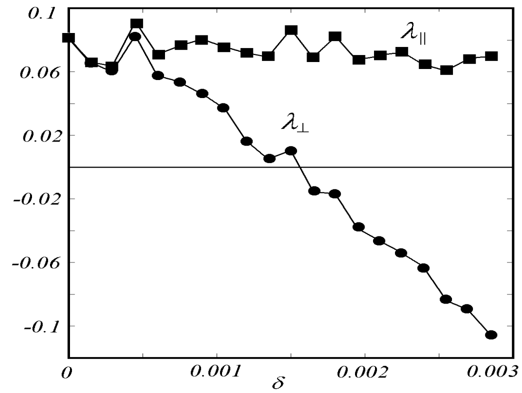

Figure 13: Largest Lyapunov exponents for the two coupled systems (29) versus δ. $\lambda_\perp$ corresponds to the transverse direction and $\lambda_{||}$ to the direction along the synchronization manifold.

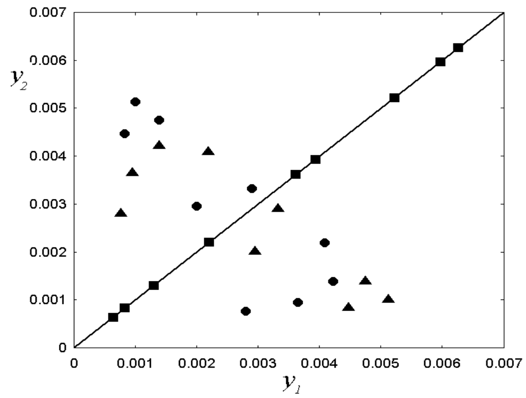

Figure 14: In-phase and out-of-phase periodic cycles for $\gamma_1 = 0.027$ and $\delta = 0.0005$ (different symbols mark three different orbits).

Another scenario for the synchronization gain in the considered system is connected with the transition to periodic regimes. Even with the same coupling parameter $\delta = 0.0005$, by increasing γ_1 we may force the systems to escape the interval of chaotic regime, cf. Fig. 7. As a result, in-phase and out-of-phase stable periodic motions appear [19]. In Fig. 14, the stable period-9 on the diagonal corresponds to in-phase synchronous motion. The symmetrical cycles out of the diagonal represents the out-of-phase oscillations which are illustrated in Fig. 15.

6 Conclusions

The proposed model for an individual region economic growth (8) and for two regions with a common trade (29) in the form of two- and four-dimensional noninvertible piecewise-smooth maps is shown to exhibit different behaviors. We have found numerically the interval of chaotic and periodic motions. We have obtained the conditions for stability loss of the equilibrium point with nonsmooth neighborhood which corresponds to the zero level

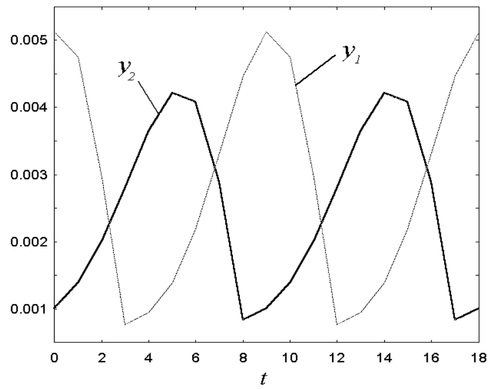

Figure 15: Out-of-phase motion. y_1 is proportional to the income of the first region, y_2 to the second.

of foreign investments. Finally, the conditions for chaotic synchronization of two trading identical regions are presented.

Being out of the scope of the present paper, the following related problems can be focused in future investigations: the influence of the regions difference on their common dynamics, the case of three and more spatially coupled systems, the effect of noise on the dynamics.

Acknowledgments

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Appendix A

In this Appendix we obtain analytical conditions for the stability of equilibrium point M. First, observe that the eigenvalues of J_1 evaluated at the fixed point are complex conjugated provided

$$s_1^2 + 4s_2 = \left(\frac{-\alpha}{(\beta + y_0)^2} + \frac{\gamma_1}{\beta + y_0} + \gamma_0 \right)^2 + 4 \left(\frac{\alpha\gamma_0}{(\beta + y_0)^2} - \frac{\gamma_1}{\beta + y_0} \right) < 0. \tag{33}$$

The following notations are necessary:

$$\psi = -\frac{\alpha}{(y_0 + \beta)^2}; \quad \bar{\varphi} = \frac{\gamma_1^2 - \alpha\gamma_0^2 + \gamma_1\gamma_0\beta}{\gamma_0 y_f + \gamma_1 + \gamma_0\beta}.$$

Proposition 1 *Let the parameters of system* (8) *are positive, satisfy condition* (33) *and, additionally,*

$$\gamma_0 < 1, \quad \alpha < \beta\nu + \nu^2, \tag{34}$$

where $\nu = \min\left\{\frac{\gamma_1+\beta\gamma_0}{1-\gamma_0}, \frac{\gamma_1}{\gamma_0}\right\}$. *Then there exists such integer* $k_0 > 0$ *that the following inequalities hold*

$$\varphi_k > \bar{\varphi} \quad \textit{for all} \quad 0 < k < k_0 \quad \textit{and} \quad \varphi_{k_0} \leq \bar{\varphi},$$

where φ_k *are determined by the mapping*

$$\varphi_k = s_1 + \frac{s_2}{\varphi_{k-1}}, \qquad \varphi_0 = \psi.$$

Here s_1 *and* s_2 *are determined from* (12). *Let also* ξ_{k_0} *be determined by the following rule:*

$$\xi_k = s_1\xi_{k-1} + s_2\xi_{k-2}, \quad \xi_0 = \psi, \quad \xi_1 = s_1\psi + s_2.$$

Then the equilibrium point M *of system* (8) *is stable if* $|\eta_{k_0}\xi_{k_0}| < 1$ *and unstable if* $|\eta_{k_0}\xi_{k_0}| > 1$. *Here*

$$\eta_{k_0} = \begin{cases} \psi, & \textit{if} \quad \varphi_{k_0} \leq 0, \\ 1, & \textit{if} \quad 0 < \varphi_{k_0} \leq \bar{\varphi}. \end{cases}$$

Note that Proposition 1 does not give us analytical expressions for the region of parameters where the equilibrium is stable or unstable. Instead, for any fixed set of parameters we may perform a procedure in a finite number of steps (analytic or numeric) in order to determine the stability of M.

Proof. For the proof, we use a coordinates system with the origin at the equilibrium point M. The same letters (y, z) will be used to denote variables. Define the following regions in the phase space, cf. Fig. 3:
$L_1 = \{(y,z) \in \Pi_1 : \ z > 0, y > 0\}$; $L_2 = \{(y,z) \in \Pi_1 : \ z < 0, y > 0\}$;
$L_3 = \{(y,z) \in \Pi_1 : \ z < 0, y < 0\}$; $L_4 = \{(y,z) \in \Pi_0 : \ z > 0, y > 0\}$.
Denote also $\bar{L}$ as the line along the eigenvector v_1 of the Jacobian (11), i.e. $\bar{L} = \{(y,z) : y = \lambda_1^0 z\}$. In the following we will use the linearized map F_L in the neighborhood of M:

$$F_L : \ F_L(y,z) = \begin{cases} J_0 \cdot (y,z)^T, & \text{if} \quad (y,z) \in \Pi_0, \\ J_1 \cdot (y,z)^T, & \text{if} \quad (y,z) \in \Pi_1. \end{cases}$$

It is convenient to introduce a separate notation $\bar{L}_1$ for the part of the line $\bar{L}$ that belongs to Π_1, i.e. $\bar{L}_1 = \bar{L} \cap \Pi_1$ In the following we split the proof into six parts.

1. All points from Π_0 are mapped into $\bar{L}$ under the action of F_L, i.e. $F_L(\Pi_0) \subset \bar{L}$. This can be shown by direct substitution: for any $(y,z) \in \Pi_0$ we have $(y_1, z_1)^T = J_0(y,z)^T$ with $y_1 = \lambda_1^0 y$ and $z_1 = y$. Hence, $(y_1, z_1) \in \bar{L}$.

2. Let us show that $F_L(\bar{L} \cap \Pi_0) \subset \bar{L}_1$. By virtue of 1, we have $F_L(\bar{L} \cap \Pi_0) \subset \bar{L}$. Next, note that $\lambda_1^0 < 0$ for the considered parameter values. Therefore for any point (y,z) from $\bar{L} \cap \Pi_0$ holds $y < 0$, cf. Fig. 3. Its image (y_1, z_1) has $z_1 = y < 0$ and, therefore, belongs to Π_1.

As a consequence of 1 and 2 we state, that $F_L^2(L_4) \subset \bar{L}_1$ and $F_L(\Pi_0 \setminus L_4) \subset \bar{L}_1$.

3. Any point from L_3 eventually escapes the region L_3. This follows from the fact that our system, restricted to L_3 has form $(y,z)^T \to J_1(y,z)^T$, where the matrix J_1 has complex

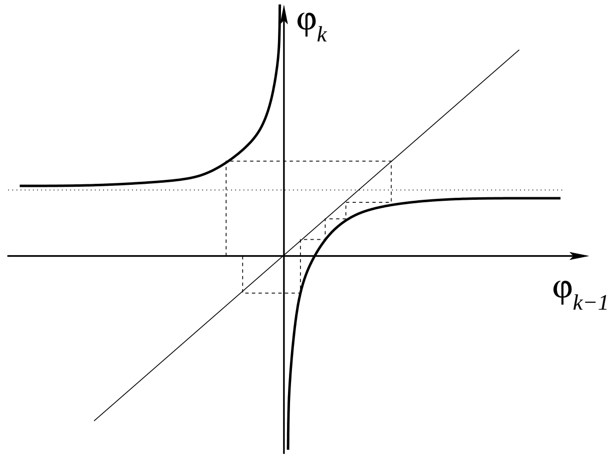

Figure 16: One dimensional mapping for φ_k.

conjugated eigenvalues due to (33). Hence, it exhibits rotation by some fixed angle (not equal to π or 0) in polar coordinates centered at the equilibrium point.

4. $F_L(L_2) \subset L_1 \cup \Pi_0$. In order to show it, note that for $(y, z) \in L_2$ the inequality $y > 0$ holds. Hence for the z-component of its image we also have $z_1 > 0$. It implies $F_L(L_1) \subset L_1 \cup \Pi_0$.

As a consequence of the above proved facts, we observe that all points from U_1 will eventually enter the region Π_0. Then, taking into account 2, they further are mapped into $\bar{L}_1$. We can thus determine a one dimensional mapping on $\bar{L}_1$: $f : (\bar{L}_1) \to (\bar{L}_1)$. Let us parametrize $\bar{L}_1$ as follows: $(\bar{L}_1) = \{(y, z) : z = \mu, y = \psi\mu, \mu < 0\}$. Here $\mu = 0$ corresponds to the equilibrium point M.

5. It is evident that the stability properties of the zero fixed point $\mu = 0$ of the above introduced one-dimensional map determine the stability of the equilibrium point M of system (8), i.e. they are either both stable or unstable.

6. Inequalities (34) imply that $\bar{\varphi} > 0$, $s_1 > 0$, and $s_2 < 0$. The proof is elementary.

7. Consider some point on $\bar{L}_1$: $y_0 = \psi\mu$, $z_0 = \mu$ where $\mu < 0$ is the parameter on $\bar{L}_1$. Then its k-th image is $(y_k, z_k) = {F_L}^k(y_0, z_0)$. It is easy to show by direct calculation that for the mapping F_L, restricted to Π_1, the ratio $\varphi_k = y_k/z_k$ is changed by the following rule

$$\varphi_k = s_1 + s_2/\varphi_{k-1}. \tag{35}$$

From part 6 of the proof it is known that the signs of s_1 and s_2 are fixed. The obtained one-dimensional mapping for φ_k is sketched in Fig. 16. The graph does not intersect the diagonal because of (33). By inspecting the form of map (35), it easy to see that there exists k_0 such that for all $0 < k < k_0$:

$$\varphi_k = \frac{y_k}{z_k} > \bar{\varphi} > 0, \quad \varphi_0 = \frac{y_0}{z_0} = \psi < 0.$$

and $\varphi_{k_0} < \bar{\varphi}$. This means that k_0-th image of (y_0, z_0) will be in Π_0. (In fact, $\bar{\varphi}$ represents the slope of $\partial\Pi$). By induction principle we have $y_k = \xi_k\mu$ and $z_k = y_{k-1}$ for all $0 < k < k_0$.

Further, consider the case $\varphi_{k_0} > 0$. Then $(y_{k_0}, z_{k_0}) \in L_4$. In virtue of part 2, we have $F_L^2(y_{k_0}, z_{k_0}) \in \bar{L}_1$. Direct calculations lead to the following expression $F_L^2(y_{k_0}, z_{k_0}) = (\psi^2\xi_{k_0}\mu, \psi\xi_{k_0}\mu)$. The obtained point on the line $\bar{L}_1$ corresponds to following value of parameter on $\bar{L}_1$: $\mu_1 = \psi\xi_{k_0}\mu$. Thus the obtained 1-dimensional map on $\bar{L}_1$ has the form $\mu \to \psi\xi_{k_0}\mu$. It is stable if and only if $|\psi\xi_{k_0}| < 1$. In the similar way we can show that in the case $\varphi_{k_0} < 0$ the criterion for the stability of the constructed 1D map is $|\xi_{k_0}| < 1$.

To complete the proof we note that in virtue of part 5 the stability of the constructed 1D map is equivalent to stability of the equilibrium point for system (8). ■

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