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Piecewise polynomial collocation for the double layer potential equation over polyhedral boundaries
Part I: The wedge Part II: The cube

A. Rathsfeld[1]

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[1] Institut für Angewandte Analysis
und Stochastik
Hausvogteiplatz 5-7
D – O 1086 Berlin
Germany

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Herausgegeben vom
Institut für Angewandte Analysis und Stochastik
Hausvogteiplatz 5-7
D – O 1086 Berlin

Fax: + 49 30 2004975
e-Mail (X.400): c=de;a=dbp;p=iaas-berlin;s=preprint
e-Mail (Internet): preprint@iaas-berlin.dbp.de

A Piecewise Polynomial Collocation Method for the Double Layer Potential Equation over the Wedge

A.Rathsfeld

Abstract

In this paper we consider a piecewise polynomial collocation method for the solution of the double layer potential equation corresponding to Laplace's equation in a three-dimensional wedge. We prove the stability for our method in case of special triangulations over the boundary.

Key words. potential equation, collocation

AMS(MOS) subject classification. 45L10, 65R20

1 Introduction

One popular method for the solution of boundary value problems for elliptic differential equations consists in the reduction to boundary integral equations. For instance, the Dirichlet problem for Laplace's equation in a bounded and simply connected polyhedron $\Omega \subseteq \mathbf{R}^3$ or the Neumann problem for the same equation on $\mathbf{R}^3 \setminus \Omega$ can be reduced to the second kind integral equation $Ax = y$ over the boundary $S := \partial\Omega$ (cf. e. g. [19]),where $A = I + 2W_S$ and

$$(1.1) \qquad W_S x(Q) := [1/2 - d_\Omega(Q)]x(Q) + \frac{1}{4\pi}\int_S \frac{n_P(Q-P)}{|P-Q|^3}x(P)d_P S,$$

$$d_\Omega(Q) := \lim_{\epsilon\to\infty} \frac{|\{P \in \Omega : |P-Q| < \epsilon\}|}{|\{P \in \mathbf{R}^3 : |P-Q| < \epsilon\}|}.$$

Here n_P denotes the unit vector of the interior normal to Ω at P and $|Z|$ is the Lebesgue measure of Z for any $Z \subseteq \mathbf{R}^3$.Note that, since the boundary S is not smooth, W_S is not compact. For the numerical solution of $Ax = (I + 2W_S)x = y$, various methods have been introduced. The first method was the so called panel method, i.e., piecewise constant collocation ([26] and cf.[16, 25, 3]). Kral and Wendland [18] (cf. also [2]) have shown that this method is stable for the case of certain rectangular domains Ω. Arbitrary polyhedral domains have been considered in [22]. Elschner [10] has analysed the Galerkin method with piecewise polynomial trial functions over arbitrary polyhedrons, and the Galerkin method together with an approximation of the Lipschitz boundary by smooth surfaces has been investigated by Dahlberg and Verchota [8]. For all these procedures, the question arises how to compute the entries of the discretized system of equations (cf. [26, 12, 24]). In order to avoid this problem, one can consider simple quadrature methods. In the papers [21, 20] Nyström methods have been analysed which are similar to those of Graham and Chandler [7], Kress [17], and Elschner [9] for the corresponding equation over polygonal boundaries. However, these quadrature methods improve the complexity only up to a certain order. The reason for this disadvantage is that the Nyström method works with one grid only. This one grid has to be adapted at once to all singularities of the kernel function

$$k(Q,P) := \frac{1}{4\pi}\frac{n_P(Q-P)}{|P-Q|^3}.$$

(Note that $k(Q,P)$ turns to infinity if $|P-Q|$ turns to 0 and P and Q lie on different faces of S.) Doing so, the number of grid points grows, and the complexity of the method cannot be reduced in the same manner as in the one-dimensional case (cf.[21, 20]).

In order to get a fully discretized numerical method which reduces the complexity similarly to the one-dimensional case ([7, 9, 17]), one needs quadrature methods, where the quadratures and the grids depend on the collocation points, in other words one needs certain discretized collocation methods. The first step in the analysis of such methods is the stability analysis of piecewise polynomial collocation. This has been done already for a piecewise constant or a piecewise linear ansatz (cf. [26, 18, 22]). For polynomials of higher degree, the stability of the collocation is still open. However, it is well known that the stability of the collocation method for the double layer equation over a polyhedron can be reduced to the stability for the case of polyhedral cones and the case of a wedge (cf.[22, 21]). In the present paper we shall prove the stability of a certain piecewise polynomial collocation procedure on a wedge. First, we introduce a space of one-dimensional spline functions and derive some properties of this space. In Section 3 we introduce a space of piecewise polynamials over the wedge using tensor products of the one-dimensional spline functions. This space is the space of ansatz functions for our collocation. Note that the underlying grid is a uniform or a certain graded one. In Section 4 we shall prove the L^2-stability of the collocation method introduced in Section 3.

Finally, let us mention that a further reduction of the complexity seems to be possible if the discretization scheme is combined with an iterative solution of the system of equations (cf. [13, 15, 4, 20]) or a fast method for the multiplication of the matrix by a vector (cf. [23, 11, 14, 1, 6, 5]).

2 The one-dimensional spline functions

Suppose we are given a grid $\triangle := \{t_k,\ k \in \mathbf{Z}\}$ satisfying $\ldots < t_k < t_{k+1} < t_{k+2} < \ldots$, and let d stand for a fixed positive integer. Then, by $IS^{2d+1}(\triangle)$ we denote the space of all piecewise polynomials φ such that:

a) The restriction of φ to the subinterval $[t_k, t_{k+1}]$ is equal to a polynomial p_k of degree less than $2d+2$.

b) For the $2d$ neighbouring grid points t_{k+j}, $j = -d, \ldots, d+1$, there holds $\varphi(t_{k+j}) = p_k(t_{k+j})$.

In other words, $IS^{2d+1}(\triangle)$ is nothing else than the image space of the local interpolation projection $K_\triangle$ (cf. property iii) below), where $K_\triangle f(t)$ for $t_k \le t \le t_{k+1}$ is defined by

$$(K_\triangle f)(t) := \sum_{j=-d}^{d+1} f(t_{k-j}) l_j(t),\ l_j(t) := \prod_{j \neq l = -d}^{d+1} \frac{(t - t_{k-l})}{(t_{k-j} - t_{k-l})}.$$

Let us suppose that $\triangle$ is a locally quasiuniform partition, i.e., that there is a constant $c_{lq} \ge 1$ with

$$c_{lq}^{-1} |t_k - t_{k-1}| \le |t_{k+1} - t_k| \le c_{lq} |t_k - t_{k-1}|,\ k \in \mathbf{Z}.$$

Furthermore, let $h_\triangle$ stand for the mesh width $\sup\{|t_k - t_{k-1}|, k \in \mathbf{Z}\}$ of $\triangle$. The following properties of $IS^{2d+1}(\triangle)$ and $K_\triangle$ are obvious:

P i). The space $IS^{2d+1}(\triangle)$ is a subset of the space of continuous function $C(\mathbf{R})$ over $\mathbf{R}$.

P ii). There is a constant $C > 0$ such that, for any $(2d+2)$ times continuously differentiable function f,

$$\|K_\triangle f - f\|_\infty \le C h_\triangle^{2d+2} \|f^{(2d+2)}\|_\infty.$$

P iii). If $\{\varphi_k\}$ is the interpolation base in $IS^{2d+1}(\triangle)$ determined by $\varphi_j(t_k) = \delta_{j,k}$, then the support of φ_k is contained in $[t_{k-d-1}, t_{k+d+1}]$ and $K_\triangle f = \sum_{k \in \mathbf{Z}} f(t_k) \varphi_k$.

P iv). If $t_k := k h_\triangle$, $k \in \mathbf{Z}$, then there holds $\varphi_k(t) = \varphi_0(t - k h_\triangle)$.

Here and in the following, by C we denote a generic constant the value of which varies from instance to instance. The properties iii) and iv) will play an important role in the stability proof in Sect.4. Let us shortly indicate the importance of iv). Suppose we are given an operator $A := I + T \in \mathcal{L}(C(\mathbf{R}))$ with $\|T\| \le q < 1$ and consider the collocation for A using $IS^{2d+1}(\triangle)$ as ansatz space and $\{t_k\}$ as collocation points. Then the discretized operator $A_\triangle$ of A takes the form $A_\triangle = K_\triangle A|_{IS^{2d+1}(\triangle)}$. We get $A_\triangle = Id + K_\triangle T|_{IS^{2d+1}(\triangle)}$, where $\|K_\triangle T|_{IS^{2d+1}(\triangle)}\| \leq q\|K_\triangle\|$. Hence, $A_\triangle$ is invertible and stable if only $\|K_\triangle\| = 1$. (Note that $A_\triangle$ is called stable if it is invertible for $h_\triangle$ small enough and if the inverse operators are uniformly bounded.) The last equation holds if and only if $d = 0$. For a higher degree of ansatz functions, we need another method of proving stability. In order to demonstrate this new method, let us suppose that T is given by

$$Tf(t) := \int_{\mathbf{R}} k(t-s)f(s)ds, \; \int_{\mathbf{R}} |k(s)|ds \leq q.$$

Then the matrix of $A_\triangle$ corresponding to the base $\{\varphi_k\}$ is the convolution matrix

$$A_\triangle = Id + (T\varphi_j(t_k))_{k,j\in\mathbf{Z}} = Id + (T\varphi_{j-k}(t_0))_{k,j\in\mathbf{Z}}$$

and its symbol function is

$$[0,2\pi) \ni \rho \mapsto \mathcal{A}(\rho) := 1 + \sum_{j\in\mathbf{Z}} \exp(ij\rho)T\varphi_j(t_0) = 1 + T\Big(\sum_{j\in\mathbf{Z}} \exp(ij\rho)\varphi_j\Big)(t_0).$$

Taking into account the next lemma, we arrive at

$$|\mathcal{A}(\rho)| \geq 1 - \|T\|_{\mathcal{L}(C(\mathbf{R}))}\|\sum_{j\in\mathbf{Z}} \exp(ij\rho)\varphi_j\|_\infty \geq 1 - q.$$

Hence, $A_\triangle$ together with $\mathcal{A}$ is invertible.

Lemma 2.1 *i) Suppose that* $\triangle = \{kh_\triangle | k \in \mathbf{Z}\}$. *Then, for any* $0 \leq \rho < 2\pi$, *there holds* $\|\sum_{j\in\mathbf{Z}} \exp(ij\rho)\varphi_j\|_\infty \leq 1$.

ii) Suppose $h_\triangle = 1$. *Then the Fourier transform* $\widehat{\varphi_0}$ *of* φ_0 *is given by*

$$\begin{aligned} \widehat{\varphi_0}(s) &= \frac{1}{\sqrt{2\pi}}\Big\{\frac{\sin s/2}{s/2}\Big\}^{2(d+1)} \sum_{j=0}^{d} \sigma_{j,d}s^{2j}, \\ \sigma_{j,d} &:= \begin{cases} 1 & \text{if } j=0, d=0, \\ \frac{[2(d-j)+1]!}{[2d+1]!^2} \sum_{1\leq l_1<l_2<\ldots<l_j\leq d} l_1^2 l_2^2 \ldots l_j^2 & \text{else.} \end{cases} \end{aligned}$$

PROOF. i) By a scaling argument we may suppose $h_\triangle = 1$. Using the inverse Fourier transform, we obtain

$$
\begin{aligned}
&\sum_{j\in\mathbf{Z}} \exp(ij\rho)\varphi_j(t)\\
= &\sum_{j\in\mathbf{Z}} \exp(ij\rho)\varphi_0(t-j) = \sum_{j\in\mathbf{Z}} \exp(ij\rho)\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}} \widehat{\varphi_0}(s)\exp\{-is(t-j)\}ds\\
= &\sum_{j\in\mathbf{Z}}\sum_{k\in\mathbf{Z}} \exp(ij\rho)\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \widehat{\varphi_0}(s+k2\pi)\exp\{-i(s+k2\pi)(t-j)\}ds\\
= &\sum_{j\in\mathbf{Z}} \exp(ij\rho)\frac{1}{2\pi}\int_0^{2\pi} \{\sqrt{2\pi}\exp(-ist)\sum_{k\in\mathbf{Z}}\widehat{\varphi_0}(s+k2\pi)\exp(-ik2\pi t)\}\exp(ijs)ds\\
= &\sqrt{2\pi}\exp\{-i(2\pi-\rho)t\}\sum_{k\in\mathbf{Z}}\widehat{\varphi_0}(2\pi-\rho+k2\pi)\exp(-ik2\pi t).
\end{aligned}
$$

Using $\sum \exp(ij\rho)\varphi_j(0) = 1$ and the last formula with $t = 0$, we may continue

$$
\begin{aligned}
\sum_{j\in\mathbf{Z}} \exp(ij\rho)\varphi_j(t) &= \frac{\sum_{j\in\mathbf{Z}} \exp(ij\rho)\varphi_j(t)}{1}\\
&= \frac{\sum_{k\in\mathbf{Z}} \widehat{\varphi_0}(2\pi-\rho+k2\pi)\exp\{-i(2\pi-\rho+k2\pi)t\}}{\sum_{k\in\mathbf{Z}} \widehat{\varphi_0}(2\pi-\rho+k2\pi)}.
\end{aligned}
$$

Together with $\widehat{\varphi_0} \geq 0$ this formula proves assertion i).

ii) Integration by parts leads to

$$
\begin{aligned}
\sqrt{2\pi}\widehat{\varphi_0}(s) &= \sum_{r=-d-1}^{d}\int_r^{r+1} exp(ist)\varphi_0(t)dt = \sum_{r=-d-1}^{d}\sum_{l=1}^{2d+2}(-1)^{l-1}\left[\frac{\exp(ist)}{(is)^l}\varphi_0^{(l-1)}(t)\right]_r^{r+1}\\
&= \sum_{l=0}^{2d+1}(-1)^{l+1}(is)^{-(l+1)}\sum_{r=-d-1}^{d+1}\exp(irs)[\varphi_0^{(l)}(t)]_{r-0}^{r+0}. \qquad (2.1)
\end{aligned}
$$

From the definition of φ_0 we know that the restrictions $p_{r-1} = \varphi_0|_{[r-1,r]}$ and $p_r = \varphi_0|_{[r,r+1]}$ are polynomials of degree less than $2(d+1)$ and that $p_r - p_{r-1}$ vanishes at the points $r-d,\ldots,r+d$. Thus $p_r(t) - p_{r-1}(t) = C_{d,r}\omega_d(t-r)$, where $C_{d,r}$ denotes a constant and $\omega_d(t) := t\prod_{j=1}^{d}(t^2-j^2)$. Setting $B_{d,l} := \omega_d^{(l)}(0)$, we get $[\varphi_0^{(l)}]_{r-0}^{r+0} = C_{d,r}B_{d,l}$, and (2.1) implies

$$
\sqrt{2\pi}\widehat{\varphi_0}(s) = \{\sum_{l=0}^{2d+1} B_{d,l}(-1)^{l+1}(is)^{-(l+1)}\}\{\sum_{r=-d-1}^{d+1} C_{d,r}\exp(irs)\}. \qquad (2.2)
$$

From

$$
\omega_d(t) = \sum_{l=0}^{2d+1}\frac{B_{d,l}}{l!}t^l = \sum_{l=0}^{d}\frac{B_{d,2l+1}}{(2l+1)!}t^{2l+1}
$$

we derive

$$(2.3)\qquad B_{d,2l+1}=\begin{cases}1 & if\ d=l=0\\ (2l+1)!(-1)^{d-l}\sum_{1\le l_1<l_2<\ldots<l_{d-l}\le d} l_1^2l_2^2\ldots l_{d-l}^2 & else.\end{cases}$$

On the other hand, we have

$$p_r^{(2d+1)}(r)-p_{r-1}^{(2d+1)}(r)=C_{d,r}\omega_d^{(2d+1)}(0)=C_{d,r}(2d+1)!.$$

Using the definition of p_r and p_{r-1}, we get

$$\begin{aligned}
C_{d,r}(2d+1)! &= \frac{\partial^{2d+1}}{\partial t^{2d+1}}\Big\{\prod_{-r\neq j=-d}^{d+1}(1-\frac{t}{j+r})-\prod_{-r+1\neq j=-d}^{d+1}(1-\frac{t}{j+r-1})\Big\}|_{t=r}\\
&= \prod_{-r\neq j=-d}^{d+1}\frac{-1}{j+r}-\prod_{-r+1\neq j=-d}^{d+1}\frac{-1}{j+r-1}\\
&= \prod_{-r\neq j=-d}^{d}\frac{1}{j+r}\{\frac{1}{r-d-1}-\frac{1}{r+d+1}\}\\
&= \frac{2(d+1)(-1)^{d-r+1}}{(d+1-r)!(d+1+r)!}=(-1)^{d-r+1}\frac{\binom{2(d+1)}{d+1+r}}{(2d+1)!},\\
C_{d,r} &= (-1)^{d-r+1}\frac{\binom{2(d+1)}{d+1+r}}{(2d+1)!^2}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sum_{r=-d-1}^{d+1}C_{d,r}\exp(irs) &= \frac{(-1)^{d+1}}{(2d+1)!^2}\sum_{r=-d-1}^{d+1}\binom{2(d+1)}{d+1+r}(-1)^r\exp(irs)\\
&= \frac{(-1)^{d+1}}{(2d+1)!^2}(1-\exp(is))^{2(d+1)}(-1)^{-(d+1)}\exp\{-i(d+1)s\}\\
&= \frac{(-1)^{d+1}}{(2d+1)!^2}\{2\sin(s/2)\}^{2(d+1)}.
\end{aligned}$$

From the last formula, (2.2), and (2.3) we get ii). □

Let us conclude this section with the definition of the corresponding spline space over the half-axis $\mathbf{R}_+$. However, we shall modify this space slightly near 0. Suppose we are given a grid $\Delta_+:=\{t_k:k=0,1,\ldots\}$ satisfying $0=t_0<t_1<\ldots,\ t_k\to\infty$, and let the positive integers d and i_* be fixed . Then, by $IS_{i_*}^{2d+1}(\Delta_+)$ we denote the space of all piecewise polynomials φ such that:

a) The restriction of φ to the subinterval $[0, t_{i_*}]$ is constant.

b) The restriction of φ to the subinterval $[t_k, t_{k+1}]$, $k = i_*, i_* + 1, \ldots$ is equal to a polynomial p_k of degree less than $2d + 2$.

c) For $k \geq i_* + d$ and the $2d$ neighbouring t_{k+j}, $j = -d, \ldots, d+1$, there holds $\varphi(t_{k+j}) = p_k(t_{k+j})$.

d) For $i_* \leq k < i_* + d$ and the $2d$ neighbouring grid points t_{k+j}, $j = -d, \ldots, d+1$, there holds $\varphi(t_{k+j}) = p_k(t_{k+j})$ for $k + j \geq i_*$ and $\varphi(t_{k+j}) = p_k(t_{i_*})$ for $k + j < i_*$.

If $\{\varphi_k\}$ is the base introduced at the beginning of this section, then a base of $IS_{i_*}^{2d+1}(\triangle_+)$ is given by

$$\varphi_0^+(t) := \begin{cases} 1 & if\ 0 \leq t \leq t_{i_*} \\ 0 & else \end{cases}, \quad \varphi_1^+(t) := \begin{cases} 0 & if\ 0 \leq t \leq t_{i_*} \\ \sum_{j=i_*-d}^{i_*} \varphi_j & else \end{cases},$$

and $\varphi_k^+ := \varphi_{k+i_*-1}$, $k = 2, 3, \ldots$. The interpolation projection onto $IS_{i_*}^{2d+1}(\triangle_+)$ is defined by $K_{\triangle_+} f := \sum_{k=0,1,\ldots} f(t_k^+)\varphi_k^+$, where $t_0^+ := 0$, $t_k^+ := t_{k+i_*-1}$, $k = 1, 2, \ldots$.

3 The collocation for the equation over the wedge

Let us consider the operator $A := I + 2W \in \mathcal{L}(L^2(S))$, where $W := W_S$ is taken from (1.1) and the wedge Ω and therewith its boundary $S := \partial\Omega$ is defined by

$$\Omega := \{(u, v, w) \in \mathbf{R}^3 : \ u \in \mathbf{R},\ v = r\cos\alpha,\ w = r\sin\alpha,\ 0 < r < \infty,\ 0 < \alpha < \gamma\}.$$

In order to set up a collocation method for the numerical solution of $Ax = y$, we need a space of ansatz functions and a set of collocation points. Since we shall look at this collocation as a model problem for the case of a bounded polyhedron, we shall introduce an infinite but countable ansatz space. First, we start with the definition of a partition and a function space over the half-plane $\mathbf{R}_+^2 := \{(s,t) \in \mathbf{R}^2,\ t > 0\}$. Suppose we have a fixed grid $\triangle_+ = \{t_k,\ k = 0, \ldots\}$ over $\mathbf{R}_+$. We assume that this grid is locally quasiuniform (cf. Sect.2) and satisfies

$$\sup\{\frac{t_{j+1} - t_j}{t_j},\ j = i_*, i_* + 1, \ldots\} \leq \epsilon, \tag{3.1}$$

where i_* is a positive integer and ϵ is a prescribed positive constant. Following Sect.2 we can introduce the base of spline functions $\{\varphi_j^+(t)\}$ over $\mathbf{R}_+$. Moreover, using the equidistant grid $\mathbf{Z}$, we can define the base of spline functions $\{\varphi_k(s)\}$ over $\mathbf{R}$. The space of tensor product splines over $\mathbf{R}_+^2$ is spanned by the set of functions $\{(s,t) \mapsto \varphi_k(s)\varphi_j^+(t)\}$. If t_j^+ is defined as in Sect.2, then the set of collocation points over $\mathbf{R}_+^2$ is given by $\{(k, t_j^+)\}$.

Now we introduce the functions and points over S with the help of affine mappings. We choose an $h>0$ and introduce $\psi_l : \mathbf{R}^2_+ \longrightarrow S$,

$$(s,t) \mapsto \psi_l(s,t) := \begin{cases} sh^{-1}(1,0,0) + th^{-1}(\cos\beta_2,\ \sin\beta_2 \sin\gamma,\ \sin\beta_2 \cos\gamma) & if\ l=2, \\ sh^{-1}(1,0,0) + th^{-1}(\cos\beta_1, \sin\beta_1, 0) & if\ l=1. \end{cases}$$

Then the set of collocation points over S is $\{\psi_l(k,t_j^+)\}$. Note that we could have set $\beta_1 = \beta_2 = \pi/2$. However, for a localized problem arising from a non-rectangular polyhedron, we need arbitrary $0<\beta_1,\beta_2<\pi$. Taking into account that $\psi_1(k,t_0^+) = \psi_2(k,t_0^+)$, we introduce the notation

$$\begin{aligned} I &:= \{\iota=(k,j,l) | j>0, k\in\mathbf{Z}, l=1,2; j=0, k\in\mathbf{Z}, l=1\}, \\ P_\iota &:= P_{k,j,l} := \psi_l(k,t_j^+), \\ \varphi_\iota(\psi_{l'}(s,t)) &:= \varphi_{k,j,l}(\psi_{l'}(s,t)) := \begin{cases} \varphi_k(s)\varphi_j^+(t) & if\ j>0\ and\ l=l' \\ & or\ if\ j=0 \\ 0 & else, \end{cases} \\ & \iota=(k,j,l)\in I,\ l=1,2,\ (s,t)\in\mathbf{R}^2_+. \end{aligned} \tag{3.2}$$

Obviously, there holds

$$\frac{1}{C}\|\sum_{\iota\in I}\xi_\iota\varphi_\iota\|_{L^2} \le h\{\sum_{\iota\in I}|\xi_\iota|^2|t_{j+1}^+ - t_j^+|\}^{1/2} \le C\|\sum_{\iota\in I}\xi_\iota\varphi_\iota\|_{L^2}. \tag{3.3}$$

Now we consider the collocation method,where the approximate solution $x_\triangle = \sum_{\iota\in I}\xi_\iota\varphi_\iota$ for the solution x of $Ax=y$ is determined by

$$Ax_\triangle(P_\kappa) = y(P_\kappa),\ \kappa\in I. \tag{3.4}$$

Theorem 3.1 *Suppose $A = I + 2W$, where W is the operator of (1.1) over the wedge with opening angle γ. Let the numbers β_1,β_2 in the definition of ψ_1,ψ_2 as well as the constant c_{lq} appearing in the property of local quasiuniformness be fixed. Then there is a small $\epsilon>0$ such that the collocation (3.4) with ansatz functions and collocation points from (3.2) is stable in L^2 if only (3.1) is satisfied.*

The matrix of the corresponding system of equations takes the form $A_\triangle := (A\varphi_\iota(P_\kappa))_{\kappa,\iota\in I}$. Using $Wf_h(P_h) = Wf(P)$ for $P=(x,y,z)$, $P_h=(hx,hy,hz)$, and the functions f and $f_h(x,y,z) := f(h^{-1}x,h^{-1}y,h^{-1}z)$, we observe that this matrix is independent of h. Since the matrix norm corresponding to

$$\|\{\xi_\iota\}\| = h\{\sum_{\iota\in I}|\xi_\iota|^2|t_{j+1}^+ - t_j^+|\}^{1/2}$$

(cf. (3.3)) is independent of h , too, we get that $A_\triangle$ is stable if and only if $A_\triangle$ is invertible for $h=1$. Thus without loss of generality we may assume $h=1$, and it will be enough to show the invertibility of $A_\triangle$. This will be done in the next section.

Finally, let us mention two special partitions $\triangle_+$ on $\mathbf{R}_+$. If $\alpha \geq 1$, $t_j := j^\alpha$, and $\epsilon > 0$, then (3.1) holds if only i_* is large enough. For $1 < q < 1+\epsilon$ and $t_j := q^j$, (3.1) is satisfied, too. The bound for the inverse of the corresponding collocation matrix will be independent of q.

4 The proof of Theorem 3.1

4.1.In the first step let us reduce the index set I. We set $I_e := \{\iota = (k,j,l) | j = 0\}$ and $I_r := I \setminus I_e$. Then the span of $\{\varphi_\iota, \iota \in I\}$ is the direct sum of the span of $\{\varphi_\iota, \iota \in I_e\}$ and of that of $\{\varphi_\iota, \iota \in I_r\}$. Corresponding to this splitting $A_\triangle$ takes the form

$$A_\triangle = \begin{pmatrix} A_\triangle^{e,e} & A_\triangle^{e,r} \\ A_\triangle^{r,e} & A_\triangle^{r,r} \end{pmatrix}, \; A_\triangle^{e,e} := (A\varphi_\iota(P_\kappa))_{\iota,\kappa\in I_e}, A_\triangle^{r,e} := (A\varphi_\iota(P_\kappa))_{\iota\in I_r,\kappa\in I_e}, \dots$$

We observe that $A_\triangle^{e,r} = 0$ and $A_\triangle^{e,e}$ is triangular, where the entries in the main diagonal are equal to $1 + 2\{1/2 - \gamma/2\pi\}$ (cf.(1.1)). Hence, for the invertibility of $A_\triangle$, it will be enough to prove that $A_\triangle^{r,e}$ and $A_\triangle^{r,r}$ are bounded and that $A_\triangle^{r,r}$ is invertible. Since the proof of the boundedness for $A_\triangle^{r,e}$ is analogous to that for $A_\triangle^{r,r}$, we shall only consider $A_\triangle^{r,r}$.

4.2.In the second step let us show the convolution structure of the matrix $A_\triangle^{r,r}$ and reduce the problem of invertibility to that for the symbol. If $J_r := \{\mu = (j,l) | j = 1,2,\ldots; l = 1,2\}$, then the vector $\xi := (\xi_\iota)_{\iota\in I_r}$ can be written as $\xi = (\xi_k)_{k\in\mathbf{Z}}$, where $\xi_k := (\xi_{k,\mu})_{\mu\in J_r}$. Now we observe $A\varphi_{k,j,l}(P_{k',j',l'}) = A\varphi_{k-k',j,l}(P_{0,j',l'})$ and conclude that $A_\triangle^{r,r}$ is a discrete convolution matrix with respect to the index k. We write

$$A_\triangle^{r,r} = (A_{\triangle,k-k'})_{k',k\in\mathbf{Z}}, \; A_{\triangle,k-k'} := (A\varphi_{k-k',\mu}(P_{0,\nu}))_{\nu,\mu\in J_r}.$$

Starting with W instead of A, we analogously define $W_\triangle^{r,r}$ and $W_{\triangle,k}$. Hence, $A_\triangle^{r,r} = Id + 2W_\triangle^{r,r}$ and $A_{\triangle,k} = Id + 2W_{\triangle,k}$, and it suffices to prove $\|W_\triangle^{r,r}\|_* < 1/2$, where the operator norm $\|\cdot\|_*$ is generated by the norm

$$\|\xi\|_* := \{\sum_{k\in\mathbf{Z}} \|\xi_k\|_*^2\}^{1/2}, \; \|\xi_k\|_* := \{\sum_{l=1}^{2} \sin\beta_l \| \sum_{j=1}^{\infty} \xi_{k,j,l}\varphi_j^+ \|_{L^2(\mathbf{R}_+)}^2\}^{1/2}$$

which is equivalent to the norms in (3.3). As it is well known, the norm of the convolution operator $W_\triangle^{r,r}$ can be estimated by the norm of its symbol:

$$\|W_\triangle^{r,r}\|_* \leq \sup_{0\leq\rho<2\pi} \|\mathcal{W}(\rho)\|_*, \; \mathcal{W}(\rho) := \sum_{k\in\mathbf{Z}} \exp(ik\rho)W_{\triangle,k}. \quad (4.1)$$

4.3.Now let us show that the symbol $\mathcal{W}(\rho)$ is a collocation operator for a one-dimensional equation. We set $\Gamma = \Gamma_1 \cup \Gamma_2$, where

$$\begin{aligned} \Gamma_1 &:= \{t(\cos\beta_1,\ \sin\beta_1,\ 0)|0 \le t < \infty\} = \{\psi_1(0,t)|0 \le t\}, \\ \Gamma_2 &:= \{t(\cos\beta_2,\ \sin\beta_2\cos\gamma, \sin\beta_2\sin\gamma)|0 \le t < \infty\} = \{\psi_2(0,t)|0 \le t\}. \end{aligned}$$

Then we consider the space $L^2(\Gamma)$ over Γ together with the norm $\|f\| := \langle f, f\rangle^{1/2}$ and the scalar product

$$\langle f, g\rangle := \sum_{l=1}^{2} \sin\beta_l \int_{\Gamma_l} f(Q)\overline{g(Q)} d_Q\Gamma_l.$$

Furthermore, we introduce the restriction Res and the prolongation Pr by

$$\begin{aligned} Res : C(S) &\longrightarrow C(\Gamma), & Res\, f &:= f|_\Gamma, \\ Pr : C(\Gamma) &\longrightarrow C(S), & Pr\, f(s,\ t,\ u) &:= \begin{cases} f(t\cot\beta_1,\ t,\ 0) & if\ u = 0 \\ f(t\cot\beta_2/\cos\gamma,\ t,\ u) & if\ u = t\tan\gamma \end{cases} \end{aligned}$$

and define $B_\rho \in \mathcal{L}(L^2(\Gamma))$ by

$$\begin{aligned} B_\rho f &:= Res\, W\, M_\rho\, Pr\, f,\ M_\rho g(P) := m_\rho(P)g(P), \\ m_\rho(\psi_l(s,t)) &:= \{\sum_{k\in\mathbf{Z}} \exp(ik\rho)\varphi_k(s)\};\ l = 1, 2;\ s \in \mathbf{R};\ 0 < t. \end{aligned}$$

Here φ_k is the function introduced in Sect.2,property iii) over the grid $\triangle = \mathbf{Z}$. If we introduce the collocation points $\{Q_\mu|\mu \in J_r\}$ and the set of ansatz functions $\{\varphi_\mu|\mu \in J_r\} \subseteq L^2(\Gamma)$ by

$$Q_\mu := P_{0,\mu},\ \varphi_{j,l}(\psi_{l'}(0,t)) := \begin{cases} \varphi_j^+(t) & if\ l = l' \\ 0 & else \end{cases},$$

then $\mathcal{W}(\rho)$ is the collocation matrix $(B_\rho\varphi_\mu(Q_\nu))_{\nu,\mu\in J_r}$ for the operator B_ρ. Thus we are left with estimating the last collocation operator. This will be done in two steps. First we shall show that the collocation matrix is a small perturbation of the Galerkin matrix, and then we shall prove that the norm of the Galerkin matrix is less than 1/2.

4.4.The Galerkin matrix corresponding to B_ρ takes the form $G^{-1}H$, where G is the Gram matrix $G := (\langle\varphi_\mu, \varphi_\nu\rangle)_{\nu,\mu\in J_r}$ and H is given by $H := (\langle B_\rho\varphi_\mu, \varphi_\nu\rangle)_{\nu,\mu\in J_r}$. We get $\mathcal{W}(\rho) - G^{-1}H = G^{-1}E$ with $E := G\mathcal{W}(\rho) - H$, and estimate the entries of E defined by

$$
(4.2) \qquad \begin{aligned} E_{\eta,\mu} &= \langle \sum_{\nu\in J_r} B_\rho\varphi_\mu(Q_\nu)\varphi_\nu, \varphi_\eta\rangle - \langle B_\rho\varphi_\mu, \varphi_\eta\rangle \\ &= \langle \sum_{\nu\in J_r} [B_\rho\varphi_\mu(Q_\nu) - B_\rho\varphi_\mu]\varphi_\nu, \varphi_\eta\rangle, \end{aligned}
$$

where we have used that $\sum \varphi_\nu = 1$ on the support of φ_η. For $Q_\nu \in supp\,\varphi_\nu$, $Q \in supp\,\varphi_\eta$ and $supp\,\varphi_\nu \cap supp\,\varphi_\eta \neq \emptyset$, we conclude

$$
\begin{aligned} |[B_\rho\varphi_\mu(Q_\nu) - B_\rho\varphi_\mu(Q)]| &= |\frac{1}{4\pi}\int_S \Big\{\frac{n_P(Q_\nu - P)}{|Q_\nu - P|^3} - \frac{n_P(Q-P)}{|Q-P|^3}\Big\} m_\rho(P) Pr\,\varphi_\mu(P) d_P S| \\ &\leq \frac{1}{4\pi}\int_S \frac{|n_P(Q-P)|}{|Q-P|^3} |Pr\,\varphi_\mu(P)| d_P S \sup_{P\in supp\,Pr\,\varphi_\mu} \left| \frac{\frac{n_P(Q_\nu-P)}{|Q_\nu-P|^3} - \frac{n_P(Q-P)}{|Q-P|^3}}{\frac{n_P(Q-P)}{|Q-P|^3}} \right|, \end{aligned}
$$

where we have used Lemma 2.1 i) for the estimation of m_ρ. Clearly, the right-hand side of the last equation vanishes if the supports of $Pr\,\varphi_\nu$ and $Pr\,\varphi_\mu$ lie on the same face of S. If they lie on different ones, it is not hard to get (cf. [21, 20])

$$
\left| \frac{\frac{n_P(Q_\nu-P)}{|Q_\nu-P|^3} - \frac{n_P(Q-P)}{|Q-P|^3}}{\frac{n_P(Q-P)}{|Q-P|^3}} \right| \leq C\frac{|Q_\nu - Q|}{|Q|} \leq C\epsilon.
$$

In view of (4.2) and of $\sum |\varphi_\nu| \leq C$ we arrive at

$$
(4.3) \qquad \begin{aligned} |E_{\eta,\mu}| &\leq C\epsilon\langle \tilde{B}|\varphi_\mu|, |\varphi_\eta|\rangle \\ \tilde{B} &:= Res\,\widetilde{W}\,Pr, \quad \widetilde{W}x(Q) := \frac{1}{4\pi}\int_S \frac{|n_P(Q-P)|}{|P-Q|^3} x(P) d_P S. \end{aligned}
$$

On the other hand, analogously to (3.3), it is not hard to show that

$$
\begin{aligned} C^{-1}\|\sum \zeta_\mu\varphi_\mu\|^2 &\leq \sum_{\mu=(j,l)\in J_r} |\zeta_\mu|^2 |t^+_{j+1} - t^+_j| \leq C\|\sum \zeta_\mu\varphi_\mu\|^2, \\ C^{-1}\|\sum \zeta_\mu|\varphi_\mu|\,\|^2 &\leq \sum_{\mu=(j,l)\in J_r} |\zeta_\mu|^2 |t^+_{j+1} - t^+_j| \leq C\|\sum \zeta_\mu|\varphi_\mu|\,\|^2. \end{aligned}
$$

For $\zeta = (\zeta_\mu)_{\mu\in J_r}$, we set $\|\zeta\|_* = \|\sum \zeta_\mu\varphi_\mu\|$ and $\|\zeta\|_{**} = \|\sum \zeta_\mu|\varphi_\mu|\,\|$. Then, for G, $\tilde{G} := (\langle|\varphi_\mu|, |\varphi_\nu|\rangle)_{\nu,\mu\in J_r}$, and $D := (\delta_{\mu,\nu}\{t^+_{j+1} - t^+_j\})_{\mu=(j,l),\nu\in J_r}$, the last estimates imply

$$
\|\zeta\|_*^2 = \langle G\zeta, \zeta\rangle = \|G^{1/2}\zeta\|^2 \sim \|D^{1/2}\zeta\|^2 \sim \|\tilde{G}^{1/2}\zeta\|^2 \sim \|\zeta\|_{**}^2,
$$

$\|G^{-1/2}D^{1/2}\| \leq C$, $\|D^{1/2}G^{-1/2}\| \leq C$, and $\|D^{-1/2}G^{1/2}\| \leq C$. Here $\|\cdot\|$ stands for the Euclidean norm and the corresponding matrix norm. If we denote the operator norms corresponding to $\|\cdot\|_*$ and $\|\cdot\|_{**}$ by the same symbols, we conclude

$$\begin{aligned}\|G^{-1}E\|_* &= \|G^{1/2}(G^{-1}E)G^{-1/2}\| = \|G^{-1/2}EG^{-1/2}\| \\ &\leq \|G^{-1/2}D^{1/2}\|\|D^{-1/2}ED^{-1/2}\|\|D^{1/2}G^{-1/2}\| \\ &\leq C\|D^{-1/2}ED^{-1/2}\|.\end{aligned}$$

Together with (4.3) this leads to

$$\|G^{-1}E\|_* \leq C\epsilon\|D^{-1/2}\tilde{H}D^{-1/2}\| \leq C\epsilon\|\tilde{G}^{-1/2}\tilde{H}\tilde{G}^{-1/2}\| \leq C\epsilon\|\tilde{G}^{-1}\tilde{H}\|_{**},$$

where $\tilde{H} := (\langle\tilde{B}|\varphi_\mu|, |\varphi_\eta|\rangle)_{\eta,\mu\in J_r}$. Thus the difference of the collocation matrix and the Galerkin matrix is bounded by $C\,\epsilon$ times the norm of the Galerkin matrix $\tilde{G}^{-1}\tilde{H}$ corresponding to the operator $\tilde{B}$. Analogously to Lemma 4.1 we conclude that $\tilde{B}$ is bounded. Hence the collocation operator $\mathcal{W}(\rho)$ is a small perturbation of the Galerkin operator.

4.5.For the Galerkin operator, we know that $\|G^{-1}H\|_* \leq \|B_\rho\|_{\mathcal{L}(L^2(\Gamma))}$. Thus the proof of Theorem 3.1 is finished if the following Lemma is proved.

Lemma 4.1 *The norm of the operator $B_\rho := Res\, W\, M_\rho\, Pr$ is less than $1/2$.*

PROOF. Since the kernel function of W does not change sign, and Lemma 2.1 i) holds, we get $\|B_\rho\| \leq \|B_0\|$ for $B_0 := Res\, W\, Pr$. In order to estimate the latter, let us introduce the strip and the points

$$Q := \psi_{l'}(0,t_Q),\ P_1 := \psi_l(0,t_{P_1}),\ P_2 := \psi_l(0,t_{P_2}),\ Str := \{\psi_l(s,t)|s \in \mathbf{R},\ t_{P_1} < t < t_{P_2}\}.$$

Of course, we assume $t_{P_1} < t_{P_2}$. Furthermore, for the sake of simplicity, let us suppose $l = 1$, $l' = 2$. For the kernel function k of the integral operator B_0, we get

$$\int_{P_1}^{P_2} k(Q,P)d_P\Gamma := \frac{1}{4\pi}\int_{Str}\frac{n_P(Q-P)}{|Q-P|^3}d_PS.$$

We observe, that the right-hand side is just the normalized solid angle under which Str is seen from Q. Obviously, this is equal to the normalized angle under which the interval $\{(t,0)|t_{P_1}\sin\beta_1 < t < t_{P_2}\sin\beta_1\} \subseteq \mathbf{R}^2$ is seen from the point $(t_Q\sin\beta_2\cos\gamma,\ t_Q\sin\beta_2\sin\gamma) \in \mathbf{R}^2$. Hence, our kernel function is nothing else than that of the double layer potential operator over the angle $\Gamma_0 := \Gamma_1 \cup \Gamma_2 \subseteq \mathbf{R}^2$, where $\Gamma_1 := \{(\tau,0)|0 \leq \tau < \infty\}$ and $\Gamma_2 := \{(\tilde{\tau}\cos\gamma,\ \tilde{\tau}\sin\gamma)|0 \leq \tilde{\tau} < \infty\}$ and where τ and $\tilde{\tau}$ are substituted by $\tau = t_P\sin\beta_1$ and $\tilde{\tau} = t_Q\sin\beta_2$, respectively. We get

$$k(Q,P)d_P\Gamma = \frac{1}{2\pi}\frac{t_Q \sin\gamma \sin\beta_2}{t_P^2 \sin^2\beta_1 + t_Q^2 \sin^2\beta_2 - 2t_P \sin\beta_1 t_Q \sin\beta_2 \cos\gamma} dt_P \sin\beta_1.$$

However, for the double layer operator over Γ_0, it is well known that its norm is less than 1/2. □

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Discretized Collocation for the Numerical Solution of the Double Layer Potential Equation over the Cube

A.Rathsfeld

Abstract

In this paper we consider a piecewise polynomial collocation method for the solution of the double layer potential equation corresponding to Laplace's equation over the cube. We give formulas for the computation of the entries in the corresponding stiffness matrix and prove the stability for our method in case of special triangulations over the boundary. Finally, we derive an asymptotic error estimate.

Key words. potential equation, collocation

AMS(MOS) subject classification. 45L10, 65R20

1 Introduction

One popular method for the solution of boundary value problems for elliptic differential equations consists in the reduction to boundary integral equations. For instance, the Dirichlet problem for Laplace's equation in a bounded and simply connected polyhedron $\Omega \subseteq \mathbf{R}^3$ or the Neumann problem for the same equation on $\mathbf{R}^3 \setminus \Omega$ can be reduced to the second kind integral equation $Ax = y$ over the boundary $S := \partial\Omega$ (cf. e. g. [21]),where $A = I + 2W_S$ and

(1.1) $$W_S x(Q) := [1/2 - d_\Omega(Q)]x(Q) + \frac{1}{4\pi}\int_S \frac{n_P(Q-P)}{|P-Q|^3} x(P) d_P S,$$

$$d_\Omega(Q) := \lim_{\epsilon\to\infty} \frac{|\{P \in \Omega : |P-Q| < \epsilon\}|}{|\{P \in \mathbf{R}^3 : |P - QU < \epsilon\}|}.$$

Here n_P denotes the unit vector of the interior normal to Ω at P and $|Z|$ is the Lebesgue measure of Z for any $Z \subseteq \mathbf{R}^3$.Note that, since the boundary S is not smooth, W_S is not compact. For the numerical solution of $Ax = (I + 2W_S)x = y$, various methods have been introduced. The first method was the so called panel method, i.e., piecewise constant collocation ([29] and cf.[18, 28, 3]). Kral and Wendland [20] (cf. also [2]) have shown that this method is stable for the case of certain rectangular domains Ω. Arbitrary polyhedral domains have been considered in [25]. Elschner [11] has analysed the Galerkin method with piecewise polynomial trial functions over arbitrary polyhedrons, and the Galerkin method together with an approximation of the Lipschitz boundary by smooth surfaces has been investigated by Dahlberg and Verchota [8]. For all these procedures, the question arises how to compute the entries of the discretized system of equations (cf. [29, 14, 27]). In order to avoid this problem, one can consider simple quadrature methods. In the papers [24, 23] Nyström methods have been analysed which are similar to those of Graham and Chandler [7], Kress [19], and Elschner [9] for the corresponding equation over polygonal boundaries. However, these quadrature methods improve the complexity only up to a certain order. The reason for this disadvantage is that the Nyström method works with one grid only. This grid has to be adapted at once to all singularities of the kernel function

$$(1.2) \qquad k(Q,P) := \frac{1}{4\pi}\frac{n_P(Q-P)}{|P-Q|^3}.$$

(Note that $k(Q,P)$ tends to infinity if $|P-Q|$ tends to 0 and P and Q lie on different faces of S.) Doing so, the number of grid points grows, and the complexity of the method cannot be reduced in the same manner as in the one-dimensional case (cf.[24, 23]).

In order to get a fully discretized numerical method which reduces the complexity similarly to the one-dimensional case ([7, 9, 19]), one needs quadrature methods, where the quadratures and the grids depend on the collocation points, in other words one needs certain discretized collocation methods. The first step in this direction is the stability analysis of piecewise constant or piecewise linear collocation due to Kral, Wendland and the author (cf. [29, 20, 25]). In the present paper we shall consider a method, where the ansatz function are taken from a certain space of higher degree tensor product splines. For this method, we shall prove the stability in the L^2-space and show nearly optimal error estimates. The method of proof requires a certain stability condition. Namely, we have to suppose that certain finite section operators are invertible and the norms of their inverses are uniformly bounded with respect to the mesh width (Applying the ideas of the proof from Theorem 2.1 in [24] we need the invertibility of the operator A' in the proof of Theorem 3.1.). However, in order to simplify the considerations, we restrict ourselves to the case of the cube $\Omega = [0,1]^3 \subseteq \mathbf{R}^3$. For this case, the assumption concerning the finite section operators is satisfied.

Finally, let us mention that, in order to get an efficient algorithm, one has to combine the presented collocation procedure with an iterative solution of the system of equations (cf. [15, 17, 4, 23]) or a fast method for the multiplication of the matrix by a vector (cf. [26, 13, 16, 1, 6, 5]). The analysis of these steps is still open.

2 The discretized collocation method

2.1. First we have to define the collocation points and the ansatz functions. In order to do this, we start with triangular parametrizations for the boundary S of the cube $[0,1]^3$. Suppose that R is the centre of a face of S and P and Q are the endpoints of an edge lying on the same face as R. By U let us denote the mid-point between P and Q. Then we define the triangular parametrization Φ_{PQR} of the triangle $Tr_{PQR} := \{P + s\vec{UR} + t\vec{PU},\ 0 \le t \le 2,\ 0 \le s \le \min(t, 2-t)\}$ by

$$\begin{aligned}\Phi_{PQR} : \qquad & Sq := \{(s,t) \in \mathbf{R}^2 : 0 \le s \le 1,\ 0 \le t \le 2\} \longrightarrow Tr_{PQR}, \\ \Phi_{PQR}(s,t) \ := \ & P + \min(t, 2-t)s\vec{UR} + t\vec{PU}.\end{aligned}$$

We choose $\nu > 1$ and, for any positive integer n, we introduce $s_0 := 0$, $s_j := (1-1/n)^{m(n)-j}$, $j = 1, 2, \dots, m(n)$ with $m(n) := \nu n[\log n]$. Furthermore, we set $t_k := s_k$, $k = 0, \dots, m(n)$ and $t_k := 2 - s_{2m(n)-k}$, $k = m(n)+1, \dots, 2m(n)$. Then the collocation points are given by $P_{PQRjk} := \Phi_{PQR}(s_j, t_k)$, where $j = 0, \dots, m(n)$, $k = 0, \dots, 2m(n)$, and R, P, Q run through all the centers of faces and corresponding endpoints of edges lying on the same face.

The definition of the spline space is more difficult. Let us start with the definition of the one-dimensional splines of degree $2d+1$ over the real half-axis corresponding to the partition $\{\tau_j := (1-1/n)^{m(n)-j},\ j \in \mathbf{Z}\}$. We shall consider the set of interpolating splines ψ defined by $\psi(\tau_j) = y_j$ such that the restriction of ψ to $[\tau_j, \tau_{j+1}]$ is equal to the polynomial p of degree less than $2d+2$ which satisfies $p(\tau_l) = y_l$, $l = j-d, \dots, j+d+1$. In other words, the basis functions ψ_k satisfying $\psi_k(\tau_j) = \delta_{j,k}$ are defined by $\psi_k|_{[\tau_j, \tau_{j+1}]} = p_{k,j}$, where $p_{k,j}$ is the unique polynomial of degree not exceeding $2d+1$ such that $p_{k,j}(\tau_l) = \delta_{l,k}$, $l = j-d, \dots, j+d+1$. Note that the support of ψ_k is $[\tau_{k-d-1}, \tau_{k+d+1}]$ and, for any sufficiently smooth function f defined over $[0,1]$, the interpolating spline $\sum f(\tau_k)\psi_k$ tends to f with order $O(n^{-(2d+2)})$ as $n \to \infty$. If we consider the interval $[0,1]$, then we cannot use the interpolation knots greater than 1. Moreover, we need a finite dimensional spline space. Hence, we set $\varphi_k^1 := \psi_k$, $k = d+2, \dots, m(n)-d-1$ and define φ_k^1, $k = m(n)-d, \dots, m(n)$ to be the function whose restriction to $[\tau_j, \tau_{j+1}]$ is the polynomial $p_{k,j}$ of degree less than $2d+2$ satisfying $p_{k,j}(\tau_l) = \delta_{l,k}$, where $l \in \{j-d, \dots, j+d+1\}$ for $j+d+1 \le m(n)$ and $l \in \{m(n)-2d-1, \dots, m(n)\}$ for $j+d+1 \ge m(n)$. We define φ_0^1 to be the characteristic function of $[0, \tau_1]$ and set $\varphi_k^1(s) := 0$, $k = 1, \dots, d+1$ for $s \in [0, \tau_1]$, $\varphi_1^1(s) := \sum_{l=1-d}^{1} \psi_l(s)$ and $\varphi_k^1(s) := \psi_k(s)$, $k = 2, \dots, d+1$ for $s \in [\tau_1, 1]$. Note that the support of φ_1^1 is $[\tau_1, \tau_{d+2}]$. Now we could have defined the tensor product splines over the boundary of the cube by

$$\Phi_{PQR}(s,t) \mapsto \varphi_j^1(s)\varphi_k^1(t),\ \Phi_{PQR}(s,t) \mapsto \varphi_j^1(s)\varphi_k^1(2-t).$$

However, to overcome some problems in the stability proof for the collocation method, we need certain modifications of the tensor product splines along the line $\{\Phi_{PQR}(s,1),\ 0 \le$

$s \leq 1\}$ and near the corner points $\Phi_{PQR}(0,0)$ and $\Phi_{PQR}(0,2)$. Therefore, let us consider the angle $\Gamma_j := \{\Phi_{PQR}(s_j,t),\ 0 \leq t \leq 2\}$ lying on one face of $S = \partial\Omega$. We identify the plane containing the latter face with the complex plane $\mathbf{C}$. Using this, we consider the polynomials as polynomials in one complex variable. Now $\varphi_k^{2,j},\ k = 1,\dots,m(n)$ is defined to be the function on Γ_j whose restriction to $[\Phi_{PQR}(s_j,\tau_r), \Phi_{PQR}(s_j,\tau_{r+1})]$ is the real part of the polynomial $p_{k,r}$ of degree less than $2d+2$ satisfying $p_{k,r}(\Phi_{PQR}(s_j,\tau_l)) = \delta_{l,k},\ l = r-d,\dots,r+d+1$. Note that $\varphi_k^{2,j}(\Phi_{PQR}(s_j,t)) = \varphi_k^1(t),\ k = d+2,\dots,m(n)-d-1$. We set $\varphi_k^{2,j}(\Phi_{PQR}(s_j,t)) = \varphi_{2m(n)-k}^{2,j}(\Phi_{PQR}(s_j,2-t)),\ k = m(n)+1,\dots,2m(n)-1$ and define

$$\varphi_0^{2,j}(\Phi_{PQR}(s_j,t)) := \begin{cases} 1 - \sum_{l=1}^{2m(n)-1} \varphi_l^{2,j}(\Phi_{PQR}(s_j,t)) = \sum_{l\leq 0}\psi_l(t) & \text{if } t<1 \\ 0 & \text{else} \end{cases},$$

$\varphi_{2m(n)}^{2,j}(\Phi_{PQR}(s_j,t)) := \varphi_0^{2,j}(\Phi_{PQR}(s_j,2-t))$, and the modified tensor product spline functions over S are given by

$$\begin{aligned} \varphi_{PQRjk}(\Phi_{PQR}(s,t)) &:= \varphi_j^1(s)\varphi_k^{2,j}(\Phi_{PQR}(s_j,t)), \\ & \quad j = 0,\dots,m(n),\ k = 0,\dots,2m(n). \end{aligned}$$

2.2. If we interchange the roles of P and Q in the notation of Section 2.1, then we can define Φ_{QPR}, Tr_{QPR}, P_{QPRjk}, and φ_{QPRjk} and get $Tr_{PQR} = Tr_{QPR}$, $P_{PQRjk} = P_{QPRj(2m(n)-k)}$, as well as $\varphi_{PQRjk} = \varphi_{QPRj(2m(n)-k)}$. Thus we suppose for the set $\{(PQRjk)\}$ of all indices that it contains either $\{(PQRjk),\ j=0,\dots,m(n),\ k=0,\dots,2m(n)\}$ or $\{(QPRjk),\ j = 0,\dots,m(n),\ k=0,\dots,2m(n)\}$. On this set $\{(PQRjk)\}$ of indices there is an equivalence relation $\sim$, where $(PQRjk) \sim (P'Q'R'j'k')$ holds if and only if $P_{PQRjk} = P_{P'Q'R'j'k'}$. Note that the latter is possible only if P_{PQRjk} is at the boundary of Tr_{PQR}. Let us denote the set of equivalence classes by $\mathcal{I}$ and its elements by $\iota,\kappa \in \mathcal{I}$. We set $P_\iota := P_{PQRjk}$ if $(PQRjk) \in \iota$ and define

$$\varphi_\iota(\tilde{P}) := \begin{cases} \varphi_{PQRjk}(\tilde{P}) & \text{if there is a } (PQRjk)\in\iota \text{ such that } \tilde{P} \in Tr_{PQR}, \\ 0 & \text{else.} \end{cases}$$

If we consider the functions as elements of the space L^2, then we can write

$$\varphi_\iota := \sum_{(PQRjk)\in\iota} \varphi_{PQRjk}.$$

Let us consider the following collocation method: Find an approximate solution $x_n := \sum_{\iota\in\mathcal{I}} \xi_\iota\varphi_\iota$ of $Ax = y$ from solving

$$(Ax_n)(P_\kappa) = y(P_\kappa),\quad \kappa \in \mathcal{I}. \tag{2.1}$$

Obviously, (2.1) is a linear system of equations for the coefficients ξ_ι of the unknown function x_n. The matrix of this system, i.e., the so called stiffness matrix takes the form

$$(a_{\kappa,\iota})_{\kappa,\iota\in\mathcal{I}}, \quad a_{\kappa,\iota} := (A\varphi_\iota)(P_\kappa).$$

The discretized collocation is the modified method (2.1), where the entries $a_{\kappa,\iota}$ are replaced by the approximate values $a'_{\kappa,\iota}$ which we shall define in the following.

2.3. Setting

$$Kx(Q) := \frac{1}{2\pi}\int_S \frac{n_P(Q-P)}{|P-Q|^3}x(P)d_PS,$$

we observe that

$$\begin{aligned}
(2.2)\qquad a_{\kappa,\iota} &= 2[1-d_\Omega(P_\kappa)]\varphi_\iota(P_\kappa) + \sum_{(PQRjk)\in\iota}(K\varphi_{PQRjk})(P_\kappa),\\
(2.3)\qquad (K\varphi_{PQRjk})(P_\kappa) &= \frac{1}{2\pi}\int_S \frac{n_P(P_\kappa-P)}{|P-P_\kappa|^3}\varphi_{PQRjk}(P)d_PS\\
&= \frac{1}{2\pi}\int_{Tr_{PQR}} \frac{n_P(P_\kappa-P)}{|P-P_\kappa|^3}\varphi_{PQRjk}(P)d_PS.
\end{aligned}$$

Thus in order to compute $a_{\kappa,\iota}$ approximately, we need cubature formulas for integrals over Tr_{PQR}. Moreover, as indicated in the introduction, we shall introduce rules depending on the collocation points P_κ. Let us start with a quadrature formula over the interval $[0,1]$

$$(2.4)\qquad \int_0^1 f(t)dt \sim \sum_{i=1}^r \omega_i f(\sigma_i), \quad 0\le\sigma_1<\ldots<\sigma_r\le 1, \quad \omega_i>0.$$

We shall assume that this formula is exact for polynomials of degree less than $2d+2$. The cubature over Tr_{PQR} will consist of an iterated product version of (2.4). Therefore, let us introduce the partition. Let $\{s_j,\ j=0,\ldots,m(n)\}$ and $\{t_k,\ k=0,\ldots,2m(n)\}$ be defined as above and consider two cases. If $P_\kappa\in Tr_{P'Q'R'}$ and $Tr_{P'Q'R'}\cap Tr_{PQR}=[P',Q']=[P,Q]$, then let $\Phi_{PQR}(0,t_\kappa)$ be the orthogonal projection of P_κ onto $[P,Q]$. We define the partition $0=u_0<u_1<\ldots<u_{m(n,\kappa)}=2$ by

$$\begin{aligned}
\{u_k,\ k=0,\ldots,m(n,\kappa)\} := &\ \{t_k,\ k=0,\ldots,m(n)\}\cup\\
&\ \{t_\kappa\pm s_j:\ 0<t_\kappa\pm s_j<2 \text{ and } j=0,\ldots,m(n)\}.
\end{aligned}$$

If $P_\kappa\in Tr_{P'Q'R'}$ and $Tr_{P'Q'R'}=Tr_{PQR}$ or $Tr_{P'Q'R'}$ and Tr_{PQR} have no more than one point in common, then set $m(n,\kappa):=2m(n)$ and $u_k:=t_k$. We arrive at the cubature formula

$$(2.5) \qquad \int_{Tr_{PQR}} f(P)d_PS = \frac{1}{4}\int_0^1\int_0^2 f\circ\Phi_{PQR}(s,t)\min(t,2-t)dtds$$
$$\sim \sum_{j=0}^{m(n)-1}\sum_{i=1}^{r}\sum_{k=0}^{m(n,\kappa)-1}\sum_{i'=1}^{r} f(\Phi_{PQR}(s_{j,i},u_{k,i'}))\omega_{j,i,k,i'},$$

where we have set

$$s_{j,i} := (1-\sigma_i)s_j + \sigma_i s_{j+1}, \quad u_{k,i'} := (1-\sigma_{i'})u_k + \sigma_{i'}u_{k+1}$$
$$\omega_{j,i,k.i'} := \frac{1}{4}\min(u_{k,i'}, 2-u_{k,i'})(s_{j+1}-s_j)\omega_i(u_{k+1}-u_k)\omega_{i'}.$$

Following the formulae (2.2)-(2.5), we define the approximate value $a'_{\kappa,\iota}$ of $a_{\kappa,\iota}$ by

$$a'_{\kappa,\iota} := 2[1-d_\Omega(P_\kappa)]\delta_{\kappa,\iota} + \sum_{(PQRjk)\in\iota}\frac{1}{2\pi}\sum_{j'=0}^{m(n)-1}\sum_{i=1}^{r}\sum_{k'=0}^{m(n,\kappa)-1}\sum_{i'=1}^{r}$$
$$\frac{n_{\Phi_{PQR}(s_{j',i},u_{k',i'})}(P_\kappa - \Phi_{PQR}(s_{j',i},u_{k',i'}))}{|\Phi_{PQR}(s_{j',i},u_{k',i'}) - P_\kappa|^3}\varphi_{PQRjk}(\Phi_{PQR}(s_{j',i},u_{k',i'}))\omega_{j',i,k',i'},$$

Now we observe that the support of φ_{PQRjk} is contained in $\{\Phi_{PQR}(s,t) : s_{\tilde{j}_1} \le s \le s_{\tilde{j}_2}, u_{\tilde{k}_1} \le t \le u_{\tilde{k}_2}\}$, where $\tilde{j}_1 := \max(0, j-d-1)$, $\tilde{j}_2 := \min(m(n), j+d+1)$ and the indices $\tilde{k}_1$, $\tilde{k}_2$ are given by $u_{\tilde{k}_1} = t_{\max(0,k-d-1)}$, $u_{\tilde{k}_2} = t_{\min(2m(n),k+d+1)}$. Especially, if $P_\kappa \in Tr_{P'Q'R'}$ and $Tr_{P'Q'R'} = Tr_{PQR}$ or $Tr_{P'Q'R'}$ and Tr_{PQR} have no more than one point in common, then $\tilde{k}_1 = \max(k-d-1,0)$ and $\tilde{k}_2 = \min(k+d+1, 2m(n))$. The summation in the definition of $a'_{\kappa,\iota}$ reduces to

$$(2.6) \qquad a'_{\kappa,\iota} := 2[1-d_\Omega(P_\kappa)]\delta_{\kappa,\iota} + \sum_{(PQRjk)\in\iota}\frac{1}{2\pi}\sum_{j'=\tilde{j}_1}^{\tilde{j}_2-1}\sum_{i=1}^{r}\sum_{k'=\tilde{k}_1}^{\tilde{k}_2-1}\sum_{i'=1}^{r}$$
$$\frac{n_{\Phi_{PQR}(s_{j',i},u_{k',i'})}(P_\kappa - \Phi_{PQR}(s_{j',i},u_{k',i'}))}{|\Phi_{PQR}(s_{j',i},u_{k',i'}) - P_\kappa|^3}\varphi_{PQRjk}(\Phi_{PQR}(s_{j',i},u_{k',i'}))\omega_{j',i,k',i'},$$

For the modification of (2.1), where the entry $a_{\kappa,\iota}$ of the stiffness matrix is replaced by the right-hand side of (2.6), we shall show the L^2-stability and derive the convergence order.

Remark 2.1 *It will turn out in Section 3 (cf. (3.4)) that the matrix $(a_{\kappa,\iota})_{\kappa,\iota}$ is of convolution type. Thus, in order to calculate its entries $a_{\kappa,\iota}$, $\kappa,\iota \in \mathcal{I}$, we may restrict ourselves to the case $(PQRjk) \in \kappa$ with fixed k, e.g., $k = m(n)$. We can approximate these entries as in (2.6) using only one additional partition $\{u_k\}$.*

3 The stability near the corner

Following [24], the first step in the stability analysis is the proof for the case that Ω is an infinite cone. Thus we start with the definition of the corresponding collocation method. Let us set $\Omega := [0,\infty)^3$, $S := \partial\Omega$, and choose $P := (0,0,0)$. By R we denote a point on a face of S such that two of its coordinates are equal to $1/2$ and one is 0. We choose Q on an edge lying at the same face with R. Moreover, let the distance between Q and P be 1. If U is the orthogonal projection of R onto the edge containing Q, then we define the triangular parametrization by

$$\begin{aligned}
\Phi_{PQR} : \quad & Str \longrightarrow Sec_{PQR}, \quad Str := \{(s,t) \in \mathbf{R}^2 : 0 \le s \le 1,\ 0 \le t < \infty\} \\
Sec_{PQR} \ :=\ & \{P + s\vec{UR} + t\vec{PU},\ 0 \le t < \infty,\ 0 \le s \le t\}, \\
\Phi_{PQR}(s,t) \ :=\ & P + ts\vec{UR} + t\vec{PU}.
\end{aligned}$$

Retaining the definition of τ_j, s_j, φ^1_j, and ψ_j from the last section and setting $t_0 := 0, t_k := \tau_k$, $k = 1,\ldots$, we introduce the collocation points $P_{PQRjk} := \Phi_{PQR}(s_j, t_k)$ for $j = 0,\ldots,m(n)$ and $k = 0,\ldots$. The spline functions are given by

$$\begin{aligned}
\varphi_{PQRjk}(\Phi_{PQR}(s,t)) \ &:=\ \varphi^1_j(s)\varphi^2_k(t) \ \ j = 0,\ldots,m(n), \ \ k = 0,\ldots \\
\varphi^2_k \ &:=\ \psi_k, \ \ k = 1,\ldots, \ \ \varphi^2_0 := 1 - \sum_{k=1}^{\infty} \varphi^2_k.
\end{aligned}$$

Analogously to the previous section, we define the corresponding set $\mathcal{I}$, φ_ι, and P_ι. Hence, if $A = I + 2W_S$ is the double layer operator defined by (1.1) over the infinite boundary S, then the collocation method is defined by (2.1), where $x_n := \sum_{\iota\in\mathcal{I}} \xi_\iota\varphi_\iota$. The entry $a_{\kappa,\iota}$ of the corresponding stiffness matrix $A_n := (a_{\kappa,\iota})_{\kappa,\iota\in\mathcal{I}}$ is given by $a_{\kappa,\iota} := (A\varphi_\iota)(P_\kappa)$, and A_n is considered as an operator acting in the space of vectors $\{\xi_\iota\}_{\iota\in\mathcal{I}}$ endowed with the norm

$$\|\{\xi_\iota\}\|_* := \|\sum_{\iota\in\mathcal{I}} \xi_\iota\varphi_\iota\|_{L^2(S)}.$$

The collocation method (2.1) is called stable if A_n is invertible for n large enough and if the norms of the inverse operators are uniformly bounded with respect to n.

Theorem 3.1 *The method (2.1) applied to the double layer operator over the boundary of $[0,\infty)^3$ is stable.*

PROOF. a) The proof follows analogously to that of Theorem 2.1 of [24]. Let $\iota_0 \in \mathcal{I}$ stand for the class of all $(PQRjk)$ such that $P_{PQRjk} = (0,0,0)$, i.e., ι_0 is the set of all

$(PQRjk)$ with $k = 0$. We define $\mathcal{J} := \mathcal{I} \setminus \{\iota_0\}$ and set $A'_n := (a_{\kappa,\iota})_{\kappa,\iota \in \mathcal{J}}$ as well as $A''_n := (a_{\kappa,\iota_0})_{\kappa \in \mathcal{J}}$. Furthermore, by $A' \in \mathcal{L}(L^2(S'))$ we denote the double layer operator $A' := I + 2W_{S'}$, where S' stands for the truncated boundary $S' := \cup_{\iota \in \mathcal{J}} supp\, \varphi_\iota$. We introduce the interpolation projection $P_n f := \sum_{\iota \in \mathcal{J}} f(P_\iota)\varphi_\iota$. Obviously, the image $im\, P_n$ can be identified with the space of functions over $\{P_\iota\}_{\iota \in \mathcal{J}}$. Consequently, A'_n maps $im\, P_n$ into $im\, P_n$. By $W'_n \in \mathcal{L}(im\, P_n)$ we denote the operator which can be obtained analogously to the definition of A'_n if we start with W_S instead of A, i. e., $W'_n := 1/2(A'_n - Id)$. For a function χ on S', we set $\chi_n := P_n \chi|_{im\, P_n}$. Finally, by χ^δ we denote the characteristic function of the set of points $P \in S$ whose distance to the set of edge points is less than $|P|\delta$. Now all we have to show is that the following assertions are valid (cf. Lemma 2.2 and the proof of Theorem 2.1 in [24]).

i) The operators A''_n and A'_n are uniformly bounded with respect to n.

ii) Let $\delta > 0, \epsilon > 0$ .Then, for n large enough, we get $\|(1 - \chi^\delta)[A' - A'_n]|_{im\, P_n}\| < \epsilon$.

iii) There exist $n_0 > 0, \delta_0 > 0$ such that , for $n \geq n_0$ and $\delta \leq \delta_0$, the operator $[Id + 2W'_n(\chi^\delta)_n] \in \mathcal{L}(im\, P_n)$ is invertible.

b) Let us show i) and start with proving that $A'_n := P_n A|_{im\, P_n}$ is bounded. On the set $\{(PQRj)\}$ there is an equivalence relation $\sim_0$, where $(PQRj) \sim_0 (P'Q'R'j')$ if and only if $P_{PQRjm(n)} = P_{P'Q'R'j'm(n)}$. We denote the set of equivalence classes by $\mathcal{J}_0$ and the elements of $\mathcal{J}_0$ by λ, μ. For $\iota \in \mathcal{J}$, $(PQRjk) \in \iota$, and $(PQRj) \in \lambda$, we write $\iota = (\lambda, k)$, and we set $P_\lambda := P_{PQRjm(n)}$. Note that these points belong to the set $\Gamma := \cup_{(PQR)}\{\Phi_{PQR}(s,1) : 0 \leq s \leq 1\}$ which is a closed polygon on S. Moreover, we introduce $\varphi_\lambda := \varphi_{(\lambda,m(n))}|_\Gamma$. Then the matrix of $P_n A|_{im\, P_n}$ corresponding to the base $\{\varphi_\iota\}$ takes the form

$$A'_n = \left((a_{(\mu,k'),(\lambda,k)})_{\mu,\lambda \in \mathcal{J}_0}\right)_{k',k=1}^{\infty}. \tag{3.1}$$

Obviously, the norm $\|\cdot\|_*$ is equivalent to

$$\|\{\xi_\iota\}_{\iota \in \mathcal{J}}\| := \frac{1}{\sqrt{n}} \sqrt{\sum_{k=1}^{\infty} (1 - 1/n)^{2(m(n)-k)} \| \sum_{\lambda \in \mathcal{J}_0} \xi_{(\lambda,k)} \varphi_\lambda \|^2_{L^2(\Gamma)}}.$$

Thus it is enough to prove the boundedness of the matrix

$$B_n := \left(((1 - 1/n)^{k-k'} a_{(\mu,k'),(\lambda,k)})_{\mu,\lambda \in \mathcal{J}_0}\right)_{k',k \in \mathbf{Z}}, \tag{3.2}$$

where the norm is the matrix norm generated by

$$\|\{\xi_\iota\}_{\iota \in \mathcal{J}}\| := \sqrt{\sum_{k \in \mathbf{Z}} \| \sum_{\lambda \in \mathcal{J}_0} \xi_{(\lambda,k)} \varphi_\lambda \|^2_{L^2(\Gamma)}} \tag{3.3}$$

and $a_{(\mu,k'),(\lambda,k)} := (A\varphi^{+}_{(\lambda,k)})(P^{+}_{(\mu,k')})$. The definition of the functions $\varphi^{+}_{(\lambda,k)}$ and the points $P^{+}_{(\lambda,k)}$ is analogous to that of $\varphi_{(\lambda,k)}$ and $P_{(\lambda,k)}$ $(k>0)$, respectively. Namely, for $(PQRj) \in \lambda$, we set

$$\varphi^{+}_{(\lambda,k)}(\Phi_{PQR}(s,t)) := \varphi^{1}_{j}(s)\psi_k(t), j=0,\dots,m(n),\ k\in\mathbf{Z}, P^{+}_{(\lambda,k)} := \Phi_{PQR}(s_j,\tau_k).$$

From homogeneity arguments we observe that

$$(3.4)\quad ((1-1/n)^{k-k'}a_{(\mu,k'),(\lambda,k)})_{\mu,\lambda\in\mathcal{J}_0} = ((1-1/n)^{k-k'}a_{(\mu,m(n)),(\lambda,m(n)+k-k')})_{\mu,\lambda\in\mathcal{J}_0}.$$

Consequently, B_n is a block convolution matrix, and B_n is bounded if and only if the symbol

$$\begin{aligned} B_n^{\rho} &:= \sum_{k\in\mathbf{Z}} e^{i\rho k}\left((1-1/n)^{-k}a_{(\mu,m(n)),(\lambda,m(n)-k)}\right)_{\mu,\lambda\in\mathcal{J}_0} \\ &= \left(A\left(\sum_{k\in\mathbf{Z}} e^{i\rho k}(1-1/n)^{-k}\varphi^{+}_{(\lambda,m(n)-k)}\right)(P^{+}_{(\mu,m(n))})\right)_{\mu,\lambda\in\mathcal{J}_0}, \quad 0\le\rho<2\pi \end{aligned}$$

is uniformly bounded with respect to ρ. Taking into account that

$$\left(\sum_{k\in\mathbf{Z}} e^{i\rho k}(1-1/n)^{-k}\varphi^{+}_{(\lambda,m(n)-k)}\right)(\Phi_{PQR}(s,t)) = \left(\sum_{k\in\mathbf{Z}} e^{i\rho k}(1-1/n)^{-k}\psi_{m(n)-k}(t)\right)\varphi^{1}_{j}(s),$$

we conclude that B_n^{ρ} is the matrix of the collocation operator $P_n^{\Gamma}A^{\rho}|imP_n^{\Gamma}$, where the interpolation projection over Γ is given by $P_n^{\Gamma}f := \sum_{\lambda\in\mathcal{J}_0} f(P_\lambda)\varphi_\lambda$, $A^{\rho} := Res\,A\,M_\rho\,Pr$, and

$$\begin{aligned} Res : C(S) &\longrightarrow C(\Gamma), & Res\,f &:= f|\Gamma, \\ Pr : C(\Gamma) &\longrightarrow C(S), & Pr\,g(\Phi_{PQR}(s,t)) &:= g(\Phi_{PQR}(s,1)), \\ M_\rho : L^2(S) &\longrightarrow L^2(S), & M_\rho f(\tilde{P}) &:= m_\rho(\tilde{P})\,f(\tilde{P}), \\ m_\rho(\Phi_{PQR}(s,t)) &:= \sum_{k\in\mathbf{Z}} e^{i\rho k}(1-1/n)^{-k}\psi_{m(n)-k}(t). \end{aligned}$$

Setting

$$P^{\Gamma}_{1,n}(f) := \sum_{\lambda\in\mathcal{J}_0:P_\lambda edge\,point} f(P_\lambda)\varphi_\lambda$$

we split P_n^Γ into $P_{1,n}^\Gamma + P_{2,n}^\Gamma$. Now $P_{1,n}^\Gamma A^\rho | im P_n^\Gamma$ is a finite rank operator and its boundedness is equivalent to the boundedness of the functionals $f \mapsto (A^\rho f)(P_\lambda)$, where P_λ is an edge point of S. However, if P_λ is an edge point, then the integral corresponding to the integral operator A is to be taken over all the faces of S not containing this edge and the boundedness of the functionals follows easily. It remains to show that $P_{2,n}^\Gamma A^\rho | im P_n^\Gamma$ is bounded. In part c) we shall show that A^ρ is a bounded operator in $L^2(\Gamma)$. Let Q_n^Γ stand for the L^2-orthogonal projection onto the space $im P_{2,n}^\Gamma$. Then the Galerkin operator $Q_n^\Gamma A^\rho|_{im P_n^\Gamma}$ is bounded, and the boundedness of $P_{2,n}^\Gamma A^\rho | im P_n^\Gamma$ follows from the fact that, for any prescribed $\epsilon > 0$,

$$(3.5) \qquad \| Q_n^\Gamma A^\rho|_{im P_n^\Gamma} - P_{2,n}^\Gamma A^\rho|_{im P_n^\Gamma} \| \leq c\,\epsilon$$

if only n is large enough. Here and in the following c denotes a generic positive constant the value of which varies from instance to instance. For the proof of (3.5) we refer to Section 4.4 of [22].

In order to prove the boundedness of $P_n A|_{span\{\varphi_{\iota_0}\}}$, we observe that

$$\varphi_{\iota_0}(\Phi_{PQR}(s,t)) := \sum_{j=0}^{m(n)} \sum_{k \leq 0} \varphi_j^1(s) \psi_k(t).$$

Hence, the matrix of $P_n A|_{span\{\varphi_{\iota_0}\}}$ takes the form $B_n b_n$ with $b_n := (entry_{(\lambda,k)})_{(\lambda,k)\in \mathcal{J}_0 \times Z}$, $entry_{(\lambda,k)} := 0$ if $k > 0$ and $entry_{(\lambda,k)} := 1$ if $k \leq 0$. Since the function $\sum entry_{(\lambda,k)} \varphi_{(\lambda,k)}^+$ is in L^2 and B_n is bounded, the operator $P_n A|_{span\{\varphi_{\iota_0}\}}$ is bounded, too.

c) Let us show that A^ρ is a bounded operator in $L^2(\Gamma)$. First we consider m_ρ. Using $\tau_{m(n)-k} = (1 - 1/n)^k$, we get

$$\begin{aligned} m_\rho(\Phi_{PQR}(s,t)) &:= t^{-1} \sum_{k \in \mathbf{Z}} e^{i\rho k} \psi_{m(n)-k}(t) + \\ & \qquad t^{-1} \sum_{k \in \mathbf{Z}} e^{i\rho k} \left[\frac{t - \tau_{m(n)-k}}{\tau_{m(n)-k}} \right] \psi_{m(n)-k}(t). \end{aligned}$$

For t in the support of $\psi_{m(n)-k}$, there holds

$$\left| \frac{t - \tau_{m(n)-k}}{\tau_{m(n)-k}} \right| \leq \frac{|\tau_{m(n)-k+d+1} - \tau_{m(n)-k}|}{\tau_{m(n)-k}} \leq c \frac{1}{n}.$$

Taking into account that (cf. Lemma 2.1 in [22]) $|\sum_{k \in \mathbf{Z}} e^{i\rho k} \psi_{m(n)-k}(t)| \leq 1 + \epsilon/2$, we conclude $|m_\rho| \leq t^{-1}(1+\epsilon)$ for any prescribed ϵ if only n is large enough.

We get $A^\rho = I + 2\, Res\, W_S\, M_\rho\, Pr$, where the norm of the operator $Res\, W_S\, M_\rho\, Pr$ can be estimated by the norm of $(1+\epsilon) Res\, W^+\, M\, Pr$ with

$$W^+x(Q) := \int_S |k(Q,P)|x(P)d_PS, \quad M\,f(Q) := c\,|Q|^{-1}f(Q)$$

The operator W^+ is a Mellin convolution operator with respect to the radial coordinate and maps homogeneous functions into homogeneous ones. Moreover, if γ denotes the orthogonal projection of Γ onto the unit sphere, then $(Res\,W^+\,M\,Pr)x(P) = c\,|P|^{-1}\{\mathcal{W}([Pr\,x]|_\gamma)\}(P/|P|)$, where $\mathcal{W} = \mathcal{W}(1) \in \mathcal{L}(L^2(\gamma))$ is the Mellin symbol of W^+ at the point 1 (cf. Theorem 2.1 in [10]). Since $|P|^{-1}$ is a bounded function on Γ and the operator $x \mapsto ([Pr\,x])|_\gamma$ is in $\mathcal{L}(L^2(\Gamma), L^2(\gamma))$, the operator $Res\,W^+\,M\,Pr$ is bounded. Thus A^ρ is bounded, too.

d) Let us consider the assumption ii). Thus we have to prove that $\|(1-\chi^\delta)[A' - P_nA]|_{im\,P_n}\| < \epsilon$. We write $(1-\chi^\delta)[A' - P_nA]|_{im\,P_n} = 2(Te_1 + Te_2)$, where $Te_1 := (1-\chi^\delta)[(P_n1)W_S - P_nW_S]|_{im\,P_n}$ and $Te_2 := (1-\chi^\delta)[1 - P_n1]W_{S'}|_{im\,P_n}$. Let $\chi := \chi(n)$, $\tilde{\chi} := \tilde{\chi}(n)$, and χ^* stand for the characteristic function of the sets $\cup_{PQR}\{\Phi_{PQR}(s,t) : 0 \le s \le 1,\ \tau_{-d} \le t \le \tau_{d+1}\}$, $\cup_{PQR}\{\Phi_{PQR}(s,t) : 0 \le s \le 1,\ \tau_{m(n)-d} \le t \le \tau_{m(n)+d+1}\}$, and $\cup_{PQR}\{\Phi_{PQR}(s,t) : 0 \le s \le 1,\ 1/2 \le t \le 2\}$, respectively. Then we observe that $(1-P_n1)$ is a bounded function and that $\chi(1-P_n1) = (1-P_n1)$ over S'. Hence, for $\|Te_2\| \le \epsilon/4$, it is enough to show that $(1-\chi^\delta)\chi W_S$ has a small norm. By homogeneity arguments, we get $\|(1-\chi^\delta)\chi W_S\| = \|(1-\chi^\delta)\tilde{\chi}W_S\|$ and $\|(1-\chi^\delta)\tilde{\chi}W_S\| = \|\tilde{\chi}(1-\chi^\delta)\chi^*W_S\|$. The last norm, however, tends to zero as $n \to \infty$ since $(1-\chi^\delta)\chi^*W_S$ is compact and the operator of multiplication by $\tilde{\chi} := \tilde{\chi}(n)$ tends strongly to zero.

Let again W^+ denote the integral operator over S with the kernel function $|k(P,Q)|$ which is the absolute value of the kernel of W_S. For Te_1, we conclude

$$\begin{aligned}
[(P_n1)W_S - P_nW_S]P_nf(P) &= \sum_{\iota\in\mathcal{J}}[W_S(P_nf)(P) - W_S(P_nf)(P_\iota)]\varphi_\iota(P),\\
&= \sum_{\iota\in\mathcal{J}} \frac{W_S(P_nf)(P) - W_S(P_nf)(P_\iota)}{W^+(|P_nf|)(P)} W^+(|P_nf|)(P)\varphi_\iota(P),
\end{aligned}$$

$$\begin{aligned}
&\|(1-\chi^\delta)[(P_n1)W_S - P_nW_S]P_nf\|\\
&\quad\le \sup_{\substack{\iota\in\mathcal{J}\\ supp\,\varphi_\iota\cap supp(1-\chi^\delta)\neq\emptyset\\ P\in supp\,\varphi_\iota}} \left|\frac{W_S(P_nf)(P) - W_S(P_nf)(P_\iota)}{W^+(|P_nf|)(P)}\right| \Big\|\sum_{\iota\in\mathcal{J}}|\varphi_\iota|W^+(|P_nf|)\Big\|\\
&\quad\le c \sup_{\substack{\iota\in\mathcal{J}\\ supp\,\varphi_\iota\cap supp(1-\chi^\delta)\neq\emptyset\\ P\in supp\,\varphi_\iota}} \left|\frac{W_S(P_nf)(P) - W_S(P_nf)(P_\iota)}{W^+(|P_nf|)(P)}\right| \|W^+(|P_nf|)\|.
\end{aligned}$$

Hence, we are left with the estimation of the supremum. We conclude

$$[W_S(P_nf)(P) - W_S(P_nf)(P_\iota)] = \int_S [k(P,Q) - k(P_\iota,Q)]P_nf(Q)d_QS,$$

$$|W_S(P_nf)(P) - W_S(P_nf)(P_\iota)| \leq \sup\left|\frac{k(P,Q) - k(P_\iota,Q)}{k(P,Q)}\right| \int_S |k(P,Q)|\,|P_nf(Q)|d_QS,$$

$$\sup_{\substack{\iota\in\mathcal{J} \\ supp\,\varphi_\iota\cap supp(1-\chi^\delta)\neq\emptyset \\ P\in supp\,\varphi_\iota}} \left|\frac{W_S(P_nf)(P) - W_S(P_nf)(P_\iota)}{W^+(|P_nf|)(P)}\right| \leq \sup\left|\frac{k(P,Q) - k(P_\iota,Q)}{k(P,Q)}\right|,$$

where the last sup is taken over any Q, P, P_ι such that $\iota \in \mathcal{J}$, $supp\,\varphi_\iota \cap supp\,(1-\chi^\delta) \neq \emptyset$, $P \in supp\,\varphi_\iota$ and Q is not on the same face of S as P_ι. Now the estimate $\|Te_1\| \leq \epsilon/4$ follows from the fact (cf. proof of Lemma 2.2 ii) in [24]) that the supremum tends to zero if $n \to \infty$.

e) Let us consider the assumption iii). By χ_e^δ we denote the characteristic function of the set of points $P \in S$ such that the distance of P to a given edge e of S is less than $|P|\delta$. Furthermore, let X_2 stand for the linear span of all φ_ι, $\iota \in \mathcal{J}$ such that $\chi_e^\delta(P_\iota) = 1$ but P_ι does not belong to the edge e. Following the proof of Lemma 2.2 iii) in [24], we only have to show $\|P_n\chi_e^\delta W_n'|_{X_2}\| < 1/2$. However, $P_n\chi_e^\delta W_n'|_{X_2} = P_n\chi_e^\delta W_S|_{X_2}$, and it remains to prove $\|P_n\chi_e^\delta W_S|_{X_2}\| < 1/2$. The last estimate follows analogously to the proof of the boundedness in part b) if one takes into account that the norm of the operator $Res\,W_S\,M_\rho\,Pr \in \mathcal{L}(L^2(\Gamma))$ restricted to a sufficiently small neighbourhood of a corner point is less than 1/2 (cf. part c) and step 2 of the proof to Theorem 2.1 in [10]). □

4 The stability at the edge

In the last section we have considered the corner point $(0,0,0)$, the tangent cone $[0,\infty)^3$ of the cube $[0,1]^3$ at (0,0,0) and the corresponding double layer operator over the cone. For this operator, we have shown that the corresponding collocation is stable. The latter collocation operator is a local representative at $(0,0,0)$ for the collocation operator of the method over the cube introduced in Section 2. Similarly, we have to analyse the stability of local representatives at all the other boundary points of the cube. However, if we consider a point in the interior of a face, then the tangent cone is a plane, the double layer operator is the identity and the stability of the collocation method is trivial. Thus it remains to check the case of an edge point. We start with the definition of the corresponding collocation method.

Let us set $\Omega := \mathbf{R} \times [0,\infty)^2$, $S := \partial\Omega$, and choose $P := (0,0,0)$. We set $Q := (1/2,0,0)$ and choose $R := (0,1/2,0)$ or $R := (0,0,1/2)$. Then

$$\Phi_{PQR}: \qquad Hpl \longrightarrow Hpl_{PQR}, \quad Hpl := \{(s,t) \in \mathbf{R}^2 : 0 \leq s < \infty,\ t \in \mathbf{R}\}$$

$$\begin{aligned} Hpl_{PQR} &:= \{P + s\vec{PR} + t\vec{PQ},\ 0 \le s < \infty,\ t \in \mathbf{R}\}, \\ \Phi_{PQR}(s,t) &:= P + s\vec{PR} + t\vec{PQ}. \end{aligned}$$

Retaining the definition of τ_j, s_j, φ_j^1, and ψ_j of Section 2 and setting

$$t_k := k/n,\ k \in \mathbf{Z},\ s_j^3 := \begin{cases} s_j & if\ j \le m(n) \\ \tau_j & else \end{cases},\ \varphi_j^3 := \begin{cases} \varphi_j^1 & if\ j \le m(n) - d - 1 \\ \psi_j & else \end{cases},$$

we introduce the collocation points $P_{PQRjk} := \Phi_{PQR}(s_j^3, t_k)$ for $j = 0, \ldots$ and $k \in \mathbf{Z}$. The spline functions are given by

$$\varphi_{PQRjk}(\Phi_{PQR}(s,t)) \ := \ \varphi_j^3(s)\varphi_k^4(t)\ j = 0, \ldots,\ \ k \in \mathbf{Z},$$

where φ_k^4 is defined for the partition $\{t_k\}$ like ψ_j for $\{\tau_j\}$. Analogously to Section 2, we define the corresponding set $\mathcal{I}$, the spline functions φ_ι, and the points P_ι. Hence, if $A = I + 2W_S$ is the double layer operator defined by (1.1) over the boundary S of the wedge, then the collocation method is defined by (2.1), where $x_n := \sum_{\iota \in \mathcal{I}} \xi_\iota \varphi_\iota$. Again we set $a_{\kappa,\iota} := (A\varphi_\iota)(P_\kappa)$, denote the matrix $(a_{\kappa,\iota})_{\kappa,\iota \in \mathcal{I}}$ by A_n and consider this matrix as an operator in the space of vectors $\{\xi_\iota\}_{\iota \in \mathcal{I}}$ endowed with the norm

$$\|\{\xi_\iota\}\|_* := \|\sum_{\iota \in \mathcal{I}} \xi_\iota \varphi_\iota\|_{L^2(S)}.$$

Theorem 4.1 *The method (2.1) applied to the double layer operator over the boundary of* $\mathbf{R} \times [0,\infty)^2$ *is stable.*

The proof is given in [22].

5 The localization principle

Let us consider the collocation and the notation of Section 2. From the stability of the local representatives in Sections 3-4 we conclude

Theorem 5.1 *The method (2.1) applied to the double layer operator over the boundary of the cube* $[0,1]^3$ *is stable. Moreover, the method (2.1) remains stable if we replace the entries* $a_{\kappa,\iota}$ *by their discretizations* $a'_{\kappa,\iota}$. *In other words, the discretized collocation is stable, too.*

PROOF. a) We start with proving that the discretized collocation operator $A'_n = (a'_{\kappa,\iota})_{\kappa,\iota\in\mathcal{I}}$ is a small perturbations of the collocation operator $A_n = (a_{\kappa,\iota})_{\kappa,\iota\in\mathcal{I}}$. If this is done, then in the following part of the proof we only have to deal with the stability for the collocation without discretization step. Thus let us estimate the error of the quadrature (2.6) applied to the integral in (2.3). Setting $Set := \{\Phi_{PQR}(s,t) : s_j \leq s \leq s_{j+1},\, u_k \leq t \leq u_{k+1}\}$ and denoting the quadrature knots and weights of (2.6) over Set by Q_i and θ_i, respectively, we arrive at

$$\int_{Set} k(P_\kappa, P)\varphi_\iota(P)d_PS - \sum_i k(P_\kappa, Q_i)\varphi_\iota(Q_i)\theta_i = \tag{5.1}$$
$$\int_{Set}[k(P_\kappa, P) - k(P_\kappa, P')]\varphi_\iota(P)d_PS - \sum_i[k(P_\kappa, Q_i) - k(P_\kappa, P')]\varphi_\iota(Q_i)\theta_i,$$

where P' is a fixed point of Set. Note that we have used that our quadrature is exact for the polynomial φ_ι over Set. The first term in (5.1) can be estimated by

$$|\int_{Set}[k(P_\kappa, P) - k(P_\kappa, P')]\varphi_\iota(P)d_PS| \leq$$
$$\int_{Set}|k(P_\kappa, P)|\,|\varphi_\iota(P)|d_PS \sup_{P\in Set}\left|\frac{k(P_\kappa, P') - k(P_\kappa, P)}{k(P_\kappa, P)}\right|.$$

For the second term, we conclude

$$\begin{aligned} |\sum_i[k(P_\kappa, Q_i) - k(P_\kappa, P')]\varphi_\iota(Q_i)\theta_i| &\leq \sup_i\left|\frac{k(P_\kappa, Q_i) - k(P_\kappa, P')}{k(P_\kappa, P')}\right| c|k(P_\kappa, P')|\,|Set|, \\ |k(P_\kappa, P')|\,|Set| &\leq c \sup_{P\in Set}\left|\frac{k(P_\kappa, P')}{k(P_\kappa, P)}\right| \int_{Set}|k(P_\kappa, P)|\,|\varphi_\iota(P)|d_PS. \end{aligned}$$

Repeating the estimations of Section 2 in [24], we get

$$\left|\frac{k(P_\kappa, P) - k(P_\kappa, P')}{k(P_\kappa, P)}\right| \leq c\frac{|P' - P|}{|P_\kappa - P|},$$

where the last ratio is small by the special choice of the partition in Section 2.2. Namely, the introduction of the additional points $t_k \pm s_j,\ j = 0, \ldots, m(n)$ in the definition of the points $u_k,\ k = 0, \ldots, m(n,\kappa)$ guarantees that the diameter of Set is small in comparison to the distance $|P_\kappa - P|$. From these and analogous arguments we derive

$$\sup_{P\in Set}\left|\frac{k(P_\kappa, P') - k(P_\kappa, P)}{k(P_\kappa, P')}\right| \leq c\epsilon, \quad \sup_{P\in Set}\left|\frac{k(P_\kappa, P')}{k(P_\kappa, P)}\right| \leq c, \tag{5.2}$$

$$\sup_{P\in Set}\left|\frac{k(P_\kappa, P') - k(P_\kappa, P)}{k(P_\kappa, P)}\right| \leq c\epsilon, \tag{5.3}$$

for arbitrary $\epsilon > 0$ if only n is sufficiently large. Consequently,

$$(5.4) \qquad |\int_{Set} k(P_\kappa, P)\varphi_\iota(P) d_P S - \sum_i k(P_\kappa, Q_i)\varphi_\iota(Q_i)\theta_i| \le c\epsilon \int_{Set} |k(P_\kappa, P)|\, |\varphi_\iota(P)| d_P S,$$

and $|a_{\kappa,\iota} - a'_{\kappa,\iota}|$ is less than $c\,\epsilon\, b_{\kappa,\iota}$ with

$$b_{\kappa,\iota} := 2\int_S |k(P_\kappa, P)|\, |\varphi_\iota(P)| d_P S.$$

Note that $b_{\kappa,\iota}$ is the entry of a collocation matrix corresponding to the integral operator with the kernel $|k(Q,P)|$. Hence, if this collocation operator is bounded, then the discretized collocation is a small perturbation of the collocation without discretization. The boundedness of the latter collocation operator follows analogously to the boundedness of the original collocation operator defined for A and using the ansatz functions $|\varphi_\iota|$ instead of φ_ι.

b) For any point U of S, we denote the tangent cone by S_U and consider the corresponding collocation for the double layer equation over S_U. We denote the collocation points by P_ι^U, $\iota \in \mathcal{I}^U$, the functions of the interpolation basis by φ_ι^U. More exactly, if $U = (0,0,0)$, then we consider the method introduced in Section 3. If U is another corner, then S_U can be identified with the boundary of $[0,\infty)^3$ (There is a translation and a rotation which maps S_U onto the boundary of $[0,\infty)^3$.). Taking into account this identification, the method of Section 2 is the corresponding collocation over S_U. For U in the interior of a face of S, we take any set of points and any interpolation basis over the plane S_U that coincides in a neighbourhood of U with $\{P_\iota\} \subseteq S$ and the spline basis $\{\varphi_\iota\} \subseteq L^2(S)$, respectively. Using these splines as ansatz functions and these points as collocation knots we get the corresponding method over S_U. Finally let us consider an edge point U. We suppose that $U = \Phi_{PQR}(0, t_U)$, that $\Phi_{PQR}(0, t_{k_U})$ is the collocation point on the edge nearest to U, and let $\Phi^U_{P'Q'R'}$ stand for the corresponding mapping onto the boundary of $\mathbf{R} \times [0,\infty)^2$ introduced in the last section. Then we can map S_U onto the boundary of $\mathbf{R} \times [0,\infty)^2$ with the help of a translation, a rotation and a dilation in such a way that $\Phi_{PQR}(s, t + t_{k_U}) = \Phi^U_{P'Q'R'}(s(t + t_{k_U})/t_{k_U}, t)$, $0 \le s \le 1$. Identifying S_U and the boundary of $\mathbf{R} \times [0,\infty)^2$ with respect to this mapping, we can consider the method of Section 4 to be the corresponding collocation method over S_U.

In any case, let $A_{U,n}$ stand for the matrix of the collocation method over S_U, $P_{U,n}$ for the interpolation operator $P_{U,n}f := \sum_{\iota \in \mathcal{I}^U} f(P_\iota^U)\varphi_\iota^U$ and $\chi_{U,n}$ for the operator $P_{U,n}\chi|_{im P_{U,n}}$ if χ_U is a function over S_U. Analogously we define P_n and χ_n over S. Moreover, we denote the orthogonal projection onto the spaces $im\, P_{U,n} \subseteq L^2(S_U)$ and $im\, P_n \subseteq L^2(S)$ by $Q_{U,n}$ and Q_n, respectively. Now our theorem follows from the proof of Lemma 3.2 in [24] and the well-known Gohberg-Krupnik localization principle ([12]). We only have to verify the following assumptions.

i) The operator $A_n Q_n$ tends strongly to A.

ii) If $U \in S$ and χ_U is a smooth function over S_U with finite limit at ∞, then there is a compact operator $T_U \in \mathcal{L}(L^2(S))$ such that

$$[A_{U,n}, \chi_{U,n}] := A_{U,n}\chi_{U,n} - \chi_{U,n}A_{U,n} = Q_{U,n}T_U|_{im\, P_{U,n}} + o(1) \;\; (n \to \infty).$$

iii) If χ is a smooth function over S, then there is a compact operator $T \in \mathcal{L}(L^2(S_U))$ such that

$$[A_n, \chi_n] := Q_n T|_{im\, P_n} + o(1) \;\; (n \to \infty).$$

iv) For any $V \in S$ and any $\epsilon > 0$, there is a neighbourhood $N_V \subseteq S \cap S_V$ of V such that

$$\|\chi_n A_n \chi_n - \chi_n A_{V,n} \chi_n\| \le \epsilon \tag{5.5}$$

if χ is a smooth function with $|\chi| \le 1$ and support in N_V.

v) For any $V \in S$, the method with the approximate operator $A_{V,n} \in \mathcal{L}(im\, P_{V,n})$ is stable.

c) Assumption v) follows from the last two sections. So let us start with proving iv). If V is a vertex, then we choose $N_V \subseteq \cup_{Q,R}\{\Phi_{VQR}(s,t) : 0 \le s \le 1,\ 0 \le t \le 1/2\}$ and the norm in (5.5) is even zero. For a point V in the interior of a face, we choose N_V on this face. Since the tangent cone S_V at V is a plane and $W_{S_V} = 0$, we get $A_{V,n} := Id$ and (5.5) holds again with $\epsilon = 0$. Thus we may suppose that V is an edge point, and without loss of generality, we assume $V = (1/2, 0, 0)$. For this special situation, we get (cf. part b)) $V = \Phi_{PQR_1}(0,1) = \Phi_{PQR_2}(0,1)$ with $P = (0,0,0)$, $Q = (1,0,0)$, $R_1 = (1/2, 1/2, 0)$, $R_2 = (1/2, 0, 1/2)$ and $V = \Phi^V_{VQ'R_1}(0,0) = \Phi^V_{VQ'R_2}(0,0)$ with $Q' = (1.5, 0, 0)$. Let P_{PQR_ljk} and φ_{PQR_ljk} be defined as in Section 2 and denote the corresponding points and functions introduced in Section 4 by $P^V_{VQ'R_ljk}$ and $\varphi^V_{VQ'R_ljk}$, respectively. Defining $\Phi : Hpl_{VQ'R_1} \cup Hpl_{VQ'R_2} \hookrightarrow Tr_{PQR_1} \cup Tr_{PQR_2}$ by

$$\begin{aligned} \Phi\left(\Phi^V_{VQ'R_1}(s,t)\right) &:= \Phi_{PQR_1}(s, \varphi(t)),\ \ \Phi\left(\Phi^V_{VQ'R_2}(s,t)\right) := \Phi_{PQR_2}(s,\varphi(t)), \\ \varphi(t) &:= \begin{cases} [(1-1/n)^n]^{-t} & if\ t \le 0 \\ 2 - [(1-1/n)^n]^t & if\ t \ge 0 \end{cases}, \end{aligned}$$

we arrive at $\Phi(P^V_{VQ'R_ljk}) = P_{PQR_lj(m(n)+k)}$. We identify $\varphi^V_{VQ'Rjk}$ with $\varphi_{PQRj(m(n)+k)}$, and the term $\chi_n A_{V,n} \chi_n$ in (5.5) is to be understood via this identification. Thus we have to estimate $a^V_{\kappa^V,\iota^V} - a_{\kappa,\iota}$, where $a^V_{\kappa^V,\iota^V} := (I + 2W_{S^V})\varphi^V_{\iota^V}(P^V_{\kappa^V})$, $a_{\kappa,\iota} := A\varphi_\iota(P_\kappa)$, and the indices $\kappa^V, \iota^V \in \mathcal{I}^{\mathcal{V}}$, $\kappa, \iota \in \mathcal{I}$ are connected by $(VQ'R_2j'k') \in \kappa^V$, $(PQR_2j'(k' + m(n))) \in \kappa$ and $(VQ'R_1jk) \in \iota^V$, $(PQR_1j(k + m(n))) \in \iota$. Moreover, since we consider the points and functions from a small neighbourhood of V, we assume $0 \le j, j' \le \delta m(n)$ and

$|k'-k| \leq \delta m(n)$ for a small number $\delta > 0$. For simplicity, let us also suppose $0 < j,\ 0 < j'$ and $k' > k$. Now we observe that the matrices $(a^V_{\kappa^V,\iota^V})_{\kappa^V,\iota^V}$ and $(a_{\kappa,\iota})_{\kappa,\iota}$ are Toeplitz matrices (cf. (3.4) and Section 4.2 of [22]). Hence, without loss of generality, we may suppose $k' = 0$ and get $P^V_{\kappa^V} = P_\kappa$. Furthermore, we conclude

$$
\begin{aligned}
a_{\kappa,\iota} &= 2\int k(P_\kappa, P)\varphi_\iota(P) d_P S = 2\int k(P_\kappa, \Phi(P))\varphi_\iota \circ \Phi(P) J_\Phi(P) d_P S, \\
\text{(5.6)}\quad a_{\kappa,\iota} - a^V_{\kappa^V,\iota^V} &= 2\int [k(P_\kappa, \Phi(P)) J_\Phi(P) - k(P_\kappa, P)]\varphi_\iota \circ \Phi(P) d_P S + \\
&\quad\ 2\int k(P^V_{\kappa^V}, P)[\varphi_\iota \circ \Phi(P) - \varphi^V_{\iota^V}(P)] d_P S.
\end{aligned}
$$

Taking into account that the derivative of Φ at V is the identity, we obtain $|J_\Phi(P)-1| \leq \epsilon$ and $|\Phi(P) - P| \leq \epsilon|P - V|$ for δ small enough. From this it is not hard to conclude (cf. Section 2 of [24]) that

$$
\sup_{P \in supp\, \varphi^V_{\iota^V}} \left| \frac{k(P_\kappa, \Phi(P)) - k(P_\kappa, P)}{k(P_\kappa, P)} \right| \leq c\,\epsilon
$$

for any prescribed $\epsilon > 0$ and δ sufficiently small. On the other hand, it is also not hard to derive

$$
\sup_{P \in supp\, \varphi^V_{\iota^V}} \left| \frac{\varphi^V_{\iota^V}(P)}{\varphi_\iota \circ \Phi(P)} \right| \leq c, \qquad \sup_{P \in supp\, \varphi^V_{\iota^V}} \left| \frac{\varphi_\iota \circ \Phi(P) - \varphi^V_{\iota^V}(P)}{\varphi_\iota \circ \Phi(P)} \right| \leq \epsilon
$$

for sufficiently small δ. Hence, both terms on the right-hand side of (5.6) are less than $c\,\epsilon \int |k(P^V_{\kappa^V}, P)|\, |\varphi^V_{\iota^V}(P)| d_P S$, i.e., by $c\,\epsilon$ times the entry of a bounded collocation matrix (for the boundedness cf. the arguments in Section 4 of [22]). Thus $A_{V,n}$ is a small perturbation of A_n in the neighbourhood of V, and iv) follows.

d) Let us show i). We first prove that the approximate operators $A_n := P_n A|_{im\, P_n}$ are uniformly bounded with respect to n. Note that the restriction of W_S acting between $S_1 \subseteq S$ and $S_2 \subseteq S$ is a compact and smoothing operator if the distance between S_1 and S_2 is positive. Thus it is enough to show that, for any $V \in S$, there is a small neighbourhood N_V of V such that A_n restricted to N_V is bounded. This, however, follows from the boundedness of the local representatives and the assertion iv) which was just proved. Knowing the uniform boundedness of A_n, it remains to show $A_n Q_n f \to Af$ for any f from a dense subset of $L^2(S)$. Hence, we may suppose that f is smooth and vanishes in a neighbourhood N_f of the edges. Again, W_S restricted to $S \setminus N_f$ is smoothing, and $A_n Q_n f \to Af$ follows easily.

e) Let us prove iii). We obtain

$$
\begin{aligned}
\text{(5.7)}\qquad [A_n, \chi_n] &= P_n A P_n \chi|_{im\, P_n} - P_n \chi P_n A|_{im\, P_n} \\
&= Q_n T|_{im\, P_n} + \{P_n - Q_n\} T|_{im\, P_n} + P_n A\{P_n - I\}\chi|_{im\, P_n},
\end{aligned}
$$

where $T := [A, \chi] = A\chi - \chi A$ is bounded from $L^2(S)$ to $C(S)$ and compact as an operator acting in $L^2(S)$. (Note that the kernel $l(P,Q)$ of $A\chi - \chi A$ is bounded by $c\delta|P-Q|^{-2}$, where δ denotes the distance from P to the plane Pl containing the face F of S such that $Q \in F$. Hence, $\int_F |l(P,Q)|^2 d_Q S$ is less than $c\,\delta^2 \int_F |P-Q|^{-4} \le c\,\delta^2 \int_{\delta\le r} r^{-3} dr \le c$.) Consequently, $\|\{P_n - Q_n\}T|_{im\,P_n}\| \to 0$. Namely, for a given $\epsilon > 0$, let χ_1 denote the characteristic function of a small neighbourhood of the set of all edge points such that $\int_S \chi_1 \le \epsilon$. Then we arrive at

$$\begin{aligned}\|\{P_n - Q_n\}TP_nf\|_{L^2}^2 &\le \int |\chi_1\{P_n - Q_n\}TP_nf|^2 + \int |[1-\chi_1]\{P_n - Q_n\}TP_nf|^2 \\ &\le \epsilon\,\|\{P_n - Q_n\}TP_nf\|_{L^\infty}^2 + \|[1-\chi_1]\{P_n - Q_n\}TP_nf\|_{L^2}^2 \\ &\le c\,\epsilon\,\|P_nf\|_{L^2}^2 + \|[1-\chi_1]\{P_n - Q_n\}TP_nf\|_{L^2}^2,\end{aligned}$$

where the second term on the right-hand side is bounded by $\|[1-\chi_1]\{P_n - Q_n\}T|_{im\,P_n}\|^2 \times \|P_nf\|_{L^2}^2$. However, $\|[1-\chi_1]\{P_n - Q_n\}T|_{im\,P_n}\|$ tends to zero since $[1-\chi_1]T$ is a compact and smoothing operator.

On the other hand, $P_nA\{P_n - I\}\chi|_{im\,P_n} = 2P_nW_S\{P_n - I\}\chi|_{im\,P_n}$ and $|\{P_n - I\}\chi P_nf(P)| \le \frac{c}{n}|P_nf(P)|$. Hence,

$$\|P_nW_S\{P_n - I\}\chi P_nf\| \le \frac{c}{n}\|P_n^+W^+|P_n^+f|\,\|,$$

where $P_n^+f := \sum_{\iota\in\mathcal{I}} f(P_\iota)|\varphi_\iota|$. Since $\|\,|P_n^+f|\,\| \le c\|P_nf\|$ and the collocation operator $P_n^+W^+|_{imP_n^+}$ is bounded (repeat the arguments of part d)), the third term on the right-hand side of (5.7) tends to zero, too. Thus (5.7) implies iii).

f) Using the arguments of the proof of Lemma 3.2 ii) in [24], assertion ii) will follow analogously to part e). □

6 The asymptotic convergence rate

Theorem 6.1 *Suppose that the right-hand side y of the boundary integral equation $Ax = y$ is continuous on S and a C^∞- function up to the boundary on each face of S. Let x_n denote the solution of the discretized collocation, i.e., x_n satisfies the modified equation (2.1), where the entry $a_{\kappa,\iota}$ is replaced by $a'_{\kappa,\iota}$ defined in (2.6). Then there is a constant independent of n such that*

$$\|x - x_n\|_{L^2(S)} \le c\,\max\{n^{-\nu/2}\sqrt{\log\,n}, n^{-(2d+2)}\}.$$

Especially, if the parameter ν in the definition of the grid is greater than $4d+4$, then we get the estimate $\|x - x_n\|_{L^2(S)} \le c\,n^{-(2d+2)}$.

Note that the number of collocation points, i.e., the number of equations in (2.1) is of order $n^2 \log^2 n$. The computation of the stiffness matrix requires no more than $O(n^4 \log^4 n)$ operations. Consequently, if we imply our discretized collocation (2.1) together with the Gaussian algorithm, we need to perform $O(n^6 \log^6 n)$ operations. In order to prove the estimate of Theorem 6.1, let us start with some lemmata.

Lemma 6.1 *For the exact solution $x = A^{-1}y$ and the triangular coordinate transformation Φ_{PQR} of Section 2, we get*

$$\sup_{(s,t)\in Sq} |s^l \partial_s^l (x \circ \Phi_{PQR})(s,t)| \leq c, \tag{6.1}$$

$$\sup_{(s,t)\in Sq,\, t\leq 1} |t^l \partial_t^l (x \circ \Phi_{PQR})(s,t)| \leq c, \; l = 0,\ldots,2d+2. \tag{6.2}$$

Proof. It is enough to prove the assertion of the lemma for the case of the tangent cone $S = S_P$ of a corner point P (Note that the restriction of W_S acting between subsets of S with positive distance is a smoothing operator. Moreover, if we consider the neighbourhood of an edge point, then the following arguments can also be applied if the Mellin transform is replaced by the Fourier transform.). First we suppose that y satisfies the estimates of x in (6.1) and (6.2) and we shall prove that $W_S y$ satisfies them , too. Let us consider the polar coordinates $t = r \cos\varphi$ and $ts = r \sin\varphi$ over $\Phi_{PQR}(Sq) \subseteq S$. Then we get $\partial_t = (1/\cos\varphi)\,\partial_r$ and $\partial_s = r \sin\varphi \cos\varphi\, \partial_r + \cos^2\varphi\, \partial_\varphi$ and it remains to prove that the derivative $(r\,\partial_r)^k (\sin\varphi\,\partial_\varphi)^l W_S y$ is bounded for $k+l \leq 2d+2$. Let R' denote the centre of the face of S which intersects the face containing R in the edge PQ. For simplicity, we assume that y vanishes outside of $\Phi_{PQR'}(Sq)$ and that $V := \Phi_{PQR}(s,t)$, $t = r\cos\varphi < 1$, $ts = r\sin\varphi$, $U := \Phi_{PQR'}(s',t')$, $t' = \rho\cos\psi < 1$, $t's' = \rho\sin\psi$, and $y(\rho,\psi) := y(U)$. We arrive at

$$\begin{aligned} W_S y(V) &= \frac{1}{4\pi}\int_0^\infty \int_0^{\pi/2} \frac{r \sin\varphi\, y(\rho,\psi)}{\sqrt{(r\cos\varphi - \rho\cos\psi)^2 + (\rho\sin\psi)^2 + (r\sin\varphi)^2}^3}\rho d\rho d\psi \\ &= \frac{1}{4\pi}\int_0^{\pi/2} \int_0^\infty \frac{r/\rho \sin\varphi\, y(\rho,\psi)}{\sqrt{(r/\rho)^2 + 1 - 2(r/\rho)[\cos\varphi\cos\psi]}^3}\frac{d\rho}{\rho}d\psi. \end{aligned}$$

Since the kernel takes the form of a Mellin convolution with respect to r and the differential operator $r\partial_r$ commutes with the Mellin convolution (cf. the proof of Theorem 3.1 in [10]), we conclude

$$(r\partial_r)^k W_S y(V) = \frac{1}{4\pi}\int_0^{\pi/2} \int_0^\infty \frac{r/\rho \sin\varphi\,(\rho\partial_\rho)^k y(\rho,\psi)}{\sqrt{(r/\rho)^2 + 1 - 2(r/\rho)[\cos\varphi\cos\psi]}^3}\frac{d\rho}{\rho}d\psi,$$

and we are left with proving the boundedness of $(\sin\varphi\,\partial_\varphi)^l W_S y$. However, if we denote the kernel of the integral operator by k, then it is not hard to show $|(\sin\varphi\,\partial_\varphi)^l k| \le c\,|k|$. Thus the boundedness of the integral operator $W^+ \in \mathcal{L}(C(S))$ with the kernel function $|k|$ implies that $(\sin\varphi\,\partial_\varphi)^l W_S y$ is bounded.

Now consider $x = A^{-1}y$. We get $A^{-1} = (I + 2W_S)^{-1} = I - 2W_S A^{-1}$ and, using the fact that $(r\,\partial_r)$ commutes with W_S and A^{-1}, we conclude

$$(r\,\partial_r)^k(\sin\varphi\,\partial_\varphi)^l x = (r\,\partial_r)^k(\sin\varphi\,\partial_\varphi)^l y - 2(\sin\varphi\,\partial_\varphi)^l W_S A^{-1}(r\,\partial_r)^k y.$$

Obviously, the first term on the right-hand side is bounded. Moreover, $A^{-1}(r\,\partial_r)^k y$ is bounded and, again, $|(\sin\varphi\,\partial_\varphi)^l k| \le c\,|k|$ implies that the second term on the right-hand side of the last equation is bounded. Thus the proof is finished. □

Lemma 6.2 *For any x satisfying (6.1) and (6.2), there holds*

$$\|x - P_n x\|_{L^2(S)} \le c n^{-\min\{\nu/2, 2d+2\}}.$$

PROOF. Let χ stand for the characteristic function of the union over PQR of all sets

$$\{\Phi_{PQR}(s,t) : 0 \le s \le s_1,\ 0 \le t \le 2 \text{ or } 0 \le s \le 1,\ 0 \le \min[t, 2-t] \le t_d\}.$$

Then we get $\|x - P_n x\|_{L^\infty(S)} \le c$ and

$$\begin{aligned} \|\chi(I - P_n)x\|_{L^2(S)} &\le \|\chi\|_{L^2(S)}\|x - P_n x\|_{L^\infty(S)} \\ &\le c\tau_0^{1/2} \le c n^{-\nu/2}. \end{aligned} \tag{6.3}$$

The remaining term $\|(1-\chi)(I-P_n)x\|_{L^2(S)}$ can be estimated by $c\,\|(1-\chi)(I-P_n)x\|_{L^\infty(S)}$. Thus we only have to estimate $|(I - P_n)x(V)|$ for $V = \Phi_{PQR}(s,t)$ with $s_j \le s \le s_{j+1}$, $t_k \le t \le t_{k+1}$ and $1 \le j,\ d \le k$. For simplicity, suppose even $d+1 \le j$ and $t_{k+1} \le 1$. We obtain

$$\begin{aligned} (I - P_n)x(V) &= x \circ \Phi_{PQR}(s,t) - \sum_{j'=j-d}^{j+d}\sum_{k'=k-d}^{k+d} x \circ \Phi_{PQR}(s_{j'}, t_{k'})\psi_{j'}(s)\varphi^{2,j}(\Phi_{PQR}(s_{j'}, t)) \\ &= Te_1 + Te_2, \end{aligned}$$

$$\begin{aligned} Te_1 &:= x \circ \Phi_{PQR}(s,t) - \sum_{j'=j-d}^{j+d} x \circ \Phi_{PQR}(s_{j'}, t)\psi_{j'}(s), \\ Te_2 &:= \sum_{j'=j-d}^{j+d} \psi_{j'}(s)\left\{ x \circ \Phi_{PQR}(s_{j'}, t) - \sum_{k'=k-d}^{k+d} x \circ \Phi_{PQR}(s_{j'}, t_{k'})\varphi^{2,j}(\Phi_{PQR}(s_{j'}, t)) \right\}. \end{aligned}$$

From the definition of $\psi_{j'}$ we conclude

$$\begin{aligned}|Te_1| &\leq c(s_{j+1}-s_j)^{2d+2}\sup_{s_{j-d}\leq s\leq s_{j+d+1}}|\partial_s^{2d+2}(x\circ\Phi_{PQR}(s,t))|\\ &\leq c\left(\frac{s_{j+1}-s_j}{s_j}\right)^{2d+2}\sup_{0\leq s\leq 1}|s^{2d+2}\partial_s^{2d+2}(x\circ\Phi_{PQR}(s,t))|\leq cn^{-(2d+2)}.\end{aligned}$$

On the other hand, let us set

$$Pro(f) := \sum_{k'=k-d}^{k+d} f(\Phi_{PQR}(s_{j'},t_{k'}))\varphi^{2,j}(\Phi_{PQR}(s_{j'},t))$$

and let $Tay(f)$ stand for the real part of the sum of the first $2d+1$ terms of the Taylor series expansion of f at the point $\Phi_{PQR}(s_{j'},t_k)$. Then Pro is a bounded projection into the space of the real parts of polynomials, and we obtain

$$\begin{aligned}|f-Pro(f)| &= |f-Tay(f)+Pro[f-Tay(f)]|\\ &\leq c\|f-Tay(f)\|_{L^\infty}\leq c(t_{k+1}-t_k)^{(2d+2)}\|\partial_t^{2d+2}f\|_{L^\infty}.\end{aligned}$$

Using this, we arrive at

$$\begin{aligned}|Te_2| &\leq c(t_{k+1}-t_k)^{2d+2}\sup_{j'=j-d,\dots,j+d}\ \sup_{t_{k_d}\leq t\leq t_{k+d+1}}|\partial_t^{2d+2}(x\circ\Phi_{PQR}(s_{j'},t))|\\ &\leq c\left(\frac{t_{k+1}-t_k}{t_k}\right)^{2d+2}\sup_{0\leq t\leq 1}|t^{2d+2}\partial_t^{2d+2}(x\circ\Phi_{PQR}(s,t))|\leq cn^{-(2d+2)},\end{aligned}$$

and the proof is finished. □

PROOF OF THEOREM 6.1. During this proof retain the notation of χ from the proof of Lemma 6.2 and let $A_n\in\mathcal{L}(imP_n)$ stand for the discretized collocation operator, i.e., for the approximation of $P_nA|_{imP_n}$, where the entries $a_{\kappa,\iota}$ of the matrix corresponding to the basis $\{\varphi_\iota\}$ are replaced by $a'_{\kappa,\iota}$ which is defined in (2.6). Then, taking into account the stability, we obtain

$$\begin{aligned}x-x_n &= x-P_nx+A_n^{-1}\{A_nP_nx-A_nx_n\}\\ \|x-x_n\|_{L^2} &\leq \|x-P_nx\|_{L^2}+c\|A_nP_nx-P_nAx\|_{L^2}\\ &\leq Te_1+c\{Te_2+Te_3+Te_4+Te_5\}\\ Te_1 &:= \|x-P_nx\|_{L^2},\ Te_2:=\|A_nP_n\chi x\|_{L^2},\ Te_3:=\|P_nA\chi x\|_{L^2},\\ Te_4 &:= \|(A_nP_n-P_nAP_n)(1-\chi)x\|_{L^2},\ Te_5:=\|P_nA(I-P_n)(1-\chi)x\|_{L^2}.\end{aligned}$$

Here Te_1 is bounded as in Lemma 6.2. The boundedness of A_n implies $Te_2 \leq c\|P_n\chi x\|_{L^2}$ which can be estimated analogously to (6.3). Furthermore, the term Te_5 can be estimated by $c\|P_nA(I-P_n)(1-\chi)x\|_{L^\infty}$, and we conclude $Te_5 \leq c\|(I-P_n)(1-\chi)x\|_{L^\infty}$, where the last norm was estimated in the proof of Lemma 6.2. For Te_3 and $(PQRjk) \in \kappa$ with $k \leq m(n)$, we obtain

$$\begin{aligned} Te_3^2 &\leq \sum_{\kappa\in\mathcal{I}} \|\varphi_\kappa\|_{L^2}^2 |(A\chi x)(P_\kappa)|^2 \\ &\leq c\sum_{\kappa\in\mathcal{I}} (\tau_{k+1}-\tau_k)\tau_k(\tau_{j+1}-\tau_j)\,\|x\|_{L^\infty} \begin{cases} \{1+2W^+\chi(P_\kappa)\}^2 & if\ P_\kappa \in supp\,\chi \\ \{2W^+\chi(P_\kappa)\}^2 & else \end{cases} \end{aligned}$$

Since $W^+\chi(P_\kappa)$ is the solid angle under which $supp\,\chi$ is seen from P_κ, i.e., the angle under which a strip of width $\leq \tau_1$ is seen from a distance $\geq \tau_k\tau_j$, we get $W^+\chi(P_\kappa) \leq c\tau_1/(\tau_k\tau_j)$. Consequently,

$$\begin{aligned} Te_3^2 &\leq c \sum_{\kappa\in\mathcal{I},\, P_\kappa\in supp\,\chi} \|\varphi_\kappa\|_{L^2}^2 + c\sum_{\kappa\in\mathcal{I}} \frac{\tau_{k+1}-\tau_k}{\tau_k}\frac{\tau_{j+1}-\tau_j}{\tau_j}\frac{\tau_1^2}{\tau_j} \\ &\leq cn^{-\nu} + c\sum_{j,k=0}^{\nu n[\log n]} \frac{1}{n}\frac{1}{n}(1-1/n)^{2\nu n[\log n]}(1-1/n)^{-\nu n[\log n]+j} \\ &\leq c\,[\log n]\,n^{-\nu}. \end{aligned}$$

Instead of Te_4 we shall estimate $\|(A_nP_n - P_nAP_n)(1-\chi)x\|_{L^\infty}$. In other words, we only have to estimate $|(A_nP_n - P_nAP_n)(1-\chi)x(P_\kappa)|$. Setting $Set_{PQRjk} := \{\Phi_{PQR}(s,t) : s_j \leq s \leq s_{j+1},\, u_k \leq t \leq u_{k+1}\}$ and denoting the quadrature knots and weights of (2.6) over Set_{PQRjk} by Q_i and θ_i, respectively, we arrive at

$$\begin{aligned} Err &:= \Big|\int_{Set_{PQRjk}} k(P_\kappa,V)P_nx(V)d_VS - \sum_i k(P_\kappa,Q_i)P_nx(Q_i)\theta_i\Big| \\ &\leq |Set_{PQRjk}|\Big\{\sup_{V\in Set_{PQRjk}} |\partial_s^{2d+2}[k(P_\kappa,V)P_nx(V)]|(s_{j+1}-s_j)^{2d+2} \\ &\qquad + \sup_{V\in Set_{PQRjk}} |\partial_t^{2d+2}[k(P_\kappa,V)P_nx(V)]|(u_{k+1}-u_k)^{2d+2}\Big\} \end{aligned}$$

Without loss of generality, let us assume that $V := \Phi_{PQR}(s,t)$, $t = r\cos\varphi < 1$, $ts = r\sin\varphi$, $P_\kappa := \Phi_{PQ'R'}(s',t')$, $t' = \rho\cos\psi$, $t's' = \rho\sin\psi$. For the kernel k of W_S, it is not hard to derive (cf. the proof of Lemma 6.1) that $|\partial_s^l k| \leq cs^{-l}|k|$ and $|\partial_t^l k| \leq ct^{-l}|k|$. Moreover, from Lemma 6.1 we infer that $|\partial_s^l P_nx| \leq c\,|\partial_s^l x| \leq c\,s^{-l}$ and $|\partial_t^l P_nx| \leq c\,|\partial_t^l x| \leq ct^{-l}$. If Set_{PQRjk} is of positive distance to the set of edge points, then we choose a point $P' \in Set_{PQRjk}$ and obtain

$$
\begin{aligned}
Err &\le c\,|Set_{PQRjk}|\,|k(P_\kappa, P')| \left\{ \left| \frac{s_{j+1}-s_j}{s_j} \right|^{2d+2} + \left| \frac{u_{k+1}-u_k}{u_k} \right|^{2d+2} \right\} \\
&\le c\,|Set_{PQRjk}|\,|k(P_\kappa, P')|\, n^{-(2d+2)}.
\end{aligned}
$$

Using (5.2) and summing up over all Set_{PQRjk} with positive distance to the edges leads to

$$
\begin{aligned}
|(A_nP_n - P_nAP_n)(1-\chi)x(P_\kappa)| &\le \sum Err \le cn^{-(2d+2)} \sum |Set_{PQRjk}|\,|k(P_\kappa, P')| \\
&\le cn^{-(2d+2)} \int_S |k(P_\kappa, \cdot)| \le c\, n^{-(2d+2)}.
\end{aligned}
$$

□

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A.Rathsfeld
Institut für Angewandte Analysis und Stochastik,
Hausvogteiplatz 5-7,
D-O-1086 Berlin, Germany

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