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Perturbation analysis of chance-constrained programs under variation of all constraint data

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Abstract

We consider stability of solutions to optimization problems with probabilistic constraints under perturbations of all constraint data (probability level, probability measure, deterministic constraints, random set mapping). Constraint qualifications ensuring stability are derived for each of the single parameters. Examples illustrating the necessity of the stated conditions as well as the limitations of the given results are provided.

1 Introduction

A fairly general shape of chance constraint programs is

$$(P) \quad \min\{g(x)|x \in X,\, \mu(H(x)) \geq p\},$$

where $g : \mathbb{R}^m \to \mathbb{R}$ is a continuous objective function, $X \subseteq \mathbb{R}^m$ is a closed subset of deterministic constraints, and the inequality defines a probabilistic constraint with $H : \mathbb{R}^m \rightrightarrows \mathbb{R}^s$ being a multifunction with closed graph, μ is a probability measure on $\mathbb{R}^s$ and $p \in (0,1)$ is some probability level. In the simplest case of linear chance constraints, g is linear, X is a polyhedron and $H(x) = \{z \in \mathbb{R}^s | Ax \geq z\}$, where A is a matrix of order (s,m) and the inequality sign has to be understood component-wise.

Since the data of optimization problems are typically uncertain or approximated by other data which are easier to handle, the question of stability of solutions arises naturally. Concerning (P), the first idea is to investigate solutions under perturbations of the right hand side p of the inequality. This reflects the modeling degree of freedom when choosing a probability at which the constraint system is supposed to be valid. Furthermore, the probability measure μ is unknown in general and has to be approximated, for instance, by empirical measures. This motivates to extend the perturbation analysis to μ. Stability of solutions of (P) with respect to p and μ is well understood now but shall be briefly reviewed in this paper for the sake of being selfcontained. Apart from these two constraint parameters, also approximations of the deterministic constraint X and of the random set mapping H in (P) may be of interest. The aim of this paper is to identify constraint qualifications for stability under partial perturbations of the single constraint parameters in (P). Due to the increasing complexity of how these parameters influence each other, the resulting constraint qualifications become more and more restrictive when passing from p over μ to X and H. Part of the result relate to convex data in (P) or even in the perturbations of (P). Special emphasis is put on a series of counter-examples highlighting the necessity and limitations of the obtained conditions.

2 Notation and basic assumptions

2.1 Stability concepts

For a multifunction $M : Z \rightrightarrows Y$ between metric spaces, we denote by $\operatorname{Gph} M$, $\operatorname{dom} M$ and M^{-1} its graph, domain and inverse, respectively. At some $x \in Z$, M will be called closed if $(x_n, y_n) \to (x, y)$ along with $y_n \in M(x_n)$ imply $y \in M(x)$. M is upper (lower) semicontinuous at x, if for all open $V \supseteq M(x)$ (with $V \cap M(x) \neq \emptyset$) there exists some open $W \ni x$ such that $V \supseteq M(x')$ ($V \cap M(x') \neq \emptyset$) for all $x' \in W$. Clearly, $\operatorname{Gph} M$ is closed if and only if M is closed at all $x \in X$. M will be called metrically regular at some $(\bar{x}, \bar{y}) \in \operatorname{Gph} M$, if there exists some $L > 0$ such that $d(x, M^{-1}(y)) \leq L d(y, M(x))$ for all (x, y) in some neighbourhood of $(\bar{x}, \bar{y})$.

For a sequence $A_n \subseteq Z$, the upper (lower) set limit in the sense of Painlevé-Kuratowski is defined as

$$\operatorname*{Limsup}_n \, (\operatorname*{Liminf}_n) \, A_n = \{x \in Z \,|\, \liminf_n \, (\limsup_n) \, (d(x, A_n) = 0\}.$$

In case that $\operatorname*{Limsup}_n A_n = \operatorname*{Liminf}_n A_n =: A$, we write $A_n \to A$. For multifunctions $M : Z \rightrightarrows Y$, corresponding upper and lower limits evaluated at some $\bar{x}$ are defined as

$$\begin{aligned} \operatorname*{Limsup}_{x \to \bar{x}} M(x) &= \{y \in Y \,|\, \exists (x_n, y_n) \in \operatorname{Gph} M : \ (x_n, y_n) \to \ (\bar{x}, y)\}, \\ \operatorname*{Liminf}_{x \to \bar{x}} M(x) &= \{y \in Y \,|\, \forall x_n \to \ \bar{x} \ \exists \ y_n \to \ y : \ y_n \in M(x_n) \text{ for } n \geq n_0\}. \end{aligned}$$

From the definitions it follows that M is closed (lower semicontinuous) at $\bar{x}$ if and only if

$$\operatorname*{Limsup}_{x \to \bar{x}} M(x) \subseteq M(\bar{x}) \quad (M(\bar{x}) \subseteq \operatorname*{Liminf}_{x \to \bar{x}} M(x)).$$

In case that both relations hold true, we write $M(\bar{x}) = \operatorname*{Lim}_{x \to \bar{x}} M(x)$. Finally, for a sequence of multifunctions $M_n : Z \rightrightarrows Y$, we introduce the following upper and lower limits evaluated at some x:

$$\begin{aligned} \left(\operatorname*{Limsup}_n M_n\right)(x) &= \bigcup_{x_n \to x} \operatorname*{Limsup}_n M_n(x_n) \\ \left(\operatorname*{Liminf}_n M_n\right)(x) &= \bigcap_{x_n \to x} \operatorname*{Liminf}_n M_n(x_n), \end{aligned}$$

We note that $\left(\operatorname*{Limsup}_n M_n\right)(x)$ coincides with the so-called graphical outer limit of M_n evaluated at x ([9], p.166) whereas $\left(\operatorname*{Liminf}_n M_n\right)(x)$ differs from the corresponding graphical inner limit in that it uses intersection in place of union.

2.2 Data spaces and metrics

The constraint data of our problem (P) are given by $(\mu,\ H,\ X, p)$. According to the assumptions above, we introduce the following data space

$$\mathcal{D} = \mathcal{P}(\mathbb{R}^s) \times \mathcal{M}(\mathbb{R}^m, \mathbb{R}^s) \times \mathcal{F}(\mathbb{R}^m) \times (0,1),$$

where $\mathcal{P}(\mathbb{R}^s)$ is the set of Borel probability measures on $\mathbb{R}^s$, $\mathcal{M}(\mathbb{R}^m, \mathbb{R}^s)$ is the set of multifunctions from $\mathbb{R}^m$ to $\mathbb{R}^s$ having closed graph and $\mathcal{F}(\mathbb{R}^m)$ denotes the hyperspace of closed subsets of $\mathbb{R}^m$. The perturbations (ν, G, Y, q) of the original data $(\mu,\ H,\ X, p)$ are supposed to belong to the same data space. Each of the factors of $\mathcal{D}$ can be endowed with a suitable metric. For $\mathcal{F}(\mathbb{R}^m)$ we take the so-called integrated set distance ([9], p. 139) between closed subsets A and B:

$$\delta(A,B) := \int_0^\infty \delta_\rho(A,B) e^{-\rho} d\rho,$$

where $\delta_\rho(A,B) := \max_{x \in B(0,\rho)} |d(x,A) - d(x,B)|$ denotes the $\rho-$ Hausdorff distance. It is known that δ metrizes the Painlevé-Kuratowski set convergence introduced above, i.e., $A_n \to A$ if and only if $\delta(A_n, A) \to 0$. Applying the same idea to graphs of multifunctions, one may define $\tau(G, \tilde{G}) := \delta(\operatorname{Gph} G, \operatorname{Gph} \tilde{G})$ as a distance on $\mathcal{M}(\mathbb{R}^m, \mathbb{R}^s)$. Then, obviously, $G_n \to G$ in the sense of $\tau(G_n, G) \to 0$, if and only if $\operatorname{Gph} G_n \to \operatorname{Gph} G$ in the sense of Painlevé-Kuratowski set convergence. Finally, on $\mathcal{P}(\mathbb{R}^s)$ we use the so-called $\mathcal{B}$- discrepancy

$$\alpha_{\mathcal{B}}(\nu, \tilde{\nu}) := \sup_{B \in \mathcal{B}} |\nu(B) - \tilde{\nu}(B)|, \quad \mathcal{B} = \{z + \mathbb{R}^s_- | z \in \mathbb{R}^s\} \cup \{H(x) | x \in X\}, \tag{1}$$

where X and H refer to the original data of problem (P). The first constituent of the collection $\mathcal{B}$ makes $\alpha_{\mathcal{B}}$ a metric on $\mathcal{P}(\mathbb{R}^s)$, while the second one is required for a suitable stability analysis.

Specific attention will be paid to convex-like problems. For this purpose, we introduce the subspace of convex problem data

$$\mathcal{D}^c = \mathcal{P}^c(\mathbb{R}^s) \times \mathcal{M}^c(\mathbb{R}^m, \mathbb{R}^s) \times \mathcal{F}^c(\mathbb{R}^m) \times (0,1),$$

where $\mathcal{M}^c(\mathbb{R}^m, \mathbb{R}^s)$ is the set of multifunctions from $\mathbb{R}^m$ to $\mathbb{R}^s$ having closed and convex graph and $\mathcal{F}^c(\mathbb{R}^m)$ denotes the hyperspace of closed and convex subsets of $\mathbb{R}^m$. By $\mathcal{P}^c(\mathbb{R}^s)$ we refer to the set of so-called $r-$ concave probability measures for some $r < 0$ ([8]) which are defined as to satisfy the inequality

$$\mu^r(\lambda B_1 + (1-\lambda)B_2) \leq \lambda \mu^r(B_1) + (1-\lambda)\mu^r(B_2) \tag{2}$$

for all Borel measurable convex subsets B_1, B_2 of $\mathbb{R}^s$ and all $\lambda \in [0,1]$ such that $\lambda B_1 + (1-\lambda)B_2$ is again Borel measurable and convex. Many of the prominent multivariate distributions (e.g. normal, Dirichlet, Student and Pareto distribution as well as uniform distribution on bounded convex sets) belong to the class $\mathcal{P}^c(\mathbb{R}^s)$

(cf. [8]). If $(\mu, H, X, p) \in \mathcal{D}^c$, then the function $\mu^r \circ H$ (with $r < 0$ from (2) is convex and, in particular, the constraint set in problem (P) is convex (after raising the inequality to the negative power r).

With problem (P) we associate the constraint set mapping $\Phi : \mathcal{D} \rightrightarrows \mathbb{R}^m$, the solution set mapping $\Psi : \mathcal{D} \rightrightarrows \mathbb{R}^m$ as well as the optimal value function $\varphi : \mathcal{D} \to \bar{\mathbb{R}}$, all of them depending on the problem data (ν, G, Y, q) which are considered as parameters:

$$\begin{aligned}
\Phi(\nu, G, Y, q) \; &: = \{x \in Y,\ \nu(G(x)) \geq q\} \\
\varphi(\nu, G, Y, q) \; &: = \inf\{g(x) | x \in \Phi(\nu, G, Y, q)\} \\
\Psi(\nu, G, Y, q) \; &: = \{x \in \Phi(\nu, G, Y, q) \, | g(x) = \varphi(\nu, G, Y, q)\}.
\end{aligned}$$

By adding a left upper index 'μ', 'H', 'X' or 'p', we refer to the respective partial mappings, when all parameters except the indexed one are fixed as the original data, e.g. ${}^X\Psi(Y) = \Psi(\mu, H, Y, p), {}^H\Phi(G) = \Phi(\mu, G, X, p)$ etc. For some open subset $Q \subseteq \mathbb{R}^m$, define the localized mappings

$$\begin{aligned}
\varphi_Q(\nu, G, Y, q) \; &: = \inf\{g(x) | x \in \Phi(\nu, G, Y, q) \cap \operatorname{cl} Q\} \\
\Psi_Q(\nu, G, Y, q) \; &: = \{x \in \Phi(\nu, G, Y, q) \cap \operatorname{cl} Q | g(x) = \varphi_Q(\nu, G, Y, q)\}.
\end{aligned}$$

The localized partial mappings are obtained by prepending the corresponding index to Φ in these definitions, e.g., ${}^p\varphi_Q(q) = \inf\{g(x) | x \in {}^p\Phi(q) \cap \operatorname{cl} Q\}$.

3 Partial Stability of Solutions and Optimal Values

In this section, we study the stability of solutions and optimal values to problem (P) with respect to single data parameters. As a basic preparatory result we need the closedness of all partial constraint mappings.

Proposition 1 *The partial constraint set mappings ${}^p\Phi$, ${}^\mu\Phi$, ${}^X\Phi$, ${}^H\Phi$ are closed at their respective original data points p, μ, X and H.*

Proof. Closedness of ${}^p\Phi$ and ${}^\mu\Phi$ follows from the upper semicontinuity of the mapping $\mu(H(\cdot))$ and from the definition of the discrepancy in (1) (cf. [10], Prop. 3.1). For closedness of ${}^X\Phi$ let $(X_n, x_n) \to (X, x)$ such that $x_n \in {}^X\Phi(X_n)$. Then, $x_n \in X_n$ and $\mu(H(x_n)) \geq p$. It follows that $x \in \operatorname{Limsup}_n X_n = X$ (by $X_n \to X$) and $\mu(H(x)) \geq \limsup_n \mu(H(x_n)) \geq p$ again by upper semicontinuity of $\mu(H(\cdot))$. This means $x \in {}^X\Phi(X)$, and, hence, closedness of ${}^X\Phi$ at X. To check ${}^H\Phi$, let $(H_n, x_n) \to (H, x)$ such that $x_n \in {}^H\Phi(H_n)$. Then, $x_n \in X$ and $\mu(H_n(x_n)) \geq p$. Closedness of X implies that $x \in X$. Furthermore, equation (9) from the appendix yields

$$\operatorname*{Limsup}_n H_n(x_n) \subseteq \left(\operatorname*{Limsup}_n H_n\right)(x) = H(x).$$

Then, equation (6) from the appendix provides the desired relation

$$\mu(H(x)) \geq \limsup_{n} \mu(H_n(x_n)) \geq p,$$

whence the closedness of ${}^H\Phi$ at H. ■

3.1 Stability with respect to the probability level

The dependence of solutions and optimal values on perturbations of the probability level is the simplest one among all data variations considered here, and the following stability results are readily derived from classical facts of parametric optimization (cf. [1], Th. 4.2.1, Th. 4.2.2, [4], Th. 1, Th. 2 and [5], Th. 2.2) upon noting that the partial constraint set mapping ${}^p\Phi$ is closed at p according to Proposition 1. We emphasize that all assumptions made to obtain stability exclusively refer to the original data (μ, H, X, p) of problem (P).

Theorem 2 *Assume that*

1. $({}^p\Phi)^{-1}$ *is metrically regular at all* $(\bar{x}, p)$ *with* $\bar{x} \in \Psi(\mu, H, X, p)$ *(solution set for the original data of* (P)*).*

Then, ${}^p\Psi$ *is closed at* p *and* ${}^p\varphi$ *is upper semicontinuous at* p*.*

2. In addition, $\Psi(\mu, H, X, p)$ *is bounded, i.e.,* $\Psi(\mu, H, X, p) \subseteq Q$ *for some bounded open* $Q \subseteq \mathbb{R}^m$*.*

Then, ${}^p\Psi_Q$ *is upper semicontinuous at* p*, and* ${}^p\varphi_Q$ *is continuous at* p*.*

3. In addition, g *(the objective in problem* (P)*) is locally Lipschitzian.*

Then, ${}^p\varphi_Q$ *is locally Lipschitzian at* p*.*

4. In addition, g *satisfies a* $k-$ *th order growth condition on the set of global solutions, i.e. (with* $\bar{x}$ *from 1. and* Q *from 2.),*

$$g(x) \geq g(\bar{x}) + d^k(x, \Psi(\mu, H, X, p)) \quad \forall x \in Q \cap \Phi(\mu, H, X, p)$$

Then, ${}^p\Psi_Q$ *is upper Hölder continuous at* p *with rate* k^{-1}*,i.e.,*

$$\sup\{d(x, {}^p\Psi_Q(p)) | x \in {}^p\Psi_Q(q)\} \leq L\,|q-p|^{k^{-1}}$$

for some $L > 0$ *and* q *close to* p*.*

The inconvenient use of localizations (by means of Q) in the stability statements 2., 3. and 4. cannot be avoided in general. However, there are some special cases where localizations are not necessary. For instance, if the set X of deterministic constraints is compact, then assumption 2. of Theorem 2 is automatically fulfilled with $Q := \{x|d(x,X) < 1\}$. Then, ${}^p\Psi_Q = {}^p\Psi$, ${}^p\varphi_Q = {}^p\varphi$ and all the results of the Theorem maybe rephrased in terms of the unlocalized mappings ${}^p\Psi$ and ${}^p\varphi$. Another instance of avoiding localizations is given in Proposition 3 below.

Re-inspection of Theorem 2 reveals that assumptions 1. and 4. are most difficult to verify. In [10] (Proof of Cor. 3.7) it was shown that for convex problem data (i.e., $(\mu, H, X, p) \in \mathcal{D}^c$) the metric regularity of $({}^p\Phi)^{-1}$ (equivalently formulated there as a Lipschitzian property of ${}^p\Phi$) is implied by the Slater-type condition

$$\text{there exists some } \hat{x} \in X \text{ such that } \mu(H(\hat{x})) > p. \tag{3}$$

For the nonconvex setting, a series of verifiable conditions was formulated in ([2]) in the special case of $H(x) = \{z \in \mathbb{R}^s | z \leq h(x)\}$ with continuous $h : \mathbb{R}^m \to \mathbb{R}^s$ (chance constraints with random right-hand side. To give a simplified idea, assume that h is locally Lipschitzian and μ has a continuous density f_μ. Then, assumption 1. of Theorem 2 will be satisfied under the following two conditions:

- If $\mu(H(\bar{x})) = p$, then there exists some $z \in h(\bar{x}) + \operatorname{bd} \mathbb{R}^s_-$ such that $f_\mu(z) > 0$ ('bd'=boundary).
- $\partial_a \langle y^*, h \rangle (\bar{x}) \cap -N_a(X; \bar{x}) = \emptyset \quad \forall y^* \in \mathbb{R}^s_- \setminus \{0\}$,

where in the second condition ∂_a and N_a refer to Mordukhovich's subdifferential and normal cone, respectively [7]. In case of differentiable h and of $\bar{x} \in \operatorname{int} X$, this second condition simply reduces to the positive linear independence of the gradients $\nabla h_i(\bar{x})$. The first condition is fulfilled, in particular, if $f_\mu(h(\bar{x})) > 0$, which is always true for the multivariate normal distribution, for instance.

Concerning assumption 4. of Theorem 2, the quadratic growth condition for g ($k = 2$) is closely related, for smooth data, to second order sufficient conditions. For convex data in the setting of our problem (P) verifiable conditions of quadratic growth are given in [2], Th. 8. Finally, we formulate a stability result for convex data avoiding any compactness or localization statements:

Proposition 3 *In problem (P), let $(\mu, H, X, p) \in \mathcal{D}^c$ and g be convex. If the unperturbed solution set $\Psi(\mu, H, X, p)$ is nonempty and bounded and if (3) is satisfied, then ${}^p\Psi$ is upper semicontinuous at p and ${}^p\varphi$ is continuous at p.*

Proof. The convexity assumption implies that the parametric constraint set ${}^p\Phi(q)$ is convex for all q. Also, one easily checks that metric regularity at all $x \in {}^p\Phi(p)$, which was noted above to be implied by (3), guarantees the lower semicontinuity of ${}^p\Phi$ at p. Furthermore, we know that ${}^p\Phi$ is closed at p according to Prop. 1. Now, apply Theorem 11 (2.) with $f := g$, $\Lambda := (0,1)$, $\lambda := q$, $\lambda_0 := p$, $M := {}^p\Phi$. ■

3.2 Stability with respect to the probability measure

Stability of program (P) with respect to variations of the probability measure μ may be partially reduced to the previously discussed case of stability with respect to the scalar probability level p. The main observation in this context was made in [10] (Proof of Th. 3.2) where (formulated in different terms there) it was shown that the metric regularity of $({}^p\Phi)^{-1}$ (see assumption 1. in Th. 2) is sufficient to guarantee (local) lower semicontinuity of $({}^\mu\Phi)^{-1}$ and thus to derive parallel results to Theorem 2. More precisely, one has

Theorem 4 *Assume that*

1. $({}^p\Phi)^{-1}$ *is metrically regular at all* $(\bar{x}, p)$ *with* $\bar{x} \in \Psi(\mu, H, X, p)$.

Then, ${}^\mu\Psi$ *is closed at* μ *and* ${}^\mu\varphi$ *is upper semicontinuous at* μ.

2. In addition, $\Psi(\mu, H, X, p)$ *is bounded, i.e.,* $\Psi(\mu, H, X, p) \subseteq Q$ *for some bounded open* $Q \subseteq \mathbb{R}^m$.

Then, ${}^\mu\Psi_Q$ *is upper semicontinuous at* μ *and* ${}^\mu\varphi_Q$ *is continuous at* μ.

3. In addition, g *is locally Lipschitzian.*

Then, there exists some bounded open set $Q' \supseteq \Psi(\mu, H, X, p)$ *(smaller than* Q*), such that* ${}^\mu\varphi_{Q'}$ *is upper Lipschitzian at* μ*, i.e., with some* $L, \delta > 0$*, one has*

$$|{}^\mu\varphi_{Q'}(\nu) - {}^\mu\varphi_{Q'}(\mu)| \leq L\alpha_{\mathcal{B}}(\nu, \mu) \quad \forall \nu \in \mathcal{P}(\mathbb{R}^s),\ \alpha_{\mathcal{B}}(\nu, \mu) < \delta.$$

4. In addition, g *satisfies a* $k-$ *th order growth condition on the set of global solutions (see Th. 2).*

Then, ${}^\mu\Psi_{Q'}$ *is upper Hölder continuous at* μ *with rate* $1/k$*, i.e., there are* $L, \delta > 0$ *such that for all* $\nu \in \mathcal{P}(\mathbb{R}^s)$ *with* $\alpha_{\mathcal{B}}(\nu, \mu) < \delta$

$$\sup\{d(x, {}^\mu\Psi_{Q'}(\mu)) | x \in {}^\mu\Psi_{Q'}(\nu)\} \leq L\,[\alpha_{\mathcal{B}}(\nu, \mu)]^{1/k}.$$

The first assertion of the Theorem relies on the local lower semicontinuity of $({}^\mu\Phi)^{-1}$ as stated above and on standard arguments of parametric programming (cf. [1]) similar as in Theorem 2. 2. and 3. are shown in Theorem 3.2 of [10] while 4. results from Theorem 2.2 in [5].

In contrast to the previous section, the first three assumptions of Theorem 4 do not guarantee the local Lipschitz property for ${}^\mu\varphi_{Q'}$ (unlike ${}^p\varphi_Q$ in Th. 2) but just the formulated weaker upper Lipschitz property. This is confirmed by the following counter-example even in case of convex-like original data ($(\mu, H, X, p) \in \mathcal{D}^c$, g convex):

Example 1 *In problem* (P) *let* $m = s = 1$, $p = 0.5$, $g(x) = x$, $X = \mathbb{R}$ *and* $H(x) = (-\infty, x]$. *We define the probability measure* μ *along with two sequences of perturbed* *probability measures ν_n, $\tilde{\nu}_n$ via the following distribution functions (recall that $\nu \in \mathcal{P}(\mathbb{R}^s)$ is uniquely defined by its distribution function $F_\nu(z) = \nu(H(z))$):*

$$\begin{aligned}
F_\mu(x) &= \max\{0, \min\{x + 0.5, 1\}\}; \\
F_{\nu_n}(x) &= \begin{cases} F_\mu(x) & x \leq 0 \\ 0.5 & x \in [0, n^{-1}] \\ x + 0.5 - n^{-1} & x \in [n^{-1}, n^{-1} + 0.5] \\ 1 & x \geq n^{-1} + 0.5 \end{cases}; \\
F_{\tilde{\nu}_n}(x) &= \begin{cases} F_\mu(x) & x \leq -n^{-2} \\ 0.5 + (x - n^{-1})/(n+1) & x \in [-n^{-2}, n^{-1}] \\ F_{\nu_n}(x) & x \geq n^{-1} \end{cases}
\end{aligned}$$

Clearly, $(\mu, H, X, p) \in \mathcal{D}^c$ (note that H has convex graph and that $\mu \in \mathcal{P}^c(\mathbb{R}^s)$ as the uniform distribution over the interval $[-0.5, 0.5]$). The original and perturbed constraint sets are given by

$$\begin{aligned}
{}^\mu\Phi(\mu) &= \{x | F_\mu(x) \geq 0.5\} = [0, \infty) = \{x | F_{\nu_n}(x) \geq 0.5\} = {}^\mu\Phi(\nu_n) \quad \forall n \in \mathbb{N}; \\
{}^\mu\Phi(\tilde{\nu}_n) &= \{x | F_{\tilde{\nu}_n}(x) \geq 0.5\} = [n^{-1}, \infty) \quad \forall n \in \mathbb{N}.
\end{aligned}$$

Consequently, ${}^\mu\Psi(\mu) = \{0\}$ and, no matter how small the open neighbourhood Q' of 0 is chosen (compare Theorem 4), one has ${}^\mu\varphi_{Q'}(\mu) = {}^\mu\varphi_{Q'}(\nu_n) = 0$ and ${}^\mu\varphi_{Q'}(\tilde{\nu}_n) = n^{-1}$ (for large n). Furthermore, due to $\mu(H(1)) = F_\mu(1) = 1 > p$, condition (3) is satisfied, which guarantees assumption 1. in Theorem 4. Summarizing, assumptions 1.-3. of Theorem 4 are satisfied. On the other hand, for the particular choice of the mapping H in this example, the collection $\mathcal{B}$ in (1) reduces to its first part. As a consequence, $\alpha_\mathcal{B}$ becomes the Kolmogorov distance

$$\alpha_\mathcal{B}(\nu, \nu') = \sup_{z \in \mathbb{R}^s} \left| \nu(z + \mathbb{R}^s_-) - \nu'(z + \mathbb{R}^s_-) \right| = \sup_{z \in \mathbb{R}^s} \left| F_\nu(z) - F_{\nu'}(z) \right|.$$

For the data in this example, one easily checks that the maximum deviation between F_μ and both of $F_{\tilde{\nu}_n}$ and F_{ν_n} is realized at $z = n^{-1}$, whereas the maximum deviation between $F_{\tilde{\nu}_n}$ and F_{ν_n} is realized at $z = 0$. Accordingly, one calculates

$$\alpha_\mathcal{B}(\nu_n, \mu) = \alpha_\mathcal{B}(\tilde{\nu}_n, \mu) = n^{-1}; \quad \alpha_\mathcal{B}(\nu_n, \tilde{\nu}_n) = [n(n+1)]^{-1},$$

hence $\nu_n, \tilde{\nu}_n \to \mu$ but $|{}^\mu\varphi_{Q'}(\nu_n) - {}^\mu\varphi_{Q'}(\tilde{\nu}_n)| = n^{-1} = (n+1)\, \alpha_\mathcal{B}(\nu_n, \tilde{\nu}_n)$. This means that ${}^\mu\varphi_{Q'}$ cannot be locally Lipschitzian at μ, although it is upper Lipschitzian at μ according to Theorem 4.

A stability result for convex data, where localizations can be ignored similar to Proposition 3, is (cf. [3], Th. 3.1):

Proposition 5 *In problem (P), let $(\mu, H, X, p) \in \mathcal{D}^c$ and g be convex. If the unperturbed solution set $\Psi(\mu, H, X, p)$ is nonempty and bounded and if condition (3) is satisfied, then, at μ, ${}^\mu\Psi$ is upper semicontinuous and ${}^\mu\varphi$ is upper Lipschitzian.*

Note that, although for the original probability measure we have the convexity requirement $\mu \in \mathcal{P}^c(\mathbb{R}^s)$, the upper semicontinuity of ${}^\mu\Psi$ relates to arbitrary perturbed probability measures $\nu \in \mathcal{P}(\mathbb{R}^s)$ here. This is important in practical applications, where the original measure μ is frequently known to be $r-$ concave for some $r < 0$ whereas its approximations (based on empirical or Kernel estimates) definitly lack this property.

3.3 Stability with respect to the deterministic constraint set

A stability analysis of problem (P) with respect to variations of the deterministic constraint set X turns out to be more restricted than in the previously discussed cases. First of all, in contrast to the previous results, stability of the constraint set mapping can no longer be reduced to stability with respect to perturbations of the right-hand side . More precisely, the following example shows that metric regularity of $({}^p\Phi)^{-1}$ does not imply closedness of ${}^X\Psi$ (whereas it implies closedness of ${}^p\Psi$ and ${}^\mu\Psi$, see Theorems 4 and 2).

Example 2 *In (P) set $m = s = 1$, $g(x) = x$, $X = \{0,1\}$, $p = 0.5$, $\mu =$ uniform distribution on $[0,1]$ and define H via $\operatorname{Gph} H = [0,1]^2$. Clearly, the unique solution of (P) is given by $\Psi(\mu, H, X, p) = {}^X\Psi(X) = \{0\}$. Since ${}^p\Phi(q) = \{0,1\}$ for q close to p (i.e., ${}^p\Phi$ is locally constant), $({}^p\Phi)^{-1}$ must be metrically regular at $(0,p)$, hence assumption 1. of Theorems 2 and 4 is fulfilled. On the other hand, defining $X_n := \{-n^{-1}, 1\}$ it is clear that $X_n \to X$ and that ${}^X\Psi(X_n) = {}^X\Phi(X_n) = \{1\}$, hence ${}^X\Psi$ is not closed at X (nor is ${}^X\varphi$ upper semicontinuous at X).*

The example suggests that it is difficult to find verifiable conditions for stability w.r.t. perturbations of X if X itself is an arbitrary closed set even if H and μ have convexity properties ($H \in \mathcal{M}^c(\mathbb{R}^m, \mathbb{R}^s)$, $\mu \in \mathcal{P}^c(\mathbb{R}^s)$). A slight modification of the example ($X := [0,1]$, $\operatorname{Gph} H := (\{0\} \times [0,1]) \cup (\{1\} \times [0,1])$, $X_n := [n^{-1}, 1]$) shows that the convexity of $\operatorname{Gph} H$ cannot be dispensed with either when expecting stability w.r.t. X. Note, that in this modified example $X \in \mathcal{F}^c(\mathbb{R}^m), \mu \in \mathcal{P}^c(\mathbb{R}^s)$ and $({}^p\Phi)^{-1}$ is again metrically regular at $(0,p)$. Furthermore, μ has to satisfy a convexity property as well, as is shown by the following example, where both X and H do satisfy the convexity requirements.

Example 3 *In (P) set $m = s = 1$, $g(x) = x$, $X = [0,1]$, $p = 1/4$, $\mu = (\delta_0 + \delta_1)/2$ (with $\delta_x =$ Dirac measure on $x \in \mathbb{R}$) and define H via*

$$\operatorname{Gph} H = \operatorname{conv}\{(0,0), (1,0.5), (1,1), (0,0.5)\}.$$

Then, $X \in \mathcal{F}^c(\mathbb{R}), H \in \mathcal{M}^c(\mathbb{R},\mathbb{R})$ but $\mu \notin \mathcal{P}^c(\mathbb{R})$ (μ is not r-concave for any $r < 0$). Elementary calculation shows that ${}^X\Phi(X) = \{0,1\}$ and ${}^X\Psi(X) = \{0\}$. With $X_n := [n^{-1}, 1]$, one has $X_n \to X$ and ${}^X\Psi(X_n) = {}^X\Phi(X_n) = \{1\}$, hence ${}^X\Psi$ is not closed at X (nor is ${}^X\varphi$ upper semicontinuous at X). On the other hand, $({}^p\Phi)^{-1}$ is metrically regular at $(0,p)$ with the same reason as in Example 2.

The following Theorem confirms that the desired stability results - even w.r.t. non-convex perturbations of X - are available in case that the original problem is a completely convex one. This is parallel to the statement concerning μ in Proposition 5. However, the constraint qualification (3) has to be strengthened.

Theorem 6 *In problem (P) assume that:*

1. $(\mu, H, X, p) \in \mathcal{D}^c$.

2. μ has a density.

3. There exists some $\hat{x} \in \operatorname{int}\operatorname{dom} H \cap X$ with $\mu(H(\hat{x})) > p$.

Then, ${}^X\Psi$ is closed at X and ${}^X\varphi$ is upper semicontinuous at X.

4. In addition, $\emptyset \neq \Psi(\mu, H, X, p) \subseteq Q$ for some bounded open $Q \subseteq \mathbb{R}^m$.

Then, ${}^X\Psi_Q$ is upper semicontinuous at X and ${}^X\varphi_Q$ is continuous at X.

5. In addition, g is convex.

Then, the restrictions ${}^X\Psi|\mathcal{F}^c(\mathbb{R}^m)$ and ${}^X\varphi|\mathcal{F}^c(\mathbb{R}^m)$ of ${}^X\Psi$ and ${}^X\varphi$ to convex perturbations of X are upper semicontinuous and continuous, respectively, at X (without localization).

Proof. First, we show that ${}^X\Phi$ is lower semicontinuous at X. If it were not, then there would exist a sequence $\mathcal{F}(\mathbb{R}^m) \ni X_n \to X$ along with an open set V such that ${}^X\Phi(X) \cap V \neq \emptyset$, but ${}^X\Phi(X_n) \cap V = \emptyset$ for all n. Rephrasing the last relation, gives

$$x \notin X_n \quad \text{for all } n \text{ and all } x \in V \text{ with } \mu(H(x)) \geq p. \tag{4}$$

Choose some $x^0 \in {}^X\Phi(X) \cap V$. Assumption 1. implies $\mu^r(H(\cdot))$ to be convex, where $r < 0$ refers to the modulus of r-concavity from $\mu \in \mathcal{P}^c$ (see (2)). Consequently, for $x_\lambda := \lambda\hat{x} + (1-\lambda)x^0$ and $\lambda \in (0,1]$, it holds that $\mu(H(x_\lambda)) > p$. Furthermore, since $x^0 \in {}^X\Phi(X)$, we have $x^0 \in X$ and $x^0 \in \operatorname{dom} H$ (otherwise the contradiction $0 < p \leq \mu(H(x^0)) = \mu(\emptyset) = 0$). After fixing some small enough $\lambda > 0$, one has $x_\lambda \in \operatorname{int}\operatorname{dom} H \cap X \cap V$ with $\mu(H(x_\lambda)) > p$ by convexity of $\operatorname{dom} H$. Now, the relation $x_\lambda \in \operatorname{int}\operatorname{dom} H$ implies H to be lower semicontinuous at x_λ (cf. [9], Th. 5.9), so $H(x_\lambda) \subseteq \operatorname{Liminf}_n H(x_n)$ for any sequence $x_n \to x_\lambda$. Now, (8) in Lemma 9 provides $\liminf \mu(H(x_n)) \geq \mu(H(x_\lambda)) > p$. In other words, since $x_n \to x_\lambda$ was arbitrary, one derives that $\mu(H(x)) > p$ for all x in an open ball around x_λ with some radius $\varepsilon > 0$ chosen small enough such that the open ball is still contained in V. But then, (4) leads to $d(x_\lambda, X_n) \geq \varepsilon > 0$ contradicting $x_\lambda \in X$ and $X_n \to X$. So, ${}^X\Phi$ is lower semicontinuous at X. Now, in Theorem 11 in the appendix (statement 1.), put $f := g$, $\Lambda := \mathcal{F}(\mathbb{R}^m)$, $\lambda_0 := X$, $M := {}^X\Phi$ (note that M is closed at λ_0 by Prop. 1) in order to verify the statement under assumption 3.

Next, select some $x^* \in \Psi(\mu, H, X, p)$ according to assumption 4. and let $\mathcal{F}(\mathbb{R}^m) \ni X_n \to X$ be an arbitrary sequence. Due to $x^* \in {}^X\Phi(X)$, the lower semicontinuity of ${}^X\Phi$ at X guarantees the existence of a sequence $x_n \to x^*$ with $x_n \in {}^X\Phi(X_n)$ and moreover, by assumption 4., with $x_n \in Q$. Denoting by M^Q the constant multifunction $M^Q(Y) \equiv \operatorname{cl} Q$, it follows that

$$x^* \in \Psi(\mu, H, X, p) \cap \operatorname*{Liminf}_{Y \to X} \left[{}^X\Phi(Y) \cap M^Q(Y)\right] .$$

Putting $f := g$, $\Lambda := \mathcal{F}(\mathbb{R}^m)$, $\lambda_0 := X$, $M := {}^X\Phi \cap M^Q$ and noting that M is closed at λ_0, we deduce from Theorem 11 (statement 1.) the assertion under assumption 4. Finally, with $f := g$, $\Lambda := \mathcal{F}^c(\mathbb{R}^m)$, $\lambda_0 := X$, $M := {}^X\Phi$, the second statement in Theorem 11 yields the last assertion of the Theorem. ■

The following lemma provides a constraint qualification alternative to 3. in Theorem 6 without requiring a density for the probability measure. Its application, however, restricts to convex perturbations of X from the very beginning.

Lemma 7 *In problem (P) let $(\mu, H, X, p) \in \mathcal{D}^c$ and assume that:*

1. $(\mu, H, X, p) \in \mathcal{D}^c$.

2. There exists some $\hat{x} \in \operatorname{int} X$ with $\mu(H(\hat{x})) \geq p$.

Then, ${}^X\Psi|\mathcal{F}^c(\mathbb{R}^m)$ is closed at X and ${}^X\varphi|\mathcal{F}^c(\mathbb{R}^m)$ is upper semicontinuous at X.

3. In addition, $\emptyset \neq \Psi(\mu, H, X, p) \subseteq Q$ for some bounded open $Q \subseteq \mathbb{R}^m$.

Then, ${}^X\Psi_Q|\mathcal{F}^c(\mathbb{R}^m)$ is upper semicontinuous at X and ${}^X\varphi_Q|\mathcal{F}^c(\mathbb{R}^m)$ is continuous at X.

4. In addition, g is convex.

Then, ${}^X\Psi|\mathcal{F}^c(\mathbb{R}^m)$ and ${}^X\varphi|\mathcal{F}^c(\mathbb{R}^m)$ are upper semicontinuous and continuous, respectively, at X (without localization).

Proof. All one has to show is lower semicontinuity of ${}^X\Phi|\mathcal{F}^c(\mathbb{R}^m)$ at X since the rest of the argumentation is identical to that in the proof of Theorem 6. Now, violation of that lower semicontinuity amounts to the existence of a sequence $\mathcal{F}^c(\mathbb{R}^m) \ni X_n \to X$ along with an open set V such that (4) holds true. We proceed in an analogous way as in the proof of Theorem 6 to find some $x_\lambda \in \operatorname{int} X \cap V$ with $\mu(H(x_\lambda)) \geq p$ on the basis of assumption 2. in this lemma. Now, the X_n being convex (in contrast to Theorem 6), relation (7) in the appendix may be invoked to show that $x_\lambda \in X_n$ for large enough n. This, however, is in contradiction to (4). ■

The next example illustrates why the constraint qualification 2. in Lemma 7 is not sufficient in order to guarantee stability with respect to non-convex perturbations of X:

Example 4 *In (P), let $m = 2, s = 1, g(x, y) = -x, X = \mathbb{R}^2, p = 0.5, \mu =$ uniform distribution on $[0, 1]$ and define H via $\operatorname{Gph} H = [0, 1] \times \{0\} \times [0, 1]$. Then,*

$(\mu, H, X, p) \in \mathcal{D}^c$, ${}^X\Phi(X) = [0,1] \times \{0\}$, ${}^X\Psi(X) = \{(1,0)\}$ *and* ${}^X\varphi(X) = -1$. *Taking* $\hat{x} = (0,0)$, *all assumptions of Lemma 7 are satisfied. However, with the non-convex perturbations* $X_n := \{(x,y) \in \mathbb{R}^2 | x \leq n\,|y|\}$ *one has* $X_n \to X$ *and* ${}^X\Phi(X_n) = {}^X\Psi(X_n) = \{(0,0)\}$, ${}^X\varphi(X_n) = 0$, *hence* ${}^X\Psi(X_n)$ *fails to be closed at* X *and* ${}^X\varphi$ *fails to be upper semicontinuous at* X.

Note, that in this example, the constraint qualification 2. of Lemma 7 is satisfied even with strict inequality and, furthermore, μ has even a density. This underlines the necessity of $\hat{x}$ belonging to $\operatorname{int}\operatorname{dom} H$ (see constraint qualification 3. in Theorem 6), as soon as one is interested in stability w.r.t. non-convex perturbations of X (note that $\operatorname{int}\operatorname{dom} H = \emptyset$ in Example 4). Another example demonstrates why μ has to have a density in the context of Theorem 6.

Example 5 *In* (P), *let* $m = 2, s = 1, g(x,y) = -x, X = [0,1] \times \{0\}, p = 0.5, \mu =$ *Dirac measure on the point* $1 \in \mathbb{R}$ *and define* H *via*

$$\operatorname{Gph} H = \operatorname{conv}\{(0,-1,0),(1,-1,0),(1,0,1),(1,1,0),(0,1,0),0,0,1)\}.$$

One easily verifies that μ *is r-concave for any* $r < 0$, *hence* $(\mu, H, X, p) \in \mathcal{D}^c$. *Furthermore,* ${}^X\Phi(X) = [0,1] \times \{0\}$, ${}^X\Psi(X) = \{(1,0)\}$ *and* ${}^X\varphi(X) = -1$. *Taking* $\hat{x} = (0.5, 0.5) \in \operatorname{int}\operatorname{dom} H$, *all assumptions of Theorem 6 except 2. are satisfied. Now, with* $X_n := \operatorname{conv}\{(0,0),(1,n^{-1})\}$, *one has* $X_n \to X$ *and* ${}^X\Phi(X_n) = {}^X\Psi(X_n) = \{(0,0)\}$, ${}^X\varphi(X_n) = 0$, *hence* ${}^X\Psi(X_n)$ *fails to be closed at* X *and* ${}^X\varphi$ *fails to be upper semicontinuous at* X.

In the last example, the perturbations of X have even been convex, so the failure of stability illustrates at the same time the necessity of $\hat{x}$ belonging to $\operatorname{int} X$ in the constraint qualification 2. of Lemma 7 (note that $\operatorname{int} X = \emptyset$ in Example 5).

3.4 Stability with respect to the random set mapping

In contrast to the previous sections, for a stability analysis relating to the random set mapping H, there is no chance to arrive at results for nonconvex perturbations under reasonable assumptions. This will be seen in Example 9 below. Therefore, the following theorem relates to the restrictions of the mappings Ψ and φ to the space $\mathcal{M}^c(\mathbb{R}^m, \mathbb{R}^s)$ of multifunctions with closed and convex graph from the very beginning.

Theorem 8 *In problem* (P) *assume the following conditions:*

1. $(\mu, H, X, p) \in \mathcal{D}^c$. *The solution set of* (P) *is nonempty:* $\Psi(\mu, H, X, p) \neq \emptyset$.

2. μ *has a density.*

3. *There exists some* $\hat{x} \in \operatorname{int}\operatorname{dom} H \cap X$ *with* $\mu(H(\hat{x})) > p$.

Then, at H, ${}^{H}\Psi|\mathcal{M}^c(\mathbb{R}^m,\mathbb{R}^s)$ is closed and ${}^{H}\varphi|\mathcal{M}^c(\mathbb{R}^m,\mathbb{R}^s)$ is upper semicontinuous.

4. In addition, $\emptyset \neq \Psi(\mu, H, X, p) \subseteq Q$ for some bounded open $Q \subseteq \mathbb{R}^m$.

Then, at H, ${}^{H}\Psi_Q|\mathcal{M}^c(\mathbb{R}^m,\mathbb{R}^s)$ is upper semicontinuous and ${}^{H}\varphi_Q|\mathcal{M}^c(\mathbb{R}^m,\mathbb{R}^s)$ is continuous.

5. In addition, g is convex.

Then, ${}^{H}\Psi|\mathcal{M}^c(\mathbb{R}^m,\mathbb{R}^s)$ and ${}^{H}\varphi|\mathcal{M}^c(\mathbb{R}^m,\mathbb{R}^s)$ are upper semicontinuous and continuous, respectively, at H (without localization).

Proof. We just have to verify that ${}^{H}\Phi|\mathcal{M}^c(\mathbb{R}^m,\mathbb{R}^s)$ is lower semicontinuous at H, since the rest of argumentation is completely analogous to the proof of Theorem 6 after having shown lower semicontinuity of the mapping ${}^{X}\Phi$ there. For brevity, we put ${}^{H}\Phi^* := {}^{H}\Phi|\mathcal{M}^c(\mathbb{R}^m,\mathbb{R}^s)$. If ${}^{H}\Phi^*$ were not lower semicontinuous at H, then there would exist a sequence $\mathcal{M}^c(\mathbb{R}^m,\mathbb{R}^s) \ni H_n \to H$ and an open set V such that ${}^{H}\Phi^*(H) \cap V \neq \emptyset$, but ${}^{H}\Phi^*(H_n) \cap V = \emptyset \, \forall n \in \mathbb{N}$. Let $x^0 \in {}^{H}\Phi^*(H) \cap V$. Exactly in the same way as in the proof of Theorem 6, one derives, for small enough $\lambda > 0$, the existence of some $x_\lambda := \lambda\hat{x} + (1-\lambda)x^0$ with $\mu(H(x_\lambda)) > p$ and $x_\lambda \in \operatorname{int}\operatorname{dom} H \cap X \cap V$. If ${}^{H}\Phi^*$ violates lower semicontinuity at H, then

$$\mu(H_n(x_\lambda)) < p \quad \forall n \in \mathbb{N}. \tag{5}$$

Now, (10) in Proposition 10 (see appendix) yields

$$H(x_\lambda) \subseteq \left(\operatorname*{Liminf}_n H_n\right)(x_\lambda) \subseteq \operatorname*{Liminf}_n H_n(x_\lambda),$$

and (8) in Lemma 9 (see appendix) gives with (5) the contradiction

$$\mu(H(x_\lambda)) \leq \liminf_n \mu(H_n(x_\lambda)) \leq p.$$

■

The following examples shall illustrate the (independent) necessity of the first three assumptions in Theorem 8. Concerning the first assumption, slight modifications of Example 2 ($\operatorname{Gph} H_n := [n^{-1}, 1] \times [0,1]$ on the one and $X := [0,1], \operatorname{Gph} H := (\{0\} \times [0,1]) \cup ([0.5,1] \times [0,1]), \operatorname{Gph} H_n := (\{-n^{-1}\} \times [0,1]) \cup ([0.5,1] \times [0,1])$ on the other hand) confirm that violating convexity of X or $\operatorname{Gph} H$ (while satisfying all the respectively remaining assumptions of Theorem 8) destroys stability. The following example shows that the same holds true for the convexity assumption $\mu \in \mathcal{P}^c(\mathbb{R}^s)$:

Example 6 *In problem (P) let m, s, p and g as in Example 2. We define $X = [0,3]$, $\operatorname{Gph} H := \operatorname{conv}\{(0,0); (3,2); (3,3); (0,1)\}$ and μ as the one-dimensional probability measure induced by the density*

$$f(x) = \begin{cases} 0.5 & \text{if } x \in [0,1] \cup [2,3] \\ 0 & \text{else} \end{cases}.$$

One easily calculates that ${}^{H}\Phi(H) = \{0,3\}$ *and* ${}^{H}\Psi(H) = \{0\}$. *Now, all assumptions of Theorem 8 are met with the exception that* μ *fails to be* $r-$ *concave for some* $r < 0$. *Defining*

$$\operatorname{Gph} H_n := \operatorname{conv}\{(n^{-1}, n^{-1}); (3,2); (3,3); (n^{-1}, 1 + n^{-1})\},$$

one verifies that $H_n \to H$ *and* ${}^{H}\Phi(H_n) = \{3\}$ *(e.g.,* $\mu(H_n(n^{-1})) = 0.5 \cdot (1 - n^{-1}) < p$*), hence* ${}^{H}\Psi(H_n) = \{3\}$ *and* ${}^{H}\Psi|\mathcal{M}^c(\mathbb{R}^m, \mathbb{R}^s)$ *cannot be closed at* H *and* ${}^{H}\varphi|\mathcal{M}^c(\mathbb{R}^m, \mathbb{R}^s)$ *is not upper semicontinuous at* H *either.*

The next example demonstrates that assumption 2. cannot be dispensed with:

Example 7 *In problem* (P) *let* m, s, g, X, p *be given as in Example 2, but now define* H *by* $\operatorname{Gph} H = [0,1]^2$ *and* μ *as the Dirac measure on the point* $1 \in \mathbb{R}$. *Then, all assumptions of Theorem 8 are met (for 3. take* $\hat{x} := 0.5$*) with the exception of 2. Furthermore,* ${}^{H}\Phi(H) = [0,1]$, *hence* ${}^{H}\Psi(H) = \{0\}$. *Defining* H_n *via* $\operatorname{Gph} H_n := \operatorname{conv}\{(0,0); (1,0); (1,1); (0, 1-n^{-1})\}$, *one verifies that* $H_n \to H$ *and that* $\mu(H_n(1)) = 1$ *but* $\mu(H_n(x)) = 0$ *for all* $x \neq 1$. *As a consequence, one gets* ${}^{H}\Phi(H_n) = {}^{H}\Psi(H_n) = \{1\}$, *hence* ${}^{H}\Psi|\mathcal{M}^c(\mathbb{R}^m, \mathbb{R}^s)$ *is not closed at* H. *Similarly,* ${}^{H}\varphi|\mathcal{M}^c(\mathbb{R}^m, \mathbb{R}^s)$ *is not upper semicontinuous at* H.

Another example highlights the role of constraint qualification 3. At the same time it (negatively) answers the question whether the alternative constraint qualification 2. of Lemma 7 could be sufficient in order to derive stability w.r.t. to convex perturbations as it was the case for the deterministic constraint set in the previous section. It turns out that even strengthening this constraint qualification towards strict inequality and insisting on μ having a density (which was not required in Lemma 7) does not yield the desired result.

Example 8 *In problem* (P) *let* $m = 2, s = 1, g(x,y) = y, p = 3/4, X = \mathbb{R}^2, \operatorname{Gph} H = \{0\} \times [0,1] \times [0,1]$ *and* $\mu =$ *uniform distribution on* $[0,1]$. *Then, all assumptions of Theorem 8 are met except 3. since* $\operatorname{int} \operatorname{dom} H = \emptyset$. *One even has*

$$\mu(H((0,0))) = \mu([0,1]) = 1 > p,$$

hence condition 2. of Lemma 7 is strictly satisfied. On the other hand,

$${}^{H}\Phi(H) = \{0\} \times [0,1], {}^{H}\Psi(H) = \{(0,0)\}, {}^{H}\varphi(H) = 0,$$

and, defining H_n *via*

$$\begin{aligned} \operatorname{Gph} H_n &:= \operatorname{conv}\{(0,1,0), (0,1,1), (n^{-1}, 0, 0), (n^{-1}, 0, 1/2), \\ &\quad (-n^{-1}, 0, 1/2), (-n^{-1}, 0, 1)\}, \end{aligned}$$

one gets $H_n \to H$, ${}^{H}\varphi(H_n) = 0.5$ *and*

$$\begin{aligned} {}^{H}\Phi(H_n) &= \operatorname{conv}\{(-(2n)^{-1}, 1/2), ((2n)^{-1}, 1/2), (0,1)\}, \\ {}^{H}\Psi(H_n) &= \left[-(2n)^{-1}, (2n)^{-1}\right] \times \{1/2\}. \end{aligned}$$

Summarizing, no stability results for ${}^{H}\Psi|\mathcal{M}^c(\mathbb{R}^m, \mathbb{R}^s)$ *and* ${}^{H}\varphi|\mathcal{M}^c(\mathbb{R}^m, \mathbb{R}^s)$ *are available at* H.

Finally, motivated by Proposition 5 and Theorem 6, one might wonder if the assumptions of Theorem 8 are sufficient in order to derive stability of the mappings ${}^H\Psi$ and ${}^H\varphi$ themselves rather than of their restrictions ${}^H\Psi|\mathcal{M}^c(\mathbb{R}^m,\mathbb{R}^s)$ and ${}^H\varphi|\mathcal{M}^c(\mathbb{R}^m,\mathbb{R}^s)$. The answer is negative:

Example 9 *Let m, s, g, X and μ be given as in Example 2, set $p := 3/4$ and define H by $\operatorname{Gph} H = [0,1]^2$. Then, $(\mu, H, X, p) \in \mathcal{D}^c$, μ has a density and $\mu(H(0.5)) > p$ with $0.5 \in \operatorname{int}\operatorname{dom} H \cap X$. Clearly ${}^H\Phi(H) = [0,1]$ and ${}^H\Psi(H) = \{0\}$. Summarizing, all assumptions of Theorem 8 are fulfilled. Now, define the following closed subsets of $[0,1]$:*

$$A_n := \bigcup_{i=0}^{2^{n-1}-1} \left[\frac{2i}{2^n}, \frac{2i+1}{2^n}\right].$$

Then, $A_n \to [0,1]$ and $\mu(A_n) = 0.5$ (recall that, on the subsets of $[0,1]$, μ is identical to the Lebesgue measure). We set

$$\operatorname{Gph} H_n := (\{1\} \times [0,1]) \cup ([0,1] \times A_n).$$

Thus, $\mathcal{M}(\mathbb{R}^m,\mathbb{R}^s) \ni H_n \to H$. Furthermore, $\mu(H_n(1)) = 1$, but $\mu(H_n(x)) = 0.5$ for $x \in [0,1)$, so ${}^H\Phi(H_n) = {}^H\Psi(H_n) = \{1\}$. Consequently, due to the fact that $H_n \notin \mathcal{M}^c(\mathbb{R}^m,\mathbb{R}^s)$, ${}^H\Psi$ fails to be closed or upper semicontinuous at H and ${}^H\varphi$ fails to be upper semicontinuous at H. Of course, the corresponding properties do hold for the restrictions ${}^H\Psi|\mathcal{M}^c(\mathbb{R}^m,\mathbb{R}^s)$ and ${}^H\varphi|\mathcal{M}^c(\mathbb{R}^m,\mathbb{R}^s)$ according to Theorem 8.

4 Appendix

In this section we collect some known or easy to prove facts. The results of the following lemma are based on [6] (for (6), see Th. 3, for (7) see Cor. 8, for (8) see Lemma 1 and Proof of Th. 4).

Lemma 9 (Lucchetti,Salinetti,Wets) *Let $A_n (n \in \mathbb{N})$, $A \subseteq \mathbb{R}^s$ be closed with $\operatorname{Limsup}_n A_n \subseteq A$ and $\mu \in \mathcal{P}(\mathbb{R}^s)$ a probability measure. Then, one has*

$$\limsup_n \mu(A_n) \leq \mu(A). \tag{6}$$

Conversely, assume that the A_n and A are closed and convex but $\operatorname{Liminf}_n A_n \supseteq A$. Then,

$$\operatorname{int} A \subseteq \operatorname{int} \bigcup_{k\in\mathbb{N}} \bigcap_{n\geq k} A_n \tag{7}$$

If, in addition to the last assumptions, μ has a density, then it holds that

$$\liminf_n \mu(A_n) \geq \mu(A). \tag{8}$$

Proposition 10 *Let $G_n (n \in \mathbb{N})$, $G : \mathbb{R}^m \rightrightarrows \mathbb{R}^s$ be multifunctions with closed graph such that $G_n \to G$. Then one has*

$$\left(\operatorname*{Limsup}_n G_n\right)(x) = G(x) \quad \textit{for all } x \in \mathbb{R}^m. \tag{9}$$

If, in addition, the G_n and G have convex graph, then it holds that

$$\left(\operatorname*{Liminf}_n G_n\right)(x) \supseteq G(x) \quad \textit{for all } x \in \operatorname{int}\operatorname{dom} G \tag{10}$$

and

$$\operatorname{int}\operatorname{dom} G \subseteq \operatorname{int} \bigcup_{k\in\mathbb{N}} \bigcap_{n\geq k} \operatorname{dom} G_n \tag{11}$$

Proof. (9) follows immediately from the definitions. In order to verify (10), let $y \in G(x)$ and a sequence $x_n \to x$ be arbitrarily given. We have to show the existence of a sequence $y_n \in G_n(x_n)$ with $y_n \to y$. To this aim, we verify the following relation:

$$\forall m \in \mathbb{N}\ \exists n_m \in \mathbb{N}\ \forall n \geq n_m\ \exists z_n^m \in G_n(x_n) \cap B^0(y, m^{-1}).$$

So, let $m \in \mathbb{N}$ be arbitrary. By $x \in \operatorname{int}\operatorname{dom} G$ and due to convexity of Gph G, G is lower semicontinuous in x (cf. [9], Th. 5.9). Consequently, there is some $\delta > 0$, such that $G(w) \cap B^0(y, 2/m) \neq \emptyset\, \forall w \in B^0(x, \delta)$. We select points $w^1, ..., w^N \in B^0(x, \delta)$ with $x \in \operatorname{int}\operatorname{conv}\{w^1, ..., w^N\}$ as well as corresponding points $\alpha^1, ..., \alpha^N$ with $\alpha^i \in G(w^i) \cap B^0(y, 2/m)$. By continuity, there is some $\Delta > 0$, such that $x \in \operatorname{int}\operatorname{conv}\{v^1, ..., v^N\}$ for all $(v^1, ..., v^N)$ with $v^i \in B^0(w^i, \Delta)$. In view of $G_n \to G$ and $(w^i, \alpha^i) \in \operatorname{Gph} G$, for each $i \in \{1, ..., N\}$ there exists some $k_i \in \mathbb{N}$, such that $\operatorname{Gph} G_n \cap [B^0(w^i, \Delta) \times B^0(\alpha^i, 2/m)] \neq \emptyset\, \forall n \geq k_i$. Hence, there is some n_m such that for all $n \geq n_m$ we may find points (v_n^i, β_n^i) with $\beta_n^i \in G_n(v_n^i) \cap B^0(\alpha^i, 2/m)$ and $v_n^i \in B^0(w^i, \Delta)$ for $i = 1, ..., N$. Thus, $x_n \in \operatorname{conv}\{v_n^1, ..., v_n^N\}$ if $n \geq n_m$. Consequently, for such n there exist $\lambda_n^1, ..., \lambda_n^N \geq 0$ with $\lambda_n^1 + ... + \lambda_n^N = 1$, such that $x_n = \lambda_n^1 v_n^1 + ... + \lambda_n^N v_n^N$. We set $z_n^m := \lambda_n^1 \beta_n^1 + ... + \lambda_n^N \beta_n^N$. Since G_n has a convex Graph, one arrives at $z_n^m \in H_n(x_n)$. Furthermore, with $z^* := \lambda_n^1 \alpha^1 + ... + \lambda_n^N \alpha^N$ it holds that

$$\|z_n^m - y\| \leq \|z_n^m - z^*\| + \|z^* - y\| \leq \sum_{i=1}^{N} \lambda_n^i \left\|\beta_n^i - \alpha^i\right\| + 2/m \leq m^{-1},$$

which proves the intermediary assertion above. In this assertion, one may assume $n_m \leq n_{m+1}\, \forall m \in \mathbb{N}$ without loss of generality. Setting $y_n := z_n^m\, \forall n \in \{n_m, ..., n_{m+1}\}\, \forall m \in \mathbb{N}$, it follows that $y_n \in G(x_n)$ and $y_n \to y$, as was to be shown.

Finally, let us prove (11). Since by closedness and convexity of $\operatorname{Gph} G$ and $\operatorname{Gph} G_n$, the sets $\operatorname{dom} G$ and $\operatorname{dom} G_n$ are closed and convex as well, it suffices to verify, according to (7), the relation $\operatorname*{Liminf}_n \operatorname{dom} G_n \supseteq \operatorname{dom} G$. To this aim, consider an arbitrary $x \in \operatorname{dom} G$ and correspondingly select some $y \in G(x)$. Then, assuming without loss of generality, the distance on $\mathbb{R}^{m+s}$ to be based on the euclidean norm, we get

$$\begin{aligned} d(x, \operatorname{dom} G_n) &= \inf\{\|x - x'\| \,|x' \in \operatorname{dom} G_n\} \\ &\leq \inf\{\sqrt{\|x - x'\|^2 + \|y - y'\|^2} |x' \in \operatorname{dom} G_n,\, y' \in G_n(x)\} \\ &= d((x, y), \operatorname{Gph} G_n) \to 0, \end{aligned}$$

where the last convergence relies on $G_n \to G$ and $(x, y) \in \operatorname{Gph} G$. Thus, $x \in \operatorname*{Liminf}_n \operatorname{dom} G_n$, as was to be shown. ■

The following Theorem (cf. [1], Th. 4.2.1, Th. 4.2.2, Th. 4.3.3) collects some classical results of parametric programming in a simplified setting sufficient for our purposes:

Theorem 11 *In the parametric problem*

$$(P_\lambda) \quad \min\{f(x)|x \in M(\lambda)\} \quad (\lambda \in \Lambda),$$

let Λ be a metric space, $M : \Lambda \rightrightarrows \mathbb{R}^n$ a multifunction which is closed at $\lambda_0 \in \Lambda$ and $f : \mathbb{R}^n \to \mathbb{R}$ a continuous function. Denote by $\Psi : \Lambda \rightrightarrows \mathbb{R}^n$ and $\varphi : \Lambda \to \mathbb{R}$ the solution set mapping and optimal value function, respectively, associated with (P_λ). Then, the following statements hold true:

1. *If M is lower semicontinuous at λ_0 or, alternatively, $\Psi(\lambda_0) \cap \operatorname*{Liminf}_{\lambda \to \lambda_0} M(\lambda) \neq \emptyset$ is satisfied, then φ is upper semicontinuous at λ_0 and Ψ is closed at λ_0. If, moreover, $\emptyset \neq \Psi(\lambda) \subseteq K$ is fulfilled for some compact set $K \subseteq \mathbb{R}^n$ and all λ close to λ_0, then φ is continuous at λ_0 and Ψ is upper semicontinuous at λ_0.*

2. *If f is convex, $\Psi(\lambda_0)$ is nonempty and bounded and $M(\lambda)$ is convex for all $\lambda \in \Lambda$ as well as closed and lower semicontinuous at λ_0, then, at λ_0, the solution set mapping $\Psi : \Lambda \rightrightarrows \mathbb{R}^n$ is upper semicontinuous and the optimal value function $\varphi : \Lambda \to \mathbb{R}$ is continuous.*

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