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Duality formula for the bridges of a Brownian diffusion. Application to gradient drifts.

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Abstract

In this paper we consider families of time Markov fields (or reciprocal classes) which have the same bridges as a Brownian diffusion. We characterize each class as the set of solutions of an integration by parts formula on the space of continuous paths $C([0;1];\mathbb{R}^d)$. Our techniques provide a characterization of gradient diffusions by a duality formula and, in case of reversibility, a generalization of a result of Kolmogorov.

1 Introduction

In this paper we characterize the bridges of a Brownian diffusion as solutions of a simple integration by parts formula (IBPF for short) on the space of continuous paths $C([0;1];\mathbb{R}^d)$. More precisely, our object of study is the class of all probabilities on the path space which have the same bridges as a reference Brownian diffusion; this class is called the *reciprocal class* of the reference diffusion. This is the continuation of the work we have undertaken in our former publication [20]; the setting of [20] was one-dimensional, in the sense that $d = 1$. We now turn to vectorial case, $d > 1$, which requires new techniques and provides broader applications.

Let us briefly describe our framework. The terminology of reciprocal class comes from reciprocal processes; these are Markovian fields with respect to the time parameter and therefore a generalization of Markov processes. The interest in these processes was motivated at first by a Conference of Schrödinger [24] about the most probable dynamics for a Brownian particle whose laws at initial and final times are given. Actually, Schrödinger was only concerned with Markovian reciprocal processes which have been called since then Schrödinger processes. His interpretation in terms of (large) deviations from an expected behavior was further developed by Föllmer, Cattiaux and Léonard, Gantert among others (cf. references [9], [3] and [10]). Schrödinger processes were also analysed by Zambrini [28] and Nagasawa [18] for their possible connections to quantum mechanics. One year after Schrödinger, Bernstein noticed the importance of non-Markovian processes with given conditional dynamics, where the conditioning is made at two fixed times. This was the beginning of the study of general reciprocal processes. Jamison [11] proved that the set of reciprocal processes is partitioned into classes called reciprocal classes. All the elements of a same class share the same Markovian bridges (or two times conditional probability distributions). Each class is characterized by two functions (F, G) (defined explicitely in Theorem 2.6 below) which take values respectively in $\mathbb{R}^d$ and $\mathbb{R}^{d\otimes d}$ called its *Reciprocal Characteristics* ([5], [14]) and can be defined starting from a reference Markovian Brownian diffusion. Krener (cf. [14]) raised the question of characterizing a reciprocal class by an equation involving (F, G). For Gaussian reciprocal processes an answer was given in [16]: the equation was a p.d.e. for the covariance function. The non Gaussian case was addressed in [25] by one of us: using the tools of Stochastic Mechanics, it was proved that the elements of a reciprocal class satisfy a stochastic Newton equation. In this equation by analogy with the Lorentz law of electromagnetism G can be interpreted as a magnetic force and F as an electric force (see also [13]).

Our main result in the present paper states that, under the assumption of finite entropy, the set of probability measures in the reciprocal class of a Brownian diffusion, coincides with the set of solutions of a functional equation the coefficients of which are F and G. Our equation is a perturbation of the duality equation satisfied by Brownian bridges, duality between the Malliavin derivation operator and the stochastic integral. The perturbation term in the equation is to be compared with the vector of Malliavin derivatives of the Hamiltonian function associated to Gibbs measures ([22]). The main difference from the one dimensional situation comes from an additional term in the IBPF. This term, which is the stochastic integral of the reciprocal characteristic G w.r.t. the coordinate process, vanishes if and only if the drift of the reference Brownian diffusion is a gradient. In [20] this term was identically zero since the gradient condition is always fulfilled in dimension $d = 1$.

The tools developed to reach the above result enable us on the one hand to characterize the laws of Brownian diffusions which are of gradient type among the set of reciprocal processes satisfying some IBPF. On the other hand we prove a generalization of Kolmogorov's theorem: the existence of a reversible law in the reciprocal class of a Brownian diffusion with drift b can only occur if b is a gradient.

The paper is divided into the following sections.

1. Introduction.
2. Brownian bridges. Reciprocal classes.
3. Integration by parts formula for a Brownian diffusion and its bridges.
4. Characterization of a reciprocal class by an IBPF.
5. Application to gradient diffusions.

2 Brownian bridges. Reciprocal classes.

2.1 Derivation operator

Let $\Omega = \mathcal{C}([0,1]; \mathbb{R}^d)$ be the canonical - polish - path space of continuous $\mathbb{R}^d$-valued functions on $[0,1]$, endowed with $\mathcal{F}$, the canonical σ-field.

Let $(X_t)_{t\in[0,1]}$ denote the family of canonical projections from Ω into $\mathbb{R}^d$.
$\mathcal{P}(\Omega)$ is the set of probability measures on Ω. We use the notation :

$$Q(f) = \int_\Omega f(\omega)Q(d\omega).$$

Let $P \in \mathcal{P}(\Omega)$ denote a fixed Wiener measure on Ω with initial measure any probability measure on $\mathbb{R}^d$. We denote by P^x the Wiener measure on Ω with initial condition $x \in \mathbb{R}^d$. More generally, for any Q in $\mathcal{P}(\Omega)$, Q^x is the conditional measure $Q(./X_0 = x)$, and $Q^{x,y}$ is the conditional measure $Q(./X_0 = x, X_1 = y)$ (bridge between x and y).

We will denote by $|\cdot|$ the euclidian norm in $\mathbb{R}^d$ and $x.y$ will denote the scalar product between x and y, two vectors in $\mathbb{R}^d$.

We now define the space of smooth cylindrical functionals on Ω :

$$\begin{aligned} \mathcal{S} &= \{\Phi; \Phi(\omega) = \varphi(\omega^i_{t_j}, 1 \le i \le d, 1 \le j \le n), \\ &\quad \varphi \in C^\infty_b(\mathbb{R}^{nd};\mathbb{R}), \quad 0 \le t_1 \le \ldots \le t_n \le 1\}. \end{aligned}$$

where $C^\infty_b(\mathbb{R}^{nd};\mathbb{R})$ denotes the set of C^∞-functions which are bounded as well as all their derivatives.

Clearly $\mathcal{S} \subset L^2(\Omega; P)$. For $0 \le \tau \le 1$, we denote by $\mathcal{S}_\tau$ the subset of $\mathcal{S}$ composed by the functionals which are $\mathcal{F}_\tau$-measurable.
On $\mathcal{S}$ we denote by D_g the *derivation operator* in the direction $g = (g^i)_{1\le i\le d} \in L^2([0,1];\mathbb{R}^d)$ defined as follows : $D_g\Phi = (D_{g^i}\Phi)_{1\le i\le d}$ where

$$\begin{aligned} D_{g^i}\Phi(\omega) &= \sum_{j=1}^n \frac{\partial\varphi}{\partial x^i_j}(\omega^1_{t_1},\ldots,\omega^d_{t_1},\ldots,\omega^1_{t_n},\ldots,\omega^d_{t_n}) \int_0^{t_j} g^i(t)dt \\ &= \int_0^1 g^i(t) D^i_t\Phi(\omega)dt \end{aligned}$$

where

$$D^i_t\Phi(\omega) = \sum_{j=1}^n \frac{\partial\varphi}{\partial x^i_j}(\omega^1_{t_1},\ldots,\omega^d_{t_n})\mathbf{1}_{t\le t_j}.$$

It is clear that $D_g\Phi$ is also equal to the Gâteaux-derivative of Φ in the direction $\int_0^\cdot g(t)dt$, which is a typical element of the Cameron-Martin space. One also defines the space $\mathbf{D}^{1,2}$ as the closure of $\mathcal{S}$ for the following norm :

$$\|\Phi\|^2_{1,2} = P(\Phi^2) + P\Big(\int_0^1 |D_t\Phi|^2 dt\Big).$$

Let us introduce the notation we will use for stochastic integrals all through the rest of the paper. For $g = (g^i)_{1\le i\le d} \in L^2([0,1];\mathbb{R}^d)$, the vectorial stochastic integral of g

under X is denoted by

$$\delta(g) = (\delta(g^i))_{1\le i\le d} := \Big(\int_0^1 g^i(t) dX_t^i \Big)_{1\le i\le d}.$$

For a process $(u^{i,j})_{i,j\in\{1,\dots,d\}}$ with values in $\mathbb{R}^{d\otimes d}$, we define whenever it exists

$$\int_0^t u_s^i dX_s := \sum_{j=1}^d \int_0^t u_s^{i,j} dX_s^j$$

Then it is well known (see for example [2]) that the operator D (also called Malliavin derivative) is the dual operator on $\mathbf{D}^{1,2}$ of the stochastic integration operator δ as stated in the following vectorial IBPF satisfied under the Wiener measure P on Ω : $\forall g \in L^2([0;1];\mathbb{R}^d)$, $\forall \Phi \in \mathcal{S}$,

$$P(D_g\Phi) = P\Big(\Phi\ \delta(g)\Big)$$

or equivalently,

$$\forall i \in \{1,\dots,d\}, \quad P(D_{g^i}\Phi) = P\Big(\Phi\ \delta(g^i)\Big). \tag{1}$$

2.2 IBPF for Brownian bridges.

In the same way as Brownian motion is the reference process in the study of Markov diffusions, it seems natural to consider Brownian bridges as reference processes in the study of Markovian bridges. For this reason we review IBPF satisfied by Brownian bridges. We first introduce the subset of the Cameron-Martin space which will contain the test functions. It is the following set :

$$\mathcal{E} = \{g, \mathbb{R}^d\text{-valued step functions on } [0;1] \text{ such that } \int_0^1 g(t)dt = 0\}.$$

We also denote by $\mathcal{E}_\tau$, for $\tau \in [0,1]$, the subset of $\mathcal{E}$ composed by step functions with support included in $[0,\tau]$.
Let us notice that the condition on the integral is of loop type: indeed if we denote by h the function $h := \int_0^\cdot g(t)dt$, we are requiring that $h(0) = h(1) = 0$.
For step functions the stochastic integral $\delta(g)$ is trivially defined for all $\omega \in \Omega$, independently of the underlying probability.

Proposition 2.1 *Let $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$ and $P^{x,y} \in \mathcal{P}(\Omega)$ be the law of the d-dimensional Brownian bridge on $[0,1]$ from x to y. Then, for all $g \in \mathcal{E}$, for any $\Phi \in \mathcal{S}$,*

$$P^{x,y}(D_g\Phi) = P^{x,y}\Big(\Phi\ \delta(g)\Big). \tag{2}$$

Proof : The duality formula (2) has been proved by Driver in [6] even for the Brownian bridge on a Riemannian manifold. His proof relies on the absolute continuity of $P^{x,y}$ with respect to P^x on $\mathcal{F}_\tau$, with $0 < \tau < 1$. However for the sake of completeness, let us sketch an alternative proof of this duality. Let us take $\Phi(\omega) = \phi_0(\omega_0)\phi_1(\omega_1)\tilde{\Phi}(\omega)$ for ϕ_0, $\phi_1 \in \mathcal{C}^\infty(\mathbb{R}^d)$, and $\tilde{\Phi} \in \mathcal{S}$ in the IBPF satisfied under P :

$$P\Big(\phi_0\,\phi_1\,\tilde{\Phi}\,\delta(g)\Big) = P(D_g(\phi_0\,\phi_1\,\tilde{\Phi})) \tag{3}$$

which holds for any $g \in L^2([0;1];\mathbb{R}^d)$ and any $\Phi \in \mathcal{S}$. One obtains from (3), for each $i \in \{1,\dots,d\}$,

$$P\Big(\phi_0(X_0)\phi_1(X_1)P(\tilde{\Phi}\delta(g^i)/X_0,X_1)\Big) =$$

$$P\Big(\phi_0(X_0)\phi_1(X_1)P(D_{g^i}\tilde{\Phi}/X_0,X_1)\Big) + P\Big(\phi_0(X_0)\partial_i\phi_1(X_1)\tilde{\Phi}\Big)\int_0^1 g^i(r)\,dr$$

so that when $\int_0^1 g(t)dt = 0$, the last term vanishes and what remains is

$$P^{x,y}\Big(\tilde{\Phi}\delta(g^i)\Big) = P\Big(\tilde{\Phi}\delta(g^i)/X_0 = x, X_1 = y\Big) = P^{x,y}\Big(D_{g^i}\tilde{\Phi}\Big)$$

for μ-a.e. (x,y) where ν denotes the law of (X_0,X_1) under P. By continuity of the map $(x,y) \mapsto P^{x,y}$ the duality formula (2) holds for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$. $\square$

Remark 2.2 In the preceding proof we deduced an IBPF for the bridge $P^{x,y}$ from an IBPF for P by choosing appropriate test functionals Φ. We will encounter this argument several times in the sequel.

2.3 Reciprocal class and reciprocal characteristics of a Brownian diffusion.

We now introduce the main object we deal with in this paper: the reciprocal class of some fixed reference diffusion P_b.

The data is then a d-dimensional Markovian diffusion solution of the stochastic differential equation:

$$dX_t = dB_t + b(t,X_t)\,dt, \quad X_0 = x, \tag{4}$$

where B is a d-dimensional Brownian motion, b is the drift function, assumed to be in $\mathcal{C}^{1,2}([0;1]\times\mathbb{R}^d;\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.
The law of this Brownian diffusion will be denoted in all the paper by P_b. It is not a restriction to fix a deterministic value for X_0 since in the present paper one only deals with the bridges of P_b.

We assume on the drift that there exists a constant $k > 0$ and an integer $N \in \mathbb{N}^*$ such that for all $t, x \in [0,1] \times \mathbb{R}^d$, for all $i, j \in \{1, \dots, d\}$,

$$\begin{aligned} x.b(t,x) &\leq k(1+|x|^2) &(5)\\ |b^i(t,x)| + |\partial_t b^i(t,x)| &\leq k(1+|x|^N) &(6)\\ \text{and} \qquad |\partial_j b^i(t,x)| + |\partial_i \partial_j b^i(t,x)| &\leq k(1+|x|^{N-1}). &(7)\end{aligned}$$

Since b is locally lipschitz continuous uniformly on time, condition (5) ensures existence and uniqueness of a strong solution to equation (4) (see for example p.234 in [4]).

Example : A typical class of functions b which satisfy (5) and (6) and (7) is given by the so called gradient diffusion, i.e. b is the gradient of a potential with polynomial growth :

$$\forall i \in \{1, \dots, d\}, \qquad b^i(x^1, \dots, x^d) = -\mathbf{P}_i'(x^i) + \sum_{j=1}^d a_{ij} x^j$$

where the $\mathbf{P}_i$ are polynomial functions of the form : $\mathbf{P}_i(x^i) = \sum_{k=0}^{N+1} \alpha_{i,k}(x^i)^k$ with $\alpha_{i,N+1} > 0$.

Lemma 2.3 *Under assumptions* (5)*, the Brownian diffusion solution of* (4) *admits finite moments of any order. In particular,*

$$P_b(\sup_{t\in[0,1]} |X_t|^{2N}) < +\infty. \tag{8}$$

Proof of Lemma 2.3 : It is enough to show that, for any $M \geq 1$, $\sup_{t\in[0,1]} |X_t|^{4M} \in L^1(P_b)$. But, by Ito formula,

$$\begin{aligned} |X_t|^{2M} &= \Big(|x|^2 + 2\sum_{i=1}^d \int_0^t X_s^i\big(dB_s^i + b^i(s,X_s)\,ds\big) + d\,t\Big)^M \\ &= |x|^{2M} + 2M \int_0^t |X_s|^{2(M-1)} X_s dB_s \\ &\quad + \int_0^t M|X_s|^{2(M-1)}\big(2X_s.b(s,X_s) + d + 2(M-1)\big)\,ds. \end{aligned}$$

Then using assumption (5) and denoting by $f(t) = P_b(\sup_{r\in[0,t]} |X_r|^{4M}) = P_b\big((\sup_{r\in[0,t]} |X_r|^{2M})^2\big)$, one obtains

$$f(t) \leq 3|x|^{4M} + 12M^2 P_b\Big(\sup_{r\in[0,t]}\Big(\int_0^r |X_s|^{2(M-1)} X_s dB_s\Big)^2\Big) + C_1\int_0^t f(s)\,ds,$$

where $C_1 > 0$.

By Doob inequality,

$$P_b\Big(\sup_{r\in[0,t]}\Big(\int_0^r |X_s|^{2(M-1)}X_s dB_s\Big)^2\Big) \le P_b\Big(\int_0^t |X_s|^{4M-2}\,ds\Big) \le C_2\int_0^t f(s)\,ds$$

for some $C_2>0$. Then, for all $t\in[0,1]$,

$$f(t)\le 3|x|^{4M}+\big(12M^2C_2+C_1\big)\int_0^t f(s)\,ds$$

which implies by Gronwall inequality that $f(1)<+\infty$. $\square$

Let us notice that Lemma 2.3 and assumption (6) imply that the usual entropy h of P_b w.r.t. the Wiener measure P^x is finite since

$$h(P_b;P^x)=P_b\big(\log(\frac{dP_b}{dP^x})\big)=\frac{1}{2}P_b\Big(\int_0^1 |b(t,X_t)|^2dt\Big)<+\infty. \tag{9}$$

In this paper, we adopt the following definition of **entropy** on $\mathcal{P}(\Omega)$ (cf. [7]) and denote it by $\mathbf{H}$:

$$\mathbf{H}(Q;P)=Q\big(h(Q^{X_0};P^{X_0})\big).$$

Let us notice that here $\mathbf{H}(P_b;P)=h(P_b;P^x)<+\infty$.
Finite entropy will be a leading assumption through the entire paper, so that we now define the following set of probability measures :

$$\mathcal{P}_{\mathbf{H}}(\Omega)=\{Q\in\mathcal{P}(\Omega):\mathbf{H}(Q;P)<+\infty\}.$$

It is indeed natural in our framework since, as already mentioned in the introduction, the Markov diffusion that Schrödinger was looking for in his paper (that he called "the most probable path"), is the unique minimizer of the entropy w.r.t. Wiener measure among a set of reciprocal processes. Finiteness of the entropy has been also crucial in subsequent papers of Föllmer [9], Wakolbinger[27], Cattiaux and Léonard [3] for instance. In the present paper two consequences of the finiteness of the entropy will play an important role. We state these two results in the following proposition and we refer the reader to [8].

Proposition 2.4 *Let Q be a probability measure in $\mathcal{P}_{\mathbf{H}}(\Omega)$. Then*
(i) *There exists an adapted process $(\beta_t)_{t\in[0;1]}$ such that the process*
$(X_t-X_0-\int_0^t\beta_s ds)_{t\in[0,1]}$ *is a Q-Brownian motion and* $Q\big(\int_0^1|\beta_t|^2\,dt\big)<+\infty$ (ii) *Let μ_0 (resp. μ) denote the law of X_0 (resp. (X_0,X_1)) under Q. Then, for μ_0 (resp. μ) a.e. x (resp. (x,y)), the entropy $\mathbf{H}(Q^x;P^x)$ (resp. $\mathbf{H}(Q^{x,y};P^{x,y})$) is finite.*

Furthermore, let us assume that $p(s,x,t,y)$, the probability transition density of P_b, satisfies the following regularity property :

$$(s,x)\mapsto p(s,x,t,y)\in C^{1,3}([0;1[\times\mathbb{R}^d;\mathbb{R}). \tag{10}$$

It is clear that for each $0 \le s < t \le 1$ and $x, y \in \mathbb{R}^d$, $p(s, x, t, y) > 0$ and also that the law of X_t is absolutely continuous w.r.t. Lebesgue measure on $\mathbb{R}^d$ with strictly positive density. We will also assume that for each $0 \le s < t \le 1$, the map

$$(x, y) \mapsto P_b(\,./X_s = x, X_t = y)$$

is continuous on $\mathbb{R}^d \times \mathbb{R}^d$.

Definition 2.5 *The reciprocal class of P_b is the subset $\mathcal{R}(P_b)$ of $\mathcal{P}(\Omega)$ defined by :*

$$\mathcal{R}(P_b) = \{Q \in \mathcal{P}(\Omega), \forall 0 \le s < t \le 1, Q(\,./\mathcal{F}_s \vee \hat{\mathcal{F}}_t) = P_b(\,./X_s, X_t)\} \tag{11}$$

where the forward (resp. backward) filtration $(\mathcal{F}_t)_{t\in[0,1]}$ (resp. $(\hat{\mathcal{F}}_t)_{t\in[0,1]}$) is given by

$$\mathcal{F}_t = \sigma(X_s, 0 \le s \le t), \quad (resp.\ \hat{\mathcal{F}}_t = \sigma(X_s, t \le s \le 1)).$$

Let us also mention the alternative definition of $\mathcal{R}(P_b)$ (see [11]) :

$$\begin{aligned} \mathcal{R}(P_b) &= \{Q \in \mathcal{P}(\Omega) : \exists\, \mu \in \mathcal{P}(\mathbb{R}^d \otimes \mathbb{R}^d), \\ &\quad Q = \int_{\mathbb{R}^d \times \mathbb{R}^d} P_b(\,/X_0 = x, X_1 = y)\mu(dx, dy)\}. \end{aligned} \tag{12}$$

which stresses the fact that any Q in $\mathcal{R}(P_b)$ is a mixture of the bridges of P_b or equivalently, that the bridges of Q coincide with the ones of P_b.

As a consequence of (11), for any $Q \in \mathcal{R}(P_b)$ and any $0 \le s \le t \le 1$, the filtrations $\mathcal{F}_s \bigvee \hat{\mathcal{F}}_t$ and $\sigma(X_r; s \le r \le t)$ are independent under Q conditionnally to $\sigma(X_s, X_t)$. Therefore the coordinate process under any element of $\mathcal{R}(P_b)$ is a *Markovian field* w.r.t. the time index; it is also called a *reciprocal process*.

It is easy to see that any Markov process is reciprocal. Nevertheless, a reciprocal process is not necessarily a Markov process; the Markov property may fail to hold unless the law of (X_0, X_1) enjoys some product decomposition. More precisely, Jamison gave in [11] the following description of the subset $\mathcal{R}_M(P_b)$ containing all the Markovian processes of $\mathcal{R}(P_b)$ (see [23] for related results):

$$\begin{aligned} \mathcal{R}_M(P_b) &= \{Q \in \mathcal{R}(P_b) : \exists \nu_0, \nu_1 \quad \sigma\text{-finite measures on } \mathbb{R}^d, \\ &\quad Q \circ (X_0, X_1)^{-1}(dx, dy) = p(0, x, 1, y)\nu_0(dx)\nu_1(dy)\}. \end{aligned} \tag{13}$$

Due to historical reasons recalled in the introduction, the elements of $\mathcal{R}_M(P_b)$ are called in the litterature *Schrödinger processes.*

The following theorem gives a necessary and sufficient condition for a Brownian diffusion to be in the reciprocal class of P_b. It was first proved by Clark following a conjecture of Krener.

Theorem 2.6 *For any* $\tilde{b} \in \mathcal{C}^{1,2}([0;1] \times \mathbb{R}^d; \mathbb{R}^d)$*, let us define the* $\mathbb{R}^d$*-valued (respectively* $\mathbb{R}^{d\otimes d}$*-valued) function* $F_{\tilde{b}}(t,x) = (F^i_{\tilde{b}}(t,x))_i$ *(resp.* $G_{\tilde{b}}(t,x) = (G^{i,j}_{\tilde{b}}(t,x))_{i,j}$*), as follows*

$$
\begin{aligned}
F^i_{\tilde{b}}(t,x) &= \left(\partial_t \tilde{b}^i + \frac{1}{2}\partial_i(|\tilde{b}|^2 + div\, \tilde{b})\right)(t,x) && (14)\\
G^{i,j}_{\tilde{b}}(t,x) &= \left(\partial_j \tilde{b}^i - \partial_i \tilde{b}^j\right)(t,x) && (15)
\end{aligned}
$$

A Brownian diffusion with drift $\tilde{b}$ *is in the reciprocal class of* P_b *if and only if*

$$(F_b, G_b) \equiv (F_{\tilde{b}}, G_{\tilde{b}}).$$

Proof of Theorem 2.6: We refer the reader to [5]. Let us simply mention that the identity $(F_b, G_b) \equiv (F_{\tilde{b}}, G_{\tilde{b}})$ is equivalent to the existence of a function $h > 0$ such that $\partial_t h + \sum_{i=1}^d b^i \partial_i h + \frac{1}{2}\Delta h = 0$ and $\tilde{b}^i - b^i = \partial_i \log h$, for all $1 \leq i \leq d$. $\square$

Definition 2.7 *The pair of functions* (F_b, G_b) *is called the* **Reciprocal Characteristics** *of* P_b.

In the sequel of the paper, b is a fixed data, and for simplicity we denote by (F, G) the reciprocal characteristics of P_b :

$$F := F_b, \quad G := G_b.$$

Let us now state a useful result. We omit the proof.

Proposition 2.8 *Under the growth conditions* (6) *and* (7) *on the drift function* b*, the reciprocal characteristics* F *and* G *satisfy the following inequality :*
$\exists K > 0, \quad \forall (t,x) \in [0;1] \times \mathbb{R}^d, \quad \forall i,j \in \{1, \ldots, d\},$

$$
\begin{aligned}
|F^i(t,x)| &\leq K(1 + |x|^{2N-1})\\
|G^{i,j}(t,x)| + |div\, G^i(t,x)| &\leq K(1 + |x|^N).
\end{aligned}
$$

Remark 2.9 The reciprocal characteristics associated to the Brownian motion, i.e. corresponding to the drift $b = 0$, are obviously $F_0 \equiv 0$ and $G_0 \equiv 0$. Let us mention the paper [1] where a subclass of $\mathcal{R}(P_0)$ has been explicitely computed.

3 Integration by parts formula for a Brownian diffusion and its bridges.

In the first part of this section we establish two integration by parts formulae (IBPF) satisfied by the d-dimensional Brownian diffusion P_b. The coefficients of the first one (identity (16)) are the reciprocal characteristics associated to this diffusion except for a term involving the value at the terminal time. The form of this IBPF differs from the one dimensional case by the presence of additional terms, especially a stochastic integral which admits for integrand the reciprocal characteristic G. It is easy to see from Theorem 2.6 that $G = 0$ if and only if b is a gradient, which is always the case in dimension 1. The second IBPF (identity (20)) is a consequence of Girsanov theorem. The second part of this section contains an IBPF satisfied by the reciprocal class of P_b.

3.1 IBPF satisfied by a Brownian diffusion.

The following statement will be a key tool both for Theorem 3.4, where we exhibit an IBPF satisfied by the reciprocal class $\mathcal{R}(P_b)$, and in the proof of Theorem 4.1.

Theorem 3.1 *Let P_b be the d-dimensional Brownian diffusion solution of* (4), *where b satisfies assumptions* (5), (6) *and* (7). *Then the following integration by parts formula is satified under P_b : for any $\tau \in [0,1]$, for any $\mathbb{R}^d$-valued step function g on $[0;\tau]$, for any $\Phi \in \mathcal{S}$ and $i \in \{1,\ldots,d\}$,*

$$\begin{aligned} P_b(D_{g^i}\Phi) \quad &= \quad P_b\Big(\Phi \; \delta(g^i)\Big) - \int_0^\tau g^i(r)dr \; P_b\Big(\Phi \, b^i(\tau, X_\tau)\Big) \\ &+ P_b\Big(\Phi \int_0^\tau g^i(r) \int_r^\tau (F^i + \frac{1}{2} div \, G^i)(t, X_t) \, dt \, dr\Big) \\ &+ P_b\Big(\Phi \int_0^\tau g^i(r) \int_r^\tau G^i(t, X_t) dX_t \, dr\Big) \end{aligned} \tag{16}$$

Proof of Theorem 3.1: The fact that each term of the RHS of (16) is finite is due to Proposition 2.8 and Lemma 2.3.
Since the proof of this theorem is almost the same as in dimension 1, we will not give all the details but rather we refer the reader to [20], Lemma 4.2. However let us recall the procedure used in dimension 1 in order to be able to point out the differences with the one dimensional case. We denote by M_b the Girsanov density of P_b w.r.t. P where $P = P_0$ is the Wiener measure whose initial law is the law of $X(0)$ under P_b:

$$M_b = \exp\Big(\sum_{i=1}^d \int_0^\tau b^i(t, X_t) dX_t^i - \frac{1}{2} \int_0^\tau |b(t, X_t)|^2 dt\Big).$$

Given a smooth truncation function χ_n with bounded derivatives on $\mathbb{R}$ satisfying

$$\begin{cases} \chi_n \mathbf{1}_{[-n-1,n+1]^c} = -(n+1)\mathbf{1}_{]-\infty,-n-1[} + (n+1)\mathbf{1}_{]n+1,+\infty[} \\ \chi_n \mathbf{1}_{[-n,n]} = Id.\mathbf{1}_{[-n,n]}. \end{cases} \tag{17}$$

we set $M_b^n = \exp\Big(\chi_n(\log M_b)\Big)$. Then $0 \le M_b^n \le M_b^n + 1$ and if P_b^n denotes the positive measure on $C([0;T] \times \mathbb{R}^d)$ with Radon Nikodym density M_b^n w.r.t. P, the integration by parts formula (1) for P yields

$$
\begin{aligned}
P\Big(M_b^n D_{g^i}\Phi\Big) &= P\Big(\Phi\, M_b^n \delta(g^i)\Big) - P\Big(\Phi D_{g^i} M_b^n\Big) \\
&= P\Big(\Phi\, M_b^n \delta(g^i)\Big) - P\Big(\Phi M_b^n D_{g^i}(\log M_b^n)\Big). \qquad (18)
\end{aligned}
$$

The difference from dimension 1 comes from the expression for $D_{g^i}(\log M_b)$ which we now compute in our d-dimensional setting. By definition of the Malliavin derivative,

$$
\begin{aligned}
D_{g^i}(\log M_b) &= \int_0^\tau g^i(r)\Big(b^i(r,X_r) + \sum_{j=1}^d \int_r^\tau \partial_i b^j(t,X_t)dX_t^j \\
&\quad - \sum_{j=1}^d \int_r^\tau b^j(t,X_t)\partial_i b^j(t,X_t)dt\Big)dr. \qquad (19)
\end{aligned}
$$

Let us express the r.h.s. using (F,G). Let us write

$$
\begin{aligned}
\int_r^\tau \partial_i b^j(t,X_t)dX_t^j &= \int_r^\tau (\partial_i b^j - \partial_j b^i)(t,X_t)dX_t^j \\
&\quad + \int_r^\tau \partial_j b^i(t,X_t)dX_t^j
\end{aligned}
$$

This last stochastic integral is part of the following Ito formula:

$$
\begin{aligned}
b^i(\tau,X_\tau) - b^i(r,X_r) &= \sum_{j=1}^d \int_r^\tau \partial_j b^i(t,X_t)dX_t^j \\
&\quad + \int_r^\tau \big(\partial_t b^i + \frac{1}{2}\Delta b^i\big)(t,X_t)dt
\end{aligned}
$$

Using the definition $G^{i,j} = \partial_j b^i - \partial_i b^j$ and the definition of F, we obtain

$$
\begin{aligned}
\partial_t b^i + \frac{1}{2}\Delta b^i &= \partial_t b^i + \frac{1}{2}\sum_{j=1}^d \partial_j \partial_j b^i \\
&= \partial_t b^i + \frac{1}{2}\partial_i div\, b + \frac{1}{2} div\, G^i \\
&= (F^i + \frac{1}{2} div\, G^i) - \sum_{j=1}^d \partial_i b^j b^j.
\end{aligned}
$$

We therefore obtain the following expression for $D_{g^i}(\log M_b)$:

$$
\begin{aligned}
D_{g^i}(\log M_b) &= \int_0^\tau g^i(r)dr\, b^i(\tau,X_\tau) - \int_0^\tau g^i(r)\Big(\sum_{j=1}^d \int_r^\tau G^{i,j}(t,X_t)dX_t^j\Big)dr \\
&\quad - \int_0^\tau g^i(r)\int_r^\tau \big(F^i + \frac{1}{2} div\, G^i\big)(t,X_t)\, dt\, dr.
\end{aligned}
$$

It remains to prove the convergence of each term of identity (18) to its respective limit. This is done by applying the dominated convergence theorem. $\square$

In the sequel we would like to use the IBPF (16) for Brownian diffusion with a drift which is not necessarily with polynomial growth. For example, in the next subsection we are interested by the bridges of P_b. If one takes $b(t,z) = -\lambda z$, which satisfies conditions (6) and (7) with $N=1$, P_b is then the Ornstein-Uhlenbeck process. The drift $\tilde{b}$ of its bridge between x and y can be explicitely computed :

$$\tilde{b}(t,z) = -\lambda z + \frac{\lambda}{\sinh(\lambda(1-t))}(y - e^{-\lambda(1-t)}z).$$

It is clear that $\tilde{b}$ does not satisfy condition (6). So let us now give a set of sufficient conditions (weaker than (6)) under which a Brownian diffusion satisfies the IBPF (16).

Proposition 3.2 *Let $\tilde{b} \in \mathcal{C}^{1,2}([0;1]\times\mathbb{R}^d;\mathbb{R}^d)$ and $\tau \in [0;1]$ such that $\mathbf{H}(P_{\tilde{b}}|_{\mathcal{F}_\tau}; P|_{\mathcal{F}_\tau}) < +\infty$. Let $F_{\tilde{b}}$ and $G_{\tilde{b}}$ be the reciprocal characteristics associated to the Brownian diffusion $P_{\tilde{b}}$. If the following conditions are satisfied :*
(A1) $\tilde{b}(\tau, X_\tau) \in L^1(P_{\tilde{b}})$
(A2) $\int_0^\tau |F_{\tilde{b}} + \frac{1}{2} div\, G_{\tilde{b}}|(t,X_t)dt \in L^1(P_{\tilde{b}})$
(A3) $\int_0^\tau |G_{\tilde{b}}^{i,j}(t,X_t)|^2 dt \in L^1(P_{\tilde{b}}), \quad \forall i,j \in \{1,\dots,d\}$
then the integration by parts formula (16) *still holds true under $P_{\tilde{b}}$.*

Let us now establish another integration by parts formula satisfied under $P_{\tilde{b}}$ where the drift $\tilde{b}$ appears instead of the reciprocal characteristics $(F_{\tilde{b}}, G_{\tilde{b}})$.

Theorem 3.3 *Let $P_{\tilde{b}} \in \mathcal{P}_{\mathbf{H}}(\Omega)$ be, as before, the Brownian diffusion whose drift $\tilde{b}$ is assumed to belong to $\mathcal{C}^{0,1}([0;1]\times\mathbb{R}^d;\mathbb{R}^d)$. Let $\tau \in [0;1]$. If for $i \in \{1,\dots,d\}$, $\int_0^\tau |\partial_i \tilde{b}(t,X_t)|^2 dt$ belong to $L^1(P_{\tilde{b}})$, then for any $\mathbb{R}^d$-valued step function g on $[0,\tau]$, for all $\Phi \in \mathcal{S}$,*

$$\begin{aligned} P_{\tilde{b}}(D_{g^i}\Phi) &= P_{\tilde{b}}\Big(\Phi\ \delta(g^i)\Big) - P_{\tilde{b}}\Big(\Phi \int_0^\tau g^i(s)\tilde{b}^i(s,X_s)ds\Big) \\ &\quad - P_{\tilde{b}}\Big(\Phi \int_0^\tau g^i(s)\int_s^\tau \sum_j \partial_i \tilde{b}^j(p,X_p)(dX_p^j - \tilde{b}^j(p,X_p)dp)ds\Big) \end{aligned} \tag{20}$$

Proof The argument runs as in the proof of Theorem 3.1 except that we do not need to develop identity (19) by Ito formula. It is sufficient to verify that each term of this identity converges by dominated convergence theorem. $\square$

A duality formula such as (20) has been proved under stronger integrability assumptions on the drift $\tilde{b}$ in [21], formula (1.8).

3.2 IBPF satisfied by the bridges of a Brownian diffusion.

We now come to an IBPF satisfied by all the elements of $\mathcal{R}(P_b)$, the reciprocal class of P_b, in which all the probabilities admit the reciprocal characteristics (F, G).

Theorem 3.4 *Let Q be a probability measure in $\mathcal{P}_{\mathbf{H}}(\Omega)$. Let us moreover assume that*

(A0) $\quad \sup_{t\in[0;1]} |X_t| \in L^{2N}(Q)$

If Q is in the reciprocal class of P_b, then for any function $g \in \mathcal{E}$, $\forall \Phi \in \mathcal{S}$, for all $i \in \{1, \ldots, d\}$, the following integration by parts formula is satisfied :

$$\begin{aligned} Q(D_{g^i}\Phi) &= Q\Big(\Phi\ \delta(g^i)\Big) \\ &\quad + Q\Big(\Phi \int_0^1 g^i(r) \int_r^1 (F^i + \frac{1}{2}\, div\, G^i)(t, X_t)\, dt\, dr\Big) \\ &\quad + Q\Big(\Phi \int_0^1 g^i(r) \int_r^1 G^i(t, X_t) dX_t\, dr\Big). \end{aligned} \tag{21}$$

Remark 3.5 1. As mentioned in Proposition 2.4 the fact that the entropy $\mathbf{H}(Q; P)$ is finite ensures that X is a Q- semi-martingale; it is therefore meaningful to consider the stochastic integral $\int_r^1 G^i(t, X_t)dX_t$ under Q. As mentioned above this was not necessary in dimension 1 since this integral did not appear.

2. Formula (21) reads like a perturbation of formula (2) for Brownian bridges. The perturbation term can also be written as

$$Q\Big(\Phi \int_0^1 g^i(r) \int_r^1 F^i(t, X_t)\, dt\, dr\Big) + Q\Big(\Phi \int_0^1 g^i(r) \int_r^1 G^i(t, X_t) \circ dX_t\, dr\Big)$$

where in the second term, the stochastic integration is of Stratonovich type; this expression reflects the symmetry of the reciprocal property under time reversal.

Proof of Theorem 3.4 : Let us denote by μ the law of (X_0, X_1) under Q. We first prove the IBPF for μ-a.e.(x, y) and the probability $Q^{x,y} := Q(\cdot/X_0 = x, X_1 = y)$. In order to do so we first prove that we can apply Proposition 3.2.
For μ-a.e. (x, y) the integrability condition (A0) still holds true under $Q^{x,y}$ as well as $\mathbf{H}(Q^{x,y}; P^{x,y}) < +\infty$ (cf. Proposition 2.4) which implies
$\mathbf{H}(Q^{x,y}|_{\mathcal{F}_\tau}; P^{x,y}|_{\mathcal{F}_\tau}) < +\infty$ for any $\tau \le 1$. Let us fix such an (x, y). Since Q belongs to the reciprocal class of P_b, for any $\tau \in [0; 1[$ the restriction of $Q^{x,y}$ to $C([0; \tau]; \mathbb{R}^d)$ is the law of the Brownian diffusion $\tilde{P}$ starting from x with drift

$$\tilde{b}(t, z) = b(t, z) + \partial_z \log p(t, z, 1, y)$$

and in particular

$$F_{\tilde{b}} = F,\ G_{\tilde{b}} = G \text{ on } [0; \tau] \times \mathbb{R}^d.$$

By assumption (10), $\tilde{b} \in C^{1,2}([0;1[\times\mathbb{R}^d;\mathbb{R})$. We also have to check assumptions (A1)-(A3) of Proposition 3.2 on $\tilde{b}$, F and G : assumption (A1) has not to be considered here since we test on functions $g \in \mathcal{E}_\tau$. Assumptions (A2) and (A3) are satisfied since (A0) is assumed and F and G satisfy Proposition 2.8. Therefore IBPF (21) holds on $[0,\tau]$ under $Q^{x,y}$ for μ-a.e. (x,y).
The second part of the proof consists in passing to $\tau = 1$. Let us simply sketch the argument. Let $\Phi \in \mathcal{S}$ be $\mathcal{F}_1$-measurable, and $g \in \mathcal{E}$. Since $\Phi \in \mathcal{S}$, there exists a function φ and a real number $\tau < 1$ such that

$$\Phi(X) = \varphi(x, X_{t_1}, \cdots, X_\tau, y), \quad Q^{x,y}\text{-}a.s..$$

Let n be large enough so that $\tau < 1 - \frac{1}{n}$ and g is constant on $[1 - \frac{2}{n}; 1[$. Let us set

$$g_n = g\mathbf{1}_{[0,1-\frac{2}{n}[} + n\Big(\int_{1-\frac{2}{n}}^1 g(r)dr\Big)\mathbf{1}_{[1-\frac{2}{n},1-\frac{1}{n}]}.$$

By construction g_n is a step function on $[0, 1-\frac{1}{n}]$ and its integral is equal to zero. We apply the IBPF (21) for $Q^{x,y}$ to the pair (Φ, g_n) on $[0, 1-\frac{1}{n}]$. It is now straightforward to verify that each term converges when n tends to infinity. By integrating in (x, y) over μ, we conclude that the desired IBPF also holds true for Q. $\square$

4 Characterization of a reciprocal class by an IBPF

Our aim is now to establish the converse statement of Theorem 3.4. More precisely, we want to show that the integration by parts formula (21) characterizes the regular reciprocal processes belonging to $\mathcal{R}(P_b)$. Actually, since we previously had to introduce the regularity condition (10) to obtain enough smoothness for the semi-martingale characteristics of bridges, we also have now to consider probabilities which a priori satisfy some regularity conditions to be able to write down the IBPF. These conditions are listed below :

(H1) *Conditional density : regularity, domination.*
(H1.1) $\forall 0 \le t < u < 1, \forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d$, there exists a function q s.t.

$$Q(X_u \in dw | X_t = z, X_1 = y) = q(t, z, u, w, 1, y)dw$$

(H1.2) $\forall 0 < u < 1, \forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad q(0, x, u, w, 1, y) > 0$
(H1.3) $\forall 0 < u < 1, \forall (w,y) \in \mathbb{R}^d \times \mathbb{R}^d$, the map $(t,z) \mapsto q(t, z, u, w, 1, y)$ is in $C^{1,2}([0,1[\times\mathbb{R}^d;\mathbb{R})$
(H1.4) for all $0 < \tau < 1, \forall (t,z) \in [0;\tau] \times \mathbb{R}^d$, there exists a neighborhood $\mathcal{V}$ of (t,z) and a function $\phi_\mathcal{V}(u, w, 1, y)$ such that whenever ∂_α denotes $\partial_s, \partial_{\xi^k}$ or $\partial_{\xi^k\xi^l}$ for $k, l \in \{1, \dots, d\}$ it holds:
$\sup_{(s,\xi)\in\mathcal{V}} |\partial_\alpha q(s, \xi, u, w, 1, y)| \le \phi_\mathcal{V}(u, w, 1, y)$,

and $\int_0^\tau \int_{\mathbb{R}^d} (1+|w|^{2N}) \phi_{\mathcal{V}}(u,w,1,y) \big(1 + \frac{\phi_{\mathcal{V}}(u,w,1,y)}{q(0,x,u,w,1,y)}\big) dwdu < +\infty$.

(H2) *Integrability condition on the derivatives of the conditional density.*
Let $0 \le s \le \tau < 1$.
(H2.1) $\int_s^\tau \int_{\mathbb{R}^d} |\partial_\alpha q(s, X_s, u, w, 1, X_1)|(1+|w|^{2N})\, dwdu \in L^1(Q)$ where ∂_α denotes ∂_s or $\partial_{\xi^k \xi^l}$ for $k, l \in \{1, \dots, d\}$
(H2.2) $\int_s^\tau \int_{\mathbb{R}^d} |\partial_{\xi^k} q(s, X_s, u, w, 1, X_1)|(1+|w|^{2N})\, dwdu \in L^2(Q)$
(H2.3) $\int_s^\tau \int_{\mathbb{R}^d} (1+|w|^{2N}) \frac{\partial_\alpha q(s,X_s,u,w,1,X_1)^2}{q(0,x,u,w,1,X_1)}\, dwdu \in L^1(Q)$ where ∂_α denotes $\partial_s, \partial_{\xi^k}$ or $\partial_{\xi^k \xi^l}$

Theorem 4.1 *Let Q be a probability measure in $\mathcal{P}_{\mathbf{H}}(\Omega)$. Let us assume that Q satisfies assumptions* (H1), (H2) *and* (A0). *If the IBPF* (21) *is satisfied under Q then Q is in the reciprocal class of P_b.*

The proof of this theorem is decomposed into the following four different steps.
Step 1 : Each bridge $Q^{x,y}$ of Q is a Brownian diffusion with drift b^{xy} given by an expression of the form (22).
Step 2 : Each drift b^{xy} is regular enough to compute the reciprocal characteristics $F^{x,y}$ and $G^{x,y}$ of $Q^{x,y}$.
Step 3 : $Q^{x,y}$ satisfies an IBPF of the type (16) with its own reciprocal characteristics $F^{x,y}$ and $G^{x,y}$ as parameters.
Step 4 : $Q^{x,y}$ also satisfies an IBPF of the type (21) but with reciprocal characteristics F and G as parameters. Therefore $F^{x,y} \equiv F$ and $G^{x,y} \equiv G$, which implies that all the bridges of Q and P_b are equal.

Proof of Theorem 4.1

Step 1: Let μ be the law of (X_0, X_1) under Q. By Proposition 2.4, for μ a.e. (x,y), X under $Q^{x,y}$ (resp. under both probabilities $Q^{x,y}(\cdot/\mathcal{F}_s)$ and $Q^{x,y}(\cdot/X_s)$) is a Brownian semi-martingale, whose drift, denoted by b^{xy}, will be now computed.
First, let us prove that $Q^{x,y}$ is Markovian. Notice that, for s fixed in $[0,1]$, applying IBPF (21) to the following functions : Φ $\mathcal{F}_s$-measurable, $g = \tilde{g} - \frac{1}{1-s}\int_s^1 \tilde{g}(r)dr\mathbf{1}_{[s,1]}$, one obtains easily that the drift of X at time $r \in [s;1]$ is under $Q^{x,y}(\cdot/\mathcal{F}_s)$ (resp. $Q^{x,y}(\cdot/X_s)$) given by $Q^{x,y}_{\mathcal{F}_s}(U_r/\mathcal{F}_r)$ (resp. $Q^{x,y}_{X_s}(U_r/\mathcal{F}_r)$) where $Q^{x,y}_{\mathcal{F}_s}$ (resp. $Q^{x,y}_{X_s}$) denotes $Q^{x,y}(\cdot/\mathcal{F}_s)$ (resp. $Q^{x,y}(\cdot/X_s)$) and

$$\begin{aligned} U^i_r &= \frac{X^i_1 - X^i_r}{1-r} - \Big(\int_r^1 u^i_p dp + \int_r^1 v^i_p dX_p\Big) \\ &\quad + \frac{1}{1-r}\int_r^1 \Big(\int_p^1 u^i_q dq + \int_p^1 v^i_q dX_q\Big) dp \end{aligned}$$

with $u^i_p := (F^i + \frac{1}{2} div\, G^i)(p, X_p)$ and $v^{i,j}_p = G^{i,j}(p, X_p)$.
But it is straightforward to check that for any $r \ge s$

$$Q^{x,y}_{\mathcal{F}_s}(\cdot/\mathcal{F}_r) = Q^{x,y}_{X_s}(\cdot/\mathcal{F}_r) = Q^x(\cdot/\mathcal{F}_r)$$

Thus $(X_r, r \in [s;1])$ has the same drift under $Q^{x,y}_{\mathcal{F}_s}$ et $Q^{x,y}_{X_s}$. It is therefore a Markovian Brownian diffusion under $Q^{x,y}$, whose drift at time t is equal to :

$$
\begin{aligned}
b^{xy,i}(t, X_t) \quad &= \quad \frac{y^i - X^i_t}{1-t} \\
&\quad - Q^{x,y}\Big(\int_t^1 u^i(p, X_p)dp + \int_t^1 v^i(p, X_p)dX_p/X_t\Big) \qquad (22) \\
&\quad + \frac{1}{1-t} Q^{x,y}\Big(\int_t^1 \Big(\int_p^1 u^i(q, X_q)dq + \int_p^1 v^i(q, X_q)dX_q\Big) dp/X_t\Big)
\end{aligned}
$$

with $u^i(p, X_p) = (F^i + \frac{1}{2} div\, G^i)(p, X_p)$ and $v^{i,j}(p, X_p) = G^{i,j}(p, X_p)$.

The other steps amount to show that each bridge $Q^{x,y}$ is equal to $(P_b)^{x,y}$ so that, once we mix them up under μ to get $Q = \int_{\mathbb{R}^d \times \mathbb{R}^d} Q^{x,y}\, \mu(dxdy)$ the probability that we obtain is indeed the law of a reciprocal process.

Step 2: We now come to an important point : to establish the regularity of b^{xy}, in such a way that we can compute F^{xy} and G^{xy}, the reciprocal characteristics of $Q^{x,y}$. More precisely, we will show that under the assumptions (H1), for μ a.e. (x, y), the map $(t, z) \mapsto b^{xy}(t, z) \in C^{1,2}([0, 1[\times \mathbb{R}^d\, ; \mathbb{R})$. For this purpose the relevant expression for b^{xy} is the following (it can be proved by the same argument as in *Step* 1 : for any $(t, z) \in [0; 1[\times \mathbb{R}^d$ and $\tau \in]t, 1[$,

$$
\begin{aligned}
b^{xy,i}(t, z) \quad &= \quad \frac{Q^{x,y}(X^i_\tau - z^i / X_t = z)}{\tau - t} \\
&\quad - Q^{x,y}\Big(\int_t^\tau u^i(p, X_p)dp + \int_t^\tau v^i(p, X_p)dX_p/X_t\Big) \qquad (23) \\
&\quad + \frac{1}{\tau - t} Q^{x,y}\Big(\int_t^\tau \Big(\int_p^\tau u^i(q, X_q)dq + \int_p^\tau v^i(q, X_q)dX_q\Big) dp/X_t\Big)
\end{aligned}
$$

Let us first notice this implies the following identity

$$
\begin{aligned}
b^{xy,i}(t, z) \quad &= \quad \frac{1}{\tau - t}\Big(\int_{\mathbb{R}^d} w^i q(t, z, \tau, w, 1, y)dw - z^i\Big) \\
&\quad - \int_t^\tau \int_{\mathbb{R}^d} \Gamma^i(u, w) q(t, z, u, w, 1, y) dwdu \qquad (24) \\
&\quad + \frac{1}{\tau - t} \int_t^\tau \int_s^\tau \int_{\mathbb{R}^d} \Gamma^i(u, w) q(t, z, u, w, 1, y) dwduds
\end{aligned}
$$

where $\Gamma^i(u, w) := (F^i + \frac{1}{2} div\, G^i + G^i.b^{xy})(u, w)$. Indeed, by the same argument as in *Step* 1, X is also Markovian under $Q^{\cdot,y} := Q(\cdot / X_1 = y)$. Therefore, for any $0 < t < u < 1$ and any regular function h,

$$
\begin{aligned}
Q^{x,y}(h(u, X_u)/X_t = z) \quad &= \quad Q^{\cdot,y}(h(u, X_u)/X_t = z) \\
&= \quad \int_{\mathbb{R}^d} h(u, w) q(t, z, u, w, 1, y) dw.
\end{aligned}
$$

We want to differentiate under the integral signs of (24). Using assumptions (H1.3) and (H1.4), it is sufficient to check that

$$\int_0^\tau \int_{\mathbb{R}^d} |\Gamma^i(u,w)| \phi_{\mathcal{V}}(u,w,1,y) dw du < +\infty$$

which under (H1.4) reduces to the condition

$$\int_0^\tau \int_{\mathbb{R}^d} |G^{i,j}(u,w)||b^{xy,j}(u,w)| \phi_{\mathcal{V}}(u,w,1,y) dw du < +\infty.$$

Let us divide and multiply the above integrand by $q(0,x,u,w,1,y)$; by Cauchy-Schwarz inequality w.r.t. the finite measure $q(0,x,u,w,1,y)dwdu$ we obtain the following upper bound :

$$\sum_{j=1}^d \Big(\int_0^\tau \int_{\mathbb{R}^d} |G^{i,j}(u,w)|^2 \frac{\phi_{\mathcal{V}}(u,w,1,y)^2}{q(0,x,u,w,1,y)} dw du \Big)^{\frac{1}{2}} I(j)$$

where

$$\begin{aligned} I(j) &= \Big(\int_0^\tau \int_{\mathbb{R}^d} \left| b^{xy,j}(u,w) \right|^2 q(0,x,u,w,1,y) dw du \Big)^{\frac{1}{2}} \\ &= \Big(Q^{x,y} \int_0^\tau \left| b^{xy,j}(u,X_u) \right|^2 du \Big)^{\frac{1}{2}} \end{aligned}$$

which is finite since $\mathbf{H}(Q^{x,y}|_{\mathcal{F}_\tau}; P^{x,y}|_{\mathcal{F}_\tau}) < +\infty$. For any j the coefficient of $I(j)$ is also finite by assumption (H1.4) and Proposition 2.8.

Step 3: We now assume that Q satisfies the set of assumptions (H1)-(H2) and (A0). Then $\forall \tau < 1$, for μ a.e. (x,y), $Q^{x,y}$ restricted to the interval $[0;\tau]$ satisfies the assumptions of Proposition 3.2. The proof of this assertion makes no difficulty using the same arguments as in *Step* 2. Details are left to the reader. Therefore the following IBPF holds true with (F^{xy}, G^{xy}) denoting the reciprocal characteristics of $Q^{x,y}$: for all $g \in \mathcal{E}_\tau$, $\forall \Phi \in \mathcal{S}_\tau$, $\forall 1 \le i \le d$,

$$\begin{aligned} Q^{x,y}(D_{g^i}\Phi) &= Q^{x,y}\Big(\Phi\ \delta(g^i)\Big) \\ &\quad + Q^{x,y}\Big(\Phi \int_0^\tau g^i(r) \int_r^\tau (F^{xy,i} + \frac{1}{2} div\, G^{xy,i})(t,X_t)\, dt dr\Big) \\ &\quad + Q^{x,y}\Big(\Phi \int_0^\tau g^i(r) \sum_{j=1}^d \int_r^\tau G^{xy,i}(t,X_t) dX_t dr\Big). \end{aligned} \tag{25}$$

Step 4: At this stage we have proved that $Q^{x,y}$ satisfies two IBPF. The first one has been obtained in *Step* 3; the other one is the conditioned version of the IBPF (21) for Q:

$$\begin{aligned} Q^{x,y}(D_{g^i}\Phi) &= Q^{x,y}\Big(\Phi\ \delta(g^i)\Big) \\ &\quad + Q^{x,y}\Big(\Phi \int_0^1 g^i(r)\int_r^1 (F^i + \frac{1}{2}div\, G^i)(t,X_t)\, dtdr\Big) \\ &\quad + Q^{x,y}\Big(\Phi \int_0^1 g^i(r)\int_r^1 G^i(t,X_t)dX_t dr\Big). \end{aligned} \tag{26}$$

Both IBPF hold true for μ-a.e. (x,y), any $\tau < 1$,$g \in \mathcal{E}_\tau$, $\forall \Phi \in \mathcal{S}_\tau$, $\forall 1 \le i \le d$. In this last step of the proof we will conclude that Q belongs to the reciprocal class of P_b. In order to do so it is sufficient to prove that for μ-a.e.(x,y) the pair of functions (F^{xy}, G^{xy}) coincides with (F,G). This will be a consequence of the following identification Proposition.

Proposition 4.2 *Let $\tilde{Q}$ be a probability measure on $C([0;\tau];\mathbb{R}^d)$ and B be a d-dimensional $\tilde{Q}$-Brownian motion. Let $u=(u^i)_i$ (resp. $v=(v^{ij})_{ij}$) be a continuous process on $[0;\tau]$ with values in $\mathbb{R}^d$ (resp. $\mathbb{R}^{d\otimes d}$). Let us assume that for all $i \in \{1,\dots,d\}$, $\int_0^\tau |u^i_s|ds + \sup_{t\in[0;\tau]} |\int_0^t v^i_s dB_s|^2 \in L^1(\tilde{Q})$ and*
$\forall g \in \mathcal{E}_\tau$, $\forall \Phi \in \mathcal{S}_\tau$, for all $i \in \{1,\dots,d\}$,

$$\tilde{Q}\Big(\Phi\int_0^\tau g^i(r)\Big(\int_r^\tau u^i_s ds + \int_r^\tau \sum_{j=1}^d v^{ij}_s dB^j_s\Big)ds\Big) = 0.$$

Then the two processes u and v are equal $\tilde{Q}$-a.s. to the constant 0 on $[0;\tau]$.

Proof of Proposition 4.2 Let us denote by $\mathcal{D}$ the set of step functions on $[0;\tau]$ with values in the set of rational numbers whose jump points are all rationals. $\mathcal{D}$ is a countable set. Let $g \in \mathcal{D}$ and $t \le \tau$ be a rational. Let us define

$$\tilde{g}(r) = \big(g(r) - \frac{1}{t}\int_0^t g(s)ds\big)\mathbf{1}_{[0;t]}(r)$$

By construction $\tilde{g}$ is a step function on $[0;t]$ and satisfies $\int_0^t \tilde{g}_s ds = 0$. Therefore

$$\tilde{Q}-a.s. \quad \int_0^t \tilde{g}^i(r)(\int_r^t u^i_s ds + \int_r^t \sum_{j=1}^d v^{ij}_s dB^j_s)dr = 0$$

By Fubini's theorem this implies

$$\tilde{Q}-a.s. \quad \int_0^t u^i_s(\int_0^s \tilde{g}^i(r)dr)ds + \int_0^t \sum_{j=1}^d v^{ij}_s(\int_0^s \tilde{g}^i(r)dr)dB^j_s = 0.$$

Ito formula implies that $\tilde{Q}$ a.s. for any $g \in \mathcal{D}$ and any t rational,$\int_0^t \frac{1}{s}(g(s) - \frac{1}{s}\int_0^s g(r)dr)\int_0^s r u^i_r dr ds$ is equal to $\int_0^t \frac{1}{s}(g(s) - \frac{1}{s}\int_0^s g(r)dr)\int_0^s r v^i_r dB_r ds$. These are

two processes continuous w.r.t. t. Thus the identity holds for any $t \in]0;\tau[$. Differentiating w.r.t. t we obtain:
$\forall g \in \mathcal{D}, \quad \forall t \in]0;\tau[$,

$$(g_t - \frac{1}{t}\int_0^t g_r dr)(\int_0^t r u_r^i dr - \sum_{j=1}^d \int_0^t r v_r^{i,j} dB_r^j) = 0. \tag{27}$$

Let us now take for $0 < a < t < \tau$, $g := \mathbf{1}_{[0;a[} + 2\mathbf{1}_{[a;\tau[}$. For such a choice $(g(t) - \frac{1}{t}\int_0^t g(r)dr) = \frac{a}{t} > 0$. Therefore the process $(\int_0^t r u_r^i dr - \int_0^t r v_r^{i,j} dB_r^j)_{t\in[0;\tau]}$ is a.s. equal to 0 which proves that $u \equiv v \equiv 0$ *a.s.*. $\square$

We must therefore check that $Q^{x,y}$ satisfies the assumptions of this theorem. Let us set

$$u_s^{(1),i} \equiv (F^{xy,i} + \frac{1}{2} div\, G^{xy,i} + G^{xy,i}.b^{xy})(s, X_s), \quad v_s^{(1),i,j} \equiv G^{xy,i,j}(s, X_s)$$

and

$$u_s^{(2),i} \equiv (F^i + \frac{1}{2} div\, G^i + G^{i,j} b^{xy,j})(s, X_s), \quad v_s^{(2),i,j} \equiv G^{i,j}(s, X_s).$$

In accordance with the notations of Proposition 4.2 we also define

$$u_s^i \equiv u_s^{(1),i} - u_s^{(2),i} \text{ and } v_s^{i,j} \equiv v_s^{(1),i,j} - v_s^{(2),i,j}.$$

As a result of the work already done in *Steps* 2 and 3, it is easy to see that Theorem 4.2 applies to (u, v) which are therefore $Q^{x,y}$-a.s. equal to the constant 0. This is equivalent to the identity

$$Q^{xy} \text{ a.s.} \quad \forall s \in [0;1[\quad (F^{xy}, G^{xy})(s, X_s) \equiv (F, G)(s, X_s). \tag{28}$$

Since any X_t has a strictly positive density w.r.t Lebesgue measure on $\mathbb{R}^d$, the functions $F^{xy}(s, x)$ (resp. $G^{xy}(s, x)$) and $F(s, x)$ (resp. $G(s, x)$) which are continuous in (s, x) coincide on $[0;1[\times\mathbb{R}^d$. This ends the proof of Theorem 4.1. $\square$

5 Application to gradient diffusions

In the previous sections our data has been a reference drift function $b(t, x)$. In the present section we characterize the fact that b is a gradient w.r.t. the space variable using the tools of IBPF satisfied by reciprocal processes which we have developed in the preceding sections. Let us notice that when b is a gradient then the drifts of all Brownian diffusions in the reciprocal class $\mathcal{R}(P_b)$ are gradients (cf. the proof of Theorem 2.6).
Our first application provides a characterization of the laws of gradient Brownian diffusions among a large class of probabilities Q which satisfy a finite entropy condition. Being solution of the specific IBPF (29), it will be proved that Q is not only a Brownian semi-martingale, but a Markovian one, and moreover the absence

of the term containing the Reciprocal Characteristic G will imply that its drift is the gradient of some function. Our second result generalizes a famous statement of Kolmogorov. In [17], Kolmogorov proved that a Markov diffusion can be reversible only if its drift is a gradient. Our extension of this result states that it is possible to find a reversible reciprocal process in the class $\mathcal{R}(P_b)$ only if b is a gradient.

As before the reference drift b belongs to $\mathcal{C}^{1,2}([0;1]\times\mathbb{R}^d;\mathbb{R}^d)$ and satisfies assumptions (5)-(7) and we consider probability measures on the path space satifying some a priori regularity to make sense to the IBPF. For Q a probability measure on the path space, we denote by μ_0 its projection at time 0 .

($\mathcal{H}1$) *Conditional density; regularity, domination*:

($\mathcal{H}$ 1.1) for μ_0 a.e .x, $\forall 0<t<u\leq 1, \forall (x,z)\in\mathbb{R}^d\times\mathbb{R}^d$ there exists a strictly positive function q^x such that

$$Q(X_u\in dw|X_0=x, X_t=z) = q^x(t,z,u,w)dw$$

and the map $(t,z)\mapsto q^x(t,z,u,w)$ is in $C^{1,2}([0,u]\times\mathbb{R}^d\,;\mathbb{R})$

($\mathcal{H}$ 1.2) $\quad\forall 0<\tau<1, \forall (t,z)\in[0;\tau]\times\mathbb{R}^d$, there exists a neighborhood $\mathcal{V}$ of (t,z) and a function $\phi_{\mathcal{V}}(u,w)$ such that whenever ∂_α denotes $\partial_t, \partial_{z^k}$ or $\partial_{z^k z^l}$ for $k,l\in\{1,\ldots,d\}$ it holds :

$$\sup_{(s,\xi)\in\mathcal{V}} |\partial_\alpha q^x(s,\xi,u,w)| \leq \phi_{\mathcal{V}}(u,w)$$

$$\int_0^1\int_{\mathbb{R}^d}\phi_{\mathcal{V}}(u,w)(1+|w|^{2N})dwdu < +\infty$$

($\mathcal{H}2$) *Integrability conditions on the derivatives of the conditional density*:

($\mathcal{H}$ 2.1) $\quad \int_0^1\int_{\mathbb{R}^d}|\partial_\alpha q^x(t,X_t,u,w)|(1+|w|^{2N})dwdu\in L^1(Q^x)$

where ∂_α denotes $\partial_t, \partial_{z^k z^l}$ for $k,l\in\{1,\ldots,d\}$

($\mathcal{H}$ 2.2) $\quad \int_0^1\int_{\mathbb{R}^d}|\partial_{z^k} q^x(t,X_t,u,w)|(1+|w|^{2N})dwdu\in L^2(Q^x)$.

Theorem 5.1 *Let Q be a probability measure in $\mathcal{P}_{\mathbf{H}}(\Omega)$ which satisfies the conditions ($\mathcal{H}1$) and ($\mathcal{H}2$) and* (A0).

If the following IBPF holds under Q :
for all g step function on $[0,1]$, $\forall\Phi\in\mathcal{S}$, for all $i\in\{1,\ldots,d\}$,

$$\begin{aligned}Q(D_{g^i}\Phi) &= Q\Big(\Phi\delta(g^i)\Big) - Q\Big(\Phi b(1,X_1)\Big)\int_0^1 g^i(r)dr \\ &\quad + Q\Big(\Phi\int_0^1 g^i(r)\int_r^1 F^i(t,X_t)dtdr\Big), \qquad (29)\end{aligned}$$

then b is a gradient and Q is in fact equal to the law of a gradient Brownian diffusion with drift b.

Remark 5.2 1. The conclusion of the above theorem is, in other words, that the canonical process under Q satisfies equation (4)

$$dX_t = dB_t + b(t, X_t)\,dt,$$

but its initial condition is not necessarily deterministic.

2. It will be proved below that, due to the "terminal term" or second term in the RHS of (29), the coordinate process under Q is not only reciprocal but Markovian. Moreover the fact that there is no term containing the stochastic integral of some function G as in the general formula (21) will imply the gradient property of the drift.

Proof of Theorem 5.1 The proof is divided in two steps.
In *Step* 1, we prove that, for μ_0-a.e. x, Q^x is a Brownian diffusion, whose drift is denoted by b^x. We also prove that its reciprocal characteristics (F^x, G^x) coincide with $(F, 0)$.
In *Step* 2 we prove that b is a gradient and conclude that X under Q is a Markov Brownian diffusion solution of $dX_t = b(t, X_t)dt + dW_t$ where W is a Brownian motion.

Step 1. We can adapt Step 1 in the proof of Theorem 4.1 in this simpler situation $(G = 0)$ and obtain that for μ_0-a.e. x, Q^x is a Brownian diffusion, whose drift b^x is given by, for any $r < 1$, by

$$\begin{aligned} b^{x,i}(r, X_r) &= \frac{Q^x(X_1^i - X_r^i / X_r)}{1-r} - Q^x\Big(\int_r^1 F^i(p, X_p)dp / X_r\Big) \\ &\quad + \frac{1}{1-r}\int_r^1 Q^x\Big(\int_p^1 F^i(q, X_q)dq / X_r\Big)dp \end{aligned} \tag{30}$$

Now, the key tool in order to identify (F^x, G^x) with $(F, 0)$ will be to apply Proposition 4.2 to Q^x. In order to do so, we must first prove that Q^x satisfies at the same time two IBPF. The first formula is an immediate consequence of identity (29) for Q. Indeed, if in (29) we take $\Phi = \varphi(X_0)\tilde{\Phi}$ and g step function on $[0;1]$, we obtain for μ_0-a.e. x:

$$\begin{aligned} Q^x(D_{g^i}\Phi) &= Q^x\Big(\Phi\ \delta(g^i)\Big) - Q^x\Big(\Phi\, b(1, X_1)\Big)\int_0^1 g^i(r)dr \\ &\quad + Q^x\Big(\Phi \int_0^1 g^i(r)\int_r^1 F^i(t, X_t)\,dtdr\Big). \end{aligned} \tag{31}$$

The second formula will be obtained when we have shown that Q^x satisfies the assumptions of Proposition 3.2 on each interval $[0;\tau]$, $\tau < 1$. Let $\tau < 1$ be fixed and $1 \le i \le d$. Let us recall that

$$\begin{aligned} b^{x,i}(\tau, X_\tau) &= \frac{Q^x(X_1^i - X_\tau^i / X_\tau)}{1-\tau} - Q^x\Big(\int_\tau^1 F^i(p, X_p)dp / X_\tau\Big) \\ &\quad + \frac{1}{1-\tau}\int_\tau^1 Q^x\Big(\int_p^1 F^i(q, X_q)dq / X_\tau\Big)dp. \end{aligned}$$

From assumption $(A0)$, which is still true under Q^x, we deduce that

$$\text{for } \mu_0 - \text{a.e.}x, \quad Q^x(|b^{x,i}(\tau, X_\tau)|) < +\infty.$$

From now on we restrict ourselves to the set of x such that this holds. In order to satisfy the assumptions of Proposition 3.2 it is sufficient that for all $i, j \in \{1, \dots, d\}$:

(i) $b^x \in C^{1,2}([0;\tau] \times \mathbb{R}^d; \mathbb{R}^d)$

(ii) the two integrals $\int_0^\tau |F^{x,i} + \frac{1}{2} div\, G^{x,i}|(t, X_t)dt$ and $\int_0^\tau |G^{x,ij}(t, X_t)|^2 dt$ belong to $L^1(Q^x)$

All the necessary arguments have already been developped in detail in the proof of Theorem 4.1, *Steps* 2 to 4. Here the situation is even simpler since there are no terms in G in the expression of b^x. For this reason we do not write down the details but refer the reader to the proof of Theorem 4.1. We conclude that for μ_0-a.e. x, any $\mathcal{F}_\tau$-measurable Φ in $\mathcal{S}$, and any step function g on $[0;\tau]$,

$$\begin{aligned} Q^x(D_{g^i}\Phi) &= Q^x\Big(\Phi\ \delta(g^i)\Big) - Q^x\Big(\Phi\, b^{x,i}(\tau, X_\tau)\Big)\int_0^\tau g^i(r)dr \\ &\quad + Q^x\Big(\Phi \int_0^\tau g^i(r) \int_r^\tau (F^{x,i} + \frac{1}{2} div\, G^{x,i})(t, X_t)\, dtdr\Big) \\ &\quad + Q^x\Big(\Phi \int_0^\tau g^i(r) \sum_{j=1}^d \int_r^\tau G^{x,ij}(t, X_t) dX_t^j dr\Big). \end{aligned} \tag{32}$$

Let us now restrict to step functions $g \in \mathcal{E}_\tau$. Then comparing identities (31) and (32) one obtains :

$$\begin{aligned} & Q^x\Big(\Phi \int_0^\tau g^i(r) \int_r^\tau F^i(t, X_t)\, dtdr\Big) \\ &\quad = Q^x\Big(\Phi \int_0^\tau g^i(r) \int_r^\tau (F^{x,i} + \frac{1}{2} div\, G^{x,i})(t, X_t)\, dtdr\Big) \\ &\quad + Q^x\Big(\Phi \int_0^\tau g^i(r) \sum_{j=1}^d \int_r^\tau G^{x,ij}(t, X_t) dX_t^j dr\Big). \end{aligned}$$

Since the processes $u_t^i(X) = (F^{x,i} + \frac{1}{2} div\, G^{x,i} + G^{x,i}.b^x - F^i)(t, X_t)$ and $v_t^{ij}(X) = G^{x,ij}(t, X_t)$ satisfy the assumptions of Proposition 4.2, we conclude that they are equal to zero $dt\, dQ^x$-a.s. These assumptions are indeed satisfied as a consequence of conditions **(i)** and **(ii)** above and Proposition 2.8 for F. This yields for μ_0-a.e. x:

$$Q^x \text{a.s.}, \ \forall t \in]0; 1[, \ \big(F^x(t, X_t), G^x(t, X_t)\big) = \big(F(t, X_t), 0\big). \tag{33}$$

We conclude as in the proof of Theorem 4.1 *Step* 4 that $G^x \equiv 0$ and $F^x \equiv F$. This implies that Q^x is a gradient diffusion, but this is not sufficient to conclude the same for Q, since we do not yet know that Q is a diffusion.

Step 2: In the present step we prove that b is a gradient, that is there exists a function φ defined on $[0;1]\times\mathbb{R}^d$, differentiable in the space variable, such that for all $i\in\{1,\dots,d\},(t,y)\in]0;1[\times\mathbb{R}^d, b^i(t,y)=\partial_i\varphi(t,y)$. The key tool will again be the identification of two IBPF for Q^x. Let us fix $\tau\in[0;1[$. The assumption of finite entropy for Q and assumption $(\mathcal{H}\,2.2)$ imply that Proposition 3.3 applies to $Q^x|_{\mathcal{F}_\tau}$ and provides the first of the two IBPF we will consider: for any $\Phi\in\mathcal{S}_\tau$ and any step function g on $[0;\tau]$,

$$\begin{aligned} Q^x(D_{g^i}\Phi) &= Q^x\Big(\Phi\ \delta(g^i)\Big) \qquad (34)\\ &\quad -Q^x\Big(\Phi\int_0^\tau g^i(s)\Big(b^{x,i}(s,X_s)+\sum_{j=1}^d\int_s^\tau \partial_i b^{x,j}(p,X_p)dB_p^j\Big)ds\Big), \end{aligned}$$

where B is the Q^x-Brownian motion equal to the martingale part of X under Q^x.

The second IBPF for Q^x is (31). Ito formula for b^i under Q^x yields for any $s<\tau$

$$\begin{aligned} b^i(1,X_1) &= b^i(s,X_s)+\sum_{j=1}^d\int_s^1 \partial_j b^i(p,X_p)dB_p^j \qquad (35)\\ &\quad +\int_s^1\Big(F^i+\frac{1}{2}div\,G^i+G^i.b^x+\sum_{j=1}^d\partial_i b^j(b^{x,j}-b^j)\Big)(p,X_p)dp \end{aligned}$$

Indeed,

$$\begin{aligned} &\partial_t b^i+\sum_{j=1}^d\partial_j b^i b^{x,j}+\frac{1}{2}\Delta b^i\\ &=\partial_t b^i+\sum_{j=1}^d(\partial_j b^i-\partial_i b^j)b^{x,j}+\sum_{j=1}^d\partial_i b^j b^{x,j}+\frac{1}{2}(div\,G^i+\partial_i div\,b)\\ &=F^i+\frac{1}{2}div\,G^i+G^i.b^x+\sum_{j=1}^d\partial_i b^j(b^{x,j}-b^j) \end{aligned}$$

We now plug (35) into (31) and look at the difference of the obtained IBPF with (34): for any $\Phi\in\mathcal{S}_\tau$ and g with support in $[0;\tau]$,

$$\begin{aligned} Q^x\Big(\Phi\int_0^\tau\ g^i(s)\ \Big((b^i-b^{x,i})(s,X_s)\\ +\int_s^1 u_p^i dp+\sum_{j=1}^d\int_s^\tau(\partial_j b^i-\partial_i b^{x,j})dB_p^j\Big)ds\Big)=0 \qquad (36) \end{aligned}$$

where $u_p^i(X)=\Big(\frac{1}{2}div\,G^i+G^i.b^x+\sum_{j=1}^d\partial_i b^j(b^{x,j}-b^j)\Big)(p,X_p)$.This implies

$$(b^i-b^{x,i})(s,X_s)+\int_s^1 Q^x(u_p^i|\mathcal{F}_\tau)dp+\sum_{j=1}^d\int_s^\tau(\partial_j b^i-\partial_i b^{x,j})dB_p^j=0 \qquad (37)$$

for any $s \in]0;1[$, Q^x-a.s. Taking the expectation w.r.t. Q^x and the filtration $\mathcal{F}_s$, yields

$$\forall i \in \{1,\ldots,d\}, \quad (b^i - b^{x,i})(s, X_s) = -\int_s^1 Q^x\left(u_p^i/\mathcal{F}_s\right) dp.$$

We thus conclude that $(b^i - b^{x,i})$ is a bounded variation process. Its martingale part is therefore equal to zero which is equivalent, using Ito formula, to

$$\forall i,j \in \{1,\ldots,d\}, \quad \partial_j b^i(s, X_s) = \partial_j b^{x,i}(s, X_s). \tag{38}$$

Let us fix (i,j). Since b^x is a gradient $\partial_j b^{x,i} = \partial_i b^{x,j}$ and therefore, for all $s \in]0,1[$,

$$\partial_j b^i(s, X_s) = \partial_i b^j(s, X_s)$$

which implies that b is also a gradient. Moreover identity (38) also implies that the function $(b^i - b^{x,i})(t,y)$ is independent of y. Let us denote it by $a^i(t,x)$. Since b is a gradient, $G \equiv 0$ and $u_p^i(X) \equiv \sum_{j=1}^d \partial_i b^j (b^{x,j} - b^j)(p, X_p)$. From (37) we then conclude that $a^i(t,x)$ solves the following integral equation :

$$\forall s \in]0,1[, Q^x a.s., \quad a^i(s,x) = -\int_s^1 Q^x(\sum_{j=1}^d \partial_i b^j(p, X_p)|\mathcal{F}_s) a^j(p,x) dp. \tag{39}$$

Equivalently we have obtained that for μ_0- a.e. x and Q^x-a.e.ω, the vector valued function $a(t,x)$ solves the linear system

$$\frac{d}{dt} a(t,x) = M(t,\omega) a(t,x), \quad (t,x) \in]0;1[\times \mathbb{R}^d,$$

where we have denoted by $M(t,\omega)$ the matrix with entries $\left(\partial_i b^j(t, X_t(\omega))\right)$.
This set of conditions is obviously satisfied when each function a^i is constant equal to zero. We now prove that this is the only possible case. This will be a consequence of the following lemma.

Lemma 5.3 *With the above notations, for any $\tau < 1$ and all $i \in \{1, ..., d\}$,*

$$Q^x\left(b^i(1, X_1) - \int_\tau^1 F^i(t, X_t) dt / \mathcal{F}_\tau\right) = b^{x,i}(\tau, X_\tau)$$

Proof of Lemma 5.3: Let g be a step function on $[0;\tau]$. We do not assume that $\int_0^\tau g(r) dr = 0$. Let $\Phi \in \mathcal{S}_\tau$. Taking into account that $(F^x, G^x) = (F, 0)$ and comparing (31) and (32), for (Φ, g), we obtain the following identity:

$$Q^x\left(\Phi\left(b^{x,i}(\tau, X_\tau) - (b^i(1, X_1) - \int_\tau^1 F^i(t, X_t) dt)\right)\right) \int_0^\tau g^i(r) dr = 0$$

We immediately conclude since this identity holds for any Φ, g. $\square$

Lemma 5.3 implies that $\lim_{\tau\nearrow 1} b^{x,i}(\tau, X_\tau) = b^i(1, X_1)$ in $L^1(Q^x)$. Since $a^i(t,x) = (b^i - b^{x,i})(t, X_t)$ and $t \mapsto b(t, X_t)$ is continuous at $t = 1$, we conclude that $\lim_{\tau\nearrow 1} a(\tau, x) = 0$ and the only solution is $a(t,x) \equiv 0$. We have now proved that

$$\text{for } \mu_0\text{-a.e. } x,\ \forall t \in]0,1[, y \in \mathbb{R}^d, \quad b(t,y) = b^x(t,y).$$

This enables us to conclude that X under Q is a Markov Brownian diffusion solution of $dX_t = b(t, X_t)dt + dW_t$ where W is a Q-Brownian motion. $\square$

Our second application deals with a generalization of a result of Kolmogorov [17]; this famous result states that a Brownian diffusion with drift b, supposed time-homogeneous, is reversible if and only if b is a gradient. Here we require weaker assumptions on the reversible law : we only require that there exists one reversible law in the reciprocal class of P_b. Furthermore, the drift b is not supposed to be time-homogeneous and may depend on time.

Theorem 5.4 *Suppose that there exists a probability measure Q in $\mathcal{P}_{\mathbf{H}}(\Omega)$ in the reciprocal class of P_b which satisfies the integrability condition* (A0). *If Q is reversible, then there exists a function φ such that*

$$\forall t \in]0,1[, x \in \mathbb{R}^d, i \in \{1, \dots, d\}, \quad b^i(t,x) = -\partial_i \varphi(t,x).$$

Furthermore, if Q is a Brownian diffusion with drift b, then b is time-homogeneous and Q is equal - up to a renormalizing factor - to
$\int_{\mathbb{R}^d} P_b(./X_0 = x) \exp(-2\varphi(x))\, dx.$

Example : Let us consider the particular case where the drift function $b_\lambda(x) = -\lambda x$ is the gradient of the potential $\varphi(x) = -\frac{1}{2}\lambda|x|^2$.
In [20] section 5 (cf. also [12] and [19]), we considered the law $Q \in \mathcal{P}(\Omega)$ of the solution of the following s.d.e.

$$dX_t = dB_t - \lambda X_t\, dt, \quad X_0 = X_1.$$

The process Q, called periodic Ornstein-Uhlenbeck process, is reciprocal and we proved in [20] that it belongs to the reciprocal class of the (Markov) Ornstein-Uhlenbeck process P_{b_λ}. Q is a particular Gaussian mixture of periodical bridges of P_{b_λ}. The probability Q is reversible since it is a zero mean Gaussian process with stationary covariance function. So it provides an example of a non Markovian reversible law in the class of the diffusion P_{b_λ}. The above example proves therefore that if b is a gradient there can exist more than one reversible process in the reciprocal class P_b, one beeing a Markovian diffusion with drift b and others which are reciprocal but not Markovian.

Furthermore, the (Markovian) stationary Ornstein-Uhlenbeck process $\overline{P}_{b_\lambda}$, which satisfies the same s.d.e. as above but with initial law on $\mathbb{R}^d$ the centered Gaussian one with variance $\frac{1}{4\lambda}$, is the unique reversible process inside of the set $\mathcal{R}_M(P_{b_\lambda})$

of Markovian reciprocal processes in the class of P_{b_λ}. Indeed, by the definition of $\mathcal{R}_M(P_{b_\lambda})$ given in (13), a Markovian reciprocal process in this set is determined by two measures ν_0 and ν_1. But if it is reversible $\nu_0 = \nu_1$, and then its distribution at time 0 determines it uniquely. Since we have already exhibited one reversible element of $\mathcal{R}_M(P_{b_\lambda})$, i.e. $\overline{P}_{b_\lambda}$, it is the unique one in $\mathcal{R}_M(P_{b_\lambda})$.

Proof of Theorem 5.4 :
By assumption, $Q \in \mathcal{R}(P_b)$ and Theorem 3.4 applies. So IBPF (21) is satisfied under Q. Since Q has a finite entropy, it is a Brownian semi-martingale and, as indicated in Remark 3.5, IBPF (21) can be rewritten as follows
$\forall \Phi \in \mathcal{S}$, for all $i \in \{1, \ldots, d\}$ and $g \in \mathcal{E}$,

$$\begin{aligned} Q(D_{g^i}\Phi) &= Q\Big(\Phi\ \delta(g^i)\Big) +\ Q\Big(\Phi \int_0^1 g^i(r) \int_r^1 F^i(t, X_t)\, dt\, dr\Big) \\ &\quad + Q\Big(\Phi \int_0^1 g^i(r) \int_r^1 G^i(t, X_t) \circ dX_t\, dr\Big). \end{aligned} \tag{40}$$

Let us denote by R the time reversal map on Ω:

$$R(X)_t = X_{1-t},\ t \in [0, 1],$$

and by $\hat{Q}$ the image of Q by R :

$$\hat{Q} = Q \circ R^{-1}.$$

Remarking that, for all $\Phi \in \mathcal{S}$ and $g \in \mathcal{E}$, $(D_g\Phi) \circ R \equiv -D_{\hat{g}}(\Phi \circ R)$ where $\hat{g} = g \circ R$, one obtains from (40):

$$\begin{aligned} \hat{Q}(D_{g^i}\Phi) &= -Q\big(D_{\hat{g}^i}(\Phi \circ R)\big) \\ &= -Q\Big((\Phi \circ R)\, \delta(\hat{g}^i)\Big) \\ &\quad - Q\Big((\Phi \circ R) \int_0^1 \hat{g}^i(r) \int_r^1 F^i(t, X_t)\, dt\, dr\Big) \\ &\quad - Q\Big((\Phi \circ R) \int_0^1 \hat{g}^i(r) \int_r^1 G^i(t, X_t) \circ dX_t\, dr\Big) \\ &= \hat{Q}\Big(\Phi\ \delta(g^i)\Big) + \hat{Q}\Big(\Phi \int_0^1 g^i(r) \int_r^1 F^i(1-t, X_t)\, dt\, dr\Big) \\ &\quad - \hat{Q}\Big(\Phi \int_0^1 g^i(r) \int_r^1 G^i(1-t, X_t) \circ dX_t\, dr\Big). \end{aligned} \tag{41}$$

Now recall that Q is supposed to be reversible, that is $\hat{Q} = Q$, which implies that Q also satisfies equation (41). So, under Q, both equalities (40) and (41) hold, which implies : $\forall \Phi \in \mathcal{S}$, for all $i \in \{1, \ldots, d\}$ and $g \in \mathcal{E}$,

$$Q\Big(\Phi \int_0^1 g^i(r) \Big(\int_r^1 F^i(t, X_t) dt + \int_r^1 G^i(t, X_t) \circ dX_t\Big) dr\Big)$$

$$= \hat{Q}\Big(\Phi \int_0^1 g^i(r)\Big(\int_r^1 F^i(1-t,X_t)dt - \int_r^1 G^i(1-t,X_t)\circ dX_t\Big)dr\Big)$$

By Proposition 4.2, this implies that :

$$\forall t \in]0,1[,\ F(t,.) = F(1-t,.) \text{ and } G(t,.) = -G(1-t,.). \tag{42}$$

In fact, the same identities remain true for any $\tau \in [0,1]$ instead of 1, since we could do the same proof as above reversing the time at the time τ instead of 1. We thus obtain :

$$\forall \tau \in]0,1], \forall t \in]0,\tau[, F(t,.) = F(\tau-t,.) \text{ and } G(t,.) = -G(\tau-t,.).$$

This means that the characteristics F is independent of time and that the characteristics G is equal to 0. This last sentence is equivalent to the fact that the function b is a gradient (not necessarily independent of time) : $b(t,x) = -\nabla\varphi(t,x)$.

Moreover, if Q is a Brownian diffusion with drift b (with finite entropy), its time reversal is a Brownian diffusion with drift $\hat{b}$ (cf. [8]). The reversibility assumption thus implies that $b = \hat{b}$ and does not depend on time. Now, it is well known that the measure with density $\exp(-2\varphi(x))$ with respect to Lebesgue measure, taken as initial law, makes the Brownian diffusion with drift $b = -\nabla\varphi$ reversible. It is furthermore the unique one, up to a multiplicative constant. The conclusion follows. □

Remark 5.5 1. The identities

$$\forall t \in]0,1[, \hat{F}(t,.) = F(1-t,.) \text{ and } \hat{G}(t,.) = -G(1-t,.). \tag{43}$$

were proved by one of us in [25], Proposition 4.5, using the explicit expression of $\hat{F}$ and $\hat{G}$ as functionals of the reversed drift. They reflect the symmetry of the reciprocal characteristics under time reversal. In the Markovian case the drift does not feature such symmetry (cf. [9]).

2. In the general case, if Q is a probability measure in $\mathcal{R}(P_b)\bigcap\mathcal{P}_{\mathbf{H}}(\Omega)$, not necessarily reversible, whose time reversal $\hat{Q}$ is regular enough to define the "reversed reciprocal characteristics" $\hat{F}$ and $\hat{G}$, we could also derive identities (43). As in the proof of Theorem 5.4, the argument would rely on the identification of two IBPF satisfied by $\hat{Q}$.

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